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Higher Koszul complexes

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Abstract In this paper we generalize the Koszul complexes and Koszul algebras, and introduce the higher Koszul (t-Koszul) complexes and higher Koszul algebras, where $t \geqslant 2$ is an integer. We prove that an algebra is t-Koszul if and only if its t-Koszul complex is augmented, i.e. the higher degree ($\geqslant 1$) homologies vanish. For arbitrary t-Koszul algebra Λ, we also give a description of the structure of the cohomology algebra $\mathrm{Ext}^{\bullet}_{\Lambda}(\Lambda_0, \Lambda_0)$ by using the t-Koszul complexes, where the Λ_0 is the direct sum of the simples.

Keywords: ***t*-Koszul complex, *t*-Koszul algebra.**

First introduced by Koszul, Koszul complex is applicable to the homology of Lie algebras. In order to compute the homology of the augmented algebras, Priddy constructed a kind of projective resolution for a large class of augmented algebras, including the Steenrold algebras and the enveloping algebras of Lie algebras. This kind of resolution which we call Koszul resolution generalizes the usual Koszul resolution[1]. An algebra is called Koszul algebra if each simple module has a Koszul resolution. Essentially, a Koszul resolution means that the ith graded projective presentation is generated in degree i. Recently Koszul algebras are widely applied to commutative algebra, algebraic topology, Lie theory and quantum groups[2–6].

In this paper, we generalize the Koszul complexes and Koszul algebras, and introduce the higher Koszul (t-Koszul) complexes and higher Koszul (t-Koszul) algebras, with $t \geqslant 2$ as an integer. The usual Koszul algebras are just the case of $t = 2$. We show that an algebra is a t-Koszul algebra if and only if the higher degree homologies ($\geqslant 2$) of its t-Koszul complex vanish.

Let Λ be an elementary $0,1$-generated algebra, it is well-known that the Yoneda algebra $\mathrm{Ext}^{\bullet}_{\Lambda}(\Lambda_0, \Lambda_0)$ is a positively graded algebra under the Yoneda product. We call it the cohomology algebra of Λ and denote it by $E(\Lambda)$. There is a unique compatible A_∞-algebra structure on $E(\Lambda)$, and $E(\Lambda)$ is formal if and only if Λ is Koszul[7]. Λ is a Koszul algebra if and only if $E(\Lambda)$ is generated by its degree 0 and degree 1 components[8], if and only if $E(\Lambda) \cong (\Lambda^!)^{\mathrm{opp}}$, where $\Lambda^!$ is the quadratic dual of Λ[4]. For t-Koszul algebra Λ, $t \geqslant 3$, we introduce the t-dual algebra $\Lambda^!$ of Λ, and show that as a Λ_0-Λ_0-bimodule, $E(\Lambda)_{2m+1} = \Lambda^!_{tm+1}$ and $E(\Lambda)_{2m} = \Lambda^!_{tm}$, $\forall\, i \geqslant 0$.

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

1 Higher Koszul algebras

1.1 Preliminary

Throughout this paper, we fix a positive integer t with $t \geqslant 2$. Let k be a field and $\Lambda =$

$\Lambda_0 \oplus \Lambda_1 \oplus \cdots$ be a graded k-algebra with each Λ_i being a finite-dimensional k-space. We assume that Λ is elementary (i.e. Λ_0 is a finite product of copies of k), and is generated in degrees 0 and 1 (i.e. $\Lambda_i \Lambda_j = \Lambda_{i+j}$ for all i, j). Such an algebra is called an elementary $0, 1$-generated algebra. It is isomorphic to kQ/I, where Q is a finite quiver and I is a two-sided ideal of the path algebra kQ generated by homogeneous length elements with length 2 or more[8]. Thus, $I \subseteq J^2$, where J is the two-sided ideal of kQ generated by the arrows of Q. Such an algebra Λ can be infinite-dimensional. Note that, since I is assumed to be generated by homogeneous elements of kQ, Λ is finite-dimensional if and only if I can be chosen to be admissible, i.e. $J^N \subseteq I$ for some positive integer $N \geqslant 2$. Denote by Λ-mod the category of finitely generated left Λ-modules. All modules considered in this paper are in Λ-mod.

A Λ-module M is said to be a graded Λ-module provided that $M = \bigoplus\limits_{i \in \mathbb{Z}} M_i$ where each M_i is a finite-dimensional k-space such that $\Lambda_i M_j \subseteq M_{i+j}$; and a graded Λ-module M is said to be generated in degree 0 if $M_i = (0)$ for $i < 0$ and $M_i = \Lambda_i M_0$ for all $i \geqslant 0$. Let M and N be graded Λ-modules. A Λ-homomorphism $f: M \to N$ is said to be of degree 0 if $f(M_i) = N_i$ for all i. Denote the category of finitely generated graded Λ-modules and degree 0 maps by $\mathrm{gr}(\Lambda)$. All graded modules considered in this paper are in $\mathrm{gr}(\Lambda)$-mod.

If $M = \bigoplus\limits_{i \in \mathbb{Z}} M_i$ is a graded Λ-module and n is an integer, we let $M[n]$ denote the graded Λ-module $N = \bigoplus\limits_{i \in \mathbb{Z}} N_i$ such that $N_i = M_{i-n}$. Note that Λ will be viewed as a graded Λ- module generated in degree 0. If P is a graded summand of Λ, then $P[n]$ is a graded projective Λ-module. Graded projective Λ-modules are just direct sum of projective modules of the form $P[n]$ for $n \in \mathbb{Z}$.

We denote the set of vertices of a quiver Q by Q_0 and the set of arrows of Q by Q_1. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting point and the ending point of the arrow α. We will write compositions of paths in Q from right to left. For any $v \in Q_0$, denote by e_v the trivial path (i.e. a path with length zero) with the same starting point and ending point v.

