IDEALS OF THE ASSOCIATIVE ALGEBRA OPERAD

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ABSTRACT. We prove a one-to-one correspondence between the operadic ideals of the operad $u\mathcal{A}ss$ and T-ideals. As a consequence, we show that $u\mathcal{A}ss$ is noetherian and that every proper operadic ideal of $u\mathcal{A}ss$ is generated by a single element.

1. INTRODUCTION

Throughout let $\Bbbk$ be a base field of characteristic zero. Most algebraic objects are over $\Bbbk$. Unless otherwise stated we consider associative algebras with unit in this paper. A *polynomial identity* of an algebra A is a noncommutative polynomial $f(x_1, \cdots, x_n)$ such that $f(a_1, \cdots, a_n) = 0$ for all $a_1, \cdots, a_n \in A$. An algebra satisfying a nontrivial polynomial identity is called a *PI-algebra*. Commutative algebras, the matrix algebra over a commutative algebra, finite-dimensional algebras, and Grassmann (or exterior) algebras are examples of PI-algebras.

Polynomial identities of a given PI-algebra were firstly investigated by Amitsur and Levisky in [AL] where they proved that the standard polynomial of degree $2m$ is an identity of minimal degree for the $m \times m$ full matrix algebra. It is well-known that the set of all identities satisfied by a PI-algebra is a T-ideal of the free algebra $\Bbbk\langle X\rangle$ in countable indeterminants $X := \{x_i\}_{n\in\mathbb{N}_+}$.

Let $u\mathcal{A}ss$ (resp. $\mathcal{A}ss$) denote the symmetric operad encoding the unital associative algebras (resp. the associative algebras without unit). It is well-known that the PI-theory such as the study of multilinear polynomial identities is related to ideals of the operad $u\mathcal{A}ss$. One motivation of this paper is to spell out explicitly some connections between the operad $u\mathcal{A}ss$ and the PI-theory.

Observe that the subspace of a T-ideal consisting of all multilinear polynomials is essentially equivalent to an operadic ideal of $u\mathcal{A}ss$ [Lemma 2.3]. Recall that $u\mathcal{A}ss(n) = \Bbbk\mathbb{S}_n$ for all $n \geq 0$. Let V_n be the space consisting of all multilinear polynomials in n variables $x_1, \cdots, x_n$. Clearly, V_n admits an action of the symmetric group $\mathbb{S}_n$, which is naturally isomorphic to the regular representation of $\mathbb{S}_n$. Let A be a PI-algebra and $V_n(A)$ the subspace of V_n of those polynomials that are identities of A. Then we have $V_n(A) \cong \mathcal{I}_A(n)$ where $\mathcal{I}_A$ is the operadic ideal of $u\mathcal{A}ss$ determined by the algebra A [Lemma 2.3]. As a consequence, $V_n/V_n(A) \cong (u\mathcal{A}ss/\mathcal{I}_A)(n)$. In this situation, we also say that A is a PI-algebra associated to the operadic ideal $\mathcal{I}_A$. Given a T-ideal J and let $A := \Bbbk\langle X\rangle/J$. Then, following the above procedure we can construct the associated operadic ideal $\mathcal{I}_A$ of $u\mathcal{A}ss$, denoted by $\Psi(J)$. Conversely, for every nonzero operadic ideal $\mathcal{I}$ of $u\mathcal{A}ss$, a $u\mathcal{A}ss/\mathcal{I}$-algebra is a PI-algebra since each nonzero element in $\mathcal{I}$ gives an identity of A. In particular, for any vector space V, the free $u\mathcal{A}ss/\mathcal{I}$-algebra $(u\mathcal{A}ss/\mathcal{I})(V)$ is a PI-algebra associated to $\mathcal{I}$.

Theorem 1.1 (Theorem 2.9). *There is a natural one-to-one correspondence*

$$\{\textit{proper T-ideals of } \Bbbk\langle X\rangle\} \longleftrightarrow \{\textit{proper operadic ideals of } u\mathcal{A}ss\}$$

via the map $J \mapsto \Psi(J)$.

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Theorem 1.1 says that there is no essential difference between T-ideals of $\Bbbk\langle X\rangle$ and operadic ideals of $u\mathcal{A}ss$. In [BYZ], the authors studied the ideal structure of 2-unitary operads similar to $u\mathcal{A}ss$. Note that $u\mathcal{A}ss$ is denoted as $\mathcal{A}ss$ in [BYZ]. Recall that an operad $\mathcal{P}$ is *artinian* (resp. *noetherian*) if the set of ideals of $\mathcal{P}$ satisfies the descending (resp. ascending) chain condition. Let $\mathcal{P}$ be a locally finite 2-unitary operad. Then

$$\operatorname{GKdim}\mathcal{P} < \infty \iff \mathcal{P} \text{ is artinian} \Longrightarrow \mathcal{P} \text{ is noetherian.}$$

It is easily seen that $u\mathcal{A}ss$ is not artinian since $\operatorname{GKdim}(u\mathcal{A}ss) = \infty$. Applying Kemer's theorem [Theorem 2.2] and the relationship between T-ideals and operadic ideals of $u\mathcal{A}ss$ [Theorem 1.1], we obtain the following.

