Recent Developments in the Theory of Higher Properads

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What is... a PROPERAD?

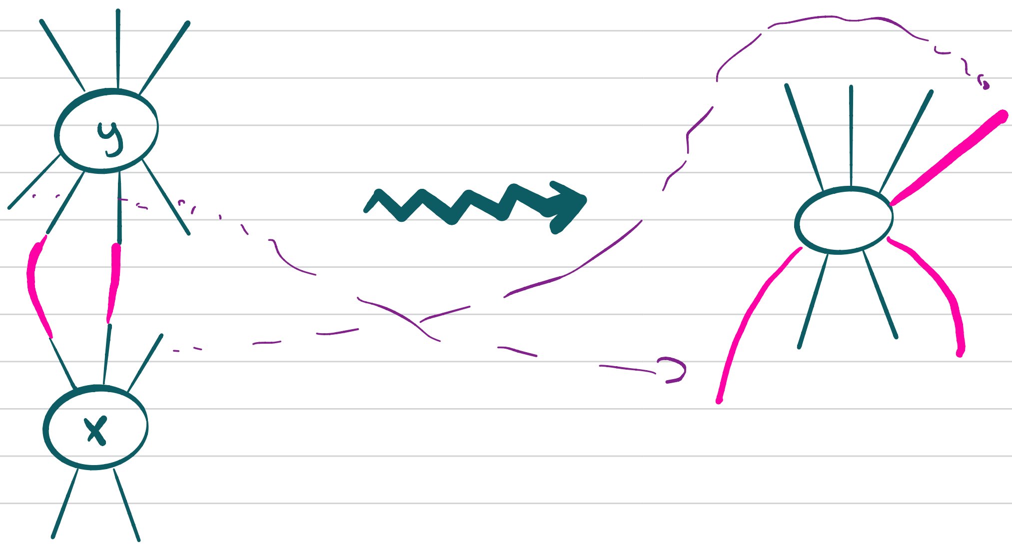
(Bruno Vallette 2003, Ross Duncan 2006)

Operad : $\mathcal{O}(n)\otimes\mathcal{O}(m) \xrightarrow{\circ_i} \mathcal{O}(n+m-1)$ $\quad 1\le i\le n$

$\mathcal{O}(n;1)\otimes\mathcal{O}(m;1)$

Properad : $P(n;k)\otimes P(m;j) \longrightarrow P(n+m-\ell\,;\,k+j-\ell)$ $\quad \ell>0$

$x \otimes y$



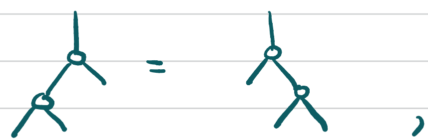
What are properads for?

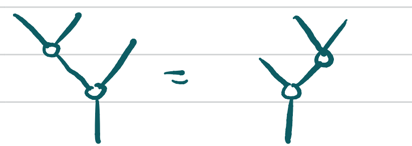
Operads can model algebras and coalgebras, but not bialgebras.

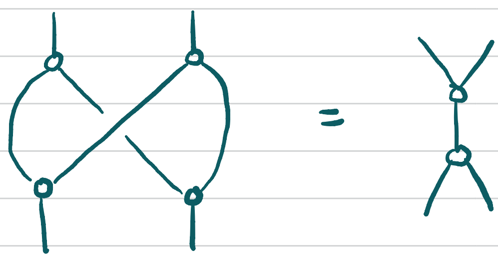
Properads can handle this job:

There is a properad with two generators: comult mult

relations:







Algebras over this properad are bialgebras

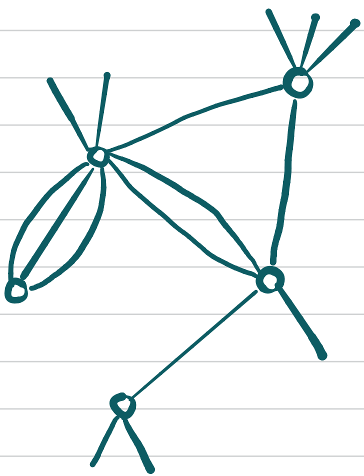
major goal

Cisinski–Moerdijk–Weiss approach to ∞-operads using “dendroidal objects”

built on a category of rooted trees

What aspects of this approach to ∞-operads can be adapted to study ∞-properads?

