GT-shadows and their action on Grothendieck's child's drawings

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This talk is loosely based on joint paper `https://arxiv.org/abs/2008.00066` *with Khanh Q. Le and Aidan Lorenz.*

The absolute Galois group $G_{\mathbb{Q}}$ of rationals and $\widehat{\mathsf{GT}}$

$G_{\mathbb{Q}}$ is the group of (field) automorphisms of the algebraic closure $\overline{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers. This group is uncountable. In fact, for every finite Galois extension $E \supset \mathbb{Q}$, any element $g \in \mathrm{Gal}(E/\mathbb{Q})$ can be extended (in infinitely many ways) to an element of $G_{\mathbb{Q}}$. The group $G_{\mathbb{Q}}$ is one of the most mysterious objects in mathematics!

In 1990, Vladimir Drinfeld introduced yet another mysterious group $\widehat{\mathsf{GT}}$ (the Grothendieck-Teichmuellller group). $\widehat{\mathsf{GT}}$ consists pairs $(\hat{m}, \hat{f})$ in $\widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ satisfying some conditions and it receives a one-to-one homomorphism

$$G_{\mathbb{Q}} \hookrightarrow \widehat{\mathsf{GT}}.$$

Only two elements of $G_{\mathbb{Q}}$ **are known explicitly**: the identity element and the complex conjugation $a + bi \mapsto a - bi$. The corresponding images in $\widehat{\mathsf{GT}}$ are $(0, 1)$ and $(-1, 1)$.

Incarnations of Grothendieck's child's drawings are . . .

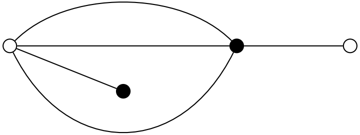
Isom. classes of (non-constant) holomorphic maps $f : \Sigma \to \mathbb{CP}^1$ from compact connected Riemann surfaces (without boundary) that do not have branching points above every $w \in \mathbb{CP}^1 - \{0, 1, \infty\}$.

Isom. classes of finite degree connected coverings of $\mathbb{CP}^1 - \{0, 1, \infty\}$.

Conjugacy classes of finite index subgroups of $\mathrm{F}_2 := \langle\, x, y \,\rangle$.

Equivalence classes of pairs (g_1, g_2) of permutations in S_d (for some d) for which the group $\langle\, g_1, g_2 \,\rangle$ acts transitively on $\{1, 2, \dots, d\}$.

Isomorphism classes of connected bipartite ribbon graphs with d edges (for some d).

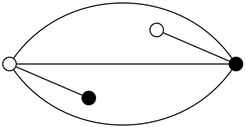


The action of $G_{\mathbb{Q}}$ on child's drawings

Given a child's drawing D, we can find a smooth projective curve X defined over $\overline{\mathbb{Q}}$ and an algebraic map $f : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ that does not have branching points above every $w \in \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. (X, f) is called a **Belyi pair** corresponding to D.

The coefficients defining the curve X and the map f lie in some finite Galois extension E of $\mathbb{Q}$. Given any $g \in \mathrm{Gal}(E/\mathbb{Q})$, the child's drawing $g(D)$ is the one corresponding to the new Belyi pair $(g(X), g(f))$. We simply act by g on the coefficients defining X and f!

The $G_{\mathbb{Q}}$-orbit of the above child's drawing has two elements. It's 'Galois conjugate' is



The action of $\widehat{\mathsf{GT}}$ on child's drawings

Let $(\hat{m}, \hat{f})$ be an element of $\widehat{\mathsf{GT}}$ and D be a child's drawing. It is convenient to represent D by a group homomorphism

$$\varphi : \mathsf{F}_2 \to S_d \,,$$

where $\varphi(\mathsf{F}_2)$ is transitive. (D corresponds to the conjugacy class of the stabilizer of 1.)