Note that Λe_v is a graded left projective Λ-module generated in degree 0, for $v \in Q_0$. If M is a graded Λ-module, then there exists a finite index set $\mathcal{I}$ and maps $\mu: \mathcal{I} \to Q_0$, $d: \mathcal{I} \to \mathbb{Z}$, and $f: \bigoplus\limits_{i \in \mathcal{I}} \Lambda e_{\mu(i)}[d(i)] \to M$, such that f is a graded projective cover of M. Moreover, if P is a graded projective Λ-module, then P decomposes into a direct sum of projectives of the form $\Lambda e_v[n]$ where $v \in Q_0$ and $n \in \mathbb{Z}$.

Denote by $\mathrm{gr}_0(\Lambda)$ the full subcategory of $\mathrm{gr}(\Lambda)$ consisting of graded modules generated in degree 0. By definition, a graded module M has a linear presentation if there is an exact sequence in $\mathrm{gr}(\Lambda)$: $P^1 \to P^0 \to M \to 0$, such that P^1 and P^0 are generated in degrees 1 and 0, respectively. Denote by $\mathcal{L}(\Lambda)$ the full subcategory of $\mathrm{gr}_0(\Lambda)$ consisting of modules with linear presentations.

The graded Jacobson radical of Λ, denoted by $\boldsymbol{r}$ is $\Lambda_1 \oplus \Lambda_2 \oplus \cdots$. We say that Λ is quadratic if $\mathbf{r}[-1] \in \mathcal{L}(\Lambda)$; or equivalently, if there is a graded exact sequence $P^2 \to P^1 \to P^0 \to \Lambda_0 \to 0$ such that P^i is generated in degree i for $i = 0, 1, 2$.

1.2 Definition of higher Koszul algebras

First for each integer $t \geqslant 2$, we introduce a function $\boldsymbol{t} : \mathbb{N}_0 \to \mathbb{N}_0$ which is given by

$$\boldsymbol{t}(2m) := tm, \quad \boldsymbol{t}(2m+1) := tm + 1 \ \forall\ m \geqslant 0.$$

Definition 1.1. Let Λ be an elementary 0, 1-generated algebra, $M \in \mathrm{gr}_0(\Lambda)$. We call M a t-Koszul module if it has a graded projective resolution

$$\cdots \longrightarrow P^{2m+1} \longrightarrow P^{2m} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

such that P^m is generated in degree $\boldsymbol{t}(m)$, for any $m \geqslant 0$. Let $\mathcal{K}_t(\Lambda)$ denote the full subcategory of $\mathrm{gr}_0(\Lambda)$ of t-Koszul modules. We call Λ a t-Koszul algebra if $\Lambda_0 \in \mathcal{K}_t(\Lambda)$.

Remark. If $t = 2$, then the t-Koszul modules are just the Koszul modules defined in ref. [4], and the t-Koszul algebras are the Koszul algebras.

1.3 Some examples

(i) Any path algebra can be regarded as a t-Koszul algebra for any $t \geqslant 2$.

(ii) Any t-truncated algebra Λ (i.e. $\Lambda = kQ/J^t$ where Q is a finite quiver and J is the ideal of kQ generated by all arrows) is a t-Koszul algebra for any $t \geqslant 2$.

(iii) Different from Koszul algebras, a t-monomial algebra Λ (i.e. $\Lambda = kQ/I$ where Q is a finite quiver and I is an ideal of kQ generated by some paths all of length t) is not necessarily a t-Koszul algebra. For example, let Λ be the algebra given by the following quiver

$$1 \cdot \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdot 2$$

with relations $\alpha\beta\alpha$. Consider the simple module S_1 corresponding to vertex 1. S_1 has graded projective resolution

$$\boldsymbol{P}^{\bullet} : \ \cdots \to \Lambda e_2[7] \to \Lambda e_2[5] \to \Lambda e_2[3] \to \Lambda e_2[1] \to \Lambda e_1 \to S_1$$

with $P^3 = \Lambda e_2$ (see ref. [5]) generated in degree 5, not in degree 4.

(iv) Let $\Lambda = kQ/I$. I is generated by homogeneous elements of lenth t. If gl.dim.$(\Lambda) \leqslant 2$, then Λ is a t-Koszul algebra.

1.4 Opposite algebras of $\boldsymbol{t}$-Koszul algebras are also t-Koszul

Lemma 1.1. Let Λ be an elementary 0, 1-generated algebra, and $\boldsymbol{P}^{\bullet}$ be a minimal graded projective resolution of Λ_0. Let i, $n(i) \in \mathbb{N}_0$. Then P^i is generated in degree $n(i)$ if and only if $\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = 0$ unless $n = n(i)$.

In particular, Λ is a t-Koszul algebra if and only if $\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = 0$ unless $n = \boldsymbol{t}(i)$ for any $i \geqslant 0$.

Proof. Since $\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n])$ is a sub-quotient of $\mathrm{Hom}_{\mathrm{gr}(\Lambda)}(P^i, \Lambda_0[n])$, the "only if" part follows.

In order to prove the "if" part, note that

$$\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = \mathrm{Hom}_{\mathrm{gr}(\Lambda)}(Z^{i-1}, \Lambda_0[n]) = 0,$$

where $Z^{i-1} = \mathrm{Ker}(P^{i-1} \longrightarrow P^{i-2})$. It follows that Z^{i-1} and hence P^i is generated in degree $n(i)$.

Similar to the Koszul case[4], we have

Proposition 1.1. If Λ is a t-Koszul algebra, then so is its opposite algebra.

