Rota's Classification Problem for Nonsymmetric Operads

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Motivation: Classification of Linear Operators

- Throughout the history, mathematical objects are often understood through studying operators defined on them.
- Well-known examples include Galois theory where fields are studied by their automorphisms (the Galois group),
- and analysis and geometry where functions and manifolds are studied through their derivations, integrals and related vector fields,
- and differential Galois theory where both operators occur.

Rota's Problem

- By the 1970s, several other operators had been discovered from studies in analysis, probability and combinatorics.

Average operator $P(x)P(y) = P(xP(y))$,

Inverse average operator $P(x)P(y) = P(P(x)y)$,

(Rota-)Baxter operator $P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy)$,

where λ is a fixed constant,

Reynolds operator $P(x)P(y) = P(xP(y)) + P(P(x)y) - P(P(x)P(y))$.

- Rota posed the problem of finding all the identities that could be satisfied by a linear operator defined on *associative algebras*. He also suggested that there should not be many such operators other than these previously known ones.

Quotation from Rota and Known Operators

- "In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable."
- Little progress was made on finding all such operators while new operators have merged from physics and combinatorial studies, such as

Nijenhuis operator	$P(x)P(y) = P(xP(y) + P(x)y - P(xy)),$
Leroux's TD operator	$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y).$

Other Post-Rota developments

- These previously known operators continued to find remarkable applications in pure and applied mathematics.
- Vast theories were established for differential algebra and difference algebra, with wide applications, including Wen-Tsun Wu's mechanical proof of geometric theorems and mathematics mechanization (based on work of Ritt).
- Rota-Baxter algebra has found applications in classical Yang-Baxter equations, operads, combinatorics, and most prominently, the renormalization of quantum field theory through the Hopf algebra framework of Connes and Kreimer.
- How to understand Rota's problem?

PI Algebras

- What is an algebraic identity that is satisfied by a linear operator?—Polynomial identity (PI) algebras gives a simplified analogue:
- A **k**-algebra R is called a PI algebra (Procesi, Rowen, ...) if there is a fixed element $f(x_1, \cdots, x_n)$ in the noncommutative polynomial algebra (that is, the free algebra) $\mathbf{k}\langle x_1, \cdots, x_n\rangle$ such that

$$f(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

  Thus an algebraic identity satisfied by an algebra is an element in the free algebra.
- Then an algebraic identity satisfied by a linear operator should be an element in a free algebra with an operator, a so called free operated algebra.

Operated algebras

- An operated $\mathbf{k}$-algebra is a $\mathbf{k}$-algebra R with a linear operator α on R.
- Examples. Differential algebras and Rota-Baxter algebras.
- We can also consider algebras with multiple operators, such as differential-difference algebras, differential Rota-Baxter algebras, Rota-Baxter families and matching Rota-Baxter algebras.
- An operated ideal of R is an ideal I of R such that $\alpha(I) \subseteq I$.
- A homomorphism from an operated $\mathbf{k}$-algebra (R, α) to an operated $\mathbf{k}$-algebra (S, β) is a $\mathbf{k}$-linear map $f : R \to S$ such that $f \circ \alpha = \beta \circ f$.
- The adjoint functor of the forgetful functor from the category of operated algebras to the category of sets gives the free operated $\mathbf{k}$-algebras.
- More precisely, a free operated $\mathbf{k}$-algebra on a set X is an operated $\mathbf{k}$-algebra $(\mathbf{k}\|X\|, \alpha_X)$ together with a map $j_X : X \to \mathbf{k}\|X\|$ with the property that, for any operated algebra (R, β) together with a map $f : X \to R$, there is a unique morphism $\bar{f} : (\mathbf{k}\|X\|, \alpha_X) \to (R, \beta)$ of operated algebras such that $f = \bar{f} \circ j_X$.