φ extends, by continuity, to a (continuous) group homomorphism $\hat{\varphi} : \widehat{\mathsf{F}}_2 \to S_d$. The child's drawing $D^{(\hat{m},\hat{f})}$ corresponds to the group homomorphism

$$\hat{\varphi} \circ \hat{T}\big|_{\mathsf{F}_2} : \mathsf{F}_2 \to S_d \,,$$

where

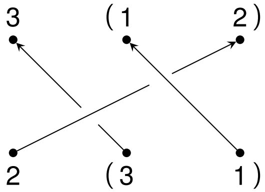
$$\hat{T}(x) := x^{2\hat{m}+1} \quad \text{and} \quad \hat{T}(y) := \hat{f}^{-1}\, y^{2\hat{m}+1} \hat{f}.$$

See Y. Ihara's paper "On the embedding of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\widehat{\mathsf{GT}}$".

The operad PaB

For every integer $n \geq 1$, PaB(n) is a groupoid, whose objects are (completely) parenthesized sequences of $1, 2, \dots, n$ (each $i \in \{1, \dots, n\}$ appears exactly once). For instance, PaB(2) has exactly two objects $(1, 2)$ and $(2, 1)$; PaB(3) has 12 objects: $(1, 2)3, (2, 1)3$, $\dots, 1(2, 3), 2(1, 3), \dots$.

Let B_n be Artin's braid group and PB_n be the kernel of the canonical homomorphism $\rho : \mathsf{B}_n \to S_n$. $\{x_{ij}\}_{1 \leq i < j \leq n}$ denote standard generators of PB_n. $\mathrm{Hom}_{\mathsf{PaB}}(\tau, \tilde{\tau}) := \rho^{-1}(\tilde{\tau}^{-1} \circ \tau) \subset \mathsf{B}_n$. For instance,

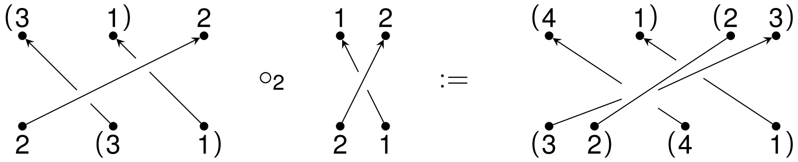


An example of computing an elementary insertion

Note that the automorphism group of every object in PaB(n) is the pure braid group PB_n on n strands. For instance,

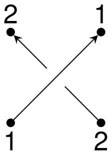
$$\mathsf{Aut}_{\mathsf{PaB}}((1,2)3) = \mathsf{PB}_3 \quad \text{and} \quad \mathsf{Aut}_{\mathsf{PaB}}((1,2)) = \mathsf{PB}_2 = \langle\, x_{12} \,\rangle.$$

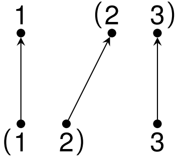
Here is an example of computing an elementary insertion:



Mac Lane's coherence theorem tells us that ...

PaB is generated by these two morphisms

β := 

α := 

Any relation involving α and β is a consequence of the pentagon relation:

$$
\begin{array}{ccccc}
 & (1(23))4 & \longrightarrow & 1((23)4) & \\
\nearrow & & & & \searrow \\
((12)3)4 & \longrightarrow & (12)(34) & \longrightarrow & 1(2(34))
\end{array}
$$

and the two hexagon relations.

The group $\widehat{\mathsf{GT}}$ is ...

the group of continuous automorphisms

$$\hat{T} : \widehat{\mathsf{PaB}} \to \widehat{\mathsf{PaB}}$$

of the profinite completion $\widehat{\mathsf{PaB}}$ of PaB.

Since β and α are topological generators of $\widehat{\mathsf{PaB}}$, every $\hat{T} \in \widehat{\mathsf{GT}}$ is uniquely determined by

$$\hat{T}(\beta) = \beta \circ x_{12}^{\hat{m}} \quad \text{and} \quad \hat{T}(\alpha) = \alpha \circ \hat{f},$$

where $\hat{f} \in \hat{\mathsf{PB}}_3 = \mathrm{Aut}_{\widehat{\mathsf{PaB}}}((1,2)3)$ and $x_{12}^{\hat{m}} \in \hat{\mathsf{PB}}_2 = \mathrm{Aut}_{\widehat{\mathsf{PaB}}}((1,2))$.

