2-UNITARY OPERADS OF GK-DIMENSION 3

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ABSTRACT. We study and classify the 2-unitary operads of Gelfand-Kirillov dimension three.

0. INTRODUCTION

Algebraic operads originated from homotopy theory in algebraic topology, and was first introduced by Boardman-Vogt [BV] and May [Ma] in 1960s-1970s. During the recent 20 years, operad theory has become an important tool in homological algebra, category theory, algebraic geometry and mathematical physics. It is well-known that every operad encodes an algebra system. For example, Ass encodes all unital associative algebras. Further, a k-linear operad itself is an algebraic object similar to an associative algebra, and algebraic structures of operads have been widely investigated by many mathematicians, see [BYZ, Dot, DK, DMR, DT, Fr1, Fr2, KP, LV, MSS, QXZZ].

The Gelfand-Kirillov dimension of an associative algebra is a useful numerical invariant in ring theory and noncommutative algebraic geometry, see [KL]. In a similar way, the Gelfand-Kirillov dimension can be defined for other algebraic objects including algebraic operads [BYZ, Fi]. Let \Bbbk is a base field. An operad \mathcal{P} is said to be *locally finite* if each $\mathcal{P}(n)$ is finite dimensional over \Bbbk . In this paper we only consider locally finite operads. The *Gelfand-Kirillov dimension* (or *GK-dimension* for short) of a locally finite operad \mathcal{P} is defined to be

GKdim
$$\mathcal{P}$$
: = lim sup log_n $\left(\sum_{i=0}^{n} \dim_{\mathbb{K}} \mathcal{P}(i) \right)$.

We refer to [BYZ, KP, QXZZ] for more information related to the GK-dimension of an operad.

Recall that an operad \mathcal{P} is unitary if $\mathcal{P}(0) = \mathbb{k}\mathbb{1}_0$ with a basis element $\mathbb{1}_0$ (which is called a 0-unit), see [Fr2, Section 2.2]. Denote by Op_+ the category of unitary operads, in which a morphism preserves the 0-unit. A 2-unitary operad \mathcal{P} is a unitary operad \mathcal{P} equipped with a morphism $\mathcal{M}ag \to \mathcal{P}$ in Op_+ , where $\mathcal{M}ag$ is the unital magmatic operad (see [BYZ, Section 8.4] or [Lo, Section 4.1.10]). The definition of a 2*a*-unitary operad is given in Definition 1.5(4). In [BYZ], the authors proved that GK-dimension of a 2-unitary operad \mathcal{P} is either an nonnegative integer or infinity and that the generating series of \mathcal{P} is a rational function when GKdim $\mathcal{P} < \infty$. The pattern of GK-dimension of a general non-2-unitary operad (or nonsymmetric operad) is very different, see Remark 5.5.

The only 2-unitary operad of GK-dimension 1 is *Com* that encodes all unital commutative algebras. All locally finite 2-unitary operads of GK-dimension 2 were classified in [BYZ, Theorem 0.6]. One way of viewing this classification is the following. We refer to [BYZ, Section 6] for the construction of 2-unitary operads of GK-dimension 2.

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Theorem 0.1. [BYZ, Theorem 0.6] There are natural equivalences between

- (1) the category of finite dimensional, not necessarily unital, k-algebras,
- (2) the category of 2-unitary operads of GK-dimension ≤ 2 ,
- (3) the category of 2a-unitary operads of GK-dimension ≤ 2 .

In the case of GK-dimension 3, our result is not as clean as Theorem 0.2. Nevertheless, we will provide a classification. Recall that an operad is called *Com-augmented* if there is an operadic unit map $u_{\mathcal{P}} : Com \to \mathcal{P}$. The morphisms in the category of *Com*-augmented operads are supposed to be compatible with operadic unit maps. Here is the main result of this paper.

Theorem 0.2. There is a natural equivalence between

- (1) the category of finite dimensional trident algebras,
- (2) the category of Com-augmented operads of GK-dimension 3.

If char $\mathbb{k} \neq 2$, we also prove that every 2-unitary operad of GK-dimension 3 is equipped with a *Com*-augmentation with possibly new 2-unit [Proposition 2.5]. Combining Theorem 0.2 with Proposition 2.5, we obtain a classification of 2-unitary operads of GK-dimension 3 in terms of finite dimensional trident algebras. However, Theorem 0.2 fails if the condition "*Com*-augmented" in part (2) is replaced by "2-unitary" [Example 5.4].

The definition of a trident algebra is given in Section 3. Roughly speaking, a trident algebra consists of a pair of k-vector spaces (A, M) equipped with some algebraic structures and a pair of k-linear maps (f, g) satisfying some identities. Note that a seemingly technical result, Theorem 0.2, makes some questions easy to solve. For example, we prove

Corollary 0.3 (Proposition 5.3). There is no Com-augmented Hopf operad of GK-dimension 2.

Corollary 0.3 motivates the following question.

Question 0.4. Is there a Com-augmented Hopf operad of finite GK-dimension large than 2.

This paper is organized as follows. In Section 1 we recall some basic definitions and properties of 2-unitary operads. We prove some properties of 2-unitary operads of GK-dimension 3 in Section 2. A key preliminary result is that a 2-unitary operad of GK-dimension 3 is *Com*-augmented after changing the 2-unit [Proposition 2.5]. We define a concept of a trident algebra in Section 3. In Section 4, we construct an operad from a trident algebra and complete the classification of 2-unitary operads of GK-dimension 3 by Theorem 0.2 and Proposition 2.5. In Section 5, we give some comments, examples, and remarks. A complete but tedious proof of Theorem 4.1 is given in Appendix (Section 6).

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1. Preliminaries

Throughout let k be a base field, and every object is over k. Let n be a nonnegative integer. Set $[n] = \{1, 2, \dots, n\}$ for n > 0 and $[0] = \emptyset$. We use \mathbb{S}_n to denote the symmetric group for $n \ge 0$. By convention, \mathbb{S}_0 is the trivial group. Following the notation introduced in [BYZ, Section 8.1], for each $\sigma \in \mathbb{S}_n$, we use the sequence (i_1, i_2, \dots, i_n) to denote a permutation $\sigma \in \mathbb{S}_n$ with $\sigma(i_k) = k$ for all $1 \le k \le n$. Denote by \mathbb{S} the disjoint union of all symmetric group \mathbb{S}_n for all $n \ge 0$. Recall that a kS-module (or S-module) means a sequence $\{\mathcal{P}(n)\}_{n\ge 0}$ of right kS_n-modules, where the right S_n-action on $\mathcal{P}(n)$ is denoted by *.

In this section, we retrospect some basic facts about operads.

1.1. **Definitions.** From different viewpoints, there are various definitions about operads. In this paper, we mainly use the *partial* definition and refer to [LV, Chapter 5] for other versions of the definition.

Definition 1.1. [Fr2, Section 2.1] An operad \mathcal{P} consists of the following data:

- (i) a kS-module $\{\mathcal{P}(n)\}_{n\geq 0}$, where an element in $\mathcal{P}(n)$ is called an *n*-ary operation.
- (ii) an element $\mathbb{1} \in \mathcal{P}(1)$, which is called the *identity*,
- (iii) for all $m \ge 1, n \ge 0$ and $1 \le i \le m$, a partial composition:

$$-\circ$$
 $-: \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$

satisfying the following coherence axioms:

(OP1) (Identity) for all $\theta \in \mathcal{P}(M)$ and all $1 \leq i \leq m$,

$$\theta \mathop{\circ}_{i} \mathbbm{1} = \theta = \mathbbm{1} \mathop{\circ}_{1} \theta;$$

(OP2) (Associativity) for all $\lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$,

(E1.1.1)
$$(\lambda \underset{i}{\circ} \mu) \underset{i-1+j}{\circ} \nu = \lambda \underset{i}{\circ} (\mu \underset{j}{\circ} \nu), \quad 1 \le i \le l, 1 \le j \le m,$$

(E1.1.2)
$$(\lambda \circ \mu) \circ _{k-1+m} \nu = (\lambda \circ \nu) \circ _{k} \mu, \quad 1 \le i < k \le l;$$

(OP3) (Equivariance) for all $\mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n)$ and $\sigma \in \mathbb{S}_n, \phi \in \mathbb{S}_m$,

(E1.1.3)
$$\mu \underset{i}{\circ} (\nu * \sigma) = (\mu \underset{i}{\circ} \nu) * \sigma',$$

(E1.1.4)
$$(\mu * \phi) \circ \nu = (\mu \circ \mu) * \phi'',$$

where $\sigma' = 1_m \underset{i}{\circ} \sigma$ and $\phi'' = \phi \underset{i}{\circ} 1_n$ are given by the partial composition in the associative algebra operad $\mathcal{A}ss$. We refer to [BYZ, Section 8] for more details concerning σ' and ϕ'' .

Let \mathcal{P} be an operad in the sense of Definition 1.1. Then one can define the composition map by

(E1.1.5)
$$\lambda \circ (\mu_1, \cdots, \mu_n) = (\cdots ((\lambda \circ \mu_n) \circ \mu_{n-1}) \circ \mu_{n-2} \mu_{n-2} \cdots) \circ \mu_1$$

for all $\lambda \in \mathcal{P}(n)$ and $\mu_i \in \mathcal{P}$ and for $1 \leq i \leq n$ [BYZ, Remark 1.3].

Example 1.2. [LV, Section 5.2.10] Let *Com* denote the commutative algebra operad. The space of *n*-ary operations of *Com* is $Com(n) = \mathbb{k}\mathbb{1}_n$ equipped with the trivial action of the symmetric group and the partial composition is given by $\mathbb{1}_m \circ_i \mathbb{1}_n = \mathbb{1}_{m+n-1}$ for all m, n, i. Note that $\mathbb{1}_1$ is the identity $\mathbb{1}$ of *Com*.

Example 1.3. [LV, Section 13.8] Suppose that $\mathcal{M}ag$ is the operad generated by the S-module

 $(\Bbbk\mu, \&1, \&S_2\nu, 0, 0, \cdots)$

and subject to relations

$$\nu \circ \mu = 1, (i = 1, 2),$$

where $\mathbb{k}S_2\nu$ is the regular $\mathbb{k}S_2$ -module with the basis ν . In this paper we use $\mathbb{1}_0$ for μ and $\mathbb{1}_2$ for ν .

Definition 1.4. [LV, Chapter 5] Let \mathcal{P} and \mathcal{P}' be operads. A morphism from \mathcal{P} to \mathcal{P}' is a sequence of \mathbb{kS}_n -homomorphism $\gamma = (\gamma_n : \mathcal{P}(n) \to \mathcal{P}'(n))_{n \ge 0}$, satisfying

$$\gamma(\mathbb{1}) = \mathbb{1}' \text{ and } \gamma(\mu \mathop{\circ}_{i} \nu) = \gamma(\mu) \mathop{\circ}_{i} \gamma(\nu),$$

where $\mathbb{1}$ and $\mathbb{1}'$ are identities of \mathcal{P} and \mathcal{P}' , respectively, and $\mu, \nu \in \mathcal{P}$. Let Op denote the category of operads.

Next we collect some definitions given in [BYZ, Fr2].

- **Definition 1.5.** (1) An operad \mathcal{P} is called *unitary* if $\mathcal{P}(0) = \mathbb{k}\mathbb{1}_0$, where $\mathbb{1}_0$ is a basis of $\mathcal{P}(0)$, and is called a 0-*unit*. The category of unitary operads is denoted by Op_+ , in which morphisms are operadic morphisms preserving 0-units.
 - (2) A unitary operad is said to be *connected*, if $\mathcal{P}(1) = \mathbb{k}\mathbb{1}$ where $\mathbb{1}$ is the identity of \mathcal{P} . In this case we also use $\mathbb{1}_1$ for $\mathbb{1}$.
 - (3) A 2-unitary operad is a unitary operad \mathcal{P} equipped with a morphism $\mathcal{M}ag \to \mathcal{P}$ in Op_+ , where $\mathcal{M}ag$ is the unital magmatic operad [Example 1.3], or equivalently, there is an element $\mathbb{1}_2 \in \mathcal{P}(2)$ (called a 2-unit) such that

(E1.5.1)
$$\mathbb{1}_{2} \circ \mathbb{1}_{0} = \mathbb{1}(=\mathbb{1}_{1}) = \mathbb{1}_{2} \circ \mathbb{1}_{0}$$

where $\mathbb{1}_0$ is a 0-unit of \mathcal{P} .

(4) A 2*a*-unitary operad is a unitary \mathcal{P} equipped with a morphism $\mathcal{A}ss \to \mathcal{P}$ in Op₊, or equivalently, \mathcal{P} is 2-unitary with a 2-unit $\mathbb{1}_2$ satisfying

$$(E1.5.2) \qquad \qquad \mathbb{1}_2 \circ \mathbb{1}_2 = \mathbb{1}_2 \circ \mathbb{1}_2.$$

In this case $\mathbb{1}_2$ is called 2*a*-unit.

(5) An operad \mathcal{P} is called *Com-augmented* if there is a morphism from $\mathcal{C}om \to \mathcal{P}$. It is clear that $\mathcal{C}om$ -augmented operads are 2*a*-unitary. In this case $\mathbb{1}_2 * (2, 1) = \mathbb{1}_2$ and $\mathbb{1}_2$ is called a *symmetric* 2*a*-unit.

Let \mathcal{P} and \mathcal{P}' be 2-unitary operads. A morphism of 2-unitary operads is a morphism $\gamma : \mathcal{P} \to \mathcal{P}'$ in Op₊ satisfying the following commutative diagram



or equivalently, the operad morphism preserves the 2-unit. The categories of 2-unitary operads, 2aunitary operads and *Com*-augmented operads, are denoted by $\mathcal{M}ag \downarrow \mathrm{Op}_+$, $\mathcal{A}ss \downarrow \mathrm{Op}_+$ and $\mathcal{C}om \downarrow \mathrm{Op}_+$, respectively. Let dim denote dim_k. **Definition 1.6.** Let $\mathcal{P} = (\mathcal{P}(n))_{n \geq 0}$ be a locally finite operad, i.e. dim $\mathcal{P}(n) < \infty$ for all $n \geq 0$.

(1) The generating series of \mathcal{P} is defined to be

$$G_{\mathcal{P}}(t) = \sum_{n=0}^{\infty} \dim \mathcal{P}(n) t^n \in \mathbb{Z}[[t]].$$

(2) The Gelfand-Kirillov dimension (GK-dimension for short) of \mathcal{P} is defined to be

$$\operatorname{GKdim}(\mathcal{P}) = \limsup_{n \to \infty} \log_n (\sum_{i=0}^n \dim \mathcal{P}(i)).$$

1.2. Truncation Ideals. Let \mathcal{P} be a unitary operad and I a subset of [n]. Recall that a *restriction* operator [Fr2, Section 2.2.1] means

(E1.6.1)
$$\pi^{I}: \mathcal{P}(n) \to \mathcal{P}(s), \qquad \pi^{I}(\theta) = \theta \circ (\mathbb{1}_{\chi_{I}(1)}, \cdots, \mathbb{1}_{\chi_{I}(n)})$$

for all $\theta \in \mathcal{P}(n)$, where χ_I is the characteristic function of I, i.e. $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ otherwise. Note that \circ is defined in (E1.1.5). If $I = \{i_1, \dots, i_s\} \subset [n]$ with $1 \leq i_1 < \dots < i_s \leq n$, we also denote π^I as π^{i_1,\dots,i_s} . We refer to [BYZ, Section 2.3] for more details.

For integers $k \geq 1$, the k-th truncation ideals ${}^k \Upsilon$ is defined by

(E1.6.2)
$${}^{k} \Upsilon_{\mathcal{P}}(n) = \bigcap_{I \subset [n], |I| \le k-1} \operatorname{Ker} \pi^{I} = \begin{cases} \bigcap_{I \subset [n], |I| = k-1} \operatorname{Ker} \pi^{I}, & \text{if } n \ge k; \\ I \subset [n], |I| = k-1 \end{cases}$$
 otherwise

By convention, let ${}^{0}\Upsilon_{\mathcal{P}} = \mathcal{P}$. If no confusion, we write ${}^{k}\Upsilon = {}^{k}\Upsilon_{\mathcal{P}}$ for brevity.

For every subset $I = \{i_1, \dots, i_s\} \subset [n]$ with $i_1 < \dots < i_s$, we denote a permutation

$$c_I: = (i_1, \cdots, i_s, 1, \cdots, i_1 - 1, i_1 + 1, \cdots, i_s - 1, i_s + 1, \cdots, n) \in \mathbb{S}_n$$

Theorem 1.7. [BYZ, Theorem 4.6] Let \mathcal{P} be a 2-unitary operad. For each $k \geq 0$, let

$$\Theta^k \colon = \{\theta_1^k, \cdots, \theta_{z_k}^k\}$$

be a k-basis for ${}^{k}\Upsilon(k)$. Let $B_{k}(n)$ be the set

$$\{\mathbb{1}_2 \circ (\theta_i^k, \mathbb{1}_{n-k}) * c_I \mid 1 \le i \le z_k, I \subset [n], |I| = k.\}$$

Then $\mathcal{P}(n)$ has a k-basis

$$\bigcup_{0 \leq k \leq n} \boldsymbol{B}_k(n) = \{\mathbbm{1}_n\} \cup \bigcup_{1 \leq k \leq n} \boldsymbol{B}_k(n),$$

and for every $k \geq 1$, ${}^k \Upsilon(n)$ has a k-basis $\bigcup_{k < i < n} B_i(n)$.

Lemma 1.8. [BYZ, Lemma 5.2] Let \mathcal{P} be a 2-unitary operad and $f_{\mathcal{P}}(k) = \dim^k \Upsilon(k)$ for each $k \ge 0$. Then

(1)
$$G_{\mathcal{P}}(t) = \sum_{k=0}^{\infty} f_{\mathcal{P}}(k) \frac{t^k}{(1-t)^{k+1}}.$$

(2) $\operatorname{GKdim} \mathcal{P} = \max\{k \mid f_{\mathcal{P}}(k) \neq 0\} + 1 = \min\{k \mid {}^k \mathcal{Y} = 0\}.$

Combining Lemma 1.8(2) with [BYZ, Proposition 0.5], if \mathcal{P} is 2-unitary, then there is a canonical morphism of 2-unitary operads

(E1.8.1)
$$\epsilon: \mathcal{P} \longrightarrow \mathcal{P}/{}^{1}\mathcal{Y} = \mathcal{C}om.$$

2. BASIC FACTS OF 2-UNITARY OPERADS OF GK-DIMENSION 3

Let \mathcal{P} be a 2-unitary operad of GK-dimension 3. By Theorem 1.7 and Lemma 1.8, we have the following basic facts:

(1)
$${}^{2}\Upsilon \neq 0$$
, and ${}^{k}\Upsilon = 0$ for all $k \geq 3$.

(2) dim
$$\mathcal{P}(n) = 1 + dn + m \frac{n(n-1)}{2}$$
, where $d := f_{\mathcal{P}}(1) = \dim^{1} \Upsilon(1)$ and $m := f_{\mathcal{P}}(2) = \dim^{2} \Upsilon(2)$.

