Twisted bi-symplectic structure of Artin-Schelter regular algebras

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Background and motivation

AS-regular algebras were introduced by Artin-Schelter in late 1980s. They have been extensively studied in the past three decades.

- (Reyes-Rogalski-Zhang) AS-regular regular algebras are twisted Calabi-Yau algebras.
- (Kontsevich, Pantev-Toën-Vaquié-Vezzosi) Calabi-Yau algebras are some kind of noncommutative symplectic spaces.

Question

Are AS-regular algebras also related to some noncommutative symplectic structure? Possibly twisted?

The goal of this talk is to give a possible answer to this question.

AS-regular algebras

Definition (Artin-Schelter)

A connected graded k-algebra A is called Artin-Schelter regular (or AS-regular for short) of dimension d if

1. A has finite global dimension n, and
2. A is Gorenstein, that is, $\mathrm{Ext}^i_A(k, A) = 0$ for $i \neq n$ and $\mathrm{Ext}^n_A(k, A) \cong k$.

Example

1. Quantum affine space: $A = k\langle x_1, \cdots, x_n\rangle/(x_j x_i - q_{ij} x_i x_j)$ where $q_{ii} = 1, q_{ij} q_{ji} = 1$ for $i \neq j$.
2. (Dubois-Violette) $A = k\langle x_1, \cdots, x_n\rangle/(f)$, where $f = (x_1, \cdots, x_n)M(x_1, \cdots, x_n)^T$ with $M \in \mathrm{GL}_n(k)$;
3. Sklyanin algebras (they are Calabi-Yau, that is $\sigma = id$)

Nakayama automorphism

Any AS-regular algebra has an automorphism $\sigma : A \to A$, called the Nakayama automorphism, which has been studied in great detail by mathematicians especially from Shanghai and Zhejiang.

To see this, let us note that in the definition, the modules k and A in $\mathrm{Ext}_A^\bullet(k, A)$ can be viewed as both left and right A-modules, but they are not necessarily the same. The difference is given by the Nakayama automorphism.

Now equip A with A-bimodule structure as follows:

$$u \circ x \circ v := u \cdot x \cdot \sigma(v), \text{for all } x, u, v \in A.$$

Denote this twisted A-bimodule by A_σ.

AS-regular algebras as twisted Calabi-Yau algebras

AS-regular algebras are highly related to Calabi-Yau algebras, a notion introduced by Ginzburg in 2007, via the following theorem:

Theorem (Reyes-Rogalski-Zhang)

Let A be a graded algebra. Then A is AS-regular of dimension n if and only if A is twisted n-Calabi-Yau, that is, A satisfies:

1. *A is homologically smooth;*
2. $\mathrm{RHom}_{A^e}(A, A \otimes A) \cong A_\sigma[n]$ *in* $D(A^e)$.

In the above theorem, if $\sigma = id$, then A is called n-Calabi-Yau, or Calabi-Yau of dimension n.

Derived noncommutative geometry

From now on, we assume R is a DG free associative algebra (if not, then take its DG free resolution; for example, if A is a Koszul algebra, then we may take $R = \Omega(A^{\text{!`}}))$. Suppose $\sigma : R \to R$ is an automorphism of R.

Definition (André, Karoubi, Le Stum-Quirós...)

Consider the following twisted product:

$$\mu_\sigma : R \otimes R \to R,\ (x, y) \mapsto \sigma(x) \cdot y,$$

for all $x, y \in R$. The (derived) twisted noncommutative 1-forms of R is

$$\Omega^1_\sigma(R) := \ker \mu_\sigma.$$

Comparison with usual noncommutative forms

In previous definition, if $\sigma = id$, then $\Omega^1_\sigma(R)$ is the classical noncommutative 1-forms of R, and is denoted by $\Omega^1(R)$. There is a (noncanonical) isomorphism of vector spaces

$$\Phi : \Omega^1(R) \xrightarrow{\cong} \Omega^1_\sigma(R),\ \sum x \otimes y \mapsto \sum x \otimes \sigma(y).$$

Proposition

If we equip $\Omega^1(R)$ and $\Omega^1_\sigma(R)$ with R-bimodule structures as

$$u \circ (\sum x \otimes y) \circ v = \sum ux \otimes yv, u \circ (\sum x \otimes y) \circ v = \sum ux \otimes y\sigma(v),$$

then Φ is a map of R-bimodules.