Theorem 1.2. (1) *The operad $u\mathcal{A}ss$ is noetherian.*
(2) *Every proper operadic ideal of $u\mathcal{A}ss$ is generated by a single element.*

We are wondering if there is a version of Theorem 1.2 for other operads such as unital Poisson operad. Recall that $\mathcal{A}ss$ is the symmetric operad encoding the associative algebras without unit. Note that Theorem 1.2(2) fails for $\mathcal{A}ss$ though Theorem 1.2(1) holds [Remark 2.15]. An operadic ideal $\mathcal{I}$ of $u\mathcal{A}ss$ may not be generated by an element in $\mathcal{I}$ of the minimal degree. For example, the k-th truncation ideal ${}^k\Upsilon$ of $u\mathcal{A}ss$ [BYZ, E0.0.2] is generated by an element in ${}^k\Upsilon(m)$ for $m > k \geq 3$, rather than in ${}^k\Upsilon(k)$, see [BFXYZZ]. Therefore, it is reasonable to consider the single generator of an operadic ideal of $\mathcal{I}$ and the corresponding multilinear polynomial in PI-theory, which will be studied in [BFXYZZ].

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2. Proofs of statements

Throughout $\Bbbk$ is a fixed field of characteristic zero and all unadorned $\otimes$ will be $\otimes_{\Bbbk}$. First we recall some basics about the operad $u\mathcal{A}ss$. Generally we refer to [LV, BYZ, QXZZ] for basic definitions and properties about operads. For convenience, we denote $[n] = \{1, 2, \cdots, n\}$.

Let $\mathbb{S}_n$ be the symmetric group of degree n. We follow that convention in [BYZ] and use the sequence $(\sigma^{-1}(1), \sigma^{-1}(2), \cdots, \sigma^{-1}(n))$ to denote an element $\sigma \in \mathbb{S}_n$. Equivalently, each $(i_1, i_2, \cdots, i_n)$ of $[n]$ corresponds to the permutation $\sigma \in \mathbb{S}_n$ given by $\sigma(i_k) = k$ for all $1 \leq k \leq n$. We also use 1_n to denote the identity element in $\mathbb{S}_n$.

Recall that $u\mathcal{A}ss(n) = \Bbbk\mathbb{S}_n$ is the right regular $\Bbbk\mathbb{S}_n$-module, and the composition map of $u\mathcal{A}ss$ is linearly extended by the following maps: for $n > 0$, $k_1, k_2, \cdots, k_n \geq 0$,

$$\begin{aligned} \mathbb{S}_n \times \mathbb{S}_{k_1} \times \cdots \times \mathbb{S}_{k_n} &\to \mathbb{S}_{\sum_{i=1}^n k_i}, \\ (\sigma, \sigma_1, \cdots, \sigma_n) &\mapsto (\tilde{B}_{\sigma^{-1}(1)}, \cdots, \tilde{B}_{\sigma^{-1}(n)}) \end{aligned}$$

for all $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$, $1 \leq i \leq n$, where

$$\tilde{B}_i = (\sum_{j=1}^{i-1} k_j + \sigma_i^{-1}(1), \cdots, \sum_{j=1}^{i-1} k_j + \sigma_i^{-1}(k_i))$$

for all $i = 1, \cdots, n$. The partial composition

$$u\mathcal{A}ss(m) \underset{i}{\circ} u\mathcal{A}ss(n) \to u\mathcal{A}ss(m+n-1)$$

is given by

$$\mu \underset{i}{\circ} \nu = \mu \circ (1_1, \cdots, \underset{i}{\nu}, \cdots, 1_1)$$

for $\mu \in u\mathcal{A}ss(m), \nu \in u\mathcal{A}ss(n)$, $m \geq 1, n \geq 0$ and $1 \leq i \leq m$.