(3)
$$G_{\mathcal{P}}(t) = \frac{1}{1-t} + d\frac{t}{(1-t)^2} + m\frac{t}{(1-t)^3}$$

Based on the above facts, we have the following lemmas, which is useful to understand the structure of a 2-unitary operad of GK-dimension 3.

Lemma 2.1. Let \mathcal{P} be a 2-unitary operad of GK-dimension 3 and $\mathbb{1}_2$ a 2-unit. Then for all $\tau, \mu \in {}^2\Upsilon(2)$,

(E2.1.1)
$$\tau \circ \mathbb{1}_{2} = \mathbb{1}_{2} \circ \tau + (\mathbb{1}_{2} \circ \tau) * (12)$$

(E2.1.2)
$$\tau \circ \mathbb{1}_2 = \mathbb{1}_2 \circ \tau + (\mathbb{1}_2 \circ \tau) * (23).$$

(E2.1.3) $\tau \circ \mu = 0 \quad (i = 1, 2).$

Proof. By a direct calculation, we have

$$\left(\tau \circ \mathbb{1}_{2} - \mathbb{1}_{2} \circ \tau - (\mathbb{1}_{2} \circ \tau) * (12)\right) \circ \mathbb{1}_{0} = 0$$

for all $1 \leq i \leq 3$. It follows that

$$\tau \circ \mathbb{1}_{2} - \mathbb{1}_{2} \circ \tau - (\mathbb{1}_{2} \circ \tau) * (12) \in {}^{3}\Upsilon(3).$$

Since \mathcal{P} is of GK-dimension 3 and ${}^{3}\Upsilon(3) = 0$, Equation (E2.1.1) holds. Similarly, (E2.1.2) and (E2.1.3) hold.

Let \mathcal{P} be a 2-unitary operad with a 2-unit $\mathbb{1}_2$. By convention, we define $\mathbb{1}'_n = \mathbb{1}_n$ for n = 0, 1, 2. Recall from [BYZ, Section 2] that, for every $n \geq 3$, one can define inductively that

$$\mathbb{1}_n = \mathbb{1}_2 \circ \mathbb{1}_{n-1}, \text{ and } \mathbb{1}'_n = \mathbb{1}_2 \circ \mathbb{1}'_{n-1}.$$

By Definition 1.5(4) a 2-unitary operad is called 2a-unitary if $\mathbb{1}_2$ is associative, or equivalently $\mathbb{1}_3 = \mathbb{1}'_3$.

Lemma 2.2. Let \mathcal{P} be a 2-unitary operad of GK-dimension 3 with 2-unit $\mathbb{1}_2$. Then $\mathbb{1}_2$ is a 2a-unit. Moreover, if $\mathbb{1}_2$ is a 2a-unit, then so is $\mathbb{1}_2 + \tau$ for any $\tau \in {}^2\Upsilon(2)$.

Proof. Suppose that $\mathbb{1}_2$ is a 2-unit of \mathcal{P} . By definition, $\mathbb{1}_3 = \mathbb{1}_{2_1^\circ}\mathbb{1}_2$ and $\mathbb{1}'_3 = \mathbb{1}_{2_2^\circ}\mathbb{1}_2$. One can easily check that $(\mathbb{1}_3 - \mathbb{1}'_3) \underset{i}{\circ} \mathbb{1}_0 = 0$ for all i = 1, 2, 3. This means that $\mathbb{1}_3 - \mathbb{1}'_3 \in {}^3\Upsilon(3)$. Since \mathcal{P} is of GK-dimension $3, {}^3\Upsilon(3) = 0$. Thus $\mathbb{1}_3 = \mathbb{1}'_3$ as required.

Clearly, for any $\tau \in {}^{2}\Upsilon(2)$, we have

$$(\mathbb{1}_2 + \mu) \underset{i}{\circ} \mathbb{1}_0 = \mathbb{1}_2 \underset{i}{\circ} \mathbb{1}_0 = \mathbb{1}_1$$

for i = 1, 2. So $\mathbb{1}_2 + \mu$ is a 2-unit. Moreover, by Lemma 2.1 (E2.1.3), we have

$$(\mathbb{1}_2 + \tau) \circ (\mathbb{1}_2 + \tau) = \mathbb{1}_2 \circ \mathbb{1}_2 + \tau \circ \mathbb{1}_2 + \mathbb{1}_2 \circ \tau$$

for i = 1, 2. Since $\mathbb{1}_2$ is a 2a-unit, $\mathbb{1}_2 \circ \mathbb{1}_2 = \mathbb{1}_2 \circ \mathbb{1}_2$. By direct computation, we have

$$\left(\mathbb{1}_{2} \circ \tau + \tau \circ \mathbb{1}_{2} - \mathbb{1}_{2} \circ \tau - \tau \circ \mathbb{1}_{2}\right) \circ \mathbb{1}_{0} = \tau - \tau = 0$$

for i = 1, 2, 3. Therefore, $(\mathbb{1}_2 \circ \tau + \tau \circ \mathbb{1}_2 - \mathbb{1}_2 \circ \tau - \tau \circ \mathbb{1}_2) \in {}^3\Upsilon(3)$. Since \mathcal{P} is of GK-dimension 3 and ${}^3\Upsilon(3) = 0$, we know $\mathbb{1}_2 \circ \tau + \tau \circ \mathbb{1}_2 = \mathbb{1}_2 \circ \tau + \tau \circ \mathbb{1}_2$. It follows that $\mathbb{1}_2 + \tau$ is a 2a-unit.

Lemma 2.3. [BYZ, Lemma 2.7] Let \mathcal{P} be a 2*a*-unitary operad. Then the following hold.

- (1) For every $n \ge 3$, $\mathbb{1}_n = \mathbb{1}'_n$.
- (2) For every $n \ge 1$ and $k_1, \dots, k_n \ge 0$, $\mathbb{1}_n \circ (\mathbb{1}_{k_1}, \dots, \mathbb{1}_{k_n}) = \mathbb{1}_{k_1 + \dots + k_n}$.

Lemma 2.4. Let \mathcal{P} be a 2-unitary operad of GK-dimension 3. Suppose that ${}^{1}\Upsilon(1)$ has a k-basis $\{\delta_{j} \mid j \in [d]\}$ and ${}^{2}\Upsilon(2)$ has a k-basis $\{\tau_{s} \mid s \in [m]\}$. Then for every $n \geq 3$, $\mathcal{P}(n)$ has a k-basis:

(E2.4.1)
$$\{\mathbb{1}_n\} \cup \{\delta_{(i),j}^n \mid i \in [n], j \in [d]\} \cup \{\tau_{(i_1i_2),s}^n \mid 1 \le i_1 < i_2 \le n, s \in [m]\},$$

where $\delta_{(i),j}^n = (\mathbb{1}_n \circ \delta_j) * c_i, \ \tau_{(i_1i_2),s}^n = (\mathbb{1}_{n-1} \circ \tau_s) * c_{i_1i_2}, \ and \ c_i = (i, 1, \cdots, i-1, \hat{i}, i+1, \cdots, n) \ and \ c_{i_1i_2} = (i_1, i_2, 1, \cdots, i_1 - 1, \hat{i}_1, i_1 + 1, \cdots, i_2 - 1, \hat{i}_2, i_2 + 1, \cdots, n).$

Proof. Since \mathcal{P} is a 2-unitary operad of GK-dimension 3, we have ${}^{k}\Upsilon(k) = 0$ for all $k \geq 3$. By Theorem 1.7, we can choose a basis of $\mathcal{P}(n)$ as follows

$$\{\mathbb{1}_n\} \cup \{\mathbb{1}_2 \circ (\delta_j, \mathbb{1}_{n-1}) * c_i \mid i \in [n], j \in [d]\} \cup \{\mathbb{1}_2 \circ (\tau_s, \mathbb{1}_{n-2}) * c_{i_1 i_2} \mid 1 \le i_1 < i_2 \le n, s \in [m]\}.$$

By Lemmas 2.2 and 2.3

$$\mathbb{1}_{n \ 0} \delta_{j} = \mathbb{1}'_{n \ 1} \delta_{j} = (\mathbb{1}_{2 \ 0} \mathbb{1}_{n-1}) \circ_{1} \delta_{j} = (\mathbb{1}_{2 \ 0} \delta_{j}) \circ_{2} \mathbb{1}_{n-1} = \mathbb{1}_{2} \circ (\delta_{j}, \mathbb{1}_{n-1}),$$

$$\mathbb{1}_{n-1} \circ_{1} \tau_{s} = \mathbb{1}'_{n-1} \circ_{1} \tau_{s} = (\mathbb{1}_{2 \ 0} \mathbb{1}_{n-2}) \circ_{1} \tau_{s} = (\mathbb{1}_{2 \ 0} \tau_{s}) \circ_{3} \mathbb{1}_{n-2} = \mathbb{1}_{2} \circ (\tau_{s}, \mathbb{1}_{n-2}),$$

we immediately obtain basis (E2.4.1) of $\mathcal{P}(n)$.

Proposition 2.5. Suppose char $\Bbbk \neq 2$. Let \mathcal{P} be a 2-unitary operad of GK-dimension 3 with a 2-unit $\mathbb{1}_2$. Let

$$\mathbb{1}'_2 := \frac{1}{2}(\mathbb{1}_2 + \mathbb{1}_2 * (12)).$$

Then $1_2'$ is also a 2a-unit. Consequently, \mathcal{P} is Com-augmented.

Proof. It is easy to check that $\mathbb{1}'_2$ is a 2-unit. By Lemma 2.2, $\mathbb{1}'_2$ is a 2*a*-unit, namely, $(\mathcal{P}, \mathbb{1}_0, \mathbb{1}, \mathbb{1}'_2)$ is a 2*a*-unitary operad.

After replacing $\mathbb{1}_2$ by $\mathbb{1}'_2$ we may assume that $\mathbb{1}_2 * (12) = \mathbb{1}_2$. It follows from induction and Lemma 2.3(1) that $\mathbb{1}_n * \sigma = \mathbb{1}_n$ for all $\sigma \in \mathbb{S}_n$. Therefore there is a canonical morphism from $\mathcal{C}om$ [Example 1.2] to \mathcal{P} sending $\mathbb{1}_n \mapsto \mathbb{1}_n$ for all $n \ge 0$.

3. TRIDENT ALGEBRAS

Let R be a untial associative algebra over k with a right action of an abelian group G satisfying (ab).g = (a.g)(b.g) and for all $a, b \in R$ and $g \in G$. Such an R is called a k*G*-module algebra. Recall that the skew group algebra R#G is the vector space $R \otimes kG$ with the multiplication

$$(a \# g)(b \# h) = ((a.h)b) \# (gh)).$$

Furthermore, a right module M over R # G means that M is a right R-module and a right & G-module satisfying $(\mu a)g = (\mu g)(a.g)$ for all $\mu \in M, g \in G$ and $a \in R$. The following lemma is easy.

Lemma 3.1. Let G be an abelian group. Let A and B be G-module algebras. Suppose M, N are right modules over the skew groups algebras A#G and B#G, respectively. Then $M \otimes N$ is a right $(A \otimes B)\#G$ -module with the action given by

$$(x \otimes y)(a \otimes b \# g) \colon = (x.(a \# g)) \otimes (y.(b \# g))$$

for $x \in M, y \in N, a \in A, b \in B, g \in G$.

Let A be a unital associative algebra. Clearly, the tensor product algebra $A \otimes A$ (also denoted by $A^{\otimes 2}$) admits a natural right \mathbb{S}_2 -action given by $(a \otimes b)(12)$: $= b \otimes a$. So we obtain a skew group algebra $(A \otimes A) \# \mathbb{S}_2$ (also denoted by $A^{\otimes 2} \# \mathbb{S}_2$). Let M be a left A- right $A^{\otimes 2} \# \mathbb{S}_2$ -bimodule. Equivalently, M is both a right \mathbb{K}_2 -module and a left A- right $(A \otimes A)$ -bimodule satisfying

- (E3.1.1) $a(\mu(12)) = (a\mu)(12),$
- (E3.1.2) $(\mu .(a \otimes b))(12) = (\mu (12)).(b \otimes a),$

for all $a, b \in A, \mu \in M$.

Remark 3.2. Observe that if M is both a right $\Bbbk S_2$ -module and a right A-module with the action $M \otimes A \to M$, $(\mu, a) \mapsto \mu : a$, then M admits another right A-module action given by

$$\mu : a = ((\mu . (12)) : a).(12),$$

which is called the *congruence action*. Therefore, a right $A^{\otimes 2} \# \mathbb{S}_2$ -module action on M is equivalent to a right \mathbb{kS}_2 -action together with a right A-module action satisfying

(E3.2.1) $(\mu \cdot a) \cdot b = (\mu \cdot b) \cdot a$

for all $\mu \in M, a \in A$.

3.1. Tridents. In this subsection we introduce a new algebraic system.

Let $A = \mathbb{k}\mathbb{1}_1 \oplus \overline{A}$ be an augmented algebra with augmentation ideal \overline{A} . Obviously, the right regular $A^{\otimes 2}$ -module $A \otimes A$ with an action of \mathbb{S}_2 given by $(a \otimes b)(12) = b \otimes a$ admits a right $A^{\otimes 2} \# \mathbb{S}_2$ -module structure. Furthermore, its quotient module $(A \otimes A)/(\overline{A} \otimes \overline{A})$ admits an $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodule structure, where the left A-action is given by

$$a \cdot [1_A \otimes 1_A] \colon = [a \otimes 1_A] + [1_A \otimes a], \quad a \cdot [b \otimes 1_A] \colon = [(ab) \otimes 1_A], \quad a \cdot [1_A \otimes b] \colon = [1_A \otimes (ab)]$$

for $a, b \in \overline{A}$, where $[x \otimes y]$ denotes the element $x \otimes y + \overline{A} \otimes \overline{A} \in (A \otimes A)/(\overline{A} \otimes \overline{A})$ for $x \otimes y \in A \otimes A$.

In fact, $(A \otimes A)/(\bar{A} \otimes \bar{A})$ is isomorphic to $\Bbbk(1_A \otimes 1_A) \oplus (\bar{A} \otimes \Bbbk 1_A) \oplus (\Bbbk 1_A \otimes \bar{A})$ as a vector space.

Let A be an augmented algebra with the augmented ideal \overline{A} and M an $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodule. Suppose that E is an extension of M by $(A \otimes A)/(\overline{A} \otimes \overline{A})$ as an $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodule. Then the triple (A, M, E)is called a *trident*.

Let (A, M, E) and (A', M', E') be two tridents. Suppose that $\alpha \colon A \to A'$ is a homomorphism of augmented algebras, and $\beta \colon M \to M'$ is a homomorphism of $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodules with the $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -actions on M' given by the algebra homomorphisms $\alpha \colon A \to A'$ and $\alpha \otimes \alpha \colon A \otimes A \to A' \otimes A'$. Clearly,

 $[\alpha \otimes \alpha]: (A \otimes A)/(\overline{A} \otimes \overline{A}) \to (A' \otimes A')/(\overline{A'} \otimes \overline{A'})$ is a homomorphism of $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodules. Then one can obtain a homomorphism of extensions of $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodules

which is also called a homomorphism of tridents. Consequently, we obtain a category \mathcal{T} , called the *trident* category. The following lemmas follows from Lemma 3.1.

Lemma 3.3. Let A, B be an associative algebras over \Bbbk . Suppose that M and N are $(A, A^{\otimes 2} \# \mathbb{S}_2)$ - and $(B, B^{\otimes 2} \# \mathbb{S}_2)$ -bimodules, respectively. Then $M \otimes N$ is $(A \otimes B, (A \otimes B)^{\otimes 2} \# \mathbb{S}_2)$ -bimodule.

3.2. Trident systems. There is another way of introducing a trident. Denote by $\text{mod}-\mathbb{S}_2$ the category of finite dimensional right $\Bbbk\mathbb{S}_2$ -modules. It is well known that $V \otimes W \in \text{mod}-\mathbb{S}_2$ with the diagonal action

$$(v \otimes w) * (12) = (v * (12)) \otimes (w * (12))$$

for $V, W \in \text{mod}-\mathbb{S}_2$. Let B be an associative algebra in the category mod- \mathbb{S}_2 . Such an algebra is also called a $\mathbb{k}S_2$ -module algebra (in Hopf algebra language). Sometimes it is called an algebra with involution (but \mathbb{k} may not be the complex field). A right (*resp.* left) module V over a $\mathbb{k}S_2$ -module algebra B means $V \in \text{mod}-\mathbb{S}_2$ and the action $V \otimes B \to V$ (*resp.* $B \otimes V \to V$) is a homomorphism of $\mathbb{k}S_2$ -modules.

Definition 3.4. A pair (A, M) with morphisms f, g, or equivalently, a quadruple (A, M, f, g), is called a *trident system* if

- (1) $A = \mathbb{k}\mathbb{1}_1 \oplus \overline{A}$ is an augmented algebra with the augmentation ideal \overline{A} ,
- (2) M is a nontrivial $(A, A \otimes A)$ -bimodule in mod- \mathbb{S}_2 ,
- (3) $f: \overline{A} \to M$ is a k-linear map in mod-S₂ where the S₂-action on \overline{A} is trivial,
- (4) $g: \overline{A} \otimes \overline{A} \to M$ is a homomorphism of right $A \otimes A$ -modules in mod- \mathbb{S}_2 ,

such that the following identities hold

$$(E3.4.1) f(ab) = af(b) + f(a) \cdot (b \otimes 1_A) + f(a) \cdot (1_A \otimes b) + g(a,b) + g(b,a),$$

(E3.4.2)
$$f(a) \cdot (b,c) = ag(b,c) - g(ab,c) - g(b,ac),$$

for all $a, b, c \in \overline{A}$.

We define morphisms between trident systems as follows. Let (A, M, f, g) and (A', M', f', g') be two trident systems. A morphism $(\alpha, \beta) \colon (A, M, f, g) \to (A', M', f', g')$ is given by an algebra homomorphism $\alpha \colon A \to A'$ and a trident A-module homomorphism $\beta \colon M \to M'$ such that the following diagrams commute

$\bar{A} \xrightarrow{f} M$		$ar{A}\otimesar{A}$	\xrightarrow{g}	M
$\alpha \downarrow \qquad $	and	$\alpha \otimes \alpha \downarrow$		$\int \beta$
$\bar{A}' \xrightarrow{f'} M'$		$\bar{A}'\otimes\bar{A}'$	$\xrightarrow{g'}$	M'

where the $(A, A \otimes A)$ -bimodule actions on M' is determined by $(A', A' \otimes A')$ -bimodule actions and the algebra homomorphisms $\alpha \colon A \to A'$ and $\alpha \otimes \alpha \colon A \otimes A \to A' \otimes A'$.

One can define a category \mathcal{C} consisting of all trident systems and morphisms defined above.