We tacitly identify F_2 with the subgroup $\langle x_{12}, x_{23} \rangle \le \mathsf{PB}_3$. The basic relations on α and β $\Rightarrow$ $\hat{f} \in \hat{\mathsf{F}}_2$. In fact, $\hat{f} \in ([\hat{\mathsf{F}}_2, \hat{\mathsf{F}}_2])^{\textit{top. closure}}$

$\widehat{\mathrm{GT}}$ as the subgroup of $\mathrm{Aut}(\hat{\mathrm{F}}_2)$

Since the automorphism group of $(1,2)3$ in $\widehat{\mathrm{PaB}}(3)$ is $\widehat{\mathrm{PB}}_3$, every $\hat{T} \in \widehat{\mathrm{GT}}$ gives us an automorphism of $\widehat{\mathrm{PB}}_3$. Restricting this automorphism to $\hat{\mathrm{F}}_2 \le \widehat{\mathrm{PB}}_3$, we get the automorphism of $\hat{F}_2$:

$$\hat{T}(x_{12}) := x_{12}^{2\hat{m}+1} \quad \text{and} \quad \hat{T}(x_{23}) := \hat{f}^{-1}\, x_{23}^{2\hat{m}+1} \hat{f}. \tag{1}$$

One can show that every element $(\hat{m}, \hat{f}) \in \widehat{\mathrm{GT}}$ is uniquely determined by the automorphism (1). Some mathematicians identify $\widehat{\mathrm{GT}}$ with the corresponding group of continuous automorphisms of $\hat{\mathrm{F}}_2$.

Truncating PaB

For our purposes, it is convenient to consider the truncation of PaB:

$$\mathsf{PaB}^{\le 4} := \mathsf{PaB}(1) \sqcup \mathsf{PaB}(2) \sqcup \mathsf{PaB}(3) \sqcup \mathsf{PaB}(4).$$

This union of groupoids is a **truncated operad** in the following sense:

- S_n acts on $\mathsf{PaB}(n)$ for every $1 \le n \le 4$,
- we have elementary insertions $\circ_i : \mathsf{PaB}(n) \times \mathsf{PaB}(m) \to \mathsf{PaB}(n+m-1)$ whenever $n+m-1 \le 4$ and
- all operad axioms for elementary insertions and the action of symmetric groups are satisfied if arities of all elements are ≤ 4.

Since PaB is generated by elements of arities 2 and 3 and the key relations are in arities 3 and 4, we have $\widehat{\mathsf{GT}} = \mathsf{Aut}(\widehat{\mathsf{PaB}}^{\le 4})$.

From now on, "operad" := "truncated operad".

Compatible equivalence relations on $\mathsf{PaB}^{\le 4}$

An equivalence relation $\sim$ on $\mathsf{PaB}^{\le 4}$ is called **compatible** if

- $\gamma \sim \tilde{\gamma} \Rightarrow$ the source (resp. the target) of γ coincides with the source (resp. the target) of $\tilde{\gamma}$;
- $\forall\, \theta \in S_n$ and $\forall\, \gamma, \tilde{\gamma} \in \mathsf{PaB}(n)$, $\gamma \sim \tilde{\gamma} \quad \Leftrightarrow \quad \theta(\gamma) \sim \theta(\tilde{\gamma})$;
- the equivalence class of $\gamma \circ \tilde{\gamma}$ depends only on the equivalence classes of γ and $\tilde{\gamma}$;
- for $\gamma \in \mathsf{PaB}(n)$, $\tilde{\gamma} \in \mathsf{PaB}(k)$, $1 \le i \le n$, and $\gamma \circ_i \tilde{\gamma}$ depends only on the equivalence classes of γ and $\tilde{\gamma}$;
- for every $2 \le n \le 4$, the groupoid $\mathsf{PaB}(n)/\sim$ is finite.

A large supply of compatible equivalence relations on $\mathsf{PaB}^{\le 4}$ comes from finite index normal subgroups $\mathsf{N} \trianglelefteq \mathsf{B}_4$ such that $\mathsf{N} \le \mathsf{PB}_4$.

$\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ is the poset of such subgroups of B_4.