The operad $u\mathcal{A}ss$ encodes unital associative algebras, namely, a unital associative algebra is exactly a $u\mathcal{A}ss$-algebra. Let (A, μ, u) be a unital associative algebra. One can define an operad morphism $\gamma = (\gamma_n)\colon u\mathcal{A}ss \to \mathcal{E}nd_A$ given by $\gamma_0(1_0) = u$ and $\gamma_2(1_2) = \mu$, where $\mathcal{E}nd_A$ is the endomorphism operad of the vector space A, see [LV, Section 5.2.11]. Each $\theta = \sum_{\sigma \in \mathbb{S}_n} c_\sigma \sigma \in \Bbbk\mathbb{S}_n$ gives an n-ary operation on A,

$$\gamma_n(\theta)\colon \quad A^{\otimes n} \to A, \quad \gamma_n(\theta)(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma \in \mathbb{S}_n} c_\sigma a_{\sigma^{-1}(1)} \cdots a_{\sigma^{-1}(n)}, \quad \text{for } a_1, \cdots, a_n \in A.$$

Next we work out some connections between T-ideals and operadic ideals of $u\mathcal{A}ss$.

Denote by $\Bbbk\langle x_1, \cdots, x_n\rangle$ the free associative algebra in noncommutative indeterminants $\{x_1, \cdots, x_n\}$ over $\Bbbk$. Observe that a PI algebra A always satisfies a multilinear polynomial identity of degree $\leq d$ if A satisfies an identity of degree d, see [MR, Proposition 13.1.9] or [GZ, Theorem 1.3.7]. A *multilinear polynomial* of degree n is a nonzero element $f(x_1, \cdots, x_n) \in \Bbbk\langle x_1, \cdots, x_n\rangle$ of the form

$$f(x_1, \cdots, x_n) = \sum_{\sigma \in \mathbb{S}_n} c_\sigma x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$$

for some $c_\sigma \in \Bbbk$. The method of multilinearization actually plays a very important role in the study of the identities of a PI-algebra.

Based on the following observation, one can study PI-algebras in the language of operads. The following lemma is a folklore.

Lemma 2.1. *Let A be an associative algebra. Then A is a PI algebra if and only if A is a $u\mathcal{A}ss/\mathcal{I}$-algebra for some nonzero operadic ideal $\mathcal{I}$ of $u\mathcal{A}ss$.*

Proof. Let $\mathcal{I} \neq 0$ be an ideal of $u\mathcal{A}ss$ and A a $u\mathcal{A}ss/\mathcal{I}$-algebra with the operadic morphism $\bar{\gamma}\colon u\mathcal{A}ss/\mathcal{I} \to \mathcal{E}nd_A$. Clearly, A is an associative algebra. Then for each nonzero element $\theta = \sum_{\sigma \in \mathbb{S}_n} c_\sigma \sigma \in \mathcal{I}(n)$, the algebra A satisfies the following multilinear polynomial

$$f_\theta(x_1, \cdots, x_n) = \sum_{\sigma \in \mathbb{S}_n} c_\sigma x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)},$$

since

$$f_\theta(a_1, \cdots, a_n) = \bar{\gamma}(\theta + \mathcal{I})(a_1, \cdots, a_n) = 0$$

for any $a_1, \cdots, a_n \in A$. Conversely, let A be a PI-algebra satisfying a multilinear polynomial of the form

$$f(x_1, \cdots, x_n) = \sum_{\sigma \in \mathbb{S}_n} c_\sigma x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)},$$

where $c_\sigma \in \Bbbk$. Clearly, A is a $u\mathcal{A}ss$-algebra. Suppose that $\gamma\colon u\mathcal{A}ss \to \mathcal{E}nd_A$ is the corresponding operadic morphism. Then $\theta_f = \sum_{\sigma \in \mathbb{S}_n} c_\sigma \sigma \in \operatorname{Ker}\gamma$ and therefore $\operatorname{Ker}\gamma$ is a nonzero operadic ideal of $u\mathcal{A}ss$. It follows that A is an algebra over the quotient operad $u\mathcal{A}ss/\operatorname{Ker}\gamma$. $\square$

Denote $\mathcal{I}_A := \operatorname{Ker}\gamma$ as in the proof of Lemma 2.1. In this case, we say that A is a PI-algebra associated to the operadic ideal $\mathcal{I}_A$. Clearly, the operadic ideal $\mathcal{I}_A$ is the maximal operadic ideal $\mathcal{I}$ of $u\mathcal{A}ss$ such that A is an algebra over $u\mathcal{A}ss/\mathcal{I}$.

