A note on the Hurwitz problem and cone spherical metrics

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Abstract: We are motivated by cone spherical metrics on compact Riemann surfaces of positive genus to solve a special case of the Hurwitz problem. Precisely speaking, letting d, g and ℓ be three positive integers and Λ be the following collection of $(\ell+2)$ partitions of a positive integer d:

$$(a_1,\cdots,a_p),\,(b_1,\cdots,b_q),\,(m_1+1,1,\cdots,1),\cdots,(m_\ell+1,1,\cdots,1),$$

where $(m_1,\cdots,m_\ell)$ is a partition of $p+q-2+2g$, we prove that there exists a branched cover from some compact Riemann surface of genus g to the Riemann sphere $\mathbb{P}^1$ with branch data Λ. An analogue for the genus-zero case was found by the first two authors (*Algebra Colloq.* **27** (2020), no. 2, 231-246), who were stimulated by such metrics on $\mathbb{P}^1$ and conjectured the veracity of the above statement there.
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1 Introduction

A branched cover f of degree $d>1$ from a compact Riemann surface to another one gives a collection Λ of finitely many partitions of d of length less than d, called the *branch data* of f. Moreover, Λ satisfies the celebrated Riemann-Hurwitz formula. On the other hand, there also exist many collections of partitions of d, called *exceptions*, which satisfy Riemann-Hurwitz formula but fail to be the branch data of any branched cover. The Hurwitz problem with a history more than 130 years asks for *enumerating branched covers with prescribed branch data with respect to some equivalence relation.* A simplified version of the problem is *whether such a branched cover* *exists with a given branch data*, which was solved [7, Proposition 3.3] in the sense that there exist no exception if the target surface has positive genus. We mainly focus in this manuscript on branched covers to the Riemann sphere $\mathbb{P}^1$. In this case, both the original problem and the simplified version are still widely open although many mathematicians have already obtained a lot of understandings about both the branch data and the exceptions. See the classical [2, 7, 10, 11, 12, 14, 15, 17, 18, 31, 33] and the more recent [1, 16, 21, 22, 23, 24, 25, 26, 27, 35, 36, 32] about this problem. The new results we obtain are all limited to the simplified version of the Hurwitz problem. We use the notations in [4, 32] in this manuscript.

Cone spherical metrics are constant curvature $+1$ conformal metrics with finitely many cone singularities on compact Riemann surfaces. The existence problem of such metrics with prescribed cone singularities on compact Riemann surfaces has been open since Troyanov [34] proposed it formally and solved the sub-critical case in 1980s. Such a metric is called *reducible* if and only if some developing map of it has monodromy in $\mathrm{U}(1)$; otherwise, it is called *irreducible*. Q. Chen, W. Wang, Y. Wu and the second author [4] characterized reducible (cone spherical) metrics in terms of meromorphic one-forms with simple poles and periods in $\sqrt{-1}\mathbb{R}$, called *unitary one-forms* on compact Riemann surfaces. In particular, the cone angles of a reducible metric are determined by the residues of poles and the multiplicities of zeros of a unitary one-form ([4, Theorem 1.5]).

A unitary one-form on $\mathbb{P}^1$ has residues in $\mathbb{R}\setminus\{0\}$ and satisfies both the residue theorem and the degree condition, i.e. the sum of multiplicities of its zeros equals the number of its simple poles minus 2. To obtain the angle constraint for reducible metrics with cone angles lying in $2\pi\mathbb{Q}_{>0}$ on $\mathbb{P}^1$, based on the draft of [4], the second author found in 2014 that it suffices to prove the following fact