Noncommutative tangent spaces

Let R be as before.

1. Then the (derived) noncommutative tangent spaces of R is

$$\mathbb{D}\mathrm{er}(R) := \mathrm{Der}(R, R \otimes R).$$

   We may similarly define the twisted noncommutative tangent spaces, but we will not use them.

2. The Karoubi-de Rham complex of R is defined to be

$$\mathrm{DR}^{\bullet}(R) := T_R\Omega^1(R)/[T_R\Omega^1(R), T_R\Omega^1(R)].$$

   It's twisted version is

$$\mathrm{DR}^{\bullet}_{\sigma}(R) := T_R\Omega^1(R)/[T_R\Omega^1(R), \sigma(T_R\Omega^1(R))].$$

The augmented version

For technical reasons, it is more convenient to take the (derived) noncommutative 1-forms as the cone

$$\mathrm{Cone}\{\Omega^1(R) \hookrightarrow R \otimes R\}.$$

as well as its twisted version. Similar convention applies to noncommutative tangent space. We have:

Proposition

Let R be as above. Then:

1. $\Omega^1_\sigma(R) \cong R_\sigma$ and $\mathbb{D}\mathrm{er}(R) \cong \mathrm{RHom}_{R^e}(R, R \otimes R)$ as R-bimodules.
2. $\Omega^1_\sigma(R)_\natural \simeq \mathrm{CH}_\bullet(R, R_\sigma)$ and $\mathbb{D}\mathrm{er}(R)_\natural \simeq \mathrm{CH}^\bullet(R)$.

We may also formulate the twisted counterpart for double derivations.

Twisted bi-symplectic structure

From the definition, we also have that

$$\mathrm{RHom}_{R^e}(R, R \otimes R) \cong R_\sigma[n] \text{ in } D(R^e)$$

is equivalent to

$$\mathbb{D}\mathrm{er}(R) \cong \Omega^1_\sigma(R)[n] \text{ in } D(R^e).$$

This leads to the following:

Definition

Let R be as before. A twisted (n-shifted) bi-symplectic structure on R is a closed 2-form $\omega = \sum da_1 da_2 \in \mathrm{DR}^2_\sigma(R)$ of degree $2-n$ s.t.

$$\iota_{(-)}\omega : \mathbb{D}\mathrm{er}(R) \to \Omega^1_\sigma(R)[-n],\ f \mapsto \sum f(a)'' \cdot \Phi(da_2) \cdot \sigma(f(a)').$$

is a quasi-isomorphism of R-bimodules.

Remarks

1. If $\sigma = id$, then the above twisted bi-symplectic structure is the usual bi-symplectic introduced by Crawley-Boevey, Etingof and Ginzburg (for associative algebra) and further studied by C.-Eshmatov and Pridham recently.
2. For an associative algebra A, let R be its DG free resolution. We say A has a derived bi-symplectic structure if R has a bi-symplectic structure.

Koszul AS-regular algebras

Now let us assume A is a Koszul AS-regular algebra of dimension n. Denote by $A^!$ and $A^¡$ the Koszul dual algebra and coalgebra of A respectively. Then

1. $A^!$ is a Frobenius algebra, that is, there exists a non-degenerate pairing on A such that $\langle a, b\rangle = \langle \sigma^*(b), a\rangle$, where σ^* is the dual Nakayama automorphism.
2. $A^¡$ is a graded coalgebra; take nonzero $\eta \in A^¡_n$, then $\Delta(\eta) = \sum \eta' \otimes \eta''$ satisfying

$$\sum \eta' \otimes \eta'' = \sum \sigma(\eta'') \otimes \eta'.$$

Moreover, $R := \Omega(A^¡)$ is a DG free resolution of A.

Main theorem I

Theorem (C.-Eshmatov-Liu)

Suppose A is a Koszul AS-regular algebra. Then $\omega := d\eta' \otimes d\eta''$ is a twisted bi-symplectic structure on R, that is, it is a closed twisted 2-form such that

$$\iota_{(-)}\omega : \mathbb{D}\mathrm{er}\, R \to \Omega^1_\sigma\, R[2-n] \tag{1}$$

is a quasi-isomorphism.