Bracketed words

- For any set Y, let $[Y] := \{\lfloor y \rfloor \,|y \in Y\}$ denote a set indexed by Y and disjoint from Y.
- For a fixed set X, let $\mathfrak{M}_0 = \mathfrak{M}(X)_0 = M(X)$ (free monoid). For $n \geq 0$, let
$$\mathfrak{M}_{n+1} := M(X \cup [\mathfrak{M}_n]).$$
- With the embedding $X \cup [\mathfrak{M}_{n-1}] \to X \cup [\mathfrak{M}_n]$, we obtain an embedding of monoids $i_n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$, giving the direct limit
$$\mathfrak{M}(X) := \varinjlim \mathfrak{M}_n.$$
- Elements of $\mathfrak{M}(X)$ are called bracketed words.
- $\mathfrak{M}(X)$ can also be identified with elements of $M(X \cup \{[,]\})$ such that [and] are paired with each other.
- $\mathfrak{M}(X)$ can also be constructed by rooted trees and Motzkin paths.

- Theorem. 1. The set $\mathfrak{M}(X)$, equipped with the concatenation product, the operator $w \mapsto \lfloor w \rfloor$, $w \in \mathfrak{M}(X)$, and the natural embedding $j_X : X \to \mathfrak{M}(X)$, is the free operated monoid on X.
  2. $\mathbf{k}\|X\| := \mathbf{k}\mathfrak{M}(X)$ ($\mathbf{k}$-span) is the free operated unitary $\mathbf{k}$-algebra on X.

Operated Polynomial Identities

- An operated $\mathbf{k}$-algebra (R, P) is called an operated PI (OPI) $\mathbf{k}$-algebra if there is a fixed element $\phi(x_1, \cdots, x_n) \in \mathbf{k}\lfloor\!\lfloor x_1, \cdots, x_n \rfloor\!\rfloor$ such that the evaluation map

$$\phi(a_1, \cdots, a_n) = 0, \quad \forall a_1, \cdots, a_n \in R.$$

  where a pair of brackets $\lfloor\ \rfloor$ is replaced by P everywhere.
- More precisely, for any $f : \{x_1, \cdots, x_n\} \to R$, the unique $\bar{f} : \mathbf{k}\lfloor\!\lfloor x_1, \cdots, x_n \rfloor\!\rfloor \to R$ of operated algebras sends ϕ to zero.
- Then (R, P) is called a ϕ-$\mathbf{k}$-algebra and P a ϕ-operator.
- Examples

  1. When $\phi = [xy] - x[y] - [x]y$, a ϕ-operator (resp. algebra) is a differential operator (resp. algebra).
  2. When $\phi = [x][y] - [x[y]] - [[x]y] - \lambda[xy]$, a ϕ-operator (resp. ϕ-algebra) is a Rota-Baxter operator (resp. algebra) of weight λ.
  3. When $\phi = [x] - x$, then a ϕ-algebra is just an associative algebra. Together with identities from the noncommutative polynomial algebra $\mathbf{k}\langle X\rangle$, we get a PI-algebra.

Free ϕ-algebras

- **Proposition** Let $\phi = \phi(x_1, \cdots, x_k) \in \mathbf{k}\lfloor X \rfloor$ be given. For any set Z, the free ϕ-algebra on Z is given by the quotient operated algebra $\mathbf{k}\lfloor Z \rfloor / I_{\phi,Z}$ where $I_{\phi,Z}$ is the operated ideal of $\mathbf{k}\lfloor Z \rfloor$ generated by the set
$$\{\phi(u_1, \cdots, u_k) \mid u_1, \cdots, u_k \in \mathbf{k}\lfloor Z \rfloor\}.$$
- Examples
  - When $\phi = [x] - x$, then the quotient $\mathbf{k}\lfloor Z \rfloor / I_{\phi,Z}$ gives the free algebra $\mathbf{k}\langle Z \rangle$ on Z.
  - When $\phi = [xy] - x[y] - [x]y$, then the quotient gives the free noncommutative differential polynomial algebra $\mathbf{k}\{Z\} := \mathbf{k}\langle \Delta(Z) \rangle$ on Z, where $\Delta(X) := \mathbb{Z}_{\geq 0} \times Z$ is the set of "differential variables".
- A major problem is to determine a canonical basis of $\mathbf{k}\lfloor Z \rfloor / I_{\phi,Z}$.

Remarks:

- A classification of linear operators can be regarded as a classification of elements in $\mathbf{k}\|X\|$.
- This problem is precise, but is too broad.
- We remind ourselves that Rota also wanted the operators to be defined on *associative algebras.*
- This means that the operated identity $\phi \in \mathbf{k}\|x_1, \cdots, x_n\|$ should be compatible with the associativity condition.
- What does this mean?