Fact 1 *Let $(a_1, \cdots, a_p)$ and $(b_1, \cdots, b_q)$ be two partitions of positive integer $d > 1$, and $(m_1, \cdots, m_\ell)$ a partition of $p+q-2 > 0$. Then there exists a unitary one-form ω on $\mathbb{P}^1$ such that it has the $p+q$ residues of $a_1, \cdots, a_p, -b_1, \cdots, -b_q$ and ℓ zeros with multiplicities $m_1, \cdots, m_\ell$, respectively, if and only if*

$$\max(m_1, \cdots, m_\ell) < \frac{d}{\mathrm{GCD}(a_1, \cdots, a_p, b_1, \cdots, b_q)}. \tag{1}$$

Given such a one-form ω, the second author observed in 2014 that the solution of the ODE

$$\frac{\mathrm{d}f}{f} = \omega$$

is a unique rational function f on $\mathbb{P}^1$ up to a multiple, from which the necessary part (1) of this fact holds. The second author reduced the sufficient part of Fact 1 to a special case of the Hurwitz problem and solved the latter affirmatively in 2015 joint with the first author. See [32, Theorem 1.1], i.e. Case 1 of Theorem 1.1. Then, taking the logarithmic differential $\frac{\mathrm{d}f}{f}$ with respect to the branched cover $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ in Case 1 of Theorem 1.1, we obtain the desired one-form ω in Fact 1. Eremenko

[8] used this case of the Hurwitz problem solved by the first and the second authors and the theory of o-minimal structures to obtain the angle constraint for reducible metrics on $\mathbb{P}^1$. Mondello-Panov [19, Theorem C] used parabolic rank two stable bundles to give the angle constraint for irreducible metrics on $\mathbb{P}^1$ a little bit earlier than Eremenko. Therefore, combining these, we obtain the angle constraint for cone spherical metrics on $\mathbb{P}^1$. X. Zhu [36] used this angle constraint to find infinitely many new exceptions for branched covers from $\mathbb{P}^1$ to itself.

Mondello-Panov [20, Theorem A] used the technique of cutting and gluing to show that the Gauss-Bonnet formula is the only angle constraint for cone spherical metrics on compact Riemann surfaces of *positive* genus. Quite recently, Q. Chen, B. Li and the first and the second authors [3] applied Jenkins-Strebel differentials to finding the angle constraint for reducible metrics on compact Riemann surfaces of *positive* genus, which was reduced to an existence theorem of unitary one-forms with both residues and zero multiplicities prescribed. A special case of the main theorem we proved in [3] reads as follows.

Fact 2 *Let p, q, g be three positive integers, $(a_1, \cdots, a_p)$ and $(b_1, \cdots, b_q)$ be two partitions of d, and $(m_1, \cdots, m_\ell)$ a partition of $p + q - 2 + 2g$. Then there exists a unitary one-form ω on some compact Riemann surface of genus g such that it has the $p+q$ residues of $a_1, \cdots, a_p, -b_1, \cdots, -b_q$ and ℓ zeros with multiplicities $m_1, \cdots, m_\ell$, respectively.*

Roughly speaking, the positive genus in Fact 2 helps us remove the algebraic restriction (1) in Fact 1 so that the residue theorem together with the degree condition guarantee the existence of ω. Nevertheless, Fact 2 is non-trivial since all the periods of ω lie in $\sqrt{-1}\mathbb{R}$. Motivated by Fact 2, we obtain Case 2 of the following theorem, whose proof is independent of this fact.

Theorem 1.1. *Let d, ℓ be two positive integers. Suppose that a collection Λ consists of $\ell + 2$ partitions of d:*

$$(a_1, a_2, \ldots, a_p),\ (b_1, b_2, \ldots, b_q),\ (m_1 + 1, 1, \ldots, 1),\ \ldots,\ (m_\ell + 1, 1, \ldots, 1).$$

Define the total branch number $v(\Lambda)$ *of Λ to be $m_1 + \cdots + m_\ell + (d - p) + (d - q)$. Then Λ is the branch data of a branched cover from a compact Riemann surface X to $\mathbb{P}^1$ if and only if it satisfies one of the following two conditions*:

1. *$v(\Lambda) = 2d - 2$ and $\max\{m_1, \ldots, m_\ell\} < \dfrac{d}{\mathrm{GCD}(a_1, \ldots, a_p, b_1, \ldots, b_q)}$;*
2. *The total branching order $v(\Lambda) \geq 2d$ is even.*

Moreover, the genus of X equals $\dfrac{v(\Lambda) + 2 - 2d}{2}$.