Let $\mathcal{I}$ be a nonzero operadic ideal of $u\mathcal{A}ss$ and $\mathcal{A}_{\mathcal{I}} = u\mathcal{A}ss/\mathcal{I}$. Suppose that V is a vector space over $\Bbbk$. Recall the free $\mathcal{A}_{\mathcal{I}}$-algebra $\mathcal{A}_{\mathcal{I}}(V)$ with

$$\mathcal{A}_{\mathcal{I}}(V) = \bigoplus_{k\geq 0} \mathcal{A}_{\mathcal{I}}(V)_k, \quad \text{with} \quad \mathcal{A}_{\mathcal{I}}(V)_k = \mathcal{A}_{\mathcal{I}}(k) \otimes_{\Bbbk\mathbb{S}_k} V^{\otimes k},$$

where the left action of $\mathbb{S}_k$ on $V^{\otimes k}$ is given by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) \colon = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)},$$

and the multiplication

$$\mathcal{A}_{\mathcal{I}}(V)_m \otimes \mathcal{A}_{\mathcal{I}}(V)_n \to \mathcal{A}_{\mathcal{I}}(V)_{m+n}$$

given by

$$[\bar{\mu}, u_1, \cdots, u_m] \cdot [\bar{\nu}, v_1, \cdots, v_n] \colon = [\overline{1_2 \circ (\mu, \nu)}, u_1, \cdots, u_m, v_1, \cdots, v_n].$$

Clearly, each nonzero element in $\mathcal{I}(n)$ gives an identity satisfied by $\mathcal{A}_{\mathcal{I}}(V)$. Moreover, the free $\mathcal{A}_{\mathcal{I}}$-algebra $\mathcal{A}_{\mathcal{I}}(V)$ is a PI-algebra associated to $\mathcal{I}$. Take different vector space V, one can obtain different PI-algebra $\mathcal{A}_{\mathcal{I}}(V)$ associated to $\mathcal{I}$.

Let $\Bbbk\langle X\rangle$ be the free algebra generated by the set $X = \{x_i\}_{i\in\mathbb{N}_+}$. Recall that an ideal H of $\Bbbk\langle X\rangle$ is called a *T-ideal* if $\varphi(H) \subset H$ for every endomorphism φ of $\Bbbk\langle X\rangle$. Let A be a PI algebra. The set $\operatorname{Id}(A)$ of all polynomial identities of A in $\Bbbk\langle X\rangle$ is a T-ideal. Conversely, if H is a T-ideal of $\Bbbk\langle X\rangle$, then $\Bbbk\langle X\rangle/H$ is a "free" or "universal" PI algebra in some sense. To be precise, if A is a PI algebra such that $\operatorname{Id}(A) \supseteq H$, then for any set mapping $\phi\colon X \to A$, there exists a unique algebra homomorphism $\bar{\phi}\colon \Bbbk\langle X\rangle/H \to A$ such that the following diagram

(E2.1.1)
$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & A \\ & \searrow & \uparrow{\scriptstyle \bar{\phi}} \\ & & \Bbbk\langle X\rangle/H \end{array}$$

commutes, see [AGPR, Theorem 2.2.17]. It is easily seen that the "free" algebra $\Bbbk\langle X\rangle/H$ is just the free $u\mathcal{A}ss/\mathcal{I}$-algebra $(u\mathcal{A}ss/\mathcal{I})(V)$, where V is the vector space spanned by $X = \{x_i\}_{i\in\mathbb{N}_+}$, and $\mathcal{I}$ is the kernel of the structure morphism $\gamma\colon u\mathcal{A}ss \to \mathcal{E}nd_{\Bbbk\langle X\rangle/H}$ of $\Bbbk\langle X\rangle/H$ as a $u\mathcal{A}ss$-algebra, since the free algebra $u\mathcal{A}ss/\mathcal{I}(V)$ also guarantees the existence of the above commutative diagram. Therefore, there is a correspondence between the classes of PI algebras and the T-ideals of $\Bbbk\langle X\rangle$.

A basic question about finite generation of T-ideals was posed by Specht [Sp] in 1950. In order to avoid confusion with finitely generated as an ideal, a finitely generated T-ideal in the class of T-ideals is usually called finitely based. In 1987 Kemer gave an affirmative answer [Ke1, Ke2]. Further discussions can be found in [AGPR, AKK, KR, Pr2].

Theorem 2.2. [Ke2, Theorem 2.4] *Every associative algebra* (*with or without unit*) *has a finite basis of identities.*

Kemer's proof is based on some structure theory of superidentities of superalgebras and certain graded tensor products with the Grassmann algebra.

For each $n \in \mathbb{N}$, we denote V_n the subspace of $\Bbbk\langle x_1, \cdots, x_n\rangle$ spanned by $x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}, \sigma \in \mathbb{S}_n$, which consists of all multilinear polynomials of degree n in the indeterminants $x_1, \cdots, x_n \in X$. Observe that V_n admits the right $\Bbbk\mathbb{S}_n$-action given by

$$(x_{i_1}\cdots x_{i_n})*\tau:\ = x_{\tau^{-1}(i_1)}\cdots x_{\tau^{-1}(i_n)}$$

for $\tau\in\mathbb{S}_n$, and

$$\Phi_n : u\mathcal{A}ss(n)\to V_n, \qquad \sigma\mapsto x_{\sigma^{-1}(1)}\cdots x_{\sigma^{-1}(n)}$$

is an isomorphism of right $\Bbbk\mathbb{S}_n$-modules.