Proposition 3.5. *Retain the above notation. The trident category is isomorphic to the category of trident systems.*

Proof. Let (A, M, f, g) be a trident system. Recall that $A^{\otimes 2}$ is a subring of $A^{\otimes 2} \# \mathbb{S}_2$. Then f and g satisfy

- (1) f(a) = f(a) * (12),
- (2) g(a,b) = g(b,a) * (12),
- (3) $f(ab) = a \cdot f(b) + f(a) \cdot (b \otimes 1_A \#(1)) + f(a) \cdot (1_A \otimes b \#(1)) + g(a, b) + g(b, a),$
- $(4) \ f(a) \cdot (b \otimes c \#(1)) = a \cdot g(b,c) g(ab,c) g(b,ac),$

for all $a, b, c \in \overline{A}$. Using these equations, one can define an extension E of M by $(A \otimes A)/(\overline{A} \otimes \overline{A})$. To be precise, $E = M \oplus (A \otimes A)/(\overline{A} \otimes \overline{A})$ as a right \mathbb{S}_2 -module with the $(A, A \otimes A)$ -bimodule action given by

$$\begin{aligned} a \cdot (x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) &= (ax + \lambda f(a) + f(a) \cdot (b \otimes 1_A) + f(a) \cdot (1_A \otimes c) \\ &+ g(b, a) + g(a, c), \lambda[a \otimes 1_A] + \lambda[1_A \otimes a] + [ab \otimes 1_A] + [1_A \otimes ac]), \\ (x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \cdot (a \otimes 1_A) &= (x \cdot (a \otimes 1_A) + g(a, c), \lambda[a \otimes 1_A] + [ba \otimes 1_A]), \\ (x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \cdot (1_A \otimes a) &= (x \cdot (1_A \otimes a) + g(b, a), \lambda[1_A \otimes a] + [1 \otimes ca]), \\ (x, \lambda[1_A \otimes 1_A] + [b \otimes 1_A] + [1_A \otimes c]) \cdot (a \otimes a') &= (x \cdot (a \otimes a') + \lambda g(a, a') + g(ba, a') + g(a, ca'), 0) \end{aligned}$$

for all $\lambda \in \mathbb{K}$, $a, a_1, a_2, b, c \in \overline{A}$ and $x \in M$. It is easy to see that (A, M, E) is a trident.

Conversely, given a trident (A, M, E), we construct a trident system as follows. Suppose that

$$0 \to M \to E \xrightarrow{\pi} (A \otimes A)/(\bar{A} \otimes \bar{A}) \to 0$$

is the corresponding short exact sequence of $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodules. Without loss of generality, we assume M is a sub-bimodule of E. Fix an element $\mathbb{1}_2 \in E$ with $\pi(\mathbb{1}_2) = [\mathbb{1}_A \otimes \mathbb{1}_A] \in (A \otimes A)/(\bar{A} \otimes \bar{A})$. For all $a, b \in \bar{A}$, we define

$$f(a): = a \cdot \mathbb{1}_2 - \mathbb{1}_2 \cdot (a \otimes \mathbb{1}_A \#(1)) - \mathbb{1}_2 \cdot (\mathbb{1}_A \otimes a \#(1)),$$

$$g(a,b): = \mathbb{1}_2 \cdot (a \otimes b \#(1))$$

in E. Clearly, in $(A \otimes A)/(\bar{A} \otimes \bar{A})$, we have

$$\pi(f(a)) = a \cdot [1_A \otimes 1_A] - [a \otimes 1_A] - [1_A \otimes a] = 0,$$

$$\pi(g(a,b)) = [1_A \otimes 1_A] \cdot (a \otimes b) = 0$$

Therefore, we obtain two k-linear maps

$$f: \overline{A} \to M$$
, and $g: \overline{A} \otimes \overline{A} \to M$.

It can be directly checked that (A, M, f, g) is a trident system.

Since both constructions above are canonical, these defines two functors that are inverse to each other. $\hfill \Box$

Definition 3.6. A trident algebra means either a trident (A, M, E) or a trident system (A, M, f, g).

3.3. Examples. We give some easy examples of trident algebras.

Example 3.7. Let A be an augmented algebra and let M be an $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodule. Consider the trivial extension of M by $(A \otimes A)/(\bar{A} \otimes \bar{A})$. Equivalently, $f : \bar{A} \to M$ and $g : \bar{A}^{\otimes 2} \to M$ are zero maps in the corresponding trident system. In this case, we call (A, M, f, g) is called a *trivial* trident algebra.

Example 3.8. This is the trident algebra corresponding to $\mathcal{D}_A \underset{H}{\otimes} \mathcal{D}_B$, where \mathcal{D}_A and \mathcal{D}_B are the 2-unitary operad defined in [BYZ, Example 2.4].

Let A and B be augmented algebras. Clearly, $A \otimes B$ is also an augmented algebra with $\overline{A \otimes B} = \overline{A} \otimes \Bbbk 1_B + \Bbbk 1_A \otimes \overline{B} + \overline{A} \otimes \overline{B}$. From Subsection 3.1, we know $(A \otimes A)/(\overline{A} \otimes \overline{A})$ and $(B \otimes B)/(\overline{B} \otimes \overline{B})$ are $(A, A^{\otimes 2} \# \mathbb{S}_2)$ -bimodule and $(B, B^{\otimes 2} \# \mathbb{S}_2)$ -bimodule, respectively. By Lemma 3.1, we obtain an $(A \otimes B, (A \otimes B)^{\otimes 2} \# \mathbb{S}_2)$ -bimodule

$$E = [(A \otimes A)/(\bar{A} \otimes \bar{A})] \otimes [(B \otimes B)/(\bar{B} \otimes \bar{B})].$$

Observe that,

$$E = (\Bbbk[1_A \otimes 1_A] \oplus (\bar{A} \otimes \Bbbk 1_A) \oplus (\Bbbk 1_A \otimes \bar{A})) \otimes (\Bbbk[1_B \otimes 1_B] \oplus (\bar{B} \otimes \Bbbk 1_B) \oplus (\Bbbk 1_B \otimes \bar{B}))$$

and

$$M = [(\bar{A} \otimes \Bbbk 1_A) \otimes (\Bbbk 1_B \otimes \bar{B})] \oplus [(\Bbbk 1_A \otimes \bar{A}) \otimes (\bar{B} \otimes \Bbbk 1_B)]$$

is a sub-bimodule of E. By easy computation, we have

$$\begin{split} E/M &\cong \Bbbk [\mathbf{1}_{A\otimes B} \otimes \mathbf{1}_{A\otimes B}] \oplus [(\Bbbk \mathbf{1}_{A} \otimes \bar{B}) \otimes \Bbbk \mathbf{1}_{A\otimes B}] \oplus [(\Bbbk \mathbf{1}_{A\otimes B} \otimes (\Bbbk \mathbf{1}_{A} \otimes \bar{B}))] \\ &\oplus [(\bar{A} \otimes \Bbbk \mathbf{1}_{B}) \otimes \Bbbk \mathbf{1}_{A\otimes B}] \oplus [(\bar{A} \otimes \bar{B}) \otimes \Bbbk \mathbf{1}_{A\otimes B}] \oplus [\Bbbk \mathbf{1}_{A\otimes B} \otimes (\bar{A} \otimes \Bbbk \mathbf{1}_{B})] \oplus [\Bbbk \mathbf{1}_{A\otimes B} \otimes (\bar{A} \otimes \bar{B})] \\ &= \Bbbk [\mathbf{1}_{A\otimes B} \otimes \mathbf{1}_{A\otimes B}] \oplus (\overline{A \otimes B} \otimes \Bbbk \mathbf{1}_{A\otimes B}) \oplus (\Bbbk \mathbf{1}_{A\otimes B} \otimes \overline{A \otimes B}) \\ &= [(A \otimes B) \otimes (A \otimes B)]/[(\overline{A \otimes B}) \otimes (\overline{A \otimes B})] \end{split}$$

Therefore, we obtain a trident $(A \otimes B, M, E)$. Finally this trident $(A \otimes B, M, E)$ is denoted by $A \odot B$. Using the language of trident system, we have

(1) $f: \overline{A \otimes B} \to M$ is determined by

$$\begin{split} f(a \otimes 1_B) &= 0, \\ f(1_A \otimes b) &= 0, \\ f(a \otimes b) &= (a \otimes 1_A) \otimes (1_B \otimes b) + (1_A \otimes a) \otimes (b \otimes 1_B) \end{split}$$

for all $a \in \overline{A}$ and $b \in \overline{B}$.

(2) $g: \overline{A \otimes B} \otimes \overline{A \otimes B} \to M$ is determined by

$$g((a \otimes 1_B) \otimes (a' \otimes 1_B)) = 0,$$

$$g((a \otimes 1_B) \otimes (1_A \otimes b')) = (a \otimes 1_A) \otimes (1_B \otimes b'),$$

$$g((a \otimes 1_B) \otimes (a' \otimes b')) = 0,$$

$$g((1_A \otimes b) \otimes (a' \otimes 1_B)) = (1_A \otimes a') \otimes (b \otimes 1_B),$$

$$g((1_A \otimes b) \otimes (1_A \otimes b')) = 0,$$

$$g((1_A \otimes b) \otimes (a' \otimes b')) = 0,$$

$$g((a \otimes b) \otimes (a' \otimes 1_B)) = 0,$$

$$g((a \otimes b) \otimes (a' \otimes b')) = 0,$$

$$g((a \otimes b) \otimes (a' \otimes b')) = 0,$$

$$g((a \otimes b) \otimes (a' \otimes b')) = 0,$$

$$g((a \otimes b) \otimes (a' \otimes b')) = 0,$$

for all $a, a' \in \overline{A}$ and $b, b' \in \overline{B}$.

4. Classification of 2-unitary operads of GK-dimension 3

4.1. An operad constructed from a trident algebra. In this part, we construct a 2-unitary operad \mathcal{P} by (A, M, E), with $\mathcal{P}(0) = \Bbbk \mathbb{1}_0$, $\mathcal{P}(1) = A$ and $\mathcal{P}(2) = E$, where the composition $\mathcal{P}(1) \circ \mathcal{P}(1) \to \mathcal{P}(1)$ is given by the multiplication of A, the compositions $\mathcal{P}(1) \circ \mathcal{P}(2) \to \mathcal{P}(2)$, $\mathcal{P}(2) \circ (\mathcal{P}(1), \mathcal{P}(1)) \to \mathcal{P}(1)$ are given by the corresponding actions of A on E.

In fact, let (A, M, f, g) be a trident system. We consider the operad \mathcal{P} generated by the kS-module $(\Bbbk \mathbb{1}_0, A, \Bbbk \mathbb{1}_2 \oplus M, 0, 0, \cdots)$ and subject to the following relations

$$\begin{aligned} a \circ \mathbb{1}_{0} &= 0, \text{ for all } a \in \bar{A}, \\ \mu_{i} \otimes \mathbb{1}_{0} &= 0, \text{ for all } \mu \in M, \\ \mathbb{1}_{2} \otimes_{i} \mathbb{1}_{0} &= \mathbb{1}_{1}, \text{ for } i = 1, 2, \\ a \circ b &= ab, \text{ for all } a, b \in \bar{A}, \\ a \circ \mu &= a \cdot \mu, \text{ for all } a \in \bar{A}, \mu \in M, \\ a \circ \mathbb{1}_{2} &= \mathbb{1}_{2} \otimes_{1} a + \mathbb{1}_{2} \otimes_{2} a + f(a), \text{ for all } a \in \bar{A}, \\ \mathbb{1}_{2} \circ (a, b) &= g(a, b), \text{ for all } a, b \in \bar{A}, \\ \mu \circ (a, b) &= \mu \cdot (a \otimes b), \text{ for all } \mu \in M, a, b \in \bar{A}, \\ \mathbb{1}_{2} \otimes_{1} \mathbb{1}_{2} &= \mathbb{1}_{2} \otimes_{2} \mathbb{1}_{2}, \\ \mu \otimes_{1} \mathbb{1}_{2} &= \mathbb{1}_{2} \otimes_{2} \mu + (\mathbb{1}_{2} \otimes_{2} \mu) * (2, 1, 3), \text{ for all } \mu \in M, \\ \mu \otimes_{2} \mathbb{1}_{2} &= \mathbb{1}_{2} \otimes_{1} \mu + (\mathbb{1}_{2} \otimes_{1} \mu) * (1, 3, 2), \text{ for all } \mu \in M, \\ \mu \otimes \mu' &= 0, \text{ for all } \mu, \mu' \in M, i = 1, 2. \end{aligned}$$

where $\mathbb{1}_2 * (2,1) = \mathbb{1}_2$, and $ab, a \cdot \mu, \mu \cdot (a \otimes b)$ are given by the multiplication of A, the left module action of A on M, and the right module action of $A \otimes A$ on M, respectively.

Next we give an explicit description of \mathcal{P} .

(C1) The vector space $\mathcal{P}(n)$:

(C11) $\mathcal{P}(0) = \mathbb{k}\mathbb{1}_0.$ (C12) $\mathcal{P}(1) = A = \mathbb{k}\mathbb{1}_1 \oplus \overline{A}.$ (C13) $\mathcal{P}(2) = \mathbb{k}\mathbb{1}_2 \oplus (\overline{A}_1^{(2)} \oplus \overline{A}_2^{(2)}) \oplus M.$ (C14) for each $n \ge 3$,

$$\mathcal{P}(n) = \mathbb{k}\mathbb{1}_n \oplus \bigoplus_{k=1}^n \bar{A}_k^{(n)} \oplus \bigoplus_{1 \le i < j \le n} M_{ij}^{(n)},$$

where $\mathbb{k}\mathbb{1}_n$ is a 1-dimensional vector space with the basis $\mathbb{1}_n$, $\overline{A}_k^{(n)}$ is a vector space isomorphic to \overline{A} for $1 \leq k \leq n$, $n \geq 2$, and $M_{ij}^{(n)}$ is a vector space isomorphic to M for $1 \leq i < j \leq n$, $n \geq 2$.

In order to write elements in $\bar{A}_k^{(n)}$ and $M_{ij}^{(n)}$, we fix two families of k-linear isomorphisms

$$\varphi_k^n\colon \bar{A}\to \bar{A}_k^{(n)} \quad \text{ and } \quad \psi_{ij}^n\colon M\to M_{ij}^{(n)},$$

for $1 \le k \le n, 1 \le i < j \le n$ and $n \ge 2$. In fact, we will see that $\bar{A}_k^{(n)} = \{\mathbb{1}_n \underset{k}{\circ} a \mid a \in \bar{A}\}$ and $M_{ij}^{(n)} = \{(\mathbb{1}_{n-1} \underset{1}{\circ} \mu) * c_{ij} \mid \mu \in M\}$, where $c_{ij} = (i, j, 1, \cdots, i-1, \hat{i}, i+1, \cdots, j-1, \hat{j}, j+1, \cdots, n)$.

(C2) The right action of $\mathbb{k}S_n$ on $\mathcal{P}(n)$: for each $\sigma \in S_n$,

$$\begin{array}{ll} (\text{C21}) & \mathbbm{1}_n \ast \sigma = \mathbbm{1}_n, \\ (\text{C22}) & \varphi_i^{(n)}(a) \ast \sigma = \varphi_{\sigma^{-1}(i)}^{(n)}(a), \\ (\text{C23}) & \psi_{ij}^{(n)}(\mu) \ast \sigma = \begin{cases} \psi_{\sigma^{-1}(i),\sigma^{-1}(j)}^{(n)}(\mu), & \sigma^{-1}(i) < \sigma^{-1}(j), \\ \psi_{\sigma^{-1}(j),\sigma^{-1}(i)}^{(n)}(\mu \ast (21)), & \sigma^{-1}(i) > \sigma^{-1}(j). \end{cases}$$

(C3) The partial composition $\mathcal{P}(m) \underset{s}{\circ} \mathcal{P}(n) \to \mathcal{P}(m+n-1)$:

$$\begin{array}{ll} (\mathrm{C31}) & \mathbbm{1}_{m} \underset{s}{\circ} \mathbbm{1}_{n} = \mathbbm{1}_{m+n-1}. \\ (\mathrm{C32}) & \mathbbm{1}_{m} \underset{s}{\circ} \varphi_{i}^{(n)}(a) = \varphi_{s+i-1}^{(m+n-1)}(a). \\ (\mathrm{C33}) & \mathbbm{1}_{m} \underset{s}{\circ} \psi_{i_{1},i_{2}}^{(n)}(\mu) = \psi_{s+i_{1}-1,s+i_{2}-1}^{(m+n-1)}(\mu). \\ (\mathrm{C34}) & \varphi_{i}^{(m)}(a) \underset{s}{\circ} \mathbbm{1}_{n} = \begin{cases} \varphi_{i}^{(m+n-1)}(a), & i < s, \\ \overset{i+n-1}{\sum} \varphi_{k}^{(m+n-1)}(a) + \sum_{i \leq k_{1} < k_{2} \leq i+n-1} \psi_{k_{1}k_{2}}^{(m+n-1)}(f(a)), & i = s, \\ \varphi_{i+n-1}^{(m+n-1)}(a), & i > s. \end{cases} \\ (\mathrm{C35}) & \varphi_{i}^{(m)}(a) \underset{s}{\circ} \varphi_{j}^{(n)}(b) = \begin{cases} \psi_{i,s+j-1}^{(m+n-1)}(g(a,b)), & i < s, \\ \overset{i+j-2}{\sum} \psi_{k,i+j-1}^{(m+n-1)}(g(a,b) + f(a) \cdot b) + \varphi_{i+j-1}^{(m+n-1)}(ab) \\ \overset{i+j-2}{\sum} \psi_{k,i+j-1}^{(m+n-1)}(g(b,a) + f(a) \cdot b), \\ \overset{i+j-2}{\sum} \psi_{k+j-1,k}^{(m+n-1)}(g(b,a) + f(a) \cdot b), \\ \psi_{i+j-1,k}^{(m+n-1)}(g(b,a)), & i > s. \end{cases} \\ (\mathrm{C36}) & \varphi_{i}^{(m)}(a) \circ \psi_{i}^{(n)}(\mu) = \begin{cases} 0, & i \neq s, \end{cases} \end{cases}$$

C36)
$$\varphi_i^{(m)}(a) \mathop{\circ}_{s} \psi_{j_1 j_2}^{(n)}(\mu) = \begin{cases} 0, & i \neq 0, \\ \psi_{i+j_1-1, i+j_2-1}^{(m+n-1)}(a\mu), & i = s. \end{cases}$$

$$(C37) \ \psi_{i_{1}i_{2}}^{(m)}(\mu) \circ \mathbb{1}_{n} = \begin{cases} \psi_{i_{1}+n-1,i_{2}+n-1}^{(m+n-1)}(\mu), & 1 \leq s < i_{1}, \\ \sum_{k=i_{1}}^{i_{1}+n-1} \psi_{k,i_{2}+n-1}^{(m+n-1)}(\mu), & s = i_{1}, \\ \psi_{i_{1}i_{2}+n-1}^{(m+n-1)}(\mu), & i_{1} < s < i_{2}, \\ \sum_{k=i_{2}}^{i_{2}+n-1} \psi_{i_{1},k}^{(m+n-1)}(\mu), & s = i_{2}, \\ \psi_{i_{1}i_{2}}^{(m+n-1)}(\mu), & i_{2} < s \leq m. \end{cases}$$

$$(C38) \ \psi_{i_{1}i_{2}}^{(m)}(\mu) \circ \varphi_{j}^{(n)}(b) = \begin{cases} 0, & s \neq i_{1}, i_{2}, \\ \psi_{i_{1}+j-1,i_{2}+n-1}^{(m+n-1)}(\mu \cdot b), & s = i_{1}, \\ \psi_{i_{1},i_{2}+j-1}^{(m+n-1)}(\mu \cdot b), & s = i_{1}, \\ \psi_{i_{1},i_{2}+j-1}^{(m+n-1)}(\mu \cdot b), & s = i_{2}. \end{cases}$$

$$(C39) \ \psi_{i_{1}i_{2}}^{(m)}(\mu) \circ \psi_{j_{1}j_{2}}^{(n)}(\nu) = 0. \end{cases}$$

Theorem 4.1. Retain the above notation. Let (A, M, f, g) be a trident algebra. Then $\mathcal{P} := F(A, M, f, g)$ is a 2-unitary Com-augmented operad of GK-dimension 3.