Examples of compatibility with associativity

- Example 1: For $\phi(x, y) = [xy] - [x]y - x[y]$, we have

$$[xy] \mapsto [x]y + x[y].$$

Thus

$$[(xy)z] \mapsto [xy]z + (xy)[z] \mapsto [x]yz + x[y]z + xy[z].$$

$$[x(yz)] \mapsto [x](yz) + x[yz] \mapsto [x]yz + x[y]z + xy[z].$$

So $[(xy)z]$ and $[x(yz)]$ have the same reduction, indicating that the differential operator is consistent with the associativity condition.

More examples

- Example 2: The same is true for the right multiplier: $\phi(x,y) = [xy] - [x]y$:

$$\lfloor x \rfloor yz \leftarrow\!\!\dashv \lfloor xy \rfloor z \leftarrow\!\!\dashv [(xy)z] = \lfloor x(yz) \rfloor \mapsto [x]yz.$$

- Example 3: Suppose $\phi(x,y) = [xy] - [y]x$. Then $[xy] \mapsto [y]x$. So

$$[w]uv \leftarrow\!\!\dashv [(uv)w] = [u(vw)] \mapsto [vw]u \mapsto [w]vu.$$

Thus a ϕ-algebra (R, δ) needs to satisfy the weak commutativity:

$$\delta(w)(uv - vu) = 0, \forall u, v, w \in Z.$$

So this operator might not be what Rota had in mind!

Differential type operators

- differential operator $[xy] = [x]y + x[y]$,
  differential operator of weight λ $[xy] = [x]y + x[y] + \lambda[x][y]$,
  homomorphism $[xy] = [x][y]$,
  semihomomorphism $[xy] = x[y]$.
- They are of the form $[xy] = N(x, y)$ where
  1. $N(x, y) \in \mathbf{k}\llbracket x, y \rrbracket$ is in DRF, namely, it does not contain $[uv], u, v \neq 1$, that is, $N(x, y)$ is in $\mathbf{k}\mathfrak{D}(x, y)$;
  2. $N(uv, w) = N(u, vw)$ is reduced to zero under the reduction $[xy] \mapsto N(x, y)$.

  An operator identity $\phi(x, y) = 0$ is said of differential type if $\phi(x, y) = [xy] - N(x, y)$ where $N(x, y)$ satisfies these properties. We call $N(x, y)$ and an operator satisfying $\phi(x, y) = 0$ of differential type.

Classification of differential type operators

- **(Rota's Problem: the Differential Case)** Find all operated polynomial identities of differential type by finding all expressions $N(x, y) \in \mathbf{k}\|x, y\|$ of differential type.
- **Conjecture (OPIs of Differential Type)** Let $\mathbf{k}$ be a field of characteristic zero. Every expression $N(x, y) \in \mathbf{k}\|x, y\|$ of differential type takes one of the forms below for some $a, b, c, e \in \mathbf{k}$:
  1. $b(x\lfloor y\rfloor + \lfloor x\rfloor y) + c\lfloor x\rfloor\lfloor y\rfloor + exy$ where $b^2 = b + ce$,
  2. $ce^2 yx + exy + c\lfloor y\rfloor\lfloor x\rfloor - ce(y\lfloor x\rfloor + \lfloor y\rfloor x)$,
  3. $axy\lfloor 1\rfloor + b\lfloor 1\rfloor xy + cxy$,
  4. $x\lfloor y\rfloor + \lfloor x\rfloor y + ax\lfloor 1\rfloor y + bxy$,
  5. $\lfloor x\rfloor y + a(x\lfloor 1\rfloor y - xy\lfloor 1\rfloor)$,
  6. $x\lfloor y\rfloor + a(x\lfloor 1\rfloor y - \lfloor 1\rfloor xy)$.