$\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4) \mapsto \sim_{\mathsf{N}}$

Let $\mathcal{G}$ be a connected groupoid and G be the automorphism group of any object $a \in Ob(\mathcal{G})$. Then every $\mathsf{N} \trianglelefteq G$ gives us an equivalence relation on $\mathcal{G}$ compatible with the composition of morphisms. Indeed, let $\gamma, \tilde{\gamma} \in \mathcal{G}(a, b)$; we declare that $\gamma \sim \tilde{\gamma}$ if $\gamma^{-1} \circ \tilde{\gamma} \in \mathsf{N}$.

Given $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, there is natural way to define $\mathsf{N}_{\mathsf{PB}_3} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$ and $\mathsf{N}_{\mathsf{PB}_2} \in \mathsf{NFI}_{\mathsf{PB}_2}(\mathsf{B}_2)$. Then N, $\mathsf{N}_{\mathsf{PB}_3}$ and $\mathsf{N}_{\mathsf{PB}_2}$ give us equivalence relations on PaB(4), PaB(3) and PaB(2), respectively. This way, we get a compatible equivalence relation $\sim_{\mathsf{N}}$ on $\mathsf{PaB}^{\leq 4}$.

Note that $\mathsf{PB}_2 = \langle\, x_{12} \,\rangle$ (infinite cyclic group) and B_2 is Abelian. So every $\mathsf{N}_{\mathsf{PB}_2} \in \mathsf{NFI}_{\mathsf{PB}_2}(\mathsf{B}_2)$ is of the form $\langle\, x_{12}^{N_{\mathrm{ord}}} \,\rangle$ for some positive integer N_{ord}.

So... what are GT-shadows???

Consider the groupoid whose objects are elements of $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ and whose morphisms are isomorphisms of operads

$$\mathsf{PaB}^{\le 4}/\mathsf{N}^{(1)} \stackrel{\cong}{\longrightarrow} \mathsf{PaB}^{\le 4}/\mathsf{N}^{(2)} . \tag{2}$$

We denote this groupoid by GTSh. GT-**shadows** are morphisms of this groupoid.

Note that every isomorphism (2) is uniquely determined by an onto morphism of operads $\mathsf{PaB}^{\le 4} \longrightarrow \mathsf{PaB}^{\le 4}/\mathsf{N}^{(2)}$. It is convenient to identify morphisms in $\mathsf{GTSh}(\mathsf{N}^{(1)}, \mathsf{N}^{(2)})$ with the onto morphisms of operads

$$\mathsf{PaB}^{\le 4} \longrightarrow \mathsf{PaB}^{\le 4}/\mathsf{N}^{(2)}$$

whose “kernel” is the compatible equivalence relation corresponding to $\mathsf{N}^{(1)}$. For $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, $\mathsf{GT}(\mathsf{N})$ denotes the set of GT-shadows with the target N.

How to “explain this to a computer”?

For every $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, we have $\mathsf{N}_{\mathsf{PB}_3} \in \mathsf{NFI}_{\mathsf{PB}_3}(\mathsf{B}_3)$ and $\mathsf{N}_{\mathsf{PB}_2} \in \mathsf{NFI}_{\mathsf{PB}_2}(\mathsf{B}_2)$ (or, equivalently, a positive integer $N_{\mathrm{ord}} := |\mathsf{PB}_2 : \mathsf{N}_{\mathsf{PB}_2}|$). These subgroups give us $\sim$ on $\mathsf{PaB}^{\le 4}$.

Since $\mathsf{PaB}^{\le 4}$ is generated by β and α, every GT-shadow $T : \mathsf{PaB}^{\le 4} \to \mathsf{PaB}^{\le 4}/\mathsf{N}$ is uniquely determined by the pair $(m + N_{\mathrm{ord}}\mathbb{Z}, f\,\mathsf{N}_{\mathsf{PB}_3}) \in \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$.