A tedious proof of Theorem 4.1 is given in the final section.

4.2. Classification of 2-unitary operads of GK-dimension 3. Now we are ready to prove the main theorem.

Theorem 4.2. The category C consisting of trident algebras (A, M, f, g) is equivalent to the category D of Com-augmented operads of GK-dimension 3.

Proof. Define a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ as follows:

(i) For any trident algebra (A, M, f, g),

$$\mathcal{F}(A, M, f, g) := F(A, M, f, g)$$

where F(A, M, f, g) is given in Theorem 4.1, namely,

$$\mathcal{F}(A, M, f, g) = \{\mathcal{P}(n)\}_{n \ge 0} = \{\Bbbk \mathbb{1}_n \oplus \bigoplus_{k=1}^n \bar{A}_k^{(n)} \oplus \bigoplus_{1 \le i < j \le n} M_{ij}^{(n)}\}_{n \ge 0}.$$

By Theorem 4.1, $\mathcal{F}(A, M, f, g)$ is a *Com*-augmented operad of GK-dimension 3.

(ii) For a morphism $(\alpha, \beta) \colon (A, M, f, g) \to (A', M', f', g')$, we define an operadic morphism

$$\Phi = \mathcal{F}(\alpha, \beta) \colon \{\mathcal{P}(n)\} \to \{\mathcal{P}'(n)\}$$

as follows:

$$\Phi_n(\mathbb{1}_n) := \mathbb{1}'_n;$$

$$\Phi_n(\varphi_k^n(a)) := \varphi'_k^n(\alpha(a)), \text{ for } a \in \overline{A};$$

$$\Phi_n(\psi_{ij}^n(\mu)) := \psi'_{ij}^n(\beta(\mu)), \text{ for } \mu \in M.$$

By a direct calculation, it follows easily from (C21)-(C23) and (C31)-(C39) that Φ is a morphism of operads since (α, β) is a morphism in the category C.

Conversely, we define a functor $\mathcal{G}: \mathcal{D} \longrightarrow \mathcal{C}$ as follows: for an object \mathcal{P} in category \mathcal{D} , we define

$$\mathcal{G}(\mathcal{P}) = (A, M, f, g)$$

where $A = \mathcal{P}(1), M = {}^{2} \Upsilon_{\mathcal{P}}(2)$, and

$$f: {}^{1}\Upsilon(1) \to {}^{2}\Upsilon(2), \qquad \qquad a \mapsto a \circ \mathbb{1}_{2} - \mathbb{1}_{2} \mathop{\circ}_{1} a - \mathbb{1}_{2} \mathop{\circ}_{2} a; \\ g: {}^{1}\Upsilon(1) \otimes {}^{1}\Upsilon(1) \to {}^{2}\Upsilon(2), \qquad \qquad (a,b) \mapsto (\mathbb{1}_{2} \mathop{\circ}_{1} a) \mathop{\circ}_{2} b.$$

We show next that (A, M, f, g) is a trident algebra. By definitions, $A := \mathcal{P}(1)$ is an associative kalgebra with identity $\mathbb{1}_1$. Considering the map $\pi^{\varnothing} : \mathcal{P}(1) \to \mathcal{P}(0) = \mathbb{k} \mathbb{1}_0, \theta \mapsto \theta \circ \mathbb{1}_0$, we know that A is an augmented algebra with the augmentation ideal Ker $\pi^{\varnothing} = {}^1 \Upsilon_{\mathcal{P}}(1)$. By the definition of truncation ideals, $M := {}^2 \Upsilon_{\mathcal{P}}(2)$ is a $\mathbb{k}\mathbb{S}_2$ -submodule of $\mathcal{P}(2)$, and is an $(A, A \otimes A)$ -bimodule with the module actions given by the related composition map. Since \mathcal{P} is a *Com*-augmented (hence 2-unitary) operad of GK-dimension 3, we have

$$(a \circ \mathbb{1}_2 - \mathbb{1}_2 \mathop{\circ}_1 a - \mathbb{1}_2 \mathop{\circ}_2 a) \mathop{\circ}_i \mathbb{1}_0 = 0,$$

for all $a \in \overline{A}$ and i = 1, 2, and

$$\left(\left(\mathbb{1}_2 \circ a \right) \circ b \right) \circ \mathbb{1}_0 = 0,$$

for all $a, b \in \overline{A}$ and i = 1, 2. Therefore $a \circ \mathbb{1}_2 - \mathbb{1}_2 \circ a - \mathbb{1}_2 \circ a$ and $(\mathbb{1}_2 \circ a) \circ b = 0$ are in M. Therefore f maps from $\overline{A} \to M$ and g maps from $\overline{A}^{\otimes 2} \to M$.

For $a, b \in A$ and $\mu \in M$, let $a \cdot \mu$ be $a \circ \mu$ and $\mu \cdot (a \otimes b) = \mu \circ (a, b)$ both of which are in M. Then, for all $a \in A$, $\mu \in M$,

$$a \cdot (\mu \ast (2,1)) = a \mathop{\circ}_{1} (\mu \ast (2,1)) = (a \mathop{\circ}_{1} \mu) \ast (\mathbbm{1}_{1} \mathop{\circ}_{1} (2,1)) = (a \cdot \mu) \ast (2,1)$$

which shows that (E3.1.1) holds. For $a, b \in A$, we have

$$\begin{aligned} (\mu * (2,1)) \cdot (a \otimes b) = & (\mu * (2,1)) \mathop{\circ}_{1} a \mathop{\circ}_{2} b = ((\mu \mathop{\circ}_{2} a) * ((2,1) \mathop{\circ}_{1} \mathbb{1}_{1})) \mathop{\circ}_{2} b \\ = & ((\mu \mathop{\circ}_{2} a) \mathop{\circ}_{1} b) * ((2,1) \mathop{\circ}_{2} \mathbb{1}_{1})) = (\mu \cdot (b \otimes a)) * (2,1), \end{aligned}$$

which shows that (E3.1.2) holds. Hence M is a trident A-module.

For $\bar{a}, \bar{b} \in \bar{A}$,

$$\begin{split} f(\bar{a}\bar{b}) =& (\bar{a}\bar{b}) \circ \mathbb{1}_{2} - \mathbb{1}_{2} \underset{1}{\circ} (\bar{a}\bar{b}) - \mathbb{1}_{2} \underset{2}{\circ} (\bar{a}\bar{b}) \\ =& \bar{a} \circ (\bar{b} \circ \mathbb{1}_{2} - \mathbb{1}_{2} \underset{1}{\circ} \bar{b} - \mathbb{1}_{2} \underset{2}{\circ} \bar{b}) + (\bar{a} \circ \mathbb{1}_{2} - \mathbb{1}_{2} \underset{1}{\circ} \bar{a} - \mathbb{1}_{2} \underset{2}{\circ} \bar{a}) \underset{1}{\circ} \bar{b} \\ &+ (\bar{a} \circ \mathbb{1}_{2} - \mathbb{1}_{2} \underset{1}{\circ} \bar{a} - \mathbb{1}_{2} \underset{2}{\circ} \bar{a}) \underset{2}{\circ} \bar{b} + (\mathbb{1}_{2} \underset{1}{\circ} \bar{a}) \underset{2}{\circ} \bar{b} + (\mathbb{1}_{2} \underset{1}{\circ} \bar{b}) \underset{2}{\circ} \bar{a} \\ &= \bar{a}f(\bar{b}) + f(\bar{a}) \underset{1}{\circ} \bar{b} + f(\bar{a}) \underset{2}{\circ} \bar{b} + g(\bar{a}, \bar{b}) + g(\bar{b}, \bar{a}). \end{split}$$

Hence (E3.4.1) holds. For $\bar{a}, \bar{b}, \bar{c} \in \bar{A}$,

$$f(\bar{a}) \cdot (\bar{b}, \bar{c}) = ((\bar{a} \circ \mathbb{1}_2 - \mathbb{1}_2 \circ \bar{a} - \mathbb{1}_2 \circ \bar{a}) \circ \bar{b}) \circ \bar{c}$$

= $\bar{a}((\mathbb{1}_2 \circ \bar{b}) \circ \bar{c}) - (\mathbb{1}_2 \circ (\bar{a}\bar{b})) \circ \bar{c} - (\mathbb{1}_2 \circ \bar{b}) \circ (\bar{a}\bar{c})$
= $\bar{a}g(\bar{b}, \bar{c}) - g(\bar{a}\bar{b}, \bar{c}) - g(\bar{b}, \bar{a}\bar{c}).$

Hence (E3.4.2) holds. Therefore (A, M, f, g) is a trident algebra. It follows that $\mathcal{G}(\mathcal{P})$ is an object in the category \mathcal{C} .

For the operad morphism $\Psi \colon \mathcal{P} \to \mathcal{P}'$, we define the morphism

$$(\alpha,\beta) = \mathcal{G}(\Psi) \colon (\mathcal{P}(1),{}^{2}\mathcal{Y}_{\mathcal{P}}(2), f,g) \to (\mathcal{P}'(1),{}^{2}\mathcal{Y}'_{\mathcal{P}'}(2), f',g')$$

as follows:

$$\mathcal{G}(\Psi) := (\Psi(1), \Psi(2)|_{{}^{2}\Upsilon(2)}).$$

Therefore \mathcal{G} is a functor from $\mathcal{D} \to \mathcal{C}$.

Finally, it is clear from the definition that \mathcal{GF} is the identity and it follows from Lemma 2.4 that \mathcal{FG} is naturally isomorphic to the identity. The assertion follows.

5. Comments, examples, and remarks

We refer to [LV] for the definition of Hadamard product $-\bigotimes_{H}$ – and a Hopf operad. Recall that the Hadamard product $\mathcal{P} \bigotimes_{H} \mathcal{Q}$ of the operads \mathcal{P} and \mathcal{Q} is defined to be

$$(\mathcal{P} \underset{\mathbf{u}}{\otimes} \mathcal{Q})(n) = \mathcal{P}(n) \otimes \mathcal{Q}(n),$$

for all $n \ge 0$ with the partial composition

$$(\mu_1\otimes \nu_1) \mathop{\circ}_{i} (\mu_2\otimes \nu_2) = (\mu_1 \mathop{\circ}_{i} \mu_2) \otimes (\nu_1 \mathop{\circ}_{i} \nu_2),$$

for $\mu_1 \otimes \nu_1 \in (\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q})(m)$, $\mu_2 \otimes \nu_2 \in (\mathcal{P} \underset{\mathrm{H}}{\otimes} \mathcal{Q})(n)$, and $m \ge 1, n \ge 0, 1 \le i \le m$. Clearly, the operad $\mathcal{C}om$ is obviously a unit for Hadamard product.

A Hopf operad is a symmetric operad \mathcal{P} with a morphism of operads $\Delta: \mathcal{P} \to \mathcal{P} \underset{H}{\otimes} \mathcal{P}$ called the coproduct of \mathcal{P} and a morphism of operads $\epsilon_{\mathcal{P}}: \mathcal{P} \to \mathcal{C}om$ called the counit, which is supposed to be coassociative and counital.

Definition 5.1. Let \mathcal{P} be a Hopf operad. We say that \mathcal{P} is a *Com-augmented Hopf operad* if

(1) \mathcal{P} is *Com*-augmented and the composition

$$\mathcal{C}om \xrightarrow{u_{\mathcal{P}}} \mathcal{P} \xrightarrow{\epsilon} \mathcal{C}om$$

is the identity map, and

(2) the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}om & \stackrel{\cong}{\longrightarrow} & \mathcal{C}om \underset{\mathrm{H}}{\otimes} \mathcal{C}om \\ & & & \\ & & & \\ & & & \\ & & & \\ \mathcal{P} & \xrightarrow{} & & \\ & & &$$

Remark 5.2. A Hopf operad satisfying the condition (1) in Definition 5.1 is also called a unital augmented connected Hopf operad in [Kh, Definition 2.5].

Proposition 5.3. Let \mathcal{P} be a Com-augmented Hopf operad of GK-dimension ≤ 2 . Then $\mathcal{P} = Com$.

Proof. Note that $\Delta : \mathcal{P} \to \mathcal{P} \underset{H}{\otimes} \mathcal{P}$ is an operadic morphism. Write \mathcal{P} as F(A, 0, 0, 0) as given by Theorem 0.2. Suppose $\overline{A} \neq 0$. Then GKdim $\mathcal{P} = 2$ and GKdim $\mathcal{P} \underset{H}{\otimes} \mathcal{P} = 3$. Since $\mathcal{P} \underset{H}{\otimes} \mathcal{P}$ has GK-dimension 3, it is of the form $F(A \odot A)$ where $A \odot A$ is given in Example 3.8. By Theorem 4.2, Δ induces a morphism

of trident algebras $(A, 0, 0, 0) \to A \odot A := (A \otimes A, M, f, g)$. Then $f\Delta \mid_{\mathcal{P}(1)} : A \to A \otimes A \to M$ is zero. We claim that $A = \Bbbk$. If not, let $0 \neq a \in \overline{A}$ and write $\Delta \mid_{\mathcal{P}(1)} (a) = 1 \otimes a + a \otimes 1 + \sum a_{(1)} \otimes a_{(2)}$ where $a_{(1)}, a_{(2)} \in \overline{A}$. Then, by three equations in Example 3.8(1),

$$0 = f\Delta(a) = f(1 \otimes a + a \otimes 1 + \sum a_{(1)} \otimes a_{(2)})$$

= $\sum ((a_{(1)} \otimes 1_A) \otimes (1_A \otimes a_{(2)}) + (1_A \otimes a_{(1)}) \otimes (a_{(2)} \otimes 1_A)).$

Therefore $\sum a_{(1)} \otimes a_{(2)} = 0$, and consequently, a is a primitive element. By Definition 5.1(2), $\mathbb{1}_n$ is group-like, i.e., $\Delta(\mathbb{1}_n) = \mathbb{1}_n \otimes \mathbb{1}_n$ for all n. Since each a is primitive, it follows from (C32) that each $\varphi_i^{(n)}(a)$ is primitive, i.e., $\Delta(\varphi_i^{(n)}(a)) = \varphi_i^{(n)}(a) \otimes \mathbb{1}_n + \mathbb{1}_n \otimes \varphi_i^{(n)}(a)$ for all i, n. Since \mathcal{P} has GK-dimension $\leq 2, \varphi_1^{(2)}(a) \circ \varphi_1^{(3)}(a) = 0$ by (C35). But

$$\begin{split} \Delta(\varphi_1^{(2)}(a) \mathop{\circ}_1^\circ \varphi_1^{(3)}(a)) &= (\varphi_1^{(2)}(a) \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \varphi_1^{(2)}(a)) \mathop{\circ}_1^\circ (\varphi_1^{(3)}(a) \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes \varphi_1^{(3)}(a)) \\ &= (\varphi_1^{(2)}(a) \otimes \mathbb{1}_2) \mathop{\circ}_1^\circ (\mathbb{1}_3 \otimes \varphi_1^{(3)}(a)) + (\mathbb{1}_2 \otimes \varphi_1^{(2)}(a)) \mathop{\circ}_1^\circ (\varphi_1^{(3)}(a) \otimes \mathbb{1}_3) \\ &= (\sum_{k=1}^2 \varphi_k^{(4)}(a)) \otimes \varphi_1^{(4)}(a) + \varphi_1^{(4)}(a) \otimes (\sum_{k=1}^2 \varphi_k^{(4)}(a)) \\ &= \varphi_2^{(4)}(a) \otimes \varphi_1^{(4)}(a) + \varphi_1^{(4)}(a) \otimes \varphi_2^{(4)}(a) + 2\varphi_1^{(4)}(a) \otimes \varphi_1^{(4)}(a) \\ &\neq 0, \end{split}$$

yielding a contradiction. Therefore $\bar{A} = 0$ and $\mathcal{P} = \mathcal{C}om$.

Unlike in the GK-dimension 2 case, the 2-unit of a 2-unitary operad needs not be unique.

Example 5.4. Let $\mathcal{P} = F(A, M, f, g)$ where $M \neq 0$. Let $\mathbb{1}_2$ be the canonical 2-unit of \mathcal{P} given in the construction of F(A, M, f, g). Let $\mathbb{1}'_2 = \mathbb{1}_2 + \psi_{12}^{(2)}(m)$. It is easy to check that $\mathbb{1}'_2$ is a 2*a*-unit. Suppose that $m * (2, 1) \neq m$. Then $(\mathcal{P}, \mathbb{1}_0, \mathbb{1}_1, \mathbb{1}'_2)$ is a 2-unitary operad, but not *Com*-augmented. As a consequence, we can not replace "*Com*-augmented" by "2-unitary" in Theorem 0.2.

Remark 5.5. For non-2-unitary operad, we have the following remarks.

- (1) By [QXZZ, Construction 7.1] there are a lot of symmetric operads of GK-dimension 3 that are not 2-unitary.
- (2) In [QXZZ], an analogue of Bergman's gap theorem of nonsymmetric operads is proved, namely, no finitely generated locally finite nonsymmetric operad has GK-dimension strictly between 1 and 2. In [LQXZZZ] the authors proved that there is no finitely generated symmetric operad with GK-dimension strictly between 1 and 2.
- (3) It is an open question if there are finitely generated symmetric operads with GK-dimension strictly between 2 and 3, see [QXZZ, Question 0.8].
- (4) For every $r \in \{0\} \cup \{1\} \cup [2, \infty)$ or $r = \infty$, the authors in [QXZZ] constructed an explicit non-symmetric operad of GK-dimension r.

The following lemma was proved in [BYZ, Theorem 6.5].

Lemma 5.6. Let \mathcal{P} be a 2-unitary operad of finite GK-dimension ≥ 3 . Then \mathcal{P} is not semiprime.