We call a branched cover from a compact Riemann surface X to $\mathbb{P}^1$ a *Belyi function* on X if it has at most three branched points. Using the same argument as the proof of Theorem 1.3 in [32], we could obtain from Theorem 1.1 the following corollary.

Corollary 1.2. *Let us use the notations in Theorem 1.1. Then there exists a Belyi function of degree ℓd on X with branch data*

$$\widetilde{\Lambda} = \{(\ell a_1, \cdots, \ell a_p), (\ell b_1, \cdots, \ell b_q), (m_1 + 1, \cdots, m_\ell + 1, 1, \cdots, 1)\}.$$

Remark 1.3. The $\ell = 1$ case of Theorem 1.1 was proved by Boccara [2] which is another motivation to us besides Facts 1-2. We already proved the first case of $v(\Lambda) = 2d - 2$ in [32], where X is $\mathbb{P}^1$. The necessary part of the second case follows from Riemann-Hurwitz. Hence, in the remaining of this note, we only need to show its sufficient part.

There exists an apparent difference between Case 1 and Case 2 in Theorem 1.1. It could be compared to some irregularity phenomena in enumeration geometry of Riemann surfaces: unstable Riemann surfaces are more irregular than stable surfaces [9, p.53].

By taking the logarithmic differential of the branched cover, Theorem 1.1 implies Fact 2 if $\max(m_1, \cdots, m_\ell) < d$. However, we could not obtain Fact 2 by merely using the theorem. Actually, we gave the proof of Fact 2 in [3] by using ribbon graphs of Strebel differentials. On the other hand, following Hurwitz, we apply the Riemann existence theorem [5, Theorem 2, p.49] to reducing Theorem 1.1 to the following proposition.

Proposition 1.4. *Let d, ℓ, g be three positive integers. If*

$$\Lambda = \big\{(a_1, a_2, \ldots, a_p),\ (b_1, b_2, \ldots, b_q),\ (m_1 + 1, 1, \ldots, 1),\ \ldots,\ (m_\ell + 1, 1, \ldots, 1)\big\}$$

is a collection of $\ell + 2$ partitions of d with $m_1 + \cdots + m_\ell = p + q - 2 + 2g$, then there exist $\ell + 2$ permutations $\tau_1, \tau_2, \sigma_1, \ldots, \sigma_\ell \in S_d$ satisfying the following three conditions:

1. *$\tau_1\tau_2\sigma_1 \cdots \sigma_\ell = id$;*
2. *τ_1 has type of $a_1^1 a_2^1 \cdots a_p^1$, τ_2 of $b_1^1 b_2^1 \cdots b_q^1$ and σ_i of $(m_i + 1)^1 1^{d-m_i-1}$ for $i = 1, 2, \ldots, \ell$;*
3. *the subgroup generated by $\tau_1, \tau_2, \sigma_1, \ldots, \sigma_\ell$ acts transitively on the set $\{1, 2, \cdots, d\}$.*

We prove the above proposition in the next section. In the last section, we propose a question on the enumeration of branched covers in Theorem 1.1 up to weak and strong equivalence relations, respectively, which salutes to Hurwitz.

2 Proof of Propostion 1.4

2.1 Case $\ell = 1$: three partitions

Actually this case was proved by Boccara [2]. In this subsection, we give an alternative proof, which seems more constructive than the original one by Boccara.

Proposition 2.1. *Proposition 1.4 holds true for $\ell = 1$.*

Proof. We use induction on d.