Vice versa, for every pair

$$(m + N_{\mathrm{ord}}\mathbb{Z}, f\,\mathsf{N}_{\mathsf{PB}_3}) \;\in\; \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$$

satisfying the versions of the hexagon relations, the version of the pentagon relation and additional conditions, we have a onto morphism of operads

$$T_{m,f} : \mathsf{PaB}^{\le 4} \to \mathsf{PaB}^{\le 4}/\mathsf{N}.$$

GT-shadows coming from elements of $\widehat{\mathrm{GT}}$

Let $\mathrm{N} \in \mathrm{NFI}_{\mathrm{PB}_4}(\mathrm{B}_4)$ and $\hat{T} \in \widehat{\mathrm{GT}}$. Composing the standard inclusion $\mathrm{PaB}^{\le 4} \to \widehat{\mathrm{PaB}}^{\le 4}$, $\hat{T}$ and the projection $\mathcal{P}_{\mathrm{N}} : \widehat{\mathrm{PaB}}^{\le 4} \to \mathrm{PaB}^{\le 4}/\mathrm{N}$, we get a GT-shadow T_{N}.

$$\begin{array}{ccc}
\widehat{\mathrm{PaB}}^{\le 4} & \xrightarrow{\ \hat{T}\ } & \widehat{\mathrm{PaB}}^{\le 4} \\
\uparrow & & \downarrow \mathcal{P}_{\mathrm{N}} \\
\mathrm{PaB}^{\le 4} & \xrightarrow{\ T_{\mathrm{N}}\ } & \mathrm{PaB}^{\le 4}/\mathrm{N}
\end{array}$$

We say that T_{N} comes from an element of $\widehat{\mathrm{GT}}$. If a GT-shadow T comes from an element of $\widehat{\mathrm{GT}}$, then T is called **genuine**. Otherwise, T is called **fake**.

The action of GT-shadows on child's drawings

Let $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ and $H \le \mathsf{F}_2$ be a subgroup that represents a child's drawing D. We say that D is **subordinate** to N if the normal core of H contains $\mathsf{N}_{\mathsf{PB}_3} \cap \mathsf{F}_2$. Dessin(N) denotes the set of child's drawings subordinate to N.

We denote by Dessin the category whose objects are elements of $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. For $\mathsf{N}^{(1)}, \mathsf{N}^{(2)} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, morphisms from $\mathsf{N}^{(1)}$ to $\mathsf{N}^{(2)}$ are functions from $\mathsf{Dessin}(\mathsf{N}^{(1)})$ to $\mathsf{Dessin}(\mathsf{N}^{(2)})$.

Theorem

Let $\mathsf{N}^{(1)}, \mathsf{N}^{(2)} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ *and* $[(m, f)] \in \mathsf{GTSh}(\mathsf{N}^{(1)}, \mathsf{N}^{(2)})$*. Let* $\varphi : \mathsf{F}_2 \to S_d$ *be a homomorphism that represents* $D \in \mathsf{Dessin}(\mathsf{N}^{(2)})$ *and* $\tilde{\varphi}$ *be a homomorphism* $\mathsf{F}_2 \to S_d$ *defined by*

$$\tilde{\varphi}(x) := \varphi(x^{2m+1}), \qquad \tilde{\varphi}(y) := \varphi(f^{-1} y^{2m+1} f).$$

The assignment $\varphi \mapsto \tilde{\varphi}$ *gives us a functor* $\mathscr{A}^\sharp : \mathsf{GTSh} \to \mathsf{Dessin}$*.*

The actions are compatible!

$\widehat{\mathsf{GT}}$ acts on $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. We denote by $\widehat{\mathsf{GT}}_{\mathsf{NFI}}$ the corresponding transformation groupoid. We have the obvious functor $\widehat{\mathsf{GT}}_{\mathsf{NFI}} \to \mathsf{GTSh}$.

Theorem

The action of $\widehat{\mathsf{GT}}$ *on child's drawings gives us a functor* $\mathscr{A} : \widehat{\mathsf{GT}}_{\mathsf{NFI}} \to \mathsf{Dessin}$*. Moreover, the diagram*

$$\begin{array}{ccc} \widehat{\mathsf{GT}}_{\mathsf{NFI}} & \longrightarrow & \mathsf{GTSh} \\ & {\scriptstyle \mathscr{A}} \searrow \quad \swarrow {\scriptstyle \mathscr{A}^{\sharp}} & \\ & \mathsf{Dessin} & \end{array}$$

commutes.