Proof. If \mathcal{P} is semiprime, by the proof of [BYZ, Theorem 6.5], ${}^{(2)}\Upsilon = 0$. So GKdim $\mathcal{P} \leq 2$, yielding a contradiction.

The following example shows that $\mathop{\otimes}_{\mathrm{H}}$ does not preserve primeness.

Example 5.7. Let $A = \Bbbk \oplus M_2(\Bbbk)$. Then $\mathcal{P} := F(A, 0, 0, 0)$ is prime of GK-dimension 2 by Theorem 0.1. Since $\mathcal{P} \bigotimes_{\mathrm{H}} \mathcal{P}$ has GK-dimension 3, it is not semiprime by Lemma 5.6.

6. Appendix: Proof of Theorem 4.1

This final section is devoted to a complete proof of Theorem 4.1.

Proof of Theorem 4.1. We need to check (OP1), (OP2), (OP3). Let $\mathbb{A} = \{\mathbb{1}_n\}, \Phi = \{\varphi_i^{(n)}(a) \mid a \in \overline{A}\},\$ and $\Psi = \{\psi_{ij}^{(n)}(y) \mid y \in M\}$. We will use these elements.

Verification of (OP1): By (C31), (C32) and (C33), $\mathbb{1} \underset{1}{\circ} \theta = \theta$ for all $\theta \in \mathcal{P}$. By (C31), (C34) and (C37), we have $\theta \underset{i}{\circ} \mathbb{1} = \theta$ for all $\theta \in \mathcal{P}$ and $1 \leq i \leq \operatorname{Ar}(\theta)$. Therefore (OP1) holds.

Verification of (OP2): There are two equations in (OP2). We only check the first one in (OP2), namely, the following equation

(E1.1.1)
$$(\lambda \circ \mu) \circ_{i-1+j} \nu = \lambda \circ (\mu \circ \nu), 1 \le i \le l, 1 \le j \le m$$

for all $\lambda \in \mathcal{P}(l), \mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$.

If two of λ, μ , and ν are $\psi_{ij}^{(n)}$, it follows from (C39) that both sides of (E1.1.1) are zero. Then there are following 20 cases to consider.

 $\mathsf{Case 1:}\ \lambda \in \Psi \text{ and } \mu, \nu \in \Phi. \quad \mathrm{Write } \ \lambda = \psi_{k_1k_2}^{(l)}(y), \ \mu = \varphi_s^{(m)}(a), \ \mathrm{and } \ \nu = \varphi_t^{(n)}(b). \ \mathrm{Then} \ \lambda \in \Psi \ \mathrm{and} \ \mu = \varphi_t^{(n)}(b).$

$$\begin{array}{l} \text{LHS of (E1.1.1)} = (\psi_{k_{1}k_{2}}^{(l)}(y) \mathop{\circ}\limits_{i} \varphi_{s}^{(m)}(a)) \mathop{\circ}\limits_{i=1+j} \varphi_{t}^{(n)}(b) \\ & = \\ & = \\ & = \\ & \text{by (C38)} \begin{cases} 0 & i \neq k_{1}, k_{2}, \\ \psi_{k_{1}k_{2}+s-1}^{(l+m-1)}(y_{1}^{-}a) & i = k_{1}, & \circ \\ \psi_{k_{1}k_{2}+s-1}^{(l+m-1)}(y_{2}^{-}a) & i = k_{2}, \end{cases} & i-l+j \\ & \varphi_{t}^{(n)}(b) \\ & = \\ & = \\ & \text{by (C38)} \begin{cases} 0 & i \neq k_{1}, k_{2}, i+j-1 \neq k_{1}+s-1, k_{2}+m-1, k_{1}, k_{2}+s-1, \\ \psi_{k_{1}k_{2}+s-1}^{(l+m+n-2)}(y_{1}^{-}a)_{1}^{-}b) & i = k_{1}, i+j-1 = k_{1}+s-1, \\ \psi_{k_{1}+s-1, k_{2}+m+n-2}^{(l+m+n-2)}((y_{1}^{-}a)_{2}^{-}b) & i = k_{1}, i+j-1 = k_{2}+m-1, (impossible) \\ \psi_{k_{1}+s-1, k_{2}+s+n-2}^{(l+m+n-2)}(y_{2}^{-}a)_{1}^{-}b) & i = k_{2}, i+j-1 = k_{1}, (impossible) \\ \psi_{k_{1}+k_{2}+s+t-2}^{(l+m+n-2)}(y_{2}^{-}a)_{2}^{-}b) & i = k_{1}, j = s, \\ \psi_{k_{1}+k_{2}+s+t-2}^{(l+m+n-2)}(y_{2}^{-}a)_{2}^{-}b) & i = k_{2}, j = s, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \text{RHS of (E1.1.1)} &= \psi_{k_{1}k_{2}}^{(l)}(y) \underset{i}{\circ} (\varphi_{s}^{(m)}(a) \underset{j}{\circ} \varphi_{t}^{(n)}(b)) \\ &= \underset{\text{by (C35)}}{=} \psi_{k_{1}k_{2}}^{(l)}(y) \underset{i}{\circ} \begin{cases} \begin{pmatrix} \psi_{s,j+t-1}^{(m+n-1)}(g(a,b)), & s < j, \\ s,t-2 \\ \sum \\ \psi_{s,s+t-1}^{(m+n-1)}(g(a,b) + f(a) \underset{j}{\cdot} b) + \varphi_{s+t-1}^{(m+n-1)}(ab) \\ + \\ \sum \\ k = s + n - 1 \\ \psi_{s+t-1,k}^{(m+n-1)}(g(b,a) + f(a) \underset{j}{\cdot} b), \\ \psi_{m+n-1}^{(m+n-1)}(\psi_{m+n-1})(g(b,a)), & s > j. \end{cases} \\ &= \underset{0}{=} \underset{\text{by (C39)}}{=} \begin{cases} 0 & s < j, \\ \psi_{k_{1}k_{2}}^{(l)}(y) \underset{i}{\circ} \varphi_{s+t-1}^{(m+n-1)}(ab) & s = j, \\ 0 & s > j, \end{cases} \\ &= \underset{0}{=} \underset{0}{=} \underset{0}{=} \begin{cases} 0 & s < j, \\ \psi_{k_{1}k_{2}}^{(l+m+n-2)}(\psi_{j} a) \underset{s > j, i = k_{1}}{y_{k_{1}+s+t-2,k_{2}+m+n-2}(y \underset{j}{\cdot} ab)} & s = j, i = k_{2} \\ \psi_{k_{1}k_{2}+s+t-2}^{(l+m+n-2)}(\psi_{j} a) \underset{j > j}{\cdot} b) & i = k_{1}, j = s, \\ \psi_{k_{1}k_{2}+s+t-2}^{(l+m+n-2)}(\psi_{j} a) \underset{j > b}{\cdot} b) & i = k_{2}, j = s, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which implies that (E1.1.1) holds.

Case 2: $\mu \in \Psi$ and $\lambda, \nu \in \Phi$. Write $\lambda = \varphi_u^{(l)}(a), \ \mu = \psi_{k_1k_2}^{(m)}(y)$, and $\nu = \varphi_v^{(n)}(c)$ where $a, c \in \overline{A}$ and $b \in M$. Then

LHS of (E1.1.1) =
$$(\varphi_u^{(l)}(a) \circ \psi_{k_1k_2}^{(m)}(y)) \circ \varphi_v^{(n)}(c)$$

$$= \begin{cases} 0 & i \neq u, \\ \psi_{k_1+u-1,k_2+u-1}^{(l+m-1)}(ay) & i = u \end{cases} \circ \varphi_v^{(n)}(c)$$

$$= \\ \text{by (C38)} \begin{cases} 0 & i \neq u, \\ 0 & j \neq k_1, k_2, \\ \psi_{k_1+u+v-2,k_2+u+n-2}^{(l+m+n-2)}(ay_{\frac{1}{2}}c) & i = u, j = k_1, \\ \psi_{k_1+u-1,k_2+u+v-2}^{(l+m+n-2)}(ay_{\frac{1}{2}}c) & i = u, j = k_2, \end{cases}$$

and

$$\begin{aligned} \text{RHS of (E1.1.1)} &= \varphi_u^{(l)}(a) \underset{i}{\circ} \left(\psi_{k_1 k_2}^{(m)}(y) \underset{j}{\circ} \varphi_v^{(n)}(c) \right) \\ &= \underset{\text{by (C38)}}{=} \varphi_u^{(l)}(a) \underset{i}{\circ} \begin{cases} 0 & j \neq k_1, k_2, \\ \psi_{k_1 + v - 1, k_2 + v - 1}^{(m+n-1)}(y \underset{i}{:} c) & j = k_1, \\ \psi_{k_1, k_2 + v - 1}^{(m+n-1)}(y \underset{i}{:} c) & j = k_2, \end{cases} \\ &= \underset{\text{by (C38)}}{=} \begin{cases} 0 & j \neq k_1, k_2, \\ 0 & i \neq u, \\ \psi_{k_1 + u + 2, k_2 + u + n - 2}^{((l+m+n-2))}(a(y \underset{i}{:} c)) & j = k_1, i = u, \\ \psi_{k_1 + u - 1, k_2 + u + v - 2}^{(l+m+n-2)}(a(y \underset{i}{:} c)) & j = k_2, i = u. \end{cases} \end{aligned}$$

Hence (E1.1.1) holds.

 $\begin{array}{l} \text{Case 3: } \nu \in \Psi \text{ and } \lambda, \mu \in \Phi. \text{ Write } \lambda = \varphi_u^{(l)}(a), \, \mu = \varphi_v^{(m)}(c), \, \text{and } \nu = \psi_{k_1k_2}^{(n)}(y). \text{ Then} \\ \text{LHS of (E1.1.1)} = (\varphi_u^{(l)}(a) \mathop{\circ}\limits_i \varphi_v^{(m)}(c)) \mathop{\circ}\limits_{i-1+j} \psi_{k_1k_2}^{(n)}(y) \\ & = \begin{cases} \psi & i \neq u, & \circ \\ \psi + \varphi_{u+v-1}^{(l+m-1)}(ac) & i = u, & i-1+j \end{cases} \psi_{k_1k_2}^{(n)}(y) \\ & = & y \text{ (C39)} \begin{cases} 0 & i \neq u, & \circ \\ \varphi_{u+v-1}^{(l+m-1)}(ac) & \stackrel{\circ}{_{i-1+j}} \psi_{k_1k_2}^{(n)}(y) & i = u, \end{cases} \\ & = & y \text{ (C36)} \begin{cases} 0 & i \neq u, & \circ \\ 0 & j \neq v, & \circ \\ \psi_{k_1+u+v-2,k_2+u+v-2}^{((ac)y)}(ac)y) & i = u, j = v, \end{cases} \end{array}$

where ψ is a linear combination of elements in Ψ . And

RHS of (E1.1.1) =
$$\varphi_u^{(l)}(a) \circ_i (\varphi_v^{(m)}(c) \circ_j \psi_{k_1 k_2}^{(n)}(y))$$

= $\varphi_u^{(l)}(a) \circ_i \begin{cases} 0 & j \neq v, \\ \psi_{k_1+v-1,k_2+v-1}^{(m+n-1)}(cy) & j = v, \end{cases}$
= $\begin{cases} 0 & j \neq v, \\ 0 & i \neq u \\ \psi_{k_1+v+u-2,k_2+v+u-2}^{(m+n-2)}(a(cy)) & j = v, i = u. \end{cases}$

Hence (E1.1.1) holds.

 $\begin{aligned} \mathsf{Case } 4: \ \lambda \in \Psi, \ \mu \in \mathbb{A}, \ \mathsf{and} \ \nu \in \Phi. \quad \mathrm{Write } \lambda = \psi_{k_1 k_2}^{(l)}(y), \ \mu = \mathbbm{m}, \ \mathsf{and} \ \nu = \varphi_t^{(n)}(c). \ \mathrm{Then} \\ \mathrm{LHS } \mathrm{of } (\mathrm{E1.1.1}) = (\psi_{k_1 k_2}^{(l)}(y) \circ \mathbbm{1} \mathbbm{m}) \underset{i=1+j}{\circ} \varphi_t^{(n)}(c) \\ & = \\ \mathrm{by } (\mathrm{C37}) \begin{cases} \psi_{k_1 + m - 1, k_2 + m - 1}^{(l+m-1)}(y), & 1 \leq i < k_1, \\ k_1 + m - 1, (l+m-1), \\ k_2 \leq i = 1 \end{cases} \\ \psi_{k_1 + k_2 + m - 1}^{(l+m-1)}(y), & k_1 < i < k_2, \\ \psi_{k_1 k_2}^{(l+m-1)}(y), & k_2 < i \leq l. \end{cases} \\ \begin{cases} \psi_{k_1 + m + 1, 2}^{(l+m-1)}(y), & k_2 < i \leq l. \\ \psi_{k_1 + m + 1, 2}^{(l+m-1)}(y), & k_2 < i \leq l. \end{cases} \\ \end{cases} \\ \begin{cases} 0, & (l+m+1) \\ \psi_{k_1 + m + 1, 2}^{(l+m-1)}(y), & k_2 < i \leq l. \end{cases} \\ \end{cases} \\ \begin{cases} 0, & (l+m+1) \\ \psi_{k_1 + m + 1, 2}^{(l+m+1)}(y), & (l+m+1) \\ \psi_{k_1 + m + 1, 2}^{(l+m+1)}(y), & (l+m+1) \\ \psi_{k_1 + m + 1, 2}^{(l+m+1)}(y) \\ \psi$

and

RHS of (E1.1.1) =
$$\psi_{k_1k_2}^{(l)}(y) \stackrel{\circ}{}_i (\mathbb{1}_m \stackrel{\circ}{}_j \varphi_t^{(n)}(c))$$

= $\psi_{k_1k_2}^{(l)}(y) \stackrel{\circ}{}_i \varphi_{j+t-1}^{(m+n-1)}(c)$
= $\psi_{k_1k_2k_2}^{(l)}(y) \stackrel{\circ}{}_i \varphi_{j+t-1}^{(m+n-2)}(c)$
by (C38) $\begin{cases} 0 & i \neq k_1, k_2, \\ \psi_{k_1+k_2+j+t-2,k_2+m+n-2}^{(l+m+n-2)}(y \stackrel{\circ}{}_i c) & i = k_1, \\ \psi_{k_1,k_2+j+t-2}^{(l+m+n-2)}(y \stackrel{\circ}{}_2 c) & i = k_2. \end{cases}$

Hence (E1.1.1) holds.

Case 5: $\lambda \in \Psi$, $\mu \in \Phi$, and $\nu \in \mathbb{A}$. Write $\lambda = \psi_{k_1k_2}^{(l)}(y)$, $\mu = \varphi_s^{(m)}(b)$, and $\nu = \mathbb{1}_n$. Then

$$\begin{split} \text{LHS of } (\text{E1.1.1}) &= \left(\psi_{k_{1}k_{2}}^{(l)}(y) \circ_{i} \varphi_{s}^{(m)}(b) \right) \circ_{i-1+j} \mathbbm{1}_{n} \\ &= \\ &= \\ &= \\ &= \\ &= \\ & \text{by } (\text{C38}) \begin{cases} 0 & i \neq k_{1}, k_{2}, \\ \psi_{k_{1}+s-1,k_{2}+m-1}(y \circ_{1}b) & i = k_{1}, \\ \psi_{k_{1}+s-1}(y \circ_{2}b) & i = k_{2}, \\ \psi_{k_{1}+s-1}(y \circ_{2}b) & i = k_{2}, \\ \psi_{k_{1}+s-1}(y \circ_{1}b) & i = k_{1}, \\ &= k_{1}, k_{2}, \\ &= k_{1}+s-1 \\ \psi_{k_{1}+s-1}(y \circ_{1}b) & k_{1}+s-1, \\ \psi_{k_{1}+s-1,k_{2}+m-1+n-1}(y \circ_{1}b), \\ &= k_{1}, \\ \psi_{k_{1}+s-1,k_{2}+m-1+n-1}(y \circ_{1}b), \\ &= k_{1}, \\ &= k_{2}, \\ \psi_{k_{1}+s-1,k_{2}+m-1+n-1}(y \circ_{1}b), \\ &= k_{1}, \\ &= k_{2}, \\ \psi_{k_{1}+s-1,k_{2}+m-1+n-1}(y \circ_{1}b), \\ &= k_{1}, \\ &= k_{2}, \\ \psi_{k_{1}+s-1,k_{2}+s-1+n-1}(y \circ_{2}b), \\ &= k_{2}, \\ &= k_{1}, \\ &= k_{2}, \\ &$$

and

which implies that (E1.1.1) holds.

Case 6: $\mu \in \Psi$, $\lambda \in \mathbb{A}$, and $\nu \in \Phi$. Write $\lambda = \mathbb{1}_l$, $\mu = \psi_{k_1k_2}^{(m)}(y)$, and $\nu = \varphi_t^{(n)}(c)$. Then

LHS of (E1.1.1) =
$$(\mathbb{1}_{l_{i}} \circ \psi_{k_{1}k_{2}}^{(m)}(y)) \circ_{i-1+j} \varphi_{t}^{(n)}(c)$$

$$= \psi_{i+k_{1}-1,i+k_{2}-1}^{(l+m-1)}(y) \circ_{i-1+j} \varphi_{t}^{(n)}(c)$$

$$= \bigcup_{\text{by (C38)}} \begin{cases} 0 & j \neq k_{1}, k_{2}, \\ \psi_{i+k_{1}+1-2,i+k_{2}+n-2}^{(l+m+n-2)}(y_{1}c) & j = k_{1}, \\ \psi_{i+k_{1}-1,i+k_{2}+1-2}^{(l+m+n-2)}(y_{2}c) & j = k_{2}, \end{cases}$$

and

$$\begin{aligned} \text{RHS of (E1.1.1)} &= \mathbbm{1}_{l} \circ \left(\psi_{k_{1}k_{2}}^{(m)}(y) \circ \varphi_{t}^{(n)}(c) \right) \\ &= \\ &= \\ &= \\ &\text{by (C38)} \ \mathbbm{1}_{l} \circ \begin{cases} 0 & j \neq k_{1}, k_{2}, \\ \psi_{k_{1}+t-1,k_{2}+n-1}^{(m+n-1)}(y_{1}^{-}c) & j = k_{1}, \\ \psi_{k_{1},k_{2}+t-1}^{(m+n-2)}(y_{2}^{-}c) & j = k_{2}, \end{cases} \\ &= \\ &= \\ &\text{by (C33)} \begin{cases} 0 & j \neq k_{1}, k_{2}, \\ \psi_{i+k_{1}+1-2,i+k_{2}+n-2}^{((m+n-2))}(y_{1}^{-}c) & j = k_{1}, \\ \psi_{i+k_{1}-1,i+k_{2}+t-2}^{(m+n-2)}(y_{2}^{-}c) & j = k_{2}, \end{cases} \end{aligned}$$

Hence (E1.1.1) holds.