Proof. By Lemma 1.1, we need to prove that $\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = 0$ unless $n = \boldsymbol{t}(i)$ for any $i \in \mathbb{N}_0$, where Λ_0 is regarded as a right Λ-module. Let

$$\cdots \longrightarrow P^{i+1} \longrightarrow P^i \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M, where P^i is generated in degree $\boldsymbol{t}(i)$ for $i \geqslant 0$. Then we get an injective resolution of the right Λ-module Λ_0

$$0 \longrightarrow \Lambda_0 \longrightarrow (P^0)^{\circledast} \longrightarrow (P^1)^{\circledast} \longrightarrow \cdots \longrightarrow (P^i)^{\circledast} \longrightarrow (P^{i+1})^{\circledast} \longrightarrow \cdots,$$

where $(P^i)^{\circledast} := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\Lambda_0}(P^i_n, \Lambda_0)$ is a right injective Λ-module. Note that for any left graded Λ-module M, the gradation of the right Λ-module $M^{\circledast}$ is given by $(M^{\circledast})_i := (M_{-i})^*$. Thus, for a right Λ-module Λ_0, $\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0[-n], \Lambda_0)$ is a subquotient of

$$\mathrm{Hom}_{\mathrm{gr}(\Lambda)}(\Lambda_0[-n], (P^i)^{\circledast}) \cong ((\Lambda_0 \otimes_\Lambda P^i)_n)^* \cong ((P^i_i)_n)^*,$$

which is zero unless $n = \boldsymbol{t}(i)$. This proves that Λ^{opp} is also t-Koszul.

1.5. t-Koszul algebras can be given by quivers with relations of length t.

Proposition 1.2. Let $\Lambda = kQ/I$ where Q is a finite quiver and I is an ideal generated by homogeneous elements of length $\geqslant 2$. Let $\boldsymbol{P}^{\bullet}$ be a graded projective resolution of Λ_0. Then P^2 is generated in degree t if and only if $\mathrm{Ext}^2_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = 0$ unless $n = t$, if and only if I is generated by homogeneous elements all of length t.

Proof. By Proposition 1.2, we know the first assertion. Also by definition we see that, if I is generated by homogeneous elements all of length t, then P^2 is generated in degree t.

Assume that $\mathrm{Ext}^2_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = 0$ unless $n = t$. Consider the exact sequence in $\mathrm{gr}(\Lambda)$:

$$0 \to W \to \Lambda \otimes_{\Lambda_0} (kQ_1) \xrightarrow{m} \Lambda \to \Lambda_0,$$

with m being the multiplication map. Then $W \subseteq \boldsymbol{r} \otimes kQ_1$, where $\boldsymbol{r}$ is the graded Jacobson radical of Λ. Thus we have

$$\mathrm{Ext}^2_{\mathrm{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = \mathrm{Hom}_{\mathrm{gr}(\Lambda)}(W, \Lambda_0[n]).$$

Now the assumption implies that W is generated in degree t. This means that I is generated by homogeneous elements all of length t as required.

Corollary 1.1. Let $\Lambda = kQ/I$ be a t-Koszul algebra. Then I is generated by homogeneous elements all of length t.

2 Higher Koszul complex

Let t be a fixed positive integer with $t \geqslant 2$. Recall the definition of the function $\boldsymbol{t}: \mathbb{N}_0 \to \mathbb{N}_0$, which is given by

$$\boldsymbol{t}(2m+1) := mt + 1, \quad \boldsymbol{t}(2m) = mt \ \forall\, m \geqslant 0. \tag{2.1}$$

Assuming that $\Lambda = kQ/I$, where Q is a finite quiver, I is an ideal of kQ generated by some combinations of paths in Q of length t, we call such an algebra a t-algebra. Rewrite Λ as $\Lambda = T_{\Lambda_0}(V)/\langle R\rangle$, where $V = kQ_1$, $T_{\Lambda_0}(V)$ is the tensor algebra of the bimodule ${}_{\Lambda_0}V_{\Lambda_0}$ over Λ_0, and $R = \mathrm{Ker}(V^{\otimes t} \to \Lambda_t)$ is a Λ_0-Λ_0-sub-bimodule of $V^{\otimes t}$.

Modifying the definition of the Koszul complex of a quadratic algebra[4], we can define the t-Koszul complex for any t-algebra.

Definition 2.1. For t-algebra $\Lambda = T_{\Lambda_0}(V)/\langle R\rangle$, define the t-Koszul complex

$$\boldsymbol{K}^{\bullet}: \cdots \longrightarrow K^{2m+1} \xrightarrow{d^{2m+1}} K^{2m} \xrightarrow{d^{2m}} K^{2m-1} \longrightarrow \cdots K^2 \xrightarrow{d^2} K^1 \xrightarrow{d^1} K^0$$

of Λ as follows. For any $i \geqslant 0$, K^i is a graded projective Λ-module given by

$$K^i = \Lambda \otimes K^i_{\boldsymbol{t}(i)},$$

where $K^i_{\boldsymbol{t}(i)}$ is a Λ_0-bimodule concentrating in degree $\boldsymbol{t}(i)$,

$$K^i_{\boldsymbol{t}(i)} = \bigcap_{0\leqslant v\leqslant \boldsymbol{t}(i)-t} V^{\otimes v}\otimes R\otimes V^{\otimes \boldsymbol{t}(i)-t-v} \subseteq V^{\otimes \boldsymbol{t}(i)}.$$

Note that $K^i_{\boldsymbol{t}(i)}$ is concentrating in degree $\boldsymbol{t}(i)$, and the $(j+\boldsymbol{t}(i))$-th component of K^i is $\Lambda_j\otimes K^i_{\boldsymbol{t}(i)}$, $\forall\, j \geqslant 0$. For brevity, we set $K^i_j = (K^i)_j$. In particular, we have

$$K^0 = \Lambda\otimes\Lambda_0 = \Lambda, \quad K^1 = \Lambda\otimes V = \Lambda\otimes\Lambda_1, \quad K^2 = \Lambda\otimes R.$$

The differential $d^i : K^i \to K^{i-1}$ is defined as the restriction of the map $\tilde{d}^i : \Lambda\otimes V^{\otimes \boldsymbol{t}(i)} \to \Lambda\otimes V^{\otimes \boldsymbol{t}(i-1)}$, where $\tilde{d}^i$ is given by

$$a\otimes(a_1\otimes\cdots\otimes a_{\boldsymbol{t}(i)}) \mapsto aa_1\cdots a_{\boldsymbol{t}(i)-\boldsymbol{t}(i-1)}\otimes(a_{\boldsymbol{t}(i)-\boldsymbol{t}(i-1)+1}\otimes\cdots\otimes a_{\boldsymbol{t}(i)}), \quad \forall\, a\in\Lambda.$$

Thus, we have

$$a\otimes(a_1\otimes\cdots\otimes a_{mt+1}) \overset{d^{2m+1}}{\longmapsto} aa_1\otimes(a_2\otimes\cdots\otimes a_{mt+1}), \quad \forall\, a\in\Lambda,$$

and

$$a\otimes(a_1\otimes\cdots\otimes a_{mt}) \overset{d^{2m}}{\longmapsto} aa_1\cdots a_{t-1}\otimes(a_t\otimes\cdots\otimes a_{mt}), \quad \forall\, a\in\Lambda.$$

Obviously d^i is well-defined and d^i is of degree 0. Thus $d^i d^{i+1} = 0$, $\forall\, i\geqslant 1$, hence $\boldsymbol{K}^{\bullet}$ is indeed a complex.