At first, note that $d \geq m_1 + 1 = p + q - 2 + 2g + 1 \geq 3$. Moreover, if $d = 3$, then $\Lambda = \{(3),\ (3),\ (3)\}$. We could choose $\tau_1 = \tau_2 = \sigma = (123)$.

Then we assume that the proposition holds true for partitions of $d \leq D - 1$, where $D \geq 4$. Recall that $\Lambda = \{(a_1, \ldots, a_p),\ (b_1, \ldots, b_q),\ (m_1 + 1, 1, \ldots, 1)\}$ is a collection of three partitions of D and $2g = m_1 - p - q + 2 \geq 2$.

Case 1 Suppose that there exist i and j such that $a_i \neq b_j$. Without loss of generality, we assume that $a_p > b_q$ and $b_q = \min\{a_1, \ldots, a_p, b_1, \ldots, b_q\}$.

Subcase 1.1 Let $m_1 - 1 < D - b_q$. Then, applying the induction hypothesis to the collection

$$\widetilde{\Lambda} = \{(a_1, \ldots, a_p - b_q),\ (b_1, \ldots, b_{q-1}),\ (m_1, 1, \ldots, 1)\},$$

of three partitions of $D - b_q$, we could find three permutations $\widetilde{\tau}_1,\ \widetilde{\tau}_2,\ \widetilde{\sigma}_1 \in S_{D-b_q}$ such that $\widetilde{\tau}_2\widetilde{\sigma}_1 = \widetilde{\tau}_1 = \tilde{\mu}_1 \cdots \tilde{\mu}_p$, where $\tilde{\mu}_i$ is a cycle of length a_i for $1 \leq i \leq p - 1$ and $\tilde{\mu}_p$ is a cycle of length $(a_p - b_q)$. Since $\langle \widetilde{\tau}_1, \widetilde{\tau}_2, \widetilde{\sigma}_1 \rangle$ acts transitively on $\{1, 2, \cdots, D - b_q\}$, by [32, Lemma 2.10], $\widetilde{\sigma}_1$ and $\tilde{\mu}_p$ have at least one common element, say a. Then we could take

$$\begin{aligned}
\tau_2 &= (D - b_q + 1, D - b_q + 2, \ldots, D)\widetilde{\tau}_2,\\
\sigma_1 &= \widetilde{\sigma}_1(a\, D),\\
\tau_1 &= \tilde{\mu}_1 \cdots \tilde{\mu}_{p-1} \cdot \Big((D - b_q + 1, D - b_q + 2, \ldots, D)\tilde{\mu}_p(a\, D)\Big).
\end{aligned}$$

Subcase 1.2 Let $m_1 - 1 \geq D - b_q$. Since $m_1 + 1 \leq D$, we have $b_q \geq 2$. Moreover, if $b_q = 2$, then $m_1 + 1 = D$. By Edmonds-Kulkarni-Stong [7, Proposition 5.2], it holds true for Case $b_q = 2$. Hence, we assume $b_q \geq 3$ and consider the following new collection

$$\widetilde{\Lambda} = \{(a_1, \ldots, a_{p-1}, a_p - 2),\ (b_1, \ldots, b_{q-1}, b_q - 2),\ (m_1 - 1, 1, \ldots, 1)\}$$

of three partitions of $D - 2$. Since $D - b_q \leq m_1 - 1 = p + q - 3 + 2g$, we have $2g \geq 3 + D - b_q - p - q$.

At first, it is easy to check that if $\max\{p,q\} \le 2$, then $2g \ge 3$. On the other hand, $D - b_q - p - q \ge \frac{D}{3} - b_q$ since $p, q \le \frac{D}{3}$. Therefore, if $\max\{p,q\} \ge 3$, then we have

$$2g \ge 3 + D - b_q - p - q \ge 3 + \frac{\max\{p,q\} \cdot b_q}{3} - b_q \ge 3.$$

Summing up, we always have $g \ge 2$ and $m_1 - 2 = p + q - 2 + 2(g-1)$.