Case 7: $\mu \in \Psi$, $\lambda \in \Phi$, and $\nu \in \mathbb{A}$. Write $\lambda = \varphi_r^{(l)}(a)$, $\mu = \psi_{k_1k_2}^{(m)}(y)$, and $\nu = \mathbb{1}_n$. Then

LHS of (E1.1.1) =
$$(\varphi_r^{(l)}(a) \circ \psi_{k_1k_2}^{(m)}(y)) \circ \mathbb{1}_{i-1+j} \mathbb{1}_n$$

$$= \begin{cases} 0 & i \neq r, \\ \psi_{k_1+r-1,k_2+r-1}^{(l+m-1)}(ay) & i = r & i-1+j \end{cases} \mathbb{1}_n$$

$$= by (C37) \begin{cases} 0 & i \neq r, \\ \psi_{k_1+r-1,k_2+r-1+n-1}^{(l+m+n-2)}(ay), & 1 \leq r-1+j < r+k_1-1, \\ k_1+r-1+n-1 & (k_1+r-1) \\ (k_1+r-1,k_2+r-1+n-1) & (k_1+r-1) \\ k_1+r-1,k_2+r-1+n-1 & (k_1+r-1) \\ k_1+r-1,k_2+r-1 & (k_1+r-1) \\ k_1+r-1,k_2+r-1 & (k_1+r-1) \\ k_1+r-1,k_2+r-1 & (k_1+r-1) \\ (k_1+r-1,k_2+r-1) & (k_1+r-1,k_1) \\ (k_1+r-1,k_2+r-1) & (k_1+r-1,k_2+r-1) \\ (k_1+r-1,k_2+r-1) & (k_1+r-1,k_1) \\ (k_1+r-1,k_2+r-1) & (k_1+r-1,k_1)$$

and

$$\begin{aligned} \text{RHS of (E1.1.1)} &= \varphi_r^{(l)}(a) \underbrace{\circ}_i \left(\psi_{k_1 k_2}^{(m)}(y) \underbrace{\circ}_j \mathbbm{1}_n \right) \\ &= \\ \text{by (C37)} \varphi_r^{(l)}(a) \underbrace{\circ}_i \begin{cases} \psi_{k_1 k_2}^{(m+n-1)}(y), &1 \leq j < k_1, \\ k_1 + n - 1, k_2 + n - 1 \end{pmatrix} \\ \psi_{k_1 k_2 + n - 1}(y), & j = k_1, \\ \psi_{k_1 k_2 + n - 1}^{(m+n-1)}(y), & j = k_2, \\ \psi_{k_1 k_2}^{(m+n-1)}(y), & j = k_2, \\ \psi_{k_1 k_2}^{(m+n-1)}(y), & k_2 < j \leq m. \end{cases} \\ \begin{cases} 0 & i \neq r, \\ \psi_{r+k_1 + n - 2, r+k_2 + n - 2}^{(m+n-1)}(y), & i = r \\ \psi_{r+k_1 + n - 2, r+k_2 + n - 2}^{(m+n-1)}(y) & i = r \\ k_1 k_2 & i \neq r, \\ \psi_{r+k_1 - 1, r+k_2 + n - 2}^{(m+n-2)}(ay) & i = r \\ \end{bmatrix} \\ \begin{cases} 0 & i \neq r, \\ k_1 k_2 & i \neq r, \\ \psi_{r+k_1 - 1, r+k_2 + n - 2}^{(m+n-2)}(ay) & i = r \\ \psi_{r+k_1 - 1, r+k_2 + n - 2}^{(m+n-2)}(ay) & i = r \\ \end{bmatrix} \\ \begin{cases} 0 & i \neq r, \\ \psi_{r+k_1 - 1, r+k_2 + n - 2}^{(m+n-2)}(ay) & i = r \\ k_1 < j < k_2, \\ 0 & i \neq r, \\ k_2 + n - 1 \\ \sum_{k=k_2}^{N} \psi_{r+k_1 - 1, r+k_2 - 1}^{(m+n-2)}(ay) & i = r \\ \end{bmatrix} \\ \begin{cases} 0 & i \neq r, \\ k_2 + n - 1 \\ \sum_{k=k_2}^{N} \psi_{r+k_1 - 1, r+k_2 - 1}^{(m+n-2)}(ay) & i = r \\ k_2 < 0 & i \neq r, \\ k_2 + n - 1 \\ \sum_{k=k_2}^{N} \psi_{r+k_1 - 1, r+k_2 - 1}^{(m+n-2)}(ay) & i = r \\ \end{cases} \\ \end{cases} \\ \end{cases} \\ \end{cases}$$

Hence (E1.1.1) holds.

Case 8: $\nu \in \Psi$, $\lambda \in \mathbb{A}$, and $\mu \in \Phi$. Write $\lambda = \mathbbm{1}_l$, $\mu = \varphi_k^{(m)}(a)$, and $\nu = \psi_{k_1k_2}^{(n)}(y)$. Then LHS of (E1.1.1) = $(\mathbbm{1}_l \circ \varphi_k^{(m)}(a)) \circ \psi_{i-1+j}^{(n)} \psi_{k_1k_2}^{(n)}(y)$ $\stackrel{=}{\underset{\text{by (C32)}}{=}} \varphi_{i+k-1}^{(l+m-1)}(a) \circ \psi_{k_1k_2}^{(n)}(y)$ $\stackrel{=}{\underset{\text{by (C36)}}{=}} \begin{cases} 0 & i+k-1 \neq i-1+j, \\ \psi_{k_1+i+j-2,k_2+i+j-2}^{(ay)} & i+k-1 = i-1+j, \end{cases}$

and

RHS of (E1.1.1) =
$$\mathbb{1}_{l} \circ_{i} (\varphi_{k}^{(m)}(a) \circ_{j} \psi_{k_{1}k_{2}}^{(n)}(y))$$

$$= \mathbb{1}_{l} \circ_{i} \left\{ \begin{array}{c} 0 & k \neq j, \\ \psi_{k_{1}+j-1,k_{2}+j-1}^{(m+n-1)}(ay) & k = j \end{array} \right.$$

$$= \left\{ \begin{array}{c} 0 & k \neq j, \\ \psi_{k_{1}+j+i-2,k_{2}+j+i-2}^{(m+n-2)}(ay) & k = j. \end{array} \right.$$

Hence (E1.1.1) holds.

Case 9: $\nu \in \Psi$, $\lambda \in \Phi$, and $\mu \in \mathbb{A}$. Write $\lambda = \varphi_r^{(l)}(a)$, $\mathbb{1}_m$, and $\nu = \psi_{k_1k_2}^{(n)}(y)$. Then LHS of (E1.1.1) = $(\varphi_r^{(l)}(a) \circ \mathbb{1}_m) \circ \varphi_{k_1k_2}^{(n)}(y)$

$$\begin{aligned} & \left(\text{E1.1.1} \right) = \left(\varphi_r^{(t)}(a) \circ \mathbb{1}_m \right) \circ \psi_{k_1 k_2}^{(t)}(y) \\ & = \\ & = \\ & \text{by (C34)} \begin{cases} \varphi_r^{(l+m-1)}(a), & r < i, \\ r+m-1 \\ \sum_{k=r} & \varphi_k^{(l+m-1)}(a) + \sum_{r \le p_1 < p_2 \le r+m-1} & \psi_{p_1 p_2}^{(l+m-1)}(f(a)), & r = i, \\ \varphi_r^{(l+m-1)}(a), & r > i. \end{cases} \\ & = \\ & \text{by (C36)} \begin{cases} 0 & r \neq i - 1 + j, \\ \psi_{r+k_1 - 1, r+k_2 - 1}^{(l+m+n-2)}(ay) & r = i - 1 + j, (impossible). \\ \psi_{i-1+j+k_1 - 1, r+k_2 - 1}^{(l+m+n-2)}(ay) & r = i - 1 + j, \\ \psi_{i-1+j+k_1 - 1, r+k_2 - 1}^{(l+m+n-2)}(ay) & r = i - 1 + j, \\ \psi_{i-1+j+k_1 - 1, r+m-1+k_1 - 1}^{(l+m+n-2)}(ay) & r = i - 1 + j, (impossible). \end{cases} \\ & r > i \end{aligned}$$

And

RHS of (E1.1.1) =
$$\varphi_r^{(l)}(a) \underset{i}{\circ} (\mathbb{1}_m \underset{j}{\circ} \psi_{k_1 k_2}^{(n)}(y))$$

= $\varphi_r^{(l)}(a) \underset{i}{\circ} \psi_{j+k_1-1,j+k_2-1}^{(m+n-1)}(y)$
= $\begin{cases} 0 & r \neq i, \\ \psi_{r+j+k_1-2,r+j+k_2-2}^{(l+m+n-2)}(ay)) & r = i. \end{cases}$

Hence (E1.1.1) holds.

LHS

Case 10: $\lambda, \mu, \nu \in \Phi$. Write $\lambda = \varphi_r^{(l)}(a)$ for $a \in \overline{A}$, $\mu = \varphi_s^{(m)}(b)$ and $\nu = \varphi_t^{(n)}(c)$. Then

$$\begin{split} \text{of } (\text{E1.1.1}) &= \left(\varphi_{r}^{(l)}(a) \circ \varphi_{s}^{(m)}(b)\right) \circ \varphi_{t}^{(m)}(c) \\ &= \begin{cases} \psi_{r,i+s-1}^{(l+m-1)}(g(a,b)), & r < i, \\ r+s-2 \\ \sum_{k=r} \psi_{k,r+s-1}^{(l+m-1)}(g(a,b) + f(a) \cdot b) + \varphi_{r+s-1}^{(l+m-1)}(ab) \\ r+m-1 \\ + \sum_{k=r+s} \psi_{r+s-1,k}^{(l+m-1)}(g(b,a) + f(a) \cdot b), & r = i, \\ \psi_{i+s-1,r+m-1}^{(l+m-1)}(g(b,a)), & r > i. \end{cases} \\ \\ &= \\ \text{by } (\text{C38}) \begin{cases} 0, & j \neq s, \\ \psi_{r,i+s-1+t-1}^{(l+m+n-2)}(g(a,b) \cdot c), & j = s. \end{cases} & r < i, \\ \{0, & j \neq s, \\ \psi_{i+s-1+t-1}^{(l+m+n-2)}(g(b,a) \cdot c), & j = s. \end{cases} & r < i, \\ \{0, & j \neq s, \\ \psi_{i+s-1+t-1,r+m-1}^{(l+m+n-2)}(g(b,a) \cdot c), & j = s. \end{cases} \end{cases}$$

where

$$Xterm = (\sum_{k=i}^{i+s-2} \psi_{k,i+s-1}^{(l+m-1)} (g(a,b) + f(a) \cdot b) + \varphi_{i+s-1}^{(l+m-1)} (ab) + \sum_{k=i+s}^{i+m-1} \psi_{i+s-1,k}^{(l+m-1)} (g(b,a) + f(a) \cdot b)) \circ_{i-1+j} \varphi_{t}^{(n)} (c)$$

$$= by (C35),(C38) \begin{cases} \begin{cases} \psi_{i+s-1}^{(l+m-1+n-1)} (g(a,b) + \psi_{i+s-1,i+j-1+t-1}^{(l+m-1+n-1)} (g(a,b) + f(a) \cdot b) \cdot c) \\ \int_{k=i}^{i+s-2} \psi_{i+s-1+t-1}^{(l+m-1+n-1)} (g(a,b) + f(a) \cdot b) \cdot c \\ \int_{k=i+s-1}^{i+s-1+t-2} \psi_{i+s-1+t-1}^{(l+m-1+n-1)} (g(a,b) + f(a) \cdot b) \cdot c \\ & ki+s-1+t-1 \end{pmatrix} (g(a,b) + f(a) \cdot b) \cdot c \\ \int_{k=i+s-1}^{i+s-1+t-2} \psi_{i+s-1+t-1,k}^{(l+m-1+n-1)} (g(c,ab) + f(a) \cdot b) \cdot c \\ & ki+s-1+t-1 \end{pmatrix} (g(c,ab) + f(a) \cdot b) \cdot c \\ & ki+s-1+t-1 \end{pmatrix} (g(c,ab) + f(a) \cdot b) \cdot c \\ & ki+s-1+t-1 \end{pmatrix} (g(c,ab) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,k+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,k+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-1,i+s-1+t-1,k+n-1} (g(a,b) + f(a) \cdot b) \cdot c \\ & \psi_{i+s-1+t-$$

and

$$\begin{aligned} \text{RHS of (E1.1.1)} &= \varphi_r^{(l)}(a) \circ_i \left(\varphi_s^{(m)}(b) \circ_j \varphi_t^{(n)}(c)\right) \\ &= \sup_{\text{by (C35)}} \varphi_r^{(l)}(a) \circ_i \begin{cases} \psi_{s,j+t-1}^{(m+n-1)}(g(b,c)), & s < j, \\ \sum_{k=s}^{s+t-2} \psi_{k,s+t-1}^{(m+n-1)}(g(b,c) + f(b) \cdot_2 c) + \varphi_{s+t-1}^{(m+n-1)}(bc) \\ + \sum_{k=s+t} \psi_{s+t-1,k}^{(m+n-1)}(g(c,b) + f(b) \cdot_1 c), \\ \psi_{j+t-1,s+n-1}^{(m+n-1)}(g(c,b)), & s > j. \end{cases} \\ &= \sup_{\text{by (C36)}} \begin{cases} \begin{cases} 0, & r \neq i, \\ \psi_{r+s-1,r+j+t-1-1}^{(m+n-2)}(ag(b,c)), & r = i. \end{cases} & s = j, \\ \begin{cases} 0, & r \neq i, \\ \psi_{r+s-1,r+j+t-1-1}^{(m+n-2)}(ag(c,b)), & r = i. \end{cases} & s > j. \end{cases} \end{aligned}$$

where

Hence (E1.1.1) holds.

Case 11: $\lambda \in \mathbb{A}$ and $\mu, \nu \in \Phi$. Write $\lambda = \mathbb{1}_l, \ \mu = \varphi_s^{(m)}(b)$, and $\nu = \varphi_t^{(n)}(c)$. Then

LHS of (E1.1.1) =
$$(\mathbbm{1}_{l} \circ \varphi_{s}^{(m)}(b)) \circ (i-1+j) \varphi_{t}^{(n)}(c)$$

$$= \varphi_{i+s-1}^{(l+m-1)}(b) \circ \varphi_{t}^{(n)}(c)$$
by (C32)
$$\begin{cases} \varphi_{i+s-1}^{(l+m+n-2)} (i-1+j+t-1)(g(b,c)), & i+s-1$$

and

$$\begin{aligned} \text{RHS of } (\text{E1.1.1}) &= \mathbbm{1}_{l} \circ (\varphi_{s}^{(m)}(b) \circ \varphi_{t}^{(n)}(c)) \\ &= \mathbbm{1}_{l} \circ (\varphi_{s}^{(m)}(b) \circ \varphi_{t}^{(n)}(c)) \\ &= \mathbbm{1}_{l} \circ (\varphi_{s}^{(m)}(b) \circ \varphi_{t}^{(m)}(c)) \\ &= \mathbbm{1}_{l} \circ (\varphi_{s}^{(m)}(b) \circ (b) \circ (b)$$

Hence (E1.1.1) holds.

Case 12: $\mu \in \mathbb{A}$ and $\lambda, \nu \in \Phi$. Write $\lambda = \varphi_r^{(l)}(a), \mu = \mathbb{1}_m$ and $\nu = \varphi_t^{(n)}(c)$. Then LHS of (E1.1.1) = $(\varphi_t^{(l)}(a) \cap \mathbb{1}_{-}) = \varphi_t^{(n)}(c)$

where

$$Xterm = \sum_{w=r}^{r+m-1} \varphi_w^{(l+m-1)}(a) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c) + \sum_{r \le k_1 < k_2 \le r+m-1} \psi_{k_1 k_2}^{(l+m-1)}(f(a)) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c)$$

$$= \begin{cases} \sum_{w=i}^{i-1+j-1} \varphi_w^{(l+m-1)}(a) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c) + \varphi_{i-1+j}^{(l+m-1)}(a) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c) + \sum_{w=i+j}^{i+m-1} \varphi_w^{(l+m-1)}(a) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c) \\ + \sum_{w=i+j}^{i-1+j-1} \psi_{i-1+j+k}^{(l+m-1)}(f(a)) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c) + \sum_{k=i}^{i-1+j-1} \psi_{k,i-1+j}^{(l+m-1)}(f(a)) \underset{i-1+j}{\circ} \varphi_t^{(n)}(c) \\ + \sum_{w=i}^{i-1+j-1} \psi_{i-1+j+t-1}^{(l+m+n-2)}(g(a,c)) + \sum_{k=i-1+j}^{i-1+j+t-2} \psi_{k,i-1+j+t-1}^{(l+m+n-2)}(g(a,c) + f(a) \underset{i}{:} c) \\ + \varphi_{i-1+j+t-1}^{(l+m+n-2)}(ac) + \sum_{k=i-1+j+t}^{i-1+j+n-1} \psi_{i-1+j+t-1,k}^{(l+m+n-2)}(g(c,a) + f(a) \underset{i}{:} c) \\ + \sum_{w=i+j}^{i+m-1} \psi_{i-1+j+t-1,w+n-1}^{(l+m+n-2)}(g(c,a)) \\ + \sum_{k=i+j}^{i+m-1} \psi_{i-1+j+t-1,w+n-1}^{(l+m-1)}(f(a) \underset{i}{:} c) + \sum_{k=i}^{i-1+j-1} \psi_{k,i-1+j+t-1}^{(l+m-1)}(f(a) \underset{i}{:} c) \end{cases}$$

and

of (E1.1.1) =
$$\varphi_r^{(l)}(a) \circ (\mathbb{1}_m \circ \varphi_t^{(n)}(c))$$

$$= \varphi_r^{(l)}(a) \circ \varphi_j^{(m+n-1)}(c)$$
by (C32) $\varphi_r^{(l+m+n-2)}(g(a,c)),$ $r < i,$

$$= \int_{k=r}^{\psi_{r,i+j+t-1-2}^{(l+m+n-2)}} \varphi_{k,r+j+t-1-1}^{(l+m+n-2)}(g(a,c) + f(a) \circ c) + \varphi_{r+j+t-1-1}^{(l+m+n-2)}(ac)$$
 $r = i,$

$$= \int_{k=r+j+t-1}^{k=r+j+t-1} \varphi_{r+j+t-1-1,k}^{(l+m+n-2)}(g(c,a) + f(a) \circ c),$$
 $r > i.$

Hence (E1.1.1) holds.