It is clear that if the Koszul complex $\boldsymbol{K}^{\bullet}$ of Λ is a projective resolution of Λ_0 (with the canonical projection $K^0(=\Lambda)\to\Lambda_0$), then Λ is a t-Koszul algebra. The main purpose of this paper is to prove that the inverse statement also holds, i.e. Λ is a t-Koszul algebra if and only if the t-Koszul complex $\boldsymbol{K}^{\bullet}$ is a projective resolution of Λ_0. Also we get a description of the structure of the Yoneda algebra $E(\Lambda)$ in this case, by using the Koszul resolution.

Set $Z^i := \mathrm{Ker} d^i$. Since the restriction $d^i : K^i_{\boldsymbol{t}(i)} \longrightarrow K^{i-1}_{\boldsymbol{t}(i)}$ is injective by construction, it follows that Z^i lives in degree $\geqslant \boldsymbol{t}(i)+1$. Also, since the restriction $d^{2m+1} : K^{2m+1}_j \longrightarrow K^{2m}_j$ is injective by construction, for $mt+1\leqslant j\leqslant mt+1+(t-2) = (m+1)t-1$, it follows that Z^{2m+1} lives in degree $\geqslant (m+1)t$. We have the following lemma.

Lemm 2.1. Let Λ be a t-Koszul algebra and $\boldsymbol{K}^{\bullet}$ its Koszul complex. Then $\boldsymbol{K}^{\bullet}\to\Lambda_0\to 0$ is a projective resolution of Λ_0 if and only if

$$Z^{2m+1}_{(m+1)t} \subseteq d^{2m+2}(K^{2m+2}_{(m+1)t}), \ \forall\, m\geqslant 0.$$

Proof. The necessity is obvious. In order to prove the sufficiency, we use induction on i to prove the cohomology $H^i(\boldsymbol{K}^{\bullet}) = 0$ for all $i\geqslant 0$. It is clear that $K^1\longrightarrow K^0\longrightarrow\Lambda_0\longrightarrow 0$ is an exact sequence. So we need to prove $H^i(\boldsymbol{K}^{\bullet}) = 0$ for all $i\geqslant 1$.

First, we claim that Z^i is generated in degree $\boldsymbol{t}(i+1)$, where $\boldsymbol{t}(i+1)$ is defined as in (2.1).

In order to prove the claim, it suffices to show that

$$\mathrm{Hom}_{\mathrm{gr}(\Lambda)}(Z^i, \Lambda_0[n]) = 0 \ \text{ if } \ n\neq \boldsymbol{t}(i+1).$$

Since by induction we have a graded projective resolution of Λ_0

$$P^{i+2} \longrightarrow P^{i+1} \longrightarrow K^i \longrightarrow K^{i-1} \longrightarrow \cdots \longrightarrow K^1 \longrightarrow K^0 \longrightarrow \Lambda_0 \longrightarrow 0,$$

where P^{i+2} and P^{i+1} are some graded projective Λ-modules, it follows that

$$\begin{aligned}\operatorname{Ext}^{i+1}_{\operatorname{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) &= \frac{\operatorname{Ker}(\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(P^{i+1}, \Lambda_0[n]) \to \operatorname{Hom}_{\operatorname{gr}(\Lambda)}(P^{i+2}, \Lambda_0[n]))}{\operatorname{Im}(\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(K^i, \Lambda_0[n]) \to \operatorname{Hom}_{\operatorname{gr}(\Lambda)}(P^{i+1}, \Lambda_0[n]))} \\ &= \frac{\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(Z^i, \Lambda_0[n])}{\operatorname{Im}(\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(K^i, \Lambda_0[n]) \to \operatorname{Hom}_{\operatorname{gr}(\Lambda)}(Z^i, \Lambda_0[n]))} \\ &= \operatorname{coker}(\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(K^i, \Lambda_0[n]) \to \operatorname{Hom}_{\operatorname{gr}(\Lambda)}(Z^i, \Lambda_0[n])).\end{aligned}$$

Note that the induced map

$$\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(K^i, \Lambda_0[n]) \to \operatorname{Hom}_{\operatorname{gr}(\Lambda)}(Z^i, \Lambda_0[n])$$

is zero. This proves

$$\operatorname{Hom}_{\operatorname{gr}(\Lambda)}(Z^i, \Lambda_0[n]) = \operatorname{Ext}^{i+1}_{\operatorname{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]).$$

But since Λ is a t-Koszul algebra, i.e. P^{i+1} is generated in degree $(i+1)$, it follows that

$$\operatorname{Ext}^{i+1}_{\operatorname{gr}(\Lambda)}(\Lambda_0, \Lambda_0[n]) = 0 \quad \text{if} \quad n \neq \boldsymbol{t}(i+1).$$

This implies that Z^i is generated in degree $\boldsymbol{t}(i+1)$, where $\boldsymbol{t}(i+1)$ is defined as in (2.1).

Secondly, we claim that $H^{2m}(\boldsymbol{K}^\bullet) = 0$, i.e. $Z^{2m} = \operatorname{Im} d^{2m+1}$, or equivalently, $Z^{2m}_{mt+1} \subseteq d^{2m+1}(K^{2m+1}_{mt+1})$, $\forall\, m \geqslant 1$.