Lemma 2.3. *Let A be a PI algebra and $V_n(A)$ be the subspace of V_n consisting of all multilinear identities of A in $\Bbbk\langle x_1,\cdots,x_n\rangle$. Denote $\mathcal{I}(n)=\Phi_n^{-1}(V_n(A))$ for each $n\in\mathbb{N}_+$. Then $\mathcal{I}=(\mathcal{I}(n))_{n\in\mathbb{N}_+}$ is an operadic ideal of $u\mathcal{A}ss$.*

Proof. It is easily seen that $V_n(A)$ is invariant under the right $\mathbb{S}_n$-action, and so is $\mathcal{I}(n)$. Let $\mu\in u\mathcal{A}ss(m),\nu\in u\mathcal{A}ss(n)$, $1\le i\le m$. If $\mu\in\mathcal{I}(m)$ or $\nu\in\mathcal{I}(n)$, then for $a_1,\cdots,a_{m+n-1}\in A$, we have

$$(\Phi_{m+n-1}(\mu\underset{i}{\circ}\nu))(a_1,\cdots,a_{m+n-1})=\Phi_m(\mu)(a_1,\cdots,a_{i-1},\Phi_n(\nu)(a_i,\cdots,a_{i+n-1}),a_{i+n},\cdots,a_{m+n-1})=0.$$

Therefore, $\Phi_{m+n-1}(\mu\underset{i}{\circ}\nu)\in V_{m+n-1}(A)$ and $\mu\underset{i}{\circ}\nu\in\mathcal{I}(m+n-1)$. It follows that $\mathcal{I}$ is an operadic ideal of $u\mathcal{A}ss$. □

Lemma 2.4. *Let $\mathcal{I}=(\mathcal{I}(n))_{n\in\mathbb{N}}$ be an operadic ideal of $u\mathcal{A}ss$ and $\Bbbk\langle X\rangle$ the free algebra generated by $X=\{x_n\}_{n\in\mathbb{N}^+}$. Put*

(E2.4.1) $$J_n:\ =\{\Phi_n(\theta)(f_1,\cdots,f_n)\mid \theta\in\mathcal{I}(n),f_1,\cdots,f_n\in\Bbbk\langle X\rangle\}$$

for each $n\in\mathbb{N}$. Suppose that J is the ideal of $\Bbbk\langle X\rangle$ generated by $\cup_{n\in\mathbb{N}}J_n$. Then J is a T-ideal of $\Bbbk\langle X\rangle$.

Proof. Let φ be an endomorphism of the free algebra $\Bbbk\langle X\rangle$. For any $\Phi_n(\theta)(f_1,\cdots,f_n)\in J_n$, we have

$$\varphi(\Phi_n(\theta)(f_1,\cdots,f_n))=\Phi_n(\theta)(\varphi(f_1),\cdots,\varphi(f_n))\in J_n.$$

Therefore, φ sends a generator onto a generator of J, and J is a T-ideal of $\Bbbk\langle X\rangle$. □

We fix the following notations.

Notation 2.5. *Let A be a PI algebra and $\mathcal{I}$ be a proper operadic ideal of $u\mathcal{A}ss$.*

1. *Let $\Psi(A)$ denote the operadic ideal of $u\mathcal{A}ss$ constructed in Lemma 2.3.*
2. *Let J be a T-ideal of $\Bbbk\langle X\rangle$. By abuse of notation, $\Psi(\Bbbk\langle X\rangle/J)$ is also denoted by $\Psi(J)$. Given a T-idea J, the operadic ideal $\Psi(J)$ of $u\mathcal{A}ss$ is defined by $\Psi(J)(n)=\Phi_n^{-1}(V_n(\Bbbk\langle X\rangle/J))$ for all n.*
3. *Let $\Omega(\mathcal{I})$ denote the T-ideal of $\Bbbk\langle X\rangle$ constructed in Lemma 2.4. Given an operadic ideal $\mathcal{I}$ of $u\mathcal{A}ss$, then $\Omega(\mathcal{I})$ is generated by $\cup_n\Phi_n(\mathcal{I}(n))$ as a T-ideal.*

Two sets of polynomials are said to be *equivalent* if they generate the same T-ideal. As usual we assume that $\Bbbk$ is a base field of characteristic zero.