Rewriting systems

- $\phi(x, y) := \lfloor xy \rfloor - N(x, y) \in \mathbf{k}\|x, y\|$ defines a rewriting system:

$$\Sigma_\phi := \{\lfloor ab \rfloor \mapsto N(a, b) \mid a, b \in \mathfrak{M}(Z) \backslash \{1\}\}, \qquad (1)$$

  where Z is a set.
- More precisely, for $g, g' \in \mathbf{k}\|Z\|$, denote $g \mapsto_{\Sigma_\phi} g'$ if g' is obtained from g by replacing a subword $\lfloor ab \rfloor$ in a monomial of g by $N(a, b)$.
- A rewriting system Σ is call
  - terminating if every reduction $g_0 \mapsto_\Sigma g_1 \mapsto \cdots$ stops after finite steps,
  - confluent if any two reductions of g can be reduced to the same element.
  - convergent if it is both terminating and confluent.
- Theorem $\phi = [xy] - N(x, y)$ defines a differential type operator if and only if the rewriting system Σ_ϕ is convergent.

Monomial well orderings

- Let Z be a set. Let $\mathfrak{M}^\star(Z)$ denote the bracketed words in $Z \cup \{\star\}$ where $\star$ appears exactly once.
- For $q \in \mathfrak{M}^\star(Z)$ and $u \in \mathfrak{M}(Z)$, let $q|_u$ denote the bracketed word in $\mathfrak{M}(Z)$ when $\star$ in q is replaced by u.
- Then $g \mapsto_{\Sigma_\phi} g'$ if there are $q \in \mathfrak{M}^\star(Z)$ and $a, b \in \mathfrak{M}(Z)$ such that
  1. $q|_{\lfloor ab \rfloor}$ is a monomial of g with coefficient $c \neq 0$,
  2. $g' = g - cq|_{[ab]-N(a,b)}$.
- A **monomial ordering** on $\mathfrak{M}(Z)$ is a well-ordering $<$ on $\mathfrak{M}(X)$ such that

$$1 \leq u \text{ and } u < v \Rightarrow q|_u < q|_v, \ \forall u, v \in \mathfrak{M}(X), q \in \mathfrak{M}^\star(X).$$

- Given a monomial ordering $<$ and a bracketed polynomial $s \in \mathbf{k}\lfloor X \rfloor$, we let $\bar{s}$ denote the leading bracketed word (monomial) of s.
- If the coefficient of $\bar{s}$ in s is 1, we call s monic with respect to the monomial order $<$.

Gröbner-Shirshov bases

- Bokut, Chen and Qiu (JPAA, 2010) determined Gröbner-Shirshov bases for free nonunitary operated algebras. This can be similarly given for free unitary operated algebras $\mathbf{k}\|Z\|$.
- Let $>$ be a monomial ordering on $\mathfrak{M}(Z)$. Let f, g be two monic bracketed polynomials.
- For $p, q \in \mathfrak{M}^\star(Z)$ and $s, t \in \mathbf{k}\|Z\|$, if $w := p|_{\bar{s}} = q|_{\bar{t}}$, then call

$$(f, g)_w^{p,q} := p|_s - q|_t$$

  a composition of f and g.
- For $S \subseteq \mathbf{k}\|Z\|$ and $u \in \mathbf{k}\|Z\|$, we call u trivial modulo (S, w) if $u = \sum_i c_i q_i|_{s_i}$, with $c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^\star(Z)$, $s_i \in S$ and $q_i|_{\overline{s_i}} < w$.
- A set $S \subseteq \mathbf{k}\|X\|$ is called a Gröbner-Shirshov basis if, for all $f, g \in S$, all compositions $(f, g)_w^{p,q}$ of f and g are trivial modulo (S, w).

Differential type, rewriting systems and Gröbner-Shirshov bases

- Theorem. (Guo-Sit-R. Zhang, 2013) For $\phi(x, y) := \lfloor xy \rfloor - N(x, y) \in \mathbf{k}\|x, y\|$, the following statements are equivalent.
- $\phi(x, y)$ is of differential type;
- The rewriting system $\Sigma_\phi = \{\lfloor ab \rfloor \mapsto N(a, b)\}$ is convergent;
- Let Z be a set with a well ordering. With a predefined monomial order $>$, the set

$$S := S_\phi := \{\phi(u, v) = \delta(uv) - N(u, v) |\ u, v \in \mathfrak{M}(Z)\backslash\{1\}\}$$

  is a Gröbner-Shirshov basis in $\mathbf{k}\|Z\|$;
- The free ϕ-algebra on a set Z is the noncommutative polynomial $\mathbf{k}$-algebra $\mathbf{k}\langle\Delta(Z)\rangle$, together with the operator $d := d_Z$ on $\mathbf{k}\langle\Delta(Z)\rangle$ defined by the following recursion:
  Let $u = u_1 u_2 \cdots u_k \in M(\Delta(Z))$, where $u_i \in \Delta(Z), 1 \leq i \leq k$.
  1. If $k = 1$, i.e., $u = \delta^i(x)$ for some $i \geq 0, x \in Z$, then define $d(u) = \delta^{(i+1)}(x)$.
  2. If $k \geq 1$, then define $d(u) = N(u_1, u_2 \cdots u_k)$.