By the induction hypothesis, there exist $\widetilde{\tau}_1$, $\widetilde{\tau}_2$, $\widetilde{\sigma}_1$ in S_{D-2} corresponding to $\widetilde{\Lambda}$ with $\widetilde{\sigma}_1 = \widetilde{\tau}_1\widetilde{\tau}_2$. Let $\widetilde{\tau}_1 = \tilde{\mu}_1 \cdots \tilde{\mu}_p$, where $\tilde{\mu}_i$ is a cycle of length a_i for $1 \le i \le p-1$ and $\tilde{\mu}_p$ is a cycle of length $(a_p - 2)$. Let $\widetilde{\tau}_2 = \tilde{\nu}_1 \cdots \tilde{\nu}_q$, where $\tilde{\nu}_i$ is a cycle of length b_i for $1 \le i \le q-1$ and $\tilde{\nu}_q$ is a cycle of length $(b_q - 2)$. By the transitive property, $\widetilde{\sigma}_1$ and $\tilde{\mu}_p$ have at least one common element, say x; $\widetilde{\sigma}_1$ and $\tilde{\nu}_q$ have at least one common element, say y. At last, we take

$$\begin{aligned}
\tau_1 &= (D,\, D-1,\, x)\widetilde{\tau}_1 \\
\tau_2 &= \widetilde{\tau}_2(D,\, D-1,\, y) \\
\sigma_1 &= \tau_1\tau_2,
\end{aligned}$$

where $\sigma_1 = (D,\, D-1,\, x)\widetilde{\sigma}_1(D,\, D-1,\, y)$ is always an (m_1+1)-cycle in S_D whether x coincides with y or not.

Case 2 Let $a_1 = \cdots = a_p = b_1 = \cdots = b_q = k$. Then we have $k \ge 3$ since

$$p \cdot k = D \ge m_1 + 1 = 2p - 1 + 2g.$$

Moreover, by Proposition 5.2 in [7], we could assume $p = q \ge 2$ and then $m_1 \ge 4$.

Subcase 2.1 Let $m_1 - 1 \le D - k$. Then, considering the following new collection

$$\widetilde{\Lambda} = \{(a_1, \ldots, a_{p-1}),\ (b_1, \ldots, b_{q-1}),\ (m_1 - 1, 1, \ldots, 1)\}$$

of three partitions of $D-k$, we find by the induction hypothesis $\widetilde{\tau}_1$, $\widetilde{\tau}_2$, $\widetilde{\sigma}_1 \in S_{D-k}$ corresponding to $\widetilde{\Lambda}$ such that $\widetilde{\sigma}_1 = \widetilde{\tau}_1\widetilde{\tau}_2$. Let $\widetilde{\tau}_1 = \tilde{\mu}_1 \cdots \tilde{\mu}_p$, where $\tilde{\mu}_i$ is a cycle of length k for $1 \le i \le p$. We claim that *each $\tilde{\mu}_i$ has at least* 2 *common elements with $\widetilde{\sigma}_1$*. In fact, if $\tilde{\mu}_i$ has exactly one common elements with $\widetilde{\sigma}_1$, say a, then the cycle in $\widetilde{\tau}_2 = \widetilde{\tau}_1^{-1}\widetilde{\sigma}_1$ containing a has length greater than k. A contradiction.

Let x_1, x_i be two common numbers lying in both $\tilde{\mu}_1 = (x_1, x_2, \ldots, x_i, \ldots, x_k)$ and $\widetilde{\sigma}_1$. Then we could take

$$\begin{aligned}
\tau_1 &= (x_1, D-k+1)(x_i, D-k+i)(D-k+1, D-k+2, \ldots, D)\widetilde{\tau}_1, \\
\tau_2 &= \widetilde{\tau}_2(D, D-1, \ldots, D-k+1), \\
\sigma_1 &= \tau_1\tau_2,
\end{aligned}$$

where $\sigma_1 = (x_1, D-k+1)(x_i, D-k+i)\widetilde{\sigma}_1$ is a cycle of length (m_1+1).