Lemma 2.6. [GZ, Theorems 1.3.7 and 1.3.8] *Every nonzero polynomial $f\in\Bbbk\langle X\rangle$ is equivalent to a finite set $\{f_1,\cdots,f_w\}$ of multilinear polynomials with $\deg f_i\le\deg f$.*

By Theorem 2.2 and Lemma 2.6, the following result is obvious.

Corollary 2.7. *Every proper T-ideal is generated by finitely many multilinear polynomials as a T-ideal.*

Let f be a multilinear polynomial of degree n in $\Bbbk\langle x_1,\cdots,x_n\rangle$. We use I_f to denote the ideal of $\Bbbk\langle X\rangle$ of the form $\{\sum_{i=1}^m g_if(u_{i1},\cdots,u_{in})h_i\mid g_i,h_i,u_{ij}\in\Bbbk\langle X\rangle, i=1,\cdots,m,j=1,\cdots,n\}$.

Lemma 2.8. *Let W be a set of multilinear polynomials $\{f\}$ where each f is a multilinear polynomial of degree n in $\Bbbk\langle x_1, \cdots, x_n\rangle$ for some n. Let $\langle W\rangle_T$ be the T-ideal of $\Bbbk\langle X\rangle$ generated by W as a T-ideal. Then $\langle W\rangle_T = \sum_{f\in W} I_f$.*

Proof. It is easy to reduce to the case when W is the singleton $\{f\}$. In this case we need to show that $\langle f\rangle_T := \langle W\rangle_T = I_f$. Clearly, $gf(u_1, \cdots, u_n)h \in \langle f\rangle_T$ for all $g, h, u_i \in \Bbbk\langle X\rangle$, $i = 1, \cdots, n$, and therefore $I_f \subset \langle f\rangle_T$. It suffices to show that I_f is a T-ideal. It is easily seen that I_f is an ideal of $\Bbbk\langle X\rangle$. Suppose that φ is an arbitrary endomorphism of $\Bbbk\langle X\rangle$. Then for all $g_i, h_i, u_{ij} \in \Bbbk\langle X\rangle, i = 1, \cdots, m, j = 1, \cdots, n$, we have

$$\varphi(\sum_{i=1}^{m} g_i f(u_{i1}, \cdots, u_{in})h_i) = \sum_{i=1}^{m} \varphi(g_i)f(\varphi(u_{i1}), \cdots, \varphi(u_{in}))\varphi(h_i) \in I_f.$$

It follows that I_f is a T-ideal and $\langle f\rangle_T = I_f$. □

Here is an intermediate step.

Theorem 2.9. *The pair (Ψ, Ω) defined in Notation 2.5 induce an inclusion-preserving one-to-one correspondence between the set of proper T-ideals of $\Bbbk\langle X\rangle$ and the set of proper operadic ideals of $u\mathcal{A}ss$.*

Proof. By Lemmas 2.3 and 2.4, we have inclusion-preserving maps

$$\Psi : \quad \{\text{proper T-ideals of } \Bbbk\langle X\rangle\} \to \{\text{proper operadic ideals of } u\mathcal{A}ss\}$$

and

$$\Omega : \quad \{\text{proper operadic ideals of } u\mathcal{A}ss\} \to \{\text{proper T-ideals of } \Bbbk\langle X\rangle\}.$$

First we prove that $\Omega(\Psi(J)) = J$ if J is a proper T-ideal of $\Bbbk\langle X\rangle$. By Corollary 2.7, J is generated by finitely many multilinear polynomials, say $\{f_1, \cdots, f_s\}$, as a T-ideal. Let $A := \Bbbk\langle X\rangle/J$. Since J is a T-ideal, every element in J is an identity of A. In particular, $f_i \in V_n(A)$ if f_i has degree n. By Lemma 2.3, $\Phi_n^{-1}(f_i) \in \mathcal{I}(n)$ where $\mathcal{I} = \Psi(J)$. By Lemma 2.4, $f_i \in \Omega(\mathcal{I})$. Consequently, $J \subseteq \Omega(\mathcal{I}) = \Omega(\Psi(J))$. Conversely, let $J' = \Omega(\Psi(J))$, we need to show that $J' \subseteq J$. Since both J and J' are T-ideals, by Lemma 2.4, it suffices to show that $J_n \subseteq J$ where J_n is defined as in (E2.4.1). Again, by the fact that J is a T-ideal, it remains to show that $\Phi_n(\theta) \in J$ for all $\theta \in \mathcal{I}(:= \Psi(J))$. By the definition of $\mathcal{I}$ in Lemma 2.3, $\Phi_n(\theta) \in V_n(A)$ where A is $\Bbbk\langle X\rangle/J$. This implies that $\Phi_n(\theta)$ is an identity of A. Therefore $\Phi_n(\theta) \in J$ as required.