Rota-Baxter type operators

- What Rota-Baxter operator, average operator, Nijenhuis operator, etc. have in common is that they are of the form

$$[u][v] = [M(u,v)]$$

where $M(u,v)$ is in $\mathbf{k}\mathfrak{M}'(Z)$.
- The expression $M(u,v)$ is formally associative:

$$M(M(u,v),w) = M(u,M(v,w))$$

modulo the relation $\phi_M := [u][v] - [M(u,v)]$.
- The rewriting rule $\lfloor u \rfloor \lfloor v \rfloor \mapsto \lfloor M(u,v) \rfloor$ is convergent.
- A $\phi(x,y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor M(x,y) \rfloor$ of the above form is called a Rota-Baxter type operator.

Conjecture on Rota-Baxter type operators

- **Conjecture.** Any Rota-Baxter type operator is of the form
$$P(x)P(y) = P(M(x,y)),$$
for an $M(x,y)$ from the following list (new types in red).

1. $xP(y)$ (average operator)
2. $P(x)y$ (reverse average operator)
3. $xP(y) + yP(x)$
4. $P(x)y + P(y)x$
5. $xP(y) + P(x)y - P(xy)$ (Nijenhuis operator)
6. $xP(y) + P(x)y + e_1 xy$ (RBA with weight e_1)
7. $xP(y) - xP(1)y + e_1 xy$
8. $P(x)y - xP(1)y + e_1 xy$
9. $xP(y) + P(x)y - xP(1)y + e_1 xy$ (TD operator with weight e_1)
10. $xP(y) + P(x)y - xyP(1) - xP(1)y + e_1 xy$
11. $xP(y) + P(x)y - P(xy) - xP(1)y + e_1 xy$
12. $xP(y) + P(x)y - xP(1)y - P(1)xy + e_1 xy$
13. $d_0 xP(1)y + e_1 xy$ (generalized endomorphisms)
14. $d_2 yP(1)x + e_0 yx$

Classification of Rota-Baxter type operators

- Theorem (Gao-Guo-Sit-S. Zheng) For $\phi(x,y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor M(x,y) \rfloor$, the following statements are equivalent.
- $\phi(x,y)$ is of Rota-Baxter type;
- The rewriting system from $\phi(x,y)$ is convergent;
- There is a Gröbner-Shirshove basis for the ideal of $\phi(x,y)$;
- Free algebras in the corresponding category have canonical bases given by the *Rota-Baxter words*.
- Corollary All operators in the above list are Rota-Baxter type operators.

General formulations for associative algebras

- (Rota's Classification Problem via rewriting systems) Determine all convergent systems of OPIs.
- Example. (Two-sided) averaging operator P is defined to satisfy

$$P(x_1)P(x_2) = P(P(x_1)x_2) = P(x_1P(x_2))$$

  It is not convergent.
- (Rota's Classification Problem via Gröbner-Shirshov bases) Determine all Gröbner-Shirshov systems of OPIs.
- A Gröbner-Shirshov system of OPIs is convergent.