Subcase 2.2 Let $m_1 - 1 > D - k$. Then we consider the following collection

$$\widetilde{\Lambda} = \{(a_1, \ldots, a_{p-1}, a_p - 2),\ (b_1, \ldots, b_{q-1}, b_q - 2),\ (m_1 - 1, 1, \ldots, 1)\}$$

of three partitions of $D - 2$. Since $g \geq 2$, we could take the same construction as in Subcase 1.2 except for Case $\Lambda = \{(3,3),(3,3),(5,1)\}$ for which we could construct the three permutations by hand.

□

2.2 Case $\ell > 1$: more than three partitions

In this subsection, we will prove Case $\ell > 1$ of Proposition 1.4 by induction on both ℓ and $m_1 + \cdots + m_\ell - p - q$. At first, we need some preliminary lemmas.

Lemma 2.2. *Let $1 \leq s,\ r \leq d$ be two integers such that*

$$s + r \geq d + 1 \quad \text{and} \quad s + r \equiv d + 1 (\text{mod } 2).$$

Then there exist an s-cycle σ_1 and an r-cycle σ_2 such that $\sigma_1\sigma_2$ is a d-cycle.

Proof. If $2k = s + r - (d+1)$, then $0 \leq 2k < r$. Let

$$\begin{aligned}\sigma_2^{-1} &= (r, r-1, \ldots, 2, 1)(2k+1, 2k, \ldots, 2, 1)\\ &= (r, r-1, \ldots, 2k+2, 2k+1, 2k-1, \ldots, 1, 2k, 2k-2, \ldots, 2)\end{aligned}$$

Hence, σ_2 is an r-cycle. Note that

$$(1, 2, \ldots, d)\sigma_2^{-1} = (1, 2k+1, 2k, \ldots, 2, r+1, r+2, \ldots, d)$$

Set $\sigma_1 = (1, 2k+1, 2k, \ldots, 2, r+1, r+2, \ldots, d)$. Then σ_1 is an s-cycle and $\sigma_1\sigma_2 = (1, 2, \ldots, d)$ is a d-cycle. □

Lemma 2.3. *Proposition 1.4 holds true if $\ell = 2$ and $p = q = m_1 = m_2$.*

Proof. Case $p = q = m_1 = m_2 = 1$ follows from [7, Proposition 5.2]. We assume $p = q = m_1 = m_2 > 1$ in what follows. We could derive $p, q < d$ from $(m_1 + 1) \leq d$. Without loss of generality, we could assume that

$$a_1 \geq a_2 \geq \cdots \geq a_p, \quad b_1 \geq b_2 \geq \cdots \geq b_q.$$

Then $a_1 \geq 2$ and $b_1 \geq 2$. Let

$$\begin{aligned}\tau_1 &= (a_1 + a_2 + \cdots + a_{p-1} + 1, a_1 + a_2 + \cdots + a_{p-2} + 1, \ldots, a_1 + 1, 1)\,(1, 2, \ldots, d)\\ &= (1, 2, \ldots, a_1)\,(a_1 + 1, a_1 + 2, \ldots, a_1 + a_2) \cdots (a_1 + \cdots + a_{p-1} + 1, \ldots, d)\\ \tau_2 &= (d, d-1, \ldots, 1)(1, b_1 + 1, \ldots, b_1 + b_2 + \cdots + b_{q-1} + 1)\\ &= (d, d-1, \ldots, b_1 + b_2 + \cdots + b_{q-1} + 1) \cdots (b_1 + b_2, \ldots, b_1 + 2, b_1 + 1)\,(b_1, \ldots, 2, 1)\end{aligned}$$