Next we show that $\Psi(\Omega(\mathcal{I})) = \mathcal{I}$ for any proper operadic ideal $\mathcal{I}$ of $u\mathcal{A}ss$. Let $J := \Omega(\mathcal{I})$. By the proof of Lemma 2.4, J is the 2-sided ideal of $\Bbbk\langle X\rangle$ generated by $\cup_{n\in\mathbb{N}} J_n$. Then, for every $\theta \in \mathcal{I}(n)$, $\Phi_n(\theta) = \Phi_n(\theta)(x_1, \cdots, x_n)$ is in $J_n \subseteq J$. Thus $\Phi_n(\theta)$ is in $V_n(\Bbbk\langle X\rangle/J)$. By Lemma 2.3, $\theta \in \Psi(J)$. This proves that $\mathcal{I} \subseteq \Psi(\Omega(\mathcal{I}))$. Conversely, let $\mathcal{I}' := \Psi(\Omega(\mathcal{I}))$, we need to show that $\mathcal{I}' \subseteq \mathcal{I}$. Let θ be in $\mathcal{I}'$. By definition, $\theta = \Phi_n^{-1}(f_\theta)$ where f_θ is some multilinear identity of $\Bbbk\langle X\rangle/J$, or equivalently, f_θ is multilinear and $f_\theta \in J$ as J is a T-ideal. By the proof of Lemma 2.4, there exist $g_{ki}, h_{ki}, u_{ki,1}, \cdots, u_{ki,n_k} \in \Bbbk\langle X\rangle$ such that

$$f_\theta = \sum_{k,i} g_{ki} f_k(u_{ki,1}, \cdots, u_{ki,n_k})h_{ki}$$

where $f_k = \Phi_{n_k}(\theta_k)$ for some $\theta_k \in \mathcal{I}(n_k)$ (and the above sum is a finite sum). Let l denote n_k. Without loss of generality, we may assume that $g_{ki}, h_{ki}, u_{ki,1}, \cdots u_{ki,l}$ are monomials. Since f_θ is a multilinear polynomial in $\Bbbk\langle x_1, \cdots, x_n\rangle$, we may further assume that each term $g_{ki}f_k(u_{ki,1}, \cdots, u_{ki,l})h_{ki}$ is a multilinear polynomial in the indeterminants $x_1, \cdots, x_n$, and for each pair (k, i), $g_{ki}, h_{ki}, u_{ki,1}, \cdots, u_{ki,l}$ are in different sets of indeterminants. We need only show that $\Phi_n^{-1}(gf_k(u_1, \cdots, u_l)h) \in \mathcal{I}(n)$, where $(g, h, u_1, \cdots, u_l)$

are sets of monomials of the form $(g_{ki}, h_{ki}, u_{ki,1}, \cdots, u_{ki,l})$, can be generated by $\theta_k \in \mathcal{I}(l)$ for all k. We denote

$$\begin{aligned} g &= a x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(r)}, \\ u_1 &= b_1 x_{\sigma^{-1}(r+1)} \cdots x_{\sigma^{-1}(r+s_1)}, \\ &\cdots, \\ u_l &= b_l x_{\sigma^{-1}(r+s_1+\cdots+s_{l-1}+1)} \cdots x_{\sigma^{-1}(r+s_1\cdots+s_l)}, \\ h &= c x_{\sigma^{-1}(r+s_1\cdots+s_l+1)} \cdots x_{\sigma^{-1}(r+s_1\cdots+s_l+t)} \end{aligned}$$

for some $\sigma \in \mathbb{S}_n$ and some nonzero scalars $a, b_1, \cdots, b_l, c$ in $\Bbbk$. Clearly, $r + s_1 + \cdots + s_l + t = n$. Then

$$\Phi_n^{-1}(g f_k(u_1, \cdots, u_l) h) = a b_1 \cdots b_l c (1_3 \circ (1_r, \theta_k \circ (1_{s_1}, \cdots, 1_{s_l}), 1_t)) * \sigma \in \langle \theta_k \rangle(n),$$

where $\langle \theta_k \rangle$ is the operadic ideal of $u\mathcal{A}ss$ generated by θ_k. It follows that $\theta \in \sum_k \langle \theta_k \rangle(n)$ where each $\theta_k \in \mathcal{I}(n_k)$. Therefore $\theta \in \mathcal{I}(n)$ as required. □