Baby model: multiplicative superalgebra

- Consider an algebra $H = H_1 \oplus H_0$ with subalgebras H_1, H_0 such that $H_i H_j \subseteq H_{ij}, i, j \in \{0, 1\}$. So H_1 is a subalgebra and H_0 is an ideal. Such an algebra is called a multiplicative superalgebra.
- Let $(A, \cdot)$ be an algebra. Let $(R, *)$ be an algebra with multiplication $*$. Let $\ell, r : A \to \mathrm{End}_{\mathbf{k}}(R)$ be two linear maps.
- We call $(R, *, \ell, r)$ or simply R an ***A*-bimodule k-algebra** if (R, ℓ, r) is an A-bimodule that is compatible with the multiplication $*$ on R:

$$\ell(x)(v * w) = (\ell(x)v) * w, (v * w)r(x) = v * (wr(x)),$$
$$(vr(x)) * w = v * (\ell(x)w), \quad \text{for all} \quad x, y \in A, v, w \in R.$$

- Every multiplicative superalgebras is of the form $A \oplus R = A(\mathbf{k}1 \oplus R)$ where A is an algebra and R is an A-bimodule algebra.
- Free multiplicative superalgebra with given H_0 is a quotient of $H_0 \oplus B(M)$, where $B(M)$ is the free A-bimodule algebra spanned by a module M.

Disconnected operads as “superoperads”

- Most studied on operad are focused on the connected ones, that is $\mathbb{S}$-modules $\mathcal{P} := (\mathcal{P}_n)_{n\geq 0}$ with $\mathcal{P}_1 = \mathbf{k}\mathrm{id}$ (and reduced: $\mathcal{P}_0 = 0$);
- A (reduced) disconnected operad $\mathcal{P}$ has a “super” decomposition $\mathcal{P} = \mathcal{P}_{=1} \oplus \mathcal{P}_{>1} = \mathcal{P}_{=1} \circ \tilde{\mathcal{P}}_{>1}$, where $\mathcal{P}_{=1}$ is the operad with $\mathcal{P}_1$ concentrated at arity 1 and $\tilde{\mathcal{P}}_{>1}$ is the connected operad $(\mathbf{k}\mathrm{id}, \mathcal{P}_2, \mathcal{P}_3, \cdots)$.
- This is similar to a multiplicative superalgebra in the sense that $\mathcal{P}_{>1}$ is closed under compositions with $\mathcal{P}_1$.
- We can regard $\mathcal{P}$ as the connected operad $\mathcal{P}_{\geq 2}$ with linear operations from $\mathcal{P}_1$, and pose an analogous Rota’s Classification Problem for operads.

Operad forms of the classification problem

- (Weak form) For a connected operad $\mathcal{P} = \mathcal{T}(M)/(S)$ with generator space spanned by M and relation space spanned by a Gröbner-Shirshov basis S. Determine operators $\mathcal{P}_{=1} = \mathcal{T}(P)/(S_P)$ on $\mathcal{P}$ such that $S \cup S_P$ is a Gröbner-Shirshov basis (for $\mathcal{T}(M \oplus P)$).
- (Strong form) Determine operators $\mathcal{P}_{=1} = \mathcal{T}(P)/(S_P)$ such that the weak form holds for every connected operad $\mathcal{P} = \mathcal{T}(M)/(S)$ with generator space spanned by M and relation space spanned by a Gröbner-Shirshov basis S.

Special cases

- Let $\mathcal{P} = \mathcal{T}(M)/(S)$ be a binary quadratic nonsymmetric operad. Define the **differential** $\mathcal{P}$ **operad** to be

$$\mathcal{DP} := \mathcal{T}(M_d)/(S \sqcup S_d),$$

  where $M_d := (M_0, M_1 \oplus \mathbf{k}\{d\}, M_2, \cdots, M_n, \cdots)$ and S_d is a set of Leibniz rules on $\mathcal{P}$.
- If S is a Gröbner-Shirshov basis in $\mathcal{T}(M)$, then $S \sqcup S_d$ is a Gröbner-Shirshov basis in $\mathcal{T}(M_d)$ for the operad $\mathcal{DP}$.
- A similar statement holds for Rota-Baxter operators.

Summary and outlook

- A long standing problem of Rota is the classification of linear operators on algebras that satisfy algebraic identities.
- This problem is made precise in the context of operated polynomial algebras and rewriting systems;
- This problem is treated in two cases: differential type and Rota-Baxter type operators, with the help of rewriting systems and Gröber-Shirshov bases;
- Similar methods can be applied to treat other classes of operators on associative algebras, and further to operads;
- Roughly speaking, the linear operators that interested Rota and maybe other mathematicians (good operators) should be the ones whose defining identities define convergent rewriting systems (good systems), or possesses Gröbner-Shirshov bases (good bases).
- Similar questions can be asked for linear operators on operads.

Thank You!