In fact, if $\sum a \otimes (a_1 \otimes \cdots \otimes a_{mt}) \in Z^{2m}_{mt+1} \subseteq \Lambda_1 \otimes K^{2m}_{mt}$, $a \in \Lambda_1$, i.e. $\sum a a_1 \cdots a_{t-1} \otimes (a_t \otimes \cdots \otimes a_{mt}$ 0, then $\sum a \otimes a_1 \otimes \cdots \otimes a_{t-1} \otimes (a_t \otimes \cdots \otimes a_{mt}) \in R \otimes V^{\otimes(m-1)t+1}$. It follows that

$$\sum 1 \otimes a \otimes a_1 \otimes \cdots \otimes a_{mt}) \in \Lambda_0 \otimes K^{2m+1}_{mt+1} = K^{2m+1}_{mt+1}$$

and

$$d^{2m+1}(\sum 1 \otimes a \otimes a_1 \otimes \cdots \otimes a_{mt}) = \sum a \otimes a_1 \otimes \cdots \otimes a_{mt}.$$

Finally, by assumption $Z^{2m+1}_{(m+1)t} \subseteq d^{2m+2}(K^{2m+2}_{(m+1)t})$, $\forall\, m \geqslant 0$, we have $Z^{2m+1} = \operatorname{Im} d^{2m+2}$, $\forall\, m$ 0, hence $H^{2m+1}(\boldsymbol{K}^\bullet) = 0$ for all $m \geqslant 0$. This completes the proof.

Before giving the main result, we make some preparations. We recall two basic lemmas in linear algebra.

Assume S to be a unitary semi-simple ring, i.e. $S = S_1 \times S_2 \times \cdots \times S_n$, where S_i's are the complete set of simple ideals of S. Let e_i be the identity of S_i. Then $1 = e_1 + e_2 + \cdots + e_n$ is a decomposition of minimal orthogonal idempotent of the identity of S. By mod-S we denote the category of finitely generated right S-modules. Similarly, S-mod and S-mod-S denote the category of finitely generated left S-modules and S-S-modules respectively.

Lemma 2.2. Let S be a semi-simple ring. Assuming $M \in$ mod-S, $N \in S$-mod, for any submodules $H, L \subseteq N$, we have

$$M \otimes (H \cap L) = (M \otimes H) \cap (M \otimes L).$$

Proof. Obviously, $M \otimes (H \cap L) \subseteq (M \otimes H) \cap (M \otimes L)$. It suffices to show that for any $x \in (M \otimes H) \cap (M \otimes L)$, $x \in M \otimes (H \cap L)$.

Decompose S as above. As a semi-simple module, M can be decomposed into $M = M_1 \oplus M_2 \oplus \cdots \oplus M_l$, where $M_i = m_i S \cong e_{s_i} S$ are simple modules such that $m_i = m_i e_{s_i} \in M$,for any

$1 \leqslant i \leqslant l$. We can write x as $x = \sum_{1\leqslant i\leqslant m} \bar{m}_i \otimes \bar{h}_i$ for $x \in M \otimes H$, where $\bar{m}_i \in M_i$, $\bar{h}_i \in H$. Since $\bar{m}_i = \sum_{1\leqslant j\leqslant l} m_j a_{ij}$, $\forall\, 1 \leqslant i \leqslant m$, where $a_{ij} \in S$, we get

$$x = \sum_{1\leqslant i\leqslant m} \bar{m}_i \otimes \bar{h}_i = \sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant l}} m_j a_{ij} \otimes \bar{h}_i = \sum_{1\leqslant j\leqslant l} m_j \otimes \sum_{1\leqslant i\leqslant m} a_{ij}\bar{h}_i.$$

Thus we can rewrite x as $x = \sum_{1\leqslant i\leqslant l} m_i \otimes h_i$ such that $h_i = e_{s_i} h_i \in H$.

Similarly, since $x \in M \otimes L$, we have $x = \sum_{1\leqslant i\leqslant l} m_i \otimes l_i$, where $l_i = e_{s_i} l_i \in L$. Because S is semi-simple, for any $x \in M \otimes N$, x has unique expression of the form $x = \sum_{1\leqslant i\leqslant l} m_i \otimes n_i$ such that $n_i = e_{s_i} n_i \in N$. Thus $h_i = l_i$, $\forall\, 1 \leqslant i \leqslant l$, hence $x \in M \otimes (H \cap L)$. This completes the proof.

Since any modules over semi-simple rings are projective and therefore flat, we have the following lemma.

Lemma 2.3. Assume that S is semi-simple, $M, N \in$ mod-S, $f \in \mathrm{Hom}_s(M, N)$, $H \in S$-mod. Considering the map $f \otimes 1\ : M \otimes H \to N \otimes H$, we have

$$\mathrm{Ker}(f \otimes 1) = (\mathrm{Ker} f) \otimes H.$$

Similarly, if $M, N \in S$-mod, $H \in$ mod-S, and $f \in \mathrm{Hom}_s(M, N)$, then

$$\mathrm{Ker}(1 \otimes f) = H \otimes (\mathrm{Ker} f).$$

Now we return to t-Koszul complexes of t-Koszul algebras. The following proposition is natural.

Proposition 2.5. Assuming that Λ is a t-Koszul algebra, we have the following exact sequence:

$$\Lambda \otimes ((R \otimes V) \cap (V \otimes R)) \xrightarrow{d^3} \Lambda \otimes R \xrightarrow{d^2} \Lambda \otimes V \xrightarrow{d^1} \Lambda \xrightarrow{d^0} \Lambda_0,$$

where $V = \Lambda_1$, $R = \mathrm{Ker}(V^{\otimes t} \xrightarrow{m} \Lambda_t)$ and m is induced by multiplication.

Proof. By the construction of the t-Koszul complex, $\mathrm{Im} d^1 = \mathrm{Ker} d^0$, i.e. the sequence is exact at Λ. Also from the proof of Lemma we know $\mathrm{Im} d^2 = \mathrm{Ker} d^1$; that is, the sequence is exact at $\Lambda \otimes V$.