Remark 2.10. If a T-ideal J is generated by multilinear identities $\{f_i\}_{i=1}^s$ as a T-ideal, then $\Psi(J)$ is generated by $\{\Phi^{-1}_{\deg f_i}(f_i)\}_{i=1}^s$ as an operadic ideal of $u\mathcal{A}ss$. To see this, let $\mathcal{I}$ be $\Psi(J)$ and $\mathcal{I}'$ be the operadic ideal of $u\mathcal{A}ss$ generated by $\{\Phi^{-1}_{\deg f_i}(f_i)\}_{i=1}^s$. It follows from Lemma 2.3 that $\mathcal{I}' \subseteq \mathcal{I}$. By Theorem 2.9, $\Omega(\mathcal{I}') \subseteq \Omega(\mathcal{I}) = J$. By Lemma 2.4, each f_i is in $\Omega(\mathcal{I}')$. Thus $\Omega(\mathcal{I}') \supseteq J$. This forces that $\mathcal{I}' = \mathcal{I}$ and consequently, $\mathcal{I}$ is generated by $\{\Phi^{-1}_{\deg f_i}(f_i)\}_{i=1}^s$ as an operadic ideal of $u\mathcal{A}ss$.

Note that a T-ideal of the free nonunital associative algebra $\Bbbk\langle X\rangle_+$ can be defined similarly. Similarly one can show the following.

Theorem 2.11. *There is an inclusion-preserving one-to-one correspondence between the set of proper T-ideals of* $\Bbbk\langle X\rangle_+$ *and the set of proper operadic ideals of* $\mathcal{A}ss$.

Kemer proved that every proper T-ideal of $\Bbbk\langle X\rangle$ is finitely generated as a T-ideal in [Ke1, Ke2]. Applying the one-to-one correspondence between T-ideals of $\Bbbk\langle X\rangle$ and operadic ideals of $u\mathcal{A}ss$ [Theorem 2.9], we have the following consequences.

Theorem 2.12. (1) *The operad* $u\mathcal{A}ss$ *is noetherian, that is, the set of operadic ideals of* $u\mathcal{A}ss$ *satisfies the ascending chain condition.*
(2) *Every ideal of* $u\mathcal{A}ss$ *is finitely generated as an operadic ideal.*

Proof. (1) Since every T-ideal of $\Bbbk\langle X\rangle$ is finitely generated as a T-ideal [Theorem 2.2], the set of T-ideals of $\Bbbk\langle X\rangle$ satisfies the ascending chain condition. The assertion follows from Theorem 2.9.

(2) This follows from part (1) by a standard noetherian argument. □

The following lemma is needed.

Lemma 2.13. [BYZ, Lemma 2.15(2)] *Let* $\mathcal{I}$ *be a finitely generated operadic ideal of* $u\mathcal{A}ss$. *Then there exists* $\theta \in \mathcal{I}(n)$ *for some* $n \geq 0$ *such that* $\mathcal{I} = \langle \theta \rangle$.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) This is Theorem 2.12(1).

(2) This follows from Lemma 2.13 and Theorem 2.12(2). □

In general the description of a T-ideal is very difficult even if every proper T-ideal is finitely generated [Ke1, Ke2]. In fact it is quite difficult to deduce the generators from a given T-ideal. An effective way is to study the multilinear polynomials in a T-ideal since every identity is equivalent to a system of multilinear polynomials. Here is a small improvement of the original Kemer's theorem [Theorem 2.2] when we consider associative algebras with unit.

Corollary 2.14. *Every proper T-ideal corresponding to an associative algebra with unit is generated by one multilinear polynomial as a T-ideal.*

Proof. Let J be a proper T-ideal of $\Bbbk\langle X\rangle$. By Theorem 2.9, $J = \Omega(\mathcal{I})$ where $\mathcal{I}$ is a proper operadic ideal of $u\mathcal{A}ss$. By part (2), $\mathcal{I}$ is generated by an element $\theta \in \mathcal{I}(n)$. Let J' be the T-ideal of $\Bbbk\langle X\rangle$ generated by $\Phi_n(\theta)$. It remains to show that $J = J'$. By the proof of Lemma 2.4, $J' \subseteq J$. Since $\Phi_n(\theta)$ is an identity of $\Bbbk\langle X\rangle/J'$, it follows from the proof of Lemma 2.3 that

$$\theta \in \Psi(J') \subseteq \Psi(J) = \Psi(\Omega(\mathcal{I})) = \mathcal{I}.$$

Since $\mathcal{I}$ is generated by θ, we obtain that $\mathcal{I} = \Psi(J') = \Psi(J)$. Now Theorem 2.9 implies that $J' = J$ as required. □

It is well-known that Corollary 2.14 fails for algebras without unit. Note that we do not provide a new proof of Kemer's theorem. Corollary 2.14 is in the same spirit as a result of Razmyslov [Ra] and Procesi [Pr1] which states that all trace identities for the full matrix algebras are generated by a single trace identity, see also [IKM].

Remark 2.15. Using Theorems 2.11 and 2.2, one sees that the operad $\mathcal{A}ss$ is also noetherian on operadic ideals.

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