Hierarchy of orbits

Consider a chain in the poset $\mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$

$$\mathsf{N}^{(1)} \supset \mathsf{N}^{(2)} \supset \mathsf{N}^{(3)} \supset \dots$$

and a child's drawing $D \in \mathrm{Dessin}(\mathsf{N}^{(1)})$. It is clear that D is subordinate to $\mathsf{N}^{(i)}$ for every $\mathsf{N}^{(i)}$ in this chain.
Recall that GT(N) denotes the set of all GT-shadows with the target N. (GT(N) is finite!)

For every child's drawing D, we have the following *hierarchy of orbits*:

$$\mathsf{GT}(\mathsf{N}^{(1)})(D) \supset \mathsf{GT}(\mathsf{N}^{(2)})(D) \supset \mathsf{GT}(\mathsf{N}^{(3)})(D) \supset \dots \supset \widehat{\mathsf{GT}}(D) \supset G_{\mathbb{Q}}(D).$$

It is **very hard** to compute $G_{\mathbb{Q}}(D)$; **there are no tools in modern mathematics** to compute orbits $\widehat{\mathsf{GT}}(D)$; **it is relatively easy** to compute orbits $\mathsf{GT}(\mathsf{N})(D)$.

GT-shadows in the Abelian setting

Proposition

For $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$, *the following conditions are equivalent:*

a) *the quotient group* PB_4/N *is Abelian;*

b) *the quotient group* $\mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$ *is Abelian;*

If N satisfies **a)** or **b)**, then we say that we are in the “Abelian setting”.

Theorem

Let $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$. *If the quotient group* PB_4/N *is Abelian then*

$$\mathsf{GT}(\mathsf{N}) = \{(\overline{m}, \underline{1}) \mid 0 \le m \le N_{\text{ord}} - 1, \ \ \gcd(2m+1, N_{\text{ord}}) = 1\}, \qquad (3)$$

where $\overline{m} := m + N_{\text{ord}}\mathbb{Z}$, $\underline{1}$ *is the identity element of* $\mathsf{PB}_3/\mathsf{N}_{\mathsf{PB}_3}$.
Furthermore, ***every*** GT*-shadow in* GT(N) *is* ***genuine****.*

Abelian child's drawings

Let $\varphi : \mathsf{F}_2 \to S_d$ be a homomorphism that represents a child's drawing D. Recall that (the conjugacy class of) the permutation group $\varphi(\mathsf{F}_2) \le S_d$ is called the **monodromy group** of D.

A child's drawing D is called **Abelian** if its monodromy group is Abelian. One can show that every Abelian child's drawing is Galois.

The following theorem is from a paper in preparation:

Theorem

Let D be an Abelian child's drawing and $\mathsf{N} \in \mathsf{NFI}_{\mathsf{PB}_4}(\mathsf{B}_4)$ such that D is subordinate to N. Then the orbits

$$\mathsf{GT}(\mathsf{N})(D) \supset \widehat{\mathsf{GT}}(D) \supset G_{\mathbb{Q}}(D)$$

*are **singletons**.*

Selected results of computer experiments

Jointly with students, I have been developing a software package ‘GT’ for working with GT-shadows and their action on child’s drawings. A beta version of this package is available at
`https://math.temple.edu/~vald/PackageGT/`

- Let D be a child’s drawing for which the $G_{\mathbb{Q}}$-orbit and the orbit with respect to the action of GT-shadows are computed. Then these orbits coincide!
- There is **no** “Furusho phenomenon” for GT-shadows: there are examples $f \in \mathrm{F}_2$ that satisfy the pentagon relation modulo N but at least one hexagon fails for (m, f) for every $m \in \{0, 1, \dots N_{\mathrm{ord}} - 1\}$.
- The connected components of GTSh we found have a very small number of objects: ≤ 2. We could **not** find a connected component of GTSh with > 2 objects!
- So far, we did **not** find **any** example of a fake GT-shadow.

Selected References

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More References?!... Sure!

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THANK YOU!