RHS

Case 13: $\nu \in \mathbb{A}$ and $\lambda, \mu \in \Phi$. Write $\lambda = \varphi_r^{(l)}(a)$ for $a \in \overline{A}$, $\mu = \varphi_s^{(m)}(b)$ and $\nu = \mathbb{1}_n$. Then

$$\begin{split} \text{LHS of } (\text{E1.1.1}) &= \left(\varphi_r^{(l)}(a) \underset{i}{\circ} \varphi_s^{(m)}(b) \right) \underset{i-1+j}{\circ} \mathbbm{1}_n \\ &= \\ \text{by (C35)} \begin{cases} \psi_{r,i+s-1}^{(l+m-1)}(g(a,b)), & r < i, \\ \frac{r+s-2}{2} \psi_{k,r+s-1}^{(l+m-1)}(g(a,b) + f(a) \underset{i}{_{2}} b) + \varphi_{r+s-1}^{(l+m-1)}(ab) \\ + \underset{k=r+s}{\sum} \psi_{r+s-1,k}^{(l+m-1)}(g(b,a) + f(a) \underset{i}{_{1}} b), & r = i, \\ \psi_{i+s-1,r+m-1}^{(l+m-1)}(g(b,a)), & r > i. \end{cases} \\ \\ &= \\ \text{by (C37)} \begin{cases} \begin{cases} \psi_{i+s-1+n-1}^{(l+m+n-2)}(g(a,b), & j < s, \\ \frac{i+s-1+n-1}{2} \psi_{r,k}^{(l+m+n-2)}(g(a,b), & j > s, \\ \frac{i+s-1+n-1}{2} \psi_{r,k}^{(l+m+n-2)}(g(a,b), & j > s, \\ \frac{i+s-1+n-1}{2} \psi_{r,k}^{(l+m+n-2)}(g(a,b), & j < s, \\ \frac{i+s-1+n-1}{2} \psi_{k,r+m-1+n-1}^{(l+m+n-2)}(g(b,a)), & j < s, \\ \frac{i+s-1+n-1}{2} \psi_{k,r+m-1+n-1}^{(l+m+n-2)}(g(b,a)), & j < s, \\ \frac{i+s-1+n-1}{2} \psi_{k,r+m-1+n-1}^{(l+m+n-2)}(g(b,a)), & j > s \end{cases} \end{cases} \end{split}$$

where

$$Xterm = \Big(\sum_{k=i}^{i+s-2} \psi_{k,i+s-1}^{(l+m-1)}(g(a,b) + f(a) ; b) + \varphi_{i+s-1}^{(l+m-1)}(ab) + \sum_{k=i+s}^{i+m-1} \psi_{i+s-1,k}^{(l+m-1)}(g(b,a) + f(a) ; b), \Big) \underset{k-1+j}{\circ} \mathbb{1}_{n}$$

$$= \left\{ \begin{cases} \sum_{k=i}^{i+s-2+n-1} \psi_{k,i+s-1+n-1}^{(l+m-1+n-1)}(g(a,b) + f(a) ; b), \\ + \varphi_{i+s-1+n-1}^{(l+m-1+n-1)}(ab), & j < s, \end{cases} \right.$$

$$= \left\{ \begin{cases} \sum_{k=i+s+n-1}^{i+s-1+n-1} \psi_{i+s-1+n-1,k}^{(l+m-1+n-1)}(g(b,a) + f(a) ; b), \\ + \sum_{k=i+s+n-1}^{i+s-1+n-1} \psi_{i+s-1+n-1}^{(l+m-1+n-1)}(g(a,b) + f(a) ; b), \\ + \sum_{k=i+s-1}^{i+s-1+n-1} \psi_{k-1,k}^{(l+m-1+n-1)}(g(a,b) + f(a) ; b), \\ + \sum_{k=i+s-1}^{i+s-1+n-1} \varphi_{k}^{(l+m-1+n-1)}(ab) + \sum_{i+s-1 \le k_{1} < k_{2} < i+s-1+n-1}^{i+s-1+n-1} \psi_{k-1,k_{2}}^{(l+m-1+n-1)}(g(b,a) + f(a) ; b), \\ + \sum_{i+s-1}^{i+s-1} \varphi_{k}^{(l+m-1+n-1)}(g(a,b) + f(a) ; b), \\ + \sum_{i+s-1}^{i+s-1} \psi_{i+s-1+n-1}^{(l+m-1+n-1)}(g(a,b) + f(a) ; b), \\ + \sum_{i+s-1}^{i+s-1} \psi_{i+s-1,k}^{(l+m-1+n-1)}(g(b,a) + f(a) ; b), \\ + \sum_{i+s-1}^{i+s-1} \psi_{i+s-1,k}^{(l+m-1+n-1)}(g(b,a) + f(a) ; b), \\ + \sum_{i+s-1}^{i+s-1} \psi_{i+s-1,k}^{(l+m-1+n-1)}(g(b,a) + f(a) ; b), \\ + \sum_{k=i+s}^{i+s-1} \psi_{i+s-1,k}^{(l+m-1+n-1)}(g(b,a) + f(a$$

and

where

$$Yterm = \varphi_{r}^{(l)}(a) \circ \left(\sum_{k=s}^{s+n-1} \varphi_{k}^{(m+n-1)}(b) + \sum_{s \le p_{1} < p_{2} \le s+n-1} \psi_{p_{1}p_{2}}^{(m+n-1)}(f(b))\right)$$

$$= \bigcup_{by (C34), (C37)} \begin{cases} \sum_{k=s}^{i+s-1+n-1} \varphi_{r,k}^{(l+m-1+n-1)}g(a,b) & r < i, \\ \sum_{k=s}^{s+n-1} \varphi_{r,k}^{(l+m-1+n-1)}g(a,b) + f(a) \ge b) + \sum_{k=s}^{s+n-1} \varphi_{r+k-1}^{(l+m-1+n-1)}(ab) & r = i, \\ \sum_{k=s}^{s+n-1} \varphi_{r+k-1}^{(l+m-1+n-1)}\psi_{r+k-1,w}^{(l+m-1+n-1)}g(b,a), & r > i. \end{cases}$$

Hence (E1.1.1) holds.

Case 14: $\lambda \in \Phi$ and $\mu, \nu \in \mathbb{A}$. Write $\lambda = \varphi_k^{(l)}(a), \ \mu = \mathbb{1}_m$ and $\nu = \mathbb{1}_n$. Then

where

$$Xterm = \left(\sum_{k \le k_1 < k_2 \le k+m-1} \psi_{k_1 k_2}^{(l+m-1)}(f(a))\right) \underset{k-1+j}{\circ} \mathbb{1}_n$$

$$= \begin{cases} \sum_{k \le k_1 < k_2 \le k+j-1} \psi_{k_1 k_2}^{(l+m+n-2)}(f(a))) \\ + \sum_{k \le k_1 < k+j-1 \le k_2 < k+j-1+n-1} \psi_{k_1 k_2}^{(l+m+n-2)}(f(a))) \\ + \sum_{k_1 < k+j-1 < k_2} \psi_{k_1, k_2+n-1}(f(a)) \\ + \sum_{k+j-1 < k_1 < k_2} \psi_{k_1 + n-1, k+j-1}^{(l+m+n-2)}(f(a))) \\ + \sum_{k \le k+j-1 < k_1 < k_2} \psi_{k_1 + n-1, k_2 + n-1}^{(l+m+n-2)}(f(a))) \end{cases}$$

and

$$\begin{aligned} \text{RHS of } (\text{E1.1.1}) &= \varphi_k^{(l)}(a) \underset{i}{\circ} \left(\mathbbm{1}_m \underset{j}{\circ} \mathbbm{1}_n \right) \\ &= \varphi_k^{(l)}(a) \underset{i}{\circ} \mathbbm{1}_{m+n-1} \\ &= \\ & \underset{\text{by (C34)}}{=} \begin{cases} \varphi_k^{(l+m+n-2)}(a), & k < i, \\ \sum_{\substack{w = k \\ \varphi_k^{(l+m+n-2)}(a)} + \sum_{k \le l_1 < l_2 \le k+m+n-2} \psi_{l_1 l_2}^{(l+m+n-2)}(f(a))) & k = i, \\ \varphi_{k+m+n-2}^{(l+m+n-2)}(a), & k > i. \end{cases} \end{aligned}$$

Hence (E1.1.1) holds.

Case 15: $\mu \in \Phi$ and $\lambda, \nu \in \mathbb{A}$. Write $\mu = \varphi_k^{(m)}(a), \lambda = \mathbb{1}_l$ and $\nu = \mathbb{1}_n$. Then

LHS of (E1.1.1) =
$$(\mathbb{1}_{l} \circ \varphi_{k}^{(m)}(a)) \circ \mathbb{1}_{i-1+j} \mathbb{1}_{n} = \varphi_{k+i-1}^{(m+l-1)}(a) \circ \mathbb{1}_{n}$$

$$= \sup_{\text{by (C34)}} \begin{cases} \varphi_{k+i-1}^{(m+l+n-2)}(a) & k < j \\ \varphi_{k+i-1}^{(m+l+n-2)}(a) + \sum_{k+i-1 \le k_{1} < k_{2} < k+i+n-2} \psi_{k_{1}k_{2}}^{(m+l+n-2)}(f(a)) & k = j \\ \varphi_{k+i+n-2}^{(m+l+n-2)}(a) & k > j \end{cases}$$

RHS of (E1.1.1) = $\mathbb{1}_l \underset{i}{\circ} (\varphi_k^{(m)}(a)) \underset{j}{\circ} \mathbb{1}_n)$

$$= \underbrace{\mathbb{1}_{l}}_{by} \underbrace{\left(\text{C34} \right)}_{(\text{C32})} \underbrace{\mathbb{1}_{i}}_{i} \underbrace{\left\{ \begin{array}{l} \varphi_{k}^{(m+n-1)}(a) \\ k+n-1 \\ s=k \\ \varphi_{k+n-1}^{(m+n-1)}(a) \\ \varphi_{k+n-1}^{(m+n-1)}(a) \\ \varphi_{k+n-1}^{(m+l+n-2)}(a) \\ s=k \\ \sum_{i=k+i-1}^{k+i+n-2} \varphi_{k-i+1}^{(m+l+n-2)}(a) + \sum_{i=k+i-1 \le k_{1} < k_{2} < k+i+n-2} \psi_{k_{1}k_{2}}^{(m+l+n-2)}(f(a)) \\ z=k \\ \varphi_{k+i+n-2}^{(m+l+n-2)}(a) \\ \varphi_{k+i+n-2}^{(m+l+n-2)}(a) \\ \varphi_{k+i+n-2}^{(m+l+n-2)}(a) \\ z=k \\$$

Hence (E1.1.1) holds.

Case 16: $\nu \in \Phi$ and $\lambda, \mu \in \mathbb{A}$. Write $\nu = \varphi_k^{(n)}(a), \lambda = \mathbb{1}_l$ and $\mu = \mathbb{1}_m$. Then

LHS of (E1.1.1) =
$$(\mathbb{1}_{l} \circ \mathbb{1}_{m}) \circ \varphi_{k}^{(n)}(a) = \mathbb{1}_{l+m-1} \circ \varphi_{k}^{(n)}(a)$$

$$= \sup_{\text{by (C32)}} \varphi_{k+i+j-2}^{(n+l+m-2)}(a)$$
RHS of (E1.1.1) = $\mathbb{1}_{l} \circ (\mathbb{1}_{m} \circ \varphi_{k}^{(n)}(a)) = \mathbb{1}_{l} \circ \varphi_{k+j+i-2}^{(n)}(a)$

$$= \varphi_{k+j+i-2}^{(n+m+l-2)}(a).$$

Hence (E1.1.1) holds.

Case 17: $\lambda \in \Psi$ and $\mu, \nu \in \mathbb{A}$. Write $\lambda = \psi_{k_1,k_2}^{(l)}(y), \ \mu = \mathbb{1}_m$ and $\nu = \mathbb{1}_n$. Then

$$\begin{split} \text{LHS of } (\text{E1.1.1}) &= \left(\psi_{k_1,k_2}^{(l)}(y) \stackrel{\circ}{\circ} \mathbbm{1}_m \right) \stackrel{\circ}{\underset{i-1+j}{\circ}} \mathbbm{1}_n \\ &= \\ \text{by } (\text{C37}) \begin{cases} \psi_{k_1+m-1}^{(l+m-1)} \stackrel{(l+m-1)}{\underset{k=k_1}{\circ}} \psi_{k,k_2+m-1}^{(l+m-1)}(y), & 1 \leq i < k_1, \\ \psi_{k_1,k_2+m-1}^{(l+m-1)}(y), & k_1 < i < k_2, \\ \psi_{k_1,k_2}^{(l+m-1)}(y), & k_2 < i \leq l. \end{cases} \\ &= \\ \text{by } (\text{C37}) \begin{cases} \begin{cases} \psi_{k_1+m-1}^{(l+m-1)}(y), & k_1 < i < k_2, \\ \psi_{k_1,k_2}^{(l+m-1)}(y), & k_2 < i \leq l. \end{cases} \\ &= \\ \begin{cases} \begin{cases} \psi_{k_1+m-1}^{(l+m+n-2)}(y), & k_2 < i \leq l. \end{cases} \\ \begin{cases} \psi_{k_1+k_2+m-1}^{(l+m+n-2)}(y), & k_2 < i \leq l. \end{cases} \\ &= \\ \end{cases} \\ &= \\ \text{by } (\text{C37}) \end{cases} \begin{cases} \begin{cases} \psi_{k_1+m-1}^{(l+m+n-2)}(y), & k_2 < i \leq l. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_2 < i < l. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_1 \leq i - 1 + j < k_1 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_1 \leq i - 1 + j < k_2 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_1 \leq i - 1 + j < k_2 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_1 \leq i - 1 + j < k_2 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_1 \leq i - 1 + j < k_2 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_1 \leq i - 1 + j < k_2 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-1}(y), & k_2 \leq i - 1 + j < k_2 + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1+n-2}(y), & k_2 \leq i - 1 + j < l + m - 1, \\ (impossible), & others. \end{cases} \\ &= \\ \begin{cases} \psi_{k_1,k_2+m-1}(y), & k_2 \leq i - 1 + j < l + m - 1, \\ (impossible), & others. \end{cases} \\ \end{cases} \end{cases} \\ &= \\ \end{cases} \end{cases} \end{cases}$$

and

$$\begin{aligned} \text{RHS of (E1.1.1)} &= \psi_{k_1,k_2}^{(l)}(y) \underset{i}{\circ} \left(\mathbbm{1}_m \underset{j}{\circ} \mathbbm{1}_n\right) \\ &= \underset{\text{by (C31)}}{=} \psi_{k_1,k_2}^{(l)}(y) \underset{i}{\circ} \mathbbm{1}_{m+n-1} \\ \\ &= \underset{\text{by (C34)}}{=} \begin{cases} \psi_{k_1,k_2}^{(l+m+n-2)}(y) \underset{i=k_1, \\ k_1+m+n-2 \\ k_1, k_2+m+n-2 \\ k_1, k_2+m+n-2 \\ (k_1, k_2+m+n-2) \\ k_1, k_2 \\ k_1, k_2 \\ k_1, k_1 \\ k_1, k_2 \\ k_1, k_2 \\ k_1, k_1 \\ k_1, k_2 \\ k_1, k_2 \\ (k_1, k_2 \\ k_1, k_2 \\ k_1, k_2 \\ (k_1, k_2 \\ k_1, k_2 \\ (k_1, k_2 \\ (k$$

Hence (E1.1.1) holds.

Case 18: $\mu \in \Psi$ and $\lambda, \nu \in \mathbb{A}$. Write $\mu = \psi_{k_1,k_2}^{(m)}(y), \lambda = \mathbb{1}_l$ and $\nu = \mathbb{1}_n$. Then

LHS of (E1.1.1) =
$$(\mathbb{1}_{l} \circ \psi_{k_{1},k_{2}}^{(m)}(y)) \circ \mathbb{1}_{i-1+j} \mathbb{1}_{n} = \psi_{k_{1}+i-1,k_{2}+i-1}^{(l+m-1)}(y) \circ \mathbb{1}_{n}$$

= $\psi_{k_{1}+i-1}^{(l+m+n-2)}(y) = (k_{1}, k_{1}+i+n-2) = (k_{1}, k_{2}+i+n-2) = k_{1}, k_{2}+i+n-2 = (k_{1}, k_{2}+i+n-2) = k_{1}, k_{2}+i+n-2 = (k_{1}, k_{2}+i+n-2) = k_{1}, k_{2}+i+n-2 = (k_{1}, k_{2}+i+n-2) = (k_{1}, k_{2}+i+n-2)$

and

RHS of (E1.1.1) =
$$\mathbb{1}_{l} \circ (\psi_{k_1,k_2}^{(m)}(y) \circ \mathbb{1}_n)$$

 $\begin{pmatrix} \psi_{k_1,k_1}^{(m+n-1)}(y) \circ \mathbb{1}_n \\ \psi_{k_1+n-1}^{(m+n-1)}(y) \circ \mathbb{1}_n \end{pmatrix}$

$$= \underset{\text{by (C37)}}{=} \mathbbm{1}_{l} \circ \begin{cases} \psi_{k_{1}+n-1,k_{2}+n-1}^{(m+n-1)}(y), & 1 \leq j < k_{1}, \\ k_{1}+n-1,k_{2}+n-1}^{k_{1}+n-1,k_{2}+n-1}(y), & j = k_{1}, \\ \psi_{k_{1},k_{2}+n-1}^{(m+n-1)}(y), & k_{1} < j < k_{2}, \\ k_{2}+n-1 \\ \sum & \psi_{k_{1},w}^{(m+n-1)}(y), & k_{2} < j \leq m \end{cases} \\ = \underset{\text{by (C33)}}{=} \begin{cases} \psi_{k_{1}+i+n-2,k_{2}+i+n-2}^{(l+m+n-1)}(y), & 1 \leq j < k_{1}, \\ k_{1}+i+n-2,k_{2}+i+n-2}(y), & 1 \leq j < k_{1}, \\ k_{1}+i+n-2,k_{2}+i+n-2}(y), & j = k_{1}, \\ \psi_{k_{1}+i-1,k_{2}+i+n-2}(y), & j = k_{1}, \\ \psi_{k_{1}+i-1,k_{2}+i+n-2}(y), & k_{1} < j < k_{2}, \\ \sum & \psi_{k_{1}+i-1,k_{2}+i+n-2}(y), & k_{1} < j < k_{2}, \\ \psi_{k_{1}+i-1,k_{2}+i-1}(y), & j = k_{2}, \\ \psi_{k_{1}+i-1,k_{2}+i-1}(y), & k_{2} < j \leq m. \end{cases} \end{cases}$$

Hence (E1.1.1) holds.

$$\begin{aligned} \text{Case 19: } \nu \in \Psi \text{ and } \lambda, \mu \in \mathbb{A}. \text{ Write } \nu = \psi_{k_1,k_2}^{(n)}(y), \lambda = \mathbbm{1}_l \text{ and } \mu = \mathbbm{1}_m. \text{ Then} \\ \text{LHS of (E1.1.1)} = (\mathbbm{1}_l \circ \mathbbm{1}_m) \circ \psi_{i-1+j}^{(n)} \psi_{k_1,k_2}^{(n)}(y) = \mathbbm{1}_{l+m-1} \circ \psi_{k_1,k_2}^{(n)}(y) \\ &= \psi_{i-1+j}^{(n+l+m-2)} \psi_{k_1+i+j-2,k_2+i+j-2}^{(n)}(y) \\ \text{RHS of (E1.1.1)} = \mathbbm{1}_l \circ (\mathbbm{1}_m \circ \psi_{k_1,k_2}^{(n)}(y) = \mathbbm{1}_l \circ \psi_{k_1+j-1,k_2+j-1}^{(n+m-1)}(y) \\ &= \psi_{i-1+j-2,k_2+i+j-2}^{(n+l+m-2)}(y) \end{aligned}$$

Hence (E1.1.1) holds.