Then we have the following equality relevant to the four permutations of τ_1, τ_2, σ_1 and σ_2:

$$\begin{aligned}\tau_1\tau_2 &= (a_1+a_2+\cdots+a_{p-1}+1,\ldots,a_1+1,1)(1,b_1+1,\ldots,b_1+b_2+\cdots+b_{q-1}+1)\\ &= \underbrace{(a_1+a_2+\cdots+a_{p-1}+1,\ldots,a_1+1,1)(12)}_{\sigma_2^{-1}}\underbrace{(12)(1,b_1+1,\ldots,b_1+b_2+\cdots+b_{q-1}+1)}_{\sigma_1^{-1}}\end{aligned}$$

Hence we complete the proof. $\square$

Lemma 2.4. *Proposition 1.4 holds true if $\ell = 2$ and $p+q = m_1+m_2$.*

Proof. Without loss of generality, we assume that $p \geq q$ and $m_1 \geq m_2$. Moreover, by Lemma 2.3, we could assume $p \neq q$ or $m_1 \neq m_2$, that is $m_2 < p$. Consider the following new collection of partitions of d

$$\widetilde{\Lambda} = \left\{(a_1+a_2+\cdots+a_{m_2+1}, a_{m_2+2},\ldots,a_p),\ (b_1,b_2,\ldots,b_q),\ (m_1+1,1,\ldots,1)\right\}.$$

Then by Proposition 2.1, we know that there exist $\widetilde{\tau}_1$, τ_2, σ_1 corresponding to $\widetilde{\Lambda}$ such that $\widetilde{\tau}_1\tau_2 = \sigma_1^{-1}$. In particular, $\widetilde{\tau}_1$ is a product of mutually disjoint $(p-m_2)$ cycles of lengths $(a_1+a_2+\cdots+a_{m_2+1})$, $a_{m_2+2},\ldots,a_p$, respectively. For simplicity, we assume that the cycle of length $(a_1+a_2+\cdots+a_{m_2+1})$ in the product is

$$(1,2,\ldots,a_1+\cdots+a_{m_2+1}).$$

Note that

$$\begin{aligned}&(a_1+\cdots+a_{m_2}+1,\ a_1+\cdots+a_{m_2-1}+1,\ \ldots,a_1+1,\ 1)\cdot(1,2,\ldots,a_1+a_2+\cdots+a_{m_2+1})\\ =&(1,2,\ldots,a_1)(a_1+1,a_1+2,\ldots,a_1+a_2)\cdots(a_1+\cdots+a_{m_2}+1,\ldots,a_1+\cdots+a_{m_2+1})\end{aligned}$$

Set

$$\begin{aligned}\sigma_2^{-1} &= (a_1+\cdots+a_{m_2}+1,\ a_1+\cdots+a_{m_2-1}+1,\ \ldots,a_1+1,\ 1),\\ \tau_1 &= \sigma_2^{-1}\widetilde{\tau}_1.\end{aligned}$$

Then we have $\tau_1\tau_2\sigma_1\sigma_2 = id$. $\square$

Lemma 2.5. *Proposition 1.4 holds true for $\ell = 2$ and $p+q \leq m_1+m_2$.*

Proof. We use induction on $m_1+m_2-(p+q)$. By Lemma 2.4, we assume it holds true for $m_1+m_2 = (p+q)+2k$. Suppose $m_1+m_2-(p+q) = 2k+2$. Then $(m_1-1)+(m_2-1) = (p+q)+2k$. By the induction hypothesis, there exist τ_1,τ_2 and m_1-cycle $\widetilde{\sigma}_1$, m_2-cycle $\widetilde{\sigma}_2$ such that $\tau_1\tau_2 = \widetilde{\sigma}_2\widetilde{\sigma}_1$. Let A_i be the set of numbers in the cycle $\widetilde{\sigma}_i (i=1,2)$. We choose two numbers $x,y \in \{1,2,\ldots,d\}$ according to the following three cases, respectively:

- If $A_1 \subseteq A_2$, then pick $x \in A_1,\ y \notin A_2$;

- If $A_2 \subseteq A_1$, then pick $x \in A_2,\ y \notin A_1$;
- If $A_1 \nsubseteq A_2$ and $A_2 \nsubseteq A_1$, then pick $x \in A_1 \setminus A_2,\ y \in A_2 \setminus A_1$.