Corollary (Van den Bergh, Reyes-Rogalski-Zhang)

Let A be as above. Then taking the commutator quotient space on both sides of (1), we obtain

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}_{n-\bullet}(A, A_\sigma).$$

Kontsevich-Rosenberg principle

In the past two decades, great progress has been made in the field of (derived) noncommutative algebraic geometry. In 1998 Kontsevich and Rosenberg proposed the following

Kontsevich-Rosenberg Principle

For a noncommutative space, say an associative algebra A, a noncommutative geometric structure (e.g. Poisson, symplectic) should induce its classical counterpart on the affine scheme $\mathrm{Rep}_V(A)$ of all representations of A in V, for any vector space V.

However, since $\mathrm{Rep}_V(A)$ are always very singular, it is difficult to compute explicitly the geometric structures on them. Passing to the homotopy category solves a lot of issues.

Representation schemes

Suppose A is a DG algebra. Then the following functor

$$\mathrm{Rep}_V(A) : \mathsf{CDGA} \to \mathsf{Set},\ \ B \mapsto \mathrm{Hom}_{\mathsf{DGA}}(A, \mathrm{End}_V \otimes B)$$

is representable, that is, there is a dg commutative algebra (CDGA) A_V such that

$$\mathrm{Hom}_{\mathsf{DGA}}(A, \mathrm{End}_V \otimes B) = \mathrm{Hom}_{\mathsf{CDGA}}(A_V, B).$$

A_V may be viewed as the functions on the representation scheme of A in V, that is, $A_V = \mathcal{O}(\mathrm{Rep}_V(A))$. In general A_V is very singular.

DRep of Berest, Khachatryan and Ramadoss

Definition (Berest-Khachatryan-Ramadoss)

The left derived functor of

$$(-)_V : \mathsf{DGA} \to \mathsf{CDGA}, \ A \mapsto A_V,$$

is called the derived represenation scheme of A in V, and is denoted by $\boldsymbol{L}(A)_V$ or $\mathrm{DRep}_V(A)$.

Berest et al showed that $\mathrm{DRep}_V(A)$ is smooth in the homotopy category of CDGA's, and moreover, the zeroth homology

$$\mathrm{H}_0(\mathrm{DRep}_V(A)) = A_V, \quad \text{for } A \in \mathsf{Alg}.$$

GL-invariant subfunctor

Observe that the general linear group $\mathrm{GL}(V)$, as a subspace of $\mathrm{End}\ V$, acts from the right on the latter by conjugation $\alpha \mapsto g^{-1}\alpha g$, for all $g \in \mathrm{GL}(V)$. We may consider the invariant subfunctor

$$\boldsymbol{L}(-)_V^{\mathrm{GL}} : \mathsf{DGA}_k \to \mathsf{CDGA}_k, \quad A \mapsto \boldsymbol{L}(A)_V^{\mathrm{GL}},$$

which classifies the isomorphism classes of derived representations of A in V, and is also denoted by $\mathrm{DRep}_V(A)^{\mathrm{GL}}$.

Twisted differential forms (André, Karoubi,...)

Suppose R is a DG commutative algebra, and $\sigma : R \to R$ is an automorphism. Then the twisted Kähler forms are given by

$$\Omega^1_\sigma(R) := I/I \cdot \Phi(I),$$

where $I = \ker \mu_\sigma$ and $\Phi : x \otimes y \mapsto x \otimes \sigma(y)$ as in the DG algebra case. The twisted differential forms of A is given by

$$\Omega^\bullet_\sigma(R) := T_R I/(u \cdot v - \sigma(v) \cdot u).$$

It is straightforward to see

$$(du)v = \sigma(v)(du), \quad dudv = -\sigma(dv)du.$$

Main theorem II

Now suppose A is a Koszul AS-regular algebra. Then the Nakayama automorphism on A (and hence on R) induces an automorphism on its derived representation schemes $\mathrm{DRep}_V(A)^{\mathrm{GL}}$.

Theorem (C.-Eshmatov-Liu)

Suppose A is a Koszul AS-regular algebra. Then the twisted bi-symplectic structure of A induces a twisted symplectic structure on $\mathrm{DRep}_V(A)^{\mathrm{GL}}$, for all vector space V.
In other words, it induces a quasi-isomorphism, up to a degree shifting, between the tangent complex and the twisted Kähler forms on $\mathrm{DRep}_V(A)^{\mathrm{GL}}$.

Here, the twisted symplectic structure is a twisted analogue of the shifted symplectic structure of Pantev, Toën, Vaquié and Vezzosi.

Thank you!