Case 20: $\lambda, \mu, \nu \in \mathbb{A}$. Equation (E1.1.1) follows from (C31).

Verification of (OP3): There are two equations in (OP3). We only check the first one in (OP3), namely, the following equation

(E1.1.3)
$$\mu \circ (\nu * \sigma) = (\mu \circ \nu) * \sigma',$$

for all $\mu \in \mathcal{P}(m), \nu \in \mathcal{P}(n)$, and $\sigma \in \mathbb{S}_n$. Note that σ' is $\lim_{i \to \infty} \sigma$. Write

(E3.6.1)
$$\sigma = \begin{pmatrix} k_1, & k_2, & \cdots & k_n \end{pmatrix}$$

where by convention $k_w = \sigma^{-1}(w)$ for all w. Then, by definition,

(E3.6.2)
$$\sigma' = (1, \dots, i-1, k_1+i-1, k_2+i-1, \dots, k_n+i-1, i+n, \dots, n+m-1).$$

By (E3.6.2), $(\sigma')^{-1}(s) = s$ if s < i and $s \ge i + n$ and $(\sigma')^{-1}(k + i - 1) = \sigma^{-1}(k) + i - 1$ for $1 \le k \le n$. We refer to [BYZ, Section 8] for more details concerning σ' .

If both μ and ν are in A, then (E1.1.3) follows easily from (OP1). We have the following 8 cases to consider.

Case 1:
$$\mu \in \Phi$$
 and $\nu \in \mathbb{A}$. Write $\mu = \varphi_k^{(m)}(a)$ and $\nu = \mathbb{1}_n$. Then
LHS of (E1.1.3) = $\varphi_k^{(m)}(a) \mathop{\circ}_i (\mathbb{1}_n * \sigma) \underset{\text{by (C33)}}{=} \varphi_k^{(m)}(a) \mathop{\circ}_i \mathbb{1}_n$

$$= \sup_{\text{by (C34)}} \begin{cases} \varphi_k^{(m+n-1)}(a), & k < i, \\ \sum_{\substack{w=k \\ \varphi_k^{(m+n-1)}(a), \\ \varphi_{k-n-1}^{(m+n-1)}(a), \\ \varphi_{k-n-1}^{(m+n-$$

and in the following computation we use the fact that f(a) * (2, 1) = f(a) [Definition 3.4(3)] and notation $(k'_1, k'_2) = ((\sigma')^{-1}(k_1), (\sigma')^{-1}(k_2))$ or $((\sigma')^{-1}(k_2), (\sigma')^{-1}(k_1))$,

$$\begin{split} \text{RHS of } (\text{E1.1.3}) &= \left(\varphi_k^{(m)}(a) \underset{i}{\circ} \mathbbm{1}_n \right) * \sigma' \\ &= \left\{ \begin{array}{ll} \sum_{\substack{k+n-1 \\ w=k}}^{\varphi_k^{(m+n-1)}(a),} & k < i, \\ \sum_{\substack{w=k \\ w=k-1}}^{w+n-1} \varphi_k^{(m+n-1)}(a) + \sum_{\substack{k \le k_1 < k_2 \le k+n-1}} \psi_{k_1 k_2}^{(m+n-1)}(f(a)), & k = i, \\ \varphi_{k+n-1}^{(m+n-1)}(a), & k > i. \end{array} \right. \end{split} \\ \\ \text{by (C22) and (C23)} \begin{cases} \varphi_{(\sigma')}^{(m+n-1)}(a), & k < i, \\ \sum_{\substack{w=k \\ \varphi_{(\sigma')}^{(m+n-1)}(a), \\ \varphi_{(\sigma')}^{(m+n-1)}(a), \\ \varphi_{(\sigma')}^{(m+n-1)}(a), \\ \varphi_{(\sigma')}^{(m+n-1)}(a), \\ \varphi_{k+n-1}^{(m+n-1)}(a), \\ \varphi_{k+n-1}^{(m+n-1)}(a),$$

Hence (E1.1.3) holds.

Case 2: $\mu \in \Psi$ and $\nu \in \mathbb{A}$. Write $\mu = \psi_{k_1k_2}^{(m)}(y)$ and $\nu = \mathbb{1}_n$. Then

LHS of (E1.1.3) =
$$\psi_{k_1k_2}^{(m)}(y) \circ (\mathbb{1}_n * \sigma) =_{\text{by (C33)}} \psi_{k_1k_2}^{(m)}(y) \circ \mathbb{1}_n$$

= $\psi_{k_1k_1}^{(m+n-1)}(y), \quad 1 \le i < k_1,$
 $\sum_{\substack{w=k_1 \\ \psi_{k_1,k_2+n-1}(y), \\ k_1, k_2+n-1}(y), \quad k_1 < i < k_2,$
 $\psi_{k_1,k_2+n-1}^{(m+n-1)}(y), \quad i = k_2,$
 $\psi_{k_1k_2}^{(m+n-1)}(y), \quad k_2 < i \le m,$

Hence (E1.1.3) holds.

Case 3: $\mu \in \mathbb{A}$ and $\nu \in \Phi$. Write $\mu = \mathbb{1}_m$ and $\nu = \varphi_k^{(n)}(a)$. Then

LHS of (E1.1.3) =
$$\mathbb{1}_m \circ (\varphi_k^{(n)}(a) * \sigma) =_{\text{by (C22)}} \mathbb{1}_m \circ \varphi_{\sigma^{-1}(k)}^{(n)}(a)$$

= $\varphi_{\sigma^{-1}(k)+i-1}^{(m+n-1)}(a),$

and

RHS of (E1.1.3) =
$$(\mathbb{1}_m \circ \varphi_k^{(n)}(a)) * \sigma' = \varphi_{k+i-1}^{(m+n-1)}(a) * \sigma'$$

= $\varphi_{(\sigma')^{-1}(k+i-1)}^{(m+n-1)}(a) = \varphi_{\sigma^{-1}(k)+i-1}^{(m+n-1)}(a).$

Hence (E1.1.3) holds.

Case 4: $\mu \in \mathbb{A}$ and $\nu \in \Psi$. Write $\mu = \mathbb{1}_m$ and $\nu = \psi_{k_1k_2}^{(n)}(y)$. Then

LHS of (E1.1.3) =
$$\mathbb{1}_{m} \circ \left(\psi_{k_{1}k_{2}}^{(n)}(y) * \sigma\right) = \underset{\text{by (C23)}}{=} \mathbb{1}_{m} \circ \begin{cases} \psi_{\sigma^{-1}(k_{1})\sigma^{-1}(k_{2})}^{(n)}(y) & \sigma^{-1}(k_{1}) < \sigma^{-1}(k_{2}) \\ \psi_{\sigma^{-1}(k_{2})\sigma^{-1}(k_{1})}^{(n)}(y * (2, 1)) & \sigma^{-1}(k_{1}) > \sigma^{-1}(k_{2}) \end{cases}$$

=
$$\begin{cases} \psi_{\sigma^{-1}(k_{1})+i-1,\sigma^{-1}(k_{2})+i-1}^{(m+n-1)}(y) & \sigma^{-1}(k_{1}) < \sigma^{-1}(k_{2}) \\ \psi_{\sigma^{-1}(k_{2})+i-1,\sigma^{-1}(k_{1})+i-1}^{(n)}(y * (2, 1)) & \sigma^{-1}(k_{1}) > \sigma^{-1}(k_{2}) \end{cases}$$

and

$$\begin{aligned} \text{RHS of } (\text{E1.1.3}) &= \left(\mathbbm{1}_m \underset{i}{\circ} \psi_{k_1 k_2}^{(n)}(y)\right) * \sigma' \\ &= \underset{\text{by (C33)}}{=} \psi_{k_1 + i - 1, k_2 + i - 1}^{(m+n-1)}(y) * \sigma' \\ &= \underset{\psi_{(\sigma')^{-1}(k_1 + i - 1), (\sigma')^{-1}(k_2 + i - 1)}{\{\psi_{(\sigma')^{-1}(k_1 + i - 1), (\sigma')^{-1}(k_1 + i - 1)}(y) - (\sigma')^{-1}(k_1 + i - 1) < (\sigma')^{-1}(k_2 + i - 1)} \\ &= \begin{cases} \psi_{(\sigma')^{-1}(k_2 + i - 1), (\sigma')^{-1}(k_1 + i - 1)}(y) - (\sigma')^{-1}(k_1 + i - 1) > (\sigma')^{-1}(k_2 + i - 1)} \\ \psi_{(\sigma')^{-1}(k_2 + i - 1), (\sigma')^{-1}(k_1 + i - 1)}(y) - (\sigma')^{-1}(k_1 + i - 1) > (\sigma')^{-1}(k_2 + i - 1)} \\ &= \begin{cases} \psi_{(m+n-1)}^{(m+n-1)} & \varphi_{(m+n-1)}^{(m+n-1)}(y) - \varphi_{(m+1)}(y) - \varphi_{(m+1)}(y) - \varphi_{(m+1)}(y)} \\ \psi_{(\sigma')^{-1}(k_2 + i - 1, \sigma^{-1}(k_1) + i - 1)}(y + (2, 1)) - \varphi_{(m+1)}(y) - \varphi_{(m+1)}(y)} \\ \end{cases} \end{aligned}$$

Hence (E1.1.3) holds.

and

Case 5: $\mu \in \Phi$ and $\nu \in \Phi$. Write $\mu = \varphi_{k_1}^{(m)}(a)$ and $\nu = \varphi_{k_2}^{(n)}(b)$. Then

$$\begin{aligned} \text{LHS of } (\text{E1.1.3}) &= \varphi_{k_1}^{(m)}(a) \circ (\varphi_{k_2}^{(n)}(b) * \sigma) \underset{\text{by (C22)}}{=} \varphi_{k_1}^{(m)}(a) \circ \varphi_{\sigma^{-1}(k_2)}^{(n)}(b) \\ &= \\ &= \\ &= \\ &= \\ & \text{by (C35)} \begin{cases} \psi_{k_1 + \sigma^{-1}(k_2) - 1}^{(m+n-1)}(g(a,b)), & k_1 < i, \\ k_1 + \sigma^{-1}(k_2) - 2} \psi_{k,k_1 + \sigma^{-1}(k_2) - 1}^{(m+n-1)}(g(a,b) + f(a) \cdot b) + \varphi_{k_1 + \sigma^{-1}(k_2) - 1}^{(m+n-1)}(ab)} \\ &= \\ & + \sum_{\substack{k = k_1 + \sigma^{-1}(k_2) - 1, k}} \psi_{k_1 + \sigma^{-1}(k_2) - 1, k}^{(m+n-1)}(g(b,a) + f(a) \cdot b) + \varphi_{k_1 + \sigma^{-1}(k_2) - 1}^{(m+n-1)}(ab)} \\ &= \\ & + \sum_{\substack{k = k_1 + \sigma^{-1}(k_2) - 1, k}} \psi_{k_1 + \sigma^{-1}(k_2) - 1, k}^{(m+n-1)}(g(b,a) + f(a) \cdot b), \\ & \psi_{i + \sigma^{-1}(k_2) - 1, k_1 + n - 1}^{(m+n-1)}(g(b,a)), & k_1 > i, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \text{RHS of } (\text{E1.1.3}) &= \left(\varphi_{k_{1}}^{(m)}(a) \underset{i}{\circ} \varphi_{k_{2}}^{(n)}(b)\right) * \sigma' \\ &= \left\{ \begin{array}{ll} \begin{cases} \psi_{k_{1},i+k_{2}-1}^{(m+n-1)}(g(a,b)), & k_{1} < i, \\ k_{1}+k_{2}-2} \\ \sum_{k=k_{1}}^{k_{1}+k_{2}-1}(g(a,b) + f(a) ; b) + \varphi_{k_{1}+k_{2}-1}^{(m+n-1)}(ab) \\ k_{1} = i, & * \sigma' \end{cases} \\ &+ \underbrace{k_{1}+k_{2}}_{k=k_{1}+k_{2}} \underbrace{\psi_{k_{1}+k_{2}-1,k}^{(m+n-1)}(g(b,a) + f(a) ; b), \\ \psi_{i+k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(b,a)), & k_{1} > i. \end{cases} \\ &= \left\{ \begin{array}{ll} \psi_{i+k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(b,a)), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(a,b)), & k_{1} > i. \end{array} \right. \end{cases} \\ &= \left\{ \begin{array}{ll} \psi_{i+k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(b,a)), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(a,b)), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(a,b)), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{2}}^{(m+n-1)}(g(a,b)), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{2}}^{(m+n-1)}(g(a,b), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{2}}^{(m+n-1)}(g(b,a)), & k_{1} > i. \\ & \psi_{i+k_{2}-1,k_{2}-1,k_{1}+n-1}^{(m+n-1)}(g(b,a)), & k_{1} > i. \end{array} \right\} \end{aligned}$$

Hence (E1.1.3) holds.

 $\begin{aligned} \mathsf{Case 6:} \ \mu \in \Phi \ \mathsf{and} \ \nu \in \Psi. & \text{Write } \mu = \varphi_h^{(m)}(a) \ \mathsf{and} \ \nu = \psi_{k_1 k_2}^{(n)}(y). \ \text{Then} \\ \text{LHS of (E1.1.3)} = \varphi_h^{(m)}(a) \mathop{\circ}_i (\psi_{k_1 k_2}^{(n)}(y) * \sigma) &= \\ & \varphi_h^{(m)}(a) \mathop{\circ}_i \psi_{\sigma^{-1}(k_1)\sigma^{-1}(k_2)}^{(m)}(y) \\ & = \\ & \varphi_{\mu^{(C36)}}^{(m+n-1)} \begin{cases} 0, & h \neq i, \\ \psi_{h+\sigma^{-1}(k_1)-1,h+\sigma^{-1}(k_1)-1}^{(m+n-1)}(ay), & h = i, \end{cases} \end{aligned}$

and

RHS of (E1.1.3) =
$$(\varphi_h^{(m)}(a) \circ \psi_{k_1k_2}^{(n)}(y)) * \sigma'$$

$$= \int_{\text{by (C36)}} \begin{cases} 0, & h \neq i, \\ \psi_{h+k_1-1,h+k_1-1}^{(m+n-1)}(ay), & h = i, \end{cases} * \sigma'$$

$$= \int_{\text{by (C23)}} \begin{cases} 0, & h \neq i, \\ \psi_{h+\sigma^{-1}(k_1)-1,h+\sigma^{-1}(k_1)-1}(ay), & h = i. \end{cases}$$

Hence (E1.1.3) holds.

$$\begin{aligned} \mathsf{Case 7:} \ \mu \in \Psi \ \mathsf{and} \ \nu \in \Phi. \quad & \text{Write} \ \mu = \psi_{k_1 k_2}^{(m)}(y) \ \text{and} \ \nu = \varphi_h^{(n)}(a). \ \text{Then} \\ \text{LHS of (E1.1.3)} = \psi_{k_1 k_2}^{(m)}(y) \mathop{\circ}_i (\varphi_h^{(n)}(a) * \sigma) = \underset{\text{by (C22)}}{=} \psi_{k_1 k_2}^{(m)}(y) \mathop{\circ}_i \varphi_{\sigma^{-1}(h)}^{(n)}(a) \\ & = \underset{\text{by (C38)}}{=} \begin{cases} 0, & i \neq k_1, k_2, \\ \psi_{k_1 k_2 - n^{-1}(h) - 1, k_2 + n - 1}^{(m+n-1)}(y \mathop{:}_i a), & i = k_1, \\ \psi_{k_1, k_2 + \sigma^{-1}(h) - 1}^{(m+n-1)}(y \mathop{:}_2 a), & i = k_2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \text{RHS of (E1.1.3)} &= (\psi_{k_1k_2}^{(m)}(y) \circ \varphi_h^{(n)}(a)) * \sigma' \\ &= \\ &= \\ &= \\ &= \\ &\text{by (C38)} \begin{cases} 0, & i \neq k_1, k_2, \\ \psi_{k_1+h-1,k_2+n-1}^{(m+n-1)}(y_{\frac{1}{2}}a), & i = k_1, \\ \psi_{k_1,k_2+h-1}^{(m+n-1)}(y_{\frac{1}{2}}a), & i = k_2. \end{cases} \\ &= \\ &= \\ &= \\ &\text{by (C23)} \begin{cases} 0, & i \neq k_1, k_2, \\ \psi_{k_1+\sigma^{-1}(h)-1,k_2+n-1}^{(m+n-1)}(y_{\frac{1}{2}}a), & i = k_1, \\ \psi_{k_1,k_2+\sigma^{-1}(h)-1}^{(m+n-1)}(y_{\frac{1}{2}}a), & i = k_2. \end{cases} \end{aligned}$$

Hence (E1.1.3) holds.

Case 8: $\mu \in \Psi$ and $\nu \in \Psi$. In this case both sides of (E1.1.3) are zero, so (E1.1.3) holds.

Combining all these cases, \mathcal{P} is a 2-unitary operad. We define a morphism $u_{\mathcal{P}} : \mathcal{C}om \to \mathcal{P}$ by sending $\mathbb{1}_m \to \mathbb{1}_m$ for all $m \ge 0$. Note that $u_{\mathcal{P}}$ is an operadic morphism by (C21) and (C31).

Let $\{a_j\}_{j=1}^d$ be a basis of \overline{A} and $\{\mu_k\}_{k=1}^m$ a basis of M where d is the dimension of \overline{A} and m is the dimension of M. By construction,

$$\{\mathbb{1}_n, \varphi_i^{(n)}(a_j) := \mathbb{1}_n \underset{i}{\circ} a_j, \psi_{i_1 i_2}^{(n)}(\mu_k) := (\mathbb{1}_{n-1} \underset{1}{\circ} \mu_k) * c_{i_1 i_2} \mid i \in [n], j \in [d], k \in [m], 1 \le i_1 < i_2 \le n\}$$

is a k-basis of $\mathcal{P}(n).$ As a consequence, the generating function of $\mathcal P$ is

$$G_{\mathcal{P}}(t) = \sum_{n=0}^{\infty} (1 + dn + m \frac{n(n-1)}{2})t^n = \frac{1}{1-t} + d\frac{t}{(1-t)^2} + m \frac{t^2}{(1-t)^3}.$$

GK-dimension 3.

Therefore \mathcal{P} has GK-dimension 3.

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