We need to prove that $\mathrm{Im} d^3 = \mathrm{Ker} d^2$. It suffices to prove that $\mathrm{Im} d^3 \supseteq \mathrm{Ker} d^2$. Since Λ is t-Koszul, by definition, $\mathrm{Ker} d^2$ is generated in degree $t+1$; thus we need only to show that $(\mathrm{Ker} d^2)_{t+1} \subseteq \mathrm{Im} d^3$.

Suppose $x \in (\mathrm{Ker} d^2)_{t+1}$, and write x as $x = \sum_i a_i \otimes R_i$, $a_i \in V$, $R_i \in R$. If $x \in ((R \otimes V) \cap (V \otimes R))$ is proved. Then, $1 \otimes x \in \Lambda \otimes ((R \otimes V) \cap (V \otimes R))$, satisfying $d^3(1 \otimes x) = x$. The proposition follows.

By the choice of x, we have $x \in V \otimes R$. It suffices to show that $x \in R \otimes V$. By the map $V^{\otimes t+1} \xrightarrow{m\otimes 1} \Lambda_t \otimes V$, we get $(m \otimes 1)(x) = d^2(x) = 0$. By Lemma 2.4, we have $x \in R \otimes V$. Thus the sequence is exact as required.

Now a key lemma follows.

Lemma 2.4. Let $\Lambda = T_{\Lambda_0}(V)/\langle R\rangle$ be a t-Koszul algebra, where $V = kQ_1$, $T_{\Lambda_0}(V)$, $R = \mathrm{Ker}(V^{\otimes t} \xrightarrow{m} \Lambda_t)$. Then for any $1 \leqslant n \leqslant t-1$, we have

$$(V^{\otimes n} \otimes R) \cap (R \otimes V^{\otimes n}) = \bigcap_{0 \leqslant i \leqslant n} (V^{\otimes i} \otimes R \otimes V^{n-i}).$$

Remark. In the case $t = 2$, the lemma is trivial.

Proof. We prove the lemma by induction on n. The case $n = 1$ is obvious. Suppose that the lemma holds in the case $\leqslant n-1$. We need to show that it also holds in the case n. Consider the following map:

$$V^{\otimes n} \otimes R \xrightarrow{m} \Lambda_{n-1} \otimes (V \otimes R) \xrightarrow{\tilde{d}^3} \Lambda_n \otimes R \xrightarrow{\tilde{d}^2} \Lambda_{n+t-1} \otimes V,$$

where m is given by the multiplication of the first $n-1$ terms; $\tilde{d}^3$ and $\tilde{d}^2$ are the restriction of the maps defined in Definition 2.1.

For $x \in (V^{\otimes n} \otimes R) \cap (R \otimes V^{\otimes n})$, we have $\tilde{d}^2(\tilde{d}^3(m(x))) = 0$. This implies that $\tilde{d}^3(m(x)) \in \mathrm{Ker}\tilde{d}^2$. Since $d^2|_{\Lambda_n \otimes R} = \tilde{d}^2|_{\Lambda_n \otimes R}$, by Proposition 2.1, there exits $\bar{x} \in \Lambda_{n-1} \otimes ((R \otimes V) \cap (V \otimes R))$ such that $d^3(\bar{x}) = \tilde{d}^3(m(x))$. But for any $1 \leqslant s \leqslant t-1$, the multiplication $v^{\otimes s} \to \Lambda_s$ is injective, thus m and $\tilde{d}^3|_{\Lambda_{n-1} \otimes V^{\otimes t+1}}$ are both injective, so we get $\bar{x} = m(x)$. On the other hand, m is obviously surjective, implying that m is an isomorphism. Hence by $\bar{x} \in \Lambda_{n-1} \otimes R \otimes V$, we get $x \in V^{\otimes n-1} \otimes R \otimes V$. Thus, we have

$$\begin{aligned} x &\in (V^{\otimes n-1} \otimes R \otimes V) \cap (R \otimes V^{\otimes n}) \cap (V^{\otimes n} \otimes R) \\ &= (((V^{\otimes n-1} \otimes R) \cap (R \otimes V^{\otimes n-1})) \otimes V) \cap (V^{\otimes n} \otimes R) \\ &= \Big(\bigcap_{0 \leqslant i \leqslant n-1} (V^i \otimes R \otimes V^{n-1-i}) \otimes V \Big) \cap (V^{\otimes n} \otimes R) \\ &= \bigcap_{0 \leqslant i \leqslant n} V^i \otimes R \otimes V^{n-i}, \end{aligned}$$

where the second step is given by Lemma, while the third one is given by induction.

Here we to give the main theorem of this section

Therem 2.1. Let Λ be an elementary 0, 1-generated algebra. Then Λ is t-Koszul if and only if its t-Koszul complex is a projective resolution of Λ_0. And in this case, the t-Koszul complex $\boldsymbol{K}^\bullet$ of Λ is also a minimal resolution of Λ_0.

Proof. The sufficiency is given by the definitions. By Lemma 2.1, to prove the necessity, it suffices to show that $Z^{2m+1}_{(m+1)t} \subseteq d^{2m+2}(K^{2m+2}_{(m+1)t})$, $\forall\, m \geqslant 0$.

Suppose $x \in Z^{2m+1}_{(m+1)t}$, $x \in \Lambda_{t-1} \otimes K^{2m+1}_{mt+1}$. Consider the following map:

$$V^{\otimes (m+1)t} \xrightarrow{m} \Lambda_{t-1} \otimes V^{\otimes mt+1} \xrightarrow{\tilde{d}^{2m+1}} \Lambda_t \otimes V^{\otimes mt},$$

where m and $\tilde{d}^{2m+1}$ are both induced by multiplication. It is easy to show that m is a Λ_0-Λ_0-bimodule isomorphism; thus there exists $\bar{x} \in V^{\otimes (m+1)t}$ such that $x = m(\bar{x})$.