Then $\sigma_2 = \widetilde{\sigma}_2(xy)$ is a $(1+m_2)$-cycle, $\sigma_1 = (xy)\widetilde{\sigma}_1$ is a $(1+m_1)$-cycle and

$$\tau_1\tau_2 = \sigma_2\sigma_1.$$

□

Now we are in a position to complete the proof of Proposition 1.4.

Proof of Proposition 1.4. Here we use induction on ℓ and $m_1 + \cdots + m_\ell - (p+q)$. Assume that Proposition 1.4 holds true for $\ell \leq k-1$ $(k \geq 3)$ and consider the case $\ell = k$. Then we use induction on $m_1 + \cdots + m_\ell - (p+q)$.

(Step 1) Let $m_1 + \cdots + m_\ell = p+q$. Then we could assume $m_1 \geq m_2 \geq \cdots \geq m_\ell$ and $p \geq q$ without loss of generality. Hence $p > m_\ell$ and the same argument as in Lemma 2.4 works.

(Step 2) Let $m_1 + \cdots + m_\ell = p+q+2n-2$ where $n > 1$. Then the same argument as in Lemma 2.5 shows that it holds true for $m_1 + \cdots + m_\ell = p+q+2n$.

□

3 Hurwitz numbers

Two branched covers $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ between compact Riemann surfaces are said to be *weakly equivalent* if there exist two biholomorphic maps $\widetilde{g}\colon X_1 \to X_2$ and $g\colon Y \to Y$ such that $f_2 \circ \widetilde{g} = g \circ f_1$, and *strongly equivalent* if the set of branched points in Y is fixed once and forever and one can take $g = id_Y$. The *weak* (*strong*) *Hurwitz number* of a branch datum is the number of weak or strong equivalence classes of branched covers realizing it. The *simple Hurwitz numbers* are those strong Hurwitz numbers with respect to branch data with form

$$(2,1,\cdots,1),\ \cdots,\ (2,1,\cdots,1),\ (a_1,\cdots,a_p).$$

Long ago Mednykh in [17, 18] gave some general formulae for the computation of the strong Hurwitz numbers, but the actual implementation of them is rather elaborate in general. Lando-Zvonkin [16, Chapter 5] made a systematic exposition on how to compute simple Hurwitz numbers for target $\mathbb{P}^1$ in terms of intersection numbers on moduli spaces of curves. Moreover, they also gave enumeration of polynomial rational functions $\mathbb{P}^1 \to \mathbb{P}^1$ in terms of the Lyashko-Looijenga mapping. Monni-Song-Song [21] used algebraic methods to compute the simple Hurwitz numbers with respect to branch data of form $\{(2,1,\cdots,1),\ \cdots,\ (2,1,\cdots,1)\}$ for arbitrary source and target Riemann surfaces and found the generating function for such simple Hurwitz numbers. Dubrovin-Yang-Zagier [6] gave a polynomial-time algorithm of computing these simple Hurwitz numbers for target surface $\mathbb{P}^1$. Quite recently, Petronio [28, 30] and Petronio-Sarti [29] computed explicitly the weak Hurwitz number for certain branch data consisting of three partitions and having target surface $\mathbb{P}^1$ by using a combinatorial method based on Grothendieck's dessin d'enfants.

Question Find a neat formula for all the weak (strong) Hurwitz numbers for the branch data in Theorem 1.1.

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