Since $x \in \Lambda_{t-1} \otimes K^{2m+1}_{mt+1}$, we have $\bar{x} \in V^{\otimes t-1} \otimes K^{2m+1}_{mt+1}$, $x \in Z^{2m+1}_{(m+1)t}$ and

$$\tilde{d}^{2m+1}|_{\Lambda_{t-1} \otimes K^{2m+1}_{mt+1}} = d^{2m+1}|_{\Lambda_{t-1} \otimes K^{2m+1}_{mt+1}}$$

implies that $\tilde{d}^{2m+1}(m(\bar{x})) = d^{2m+1}(x) = 0$. Applying Lemma 2.3, we have $\bar{x} \in R \otimes V^{\otimes mt}$. Thus

$$\begin{aligned}
\bar{x} &\in (R \otimes V^{\otimes mt}) \cap (V^{\otimes t-1} \otimes K^{2m+1}_{mt+1}) \\
&= (R \otimes V^{\otimes mt}) \cap \left(V^{\otimes t-1} \otimes \bigcap_{i=0}^{(m-1)t+1} (V^{\otimes i} \otimes R \otimes V^{\otimes (m-1)t+1-i}) \right) \\
&= (R \otimes V^{\otimes mt}) \cap \bigcap_{t-1 \leqslant i \leqslant mt} (V^{\otimes i} \otimes R \otimes V^{\otimes mt-i}) \\
&\subseteq (R \otimes V^{\otimes mt}) \cap (V^{\otimes t-1} \otimes R \otimes V^{\otimes (m-1)t+1}) \\
&= \big((R \otimes V^{\otimes t-1}) \cap (V^{\otimes t-1} \otimes R)\big) \otimes V^{\otimes (m-1)t+1} \\
&= \bigcap_{0 \leqslant i \leqslant t-1} (V^{\otimes i} \otimes R \otimes V^{\otimes t-1-i}) \otimes V^{\otimes (m-1)t+1},
\end{aligned}$$

where the last step is given by Lemma 2.4. Hence

$$\bar{x} \in \bigcap_{0 \leqslant i \leqslant mt} (V^{\otimes i} \otimes R \otimes V^{\otimes (m+1)t-i}) = K^{2m+2}_{(m+1)t}.$$

Considering the element $1 \otimes \bar{x} \in \Lambda_0 \otimes (K^{2m+2}_{(m+1)t})$, we get $x = d^{2m+2}(1 \otimes \bar{x}) \in d^{2m+2}(K^{2m+2}_{(m+1)t})$, as required.

3 Cohomology algebra of higher Koszul algebras

For any k-algebra Λ and any Λ-module M, $\mathrm{Ext}^\bullet(M, M) := \bigoplus_{n \geqslant 0} \mathrm{Ext}^n(M, M)$ becomes a positively graded algebra under the Yoneda product, where $\mathrm{Ext}^\bullet(-,-)$ denotes $\mathrm{Ext}^\bullet_{\Lambda_0}(-,-)$. An elementary 0, 1-generated algebra Λ is Koszul if and only if its cohomology algebra $\mathrm{Ext}^\bullet(\Lambda_0, \Lambda_0)$ can be generated in degree 0, and 1(see ref. [8]); if and only if $\mathrm{Ext}^\bullet(\Lambda_0, \Lambda_0) \cong (\Lambda^!)^{\mathrm{opp}}$, where $\Lambda^!$ is the quadratic dual of Λ (see ref. [4]). In this section, we introduce the t-dual algebra for any $t \geqslant 3$ and any t-algebra. For a t-Koszul algebra Λ, we give a description of its cohomology algebra by using its t-dual algebra. Concretely, we show that $\mathrm{Ext}^i(\Lambda_0, \Lambda_0)$, the i-degree component of $\mathrm{Ext}^\bullet(\Lambda_0, \Lambda_0)$, is just the $\boldsymbol{t}(i)$-degree component of the t-dual algebra of Λ, where $\boldsymbol{t}(i)$ is given by (2.1).

The following general lemma is well-known and very useful in the study of graded algebras and graded modules.

Lemma 3.1. Let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ be an arbitrary positively graded algebra, and M and N be finitely generated graded Λ-modules. Then we have

$$\mathrm{Hom}_\Lambda(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}(\Lambda)}(M, N[n]) \tag{3.1}$$

and

$$\mathrm{Ext}^i_\Lambda(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(M, N[n]) . \tag{3.2}$$

Proposition 3.1. Let Λ be an arbitrary elementary 0, 1-generated algebra, and let $\boldsymbol{P}^\bullet$ be a minimal graded resolution of Λ_0. Assume that P^i, P^j and P^{i+j} are generated in degree $n(i)$, $n(j)$ and $n(i+j)$ respectively. If $n(i) + n(j) \neq n(i+j)$. Then we have the Yoneda product

$$\mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0) \cdot \mathrm{Ext}^j_\Lambda(\Lambda_0, \Lambda_0) = 0.$$

Proof. By (3.2) and Lemma 1.1, we have

$$\begin{aligned}&\mathrm{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)\cdot\mathrm{Ext}^j_\Lambda(\Lambda_0,\Lambda_0)\\ &\subseteq \mathrm{Ext}^{i+j}_\Lambda(\Lambda_0,\Lambda_0)=\bigoplus_{n\in\mathbb{Z}}\mathrm{Ext}^{i+j}_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n])=\mathrm{Ext}^{i+j}_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(i+j)])\end{aligned}$$

$$\mathrm{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)=\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(i)]),\quad \mathrm{Ext}^j_\Lambda(\Lambda_0,\Lambda_0)=\mathrm{Ext}^j_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(j)]).$$

It follows that

$$\begin{aligned}\mathrm{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)\cdot\mathrm{Ext}^j_\Lambda(\Lambda_0,\Lambda_0)&=\mathrm{Ext}^i_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(i)])\cdot\mathrm{Ext}^j_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(j)])\\ &\subseteq \mathrm{Ext}^{i+j}_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(i)+n(j)]).\end{aligned}$$

Since $n(i+j)\neq n(i)+n(j)$, we have

$$\begin{aligned}&\mathrm{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)\cdot\mathrm{Ext}^j_\Lambda(\Lambda_0,\Lambda_0)\\ &\subseteq \mathrm{Ext}^{i+j}_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(i+j)])\cap\mathrm{Ext}^{i+j}_{\mathrm{gr}(\Lambda)}(\Lambda_0,\Lambda_0[n(i)+n(j)])=0,\end{aligned}$$

as required.

From Proposition 3.1, we have

Corollary 3.1. Let Λ be a t-Koszul algebra, $t\geqslant 3$. Then we have

$$\mathrm{Ext}^{2i+1}_\Lambda(\Lambda_0,\Lambda_0)\cdot\mathrm{Ext}^{2j+1}_\Lambda(\Lambda_0,\Lambda_0)=0.$$

Proof. In the t-Koszul , $t\geqslant 3$ case, P^{2i+1} is generated in degree $it+1$, $P^{2(i+j+1)}$ in degree $(i+j+1)t$. Since $t\geqslant 3$, $(it+1)+(jt+1)\neq(i+j+1)t$, the conclusion follows.

Also, with Proposition 1.4 and Proposition 3.2, we get

Corollary 3.2. Let Λ be a t-algebra (i.e. $\Lambda=kQ/I$, I can be generated by homogeneous elements all of length t), where $t\geqslant 3$. Then we have the Yoneda product

$$\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)\cdot\mathrm{Ext}^1(\Lambda_0,\Lambda_0)=0.$$

Let $\Lambda=kQ/I$ be a t-algebra. Rewrite Λ as $\Lambda=T_{\Lambda_0}(V)/\langle R\rangle$, where $V=kQ_1$, and $T_{\Lambda_0}(V)$ is the tensor algebra of Λ_0-Λ_0-bi-module ${}_{\Lambda_0}V_{\Lambda_0}$ over Λ_0. $R=\mathrm{Ker}(V^{\otimes t}\xrightarrow{m}\Lambda_t)$ is a sub-bimodule of $V^{\otimes t}$. For brevity, we write $V^{\otimes t}$ as V^t. Define

$$R^\perp:=\{\,f\in(V^*)^t=(V^t)^*\mid f(R)=0\,\}$$

and

$$\Lambda^!:=T_{\Lambda_0}(V^*)/\langle R^\perp\rangle, \tag{3.3}$$

where $V^*:=\mathrm{Hom}_{\Lambda_0}(V,\Lambda_0)$, and $\langle R^\perp\rangle$ is the ideal of $T_{\Lambda_0}(V^*)$ generated by $R^\perp$. We call $\Lambda^!$ the t-dual algebra of Λ. Let $\Lambda^!_i=(\Lambda^!)_i$. We have

$$\Lambda^!_{t(i)}:=\frac{(V^*)^{t(i)}}{\sum\limits_{0\leqslant v\leqslant t(i)-t}(V^*)^{t(i)-t-v}\otimes R^\perp\otimes(V^*)^v}.$$

By using the equality $(V/W)^*=\{f\in V^*|f(w)=0\}$, we get

$$\begin{aligned}(\Lambda^!_{t(i)})^*&=\Big(\frac{(V^*)^{t(i)}}{\sum\limits_{0\leqslant v\leqslant t(i)-t}(V^*)^{t(i)-t-v}\otimes R^\perp\otimes(V^*)^v}\Big)^*\\ &=\{\,f\in((V^*)^{t(i)})^*=V^{t(i)}\mid f(\sum_{0\leqslant v\leqslant t(i)-t}(V^*)^{t(i)-t-v}\otimes R^\perp\otimes(V^*)^v)=0\,\}\\ &=\bigcap_{0\leqslant v\leqslant t(i)-t}V^v\otimes R\otimes V^{t(i)-v-t}=K^i_{t(i)},\end{aligned}$$

where $K^i_{t(i)}$ is given as in Definition 2.5. This proves

Lemma 3.2. Let $\Lambda^!$ be as in (3.3). Then we have $(\Lambda^!_{t(i)})^* = K^i_{t(i)}, \ \forall \ i \geqslant 0$.

The following theorem says that for arbitrary t-Koszul Λ, the i-degree component of its cohomology algebra $\mathrm{Ext}^\bullet_\Lambda(\Lambda_0, \Lambda_0)$ is just the $\boldsymbol{t}(i)$-degree component of its dual algebra $\Lambda^!$. This generalizes the corresponding result of the usual Koszul case.

Theorem 3.1. Let Λ be an arbitrary t-Koszul algebra. Then we have

$$\mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0) = \Lambda^!_{t(i)}, \quad \forall \ i \geqslant 0.$$

Proof. Since Λ is t-Koszul, by Theorem 2.1, the t-Koszul complex $\boldsymbol{K}^\bullet$ of Λ is a projective resolution of Λ_0. Thus, applying Lemma 3.2, we have

$$\begin{aligned}\mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0) &= \frac{\mathrm{Ker}(\mathrm{Hom}_\Lambda(K^i, \Lambda_0) \longrightarrow \mathrm{Hom}_\Lambda(K^{i+1}, \Lambda_0))}{\mathrm{Im}(\mathrm{Hom}_\Lambda(K^{i-1}, \Lambda_0) \longrightarrow \mathrm{Hom}_\Lambda(K^i, \Lambda_0))} = \mathrm{Hom}_\Lambda(K^i, \Lambda_0) \\ &= \mathrm{Hom}_\Lambda(\Lambda \otimes K^i_{t(i)}, \Lambda_0) = \mathrm{Hom}_{\Lambda_0}(K^i_{t(i)}, \mathrm{Hom}_\Lambda(\Lambda, \Lambda_0)) \\ &= \mathrm{Hom}_{\Lambda_0}((\Lambda^!_{t(i)})^*, \Lambda_0) = \Lambda^!_{t(i)},\end{aligned}$$

where we use the fact that the functors Hom and $\otimes$ are adjoint: let A and B be arbitrary rings, and let ${}_AL$, ${}_BM_A$ and ${}_BN$ be left A-module, B-A-bimodule and left B-module respectively. Then

$$\mathrm{Hom}_A(L, \mathrm{Hom}_B(M, N)) \cong \mathrm{Hom}_B(M \otimes_A L, N).$$

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