(directed) graphs (with legs)



Data:

E(G) edges

V(G) vertices

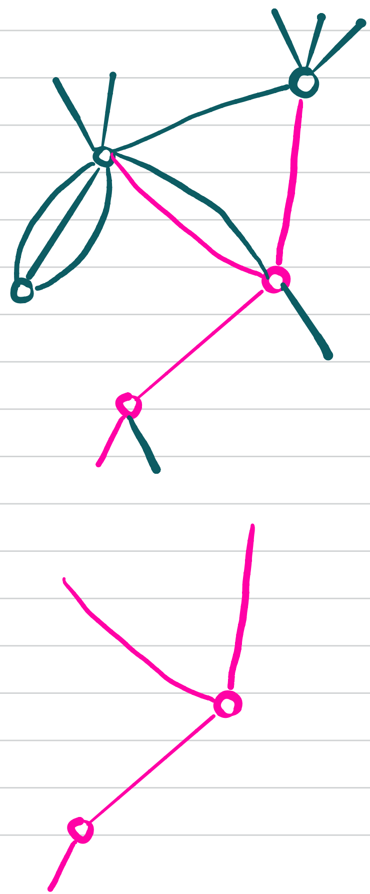
in, out: $V(G) \to \wp(E(G))$

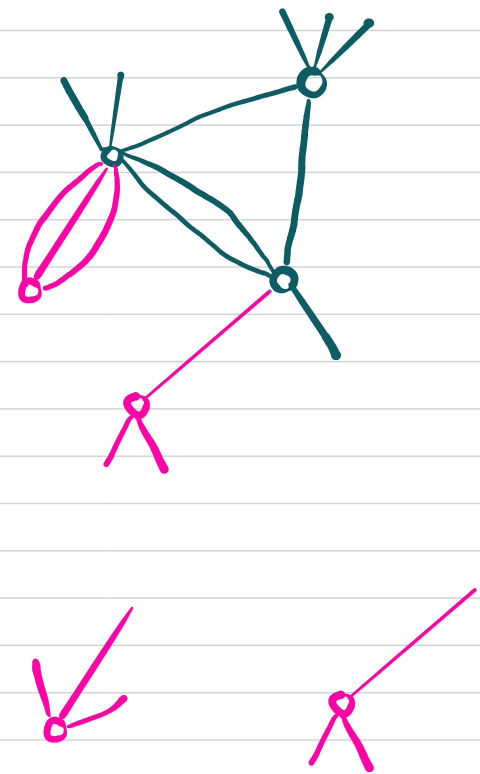
$in(v) = \{e_1, e_2, e_3\}$.

Conventions:

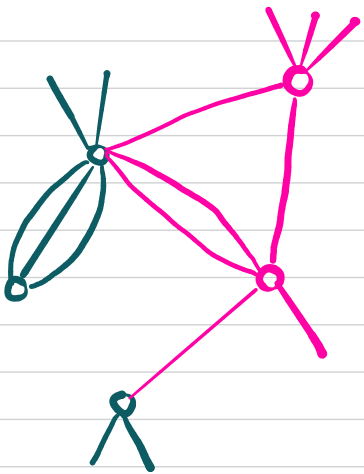
- connected
- acyclic (in directed sense)
- all edges point down

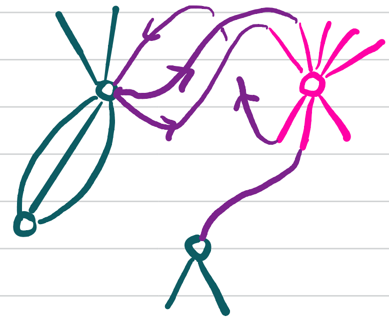
bad subgraphs

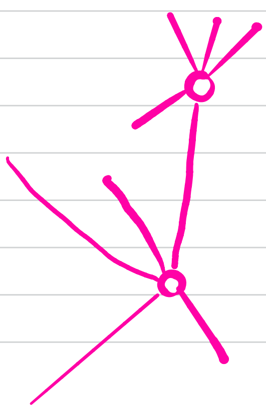


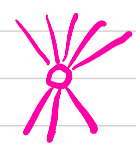


bad subgraphs

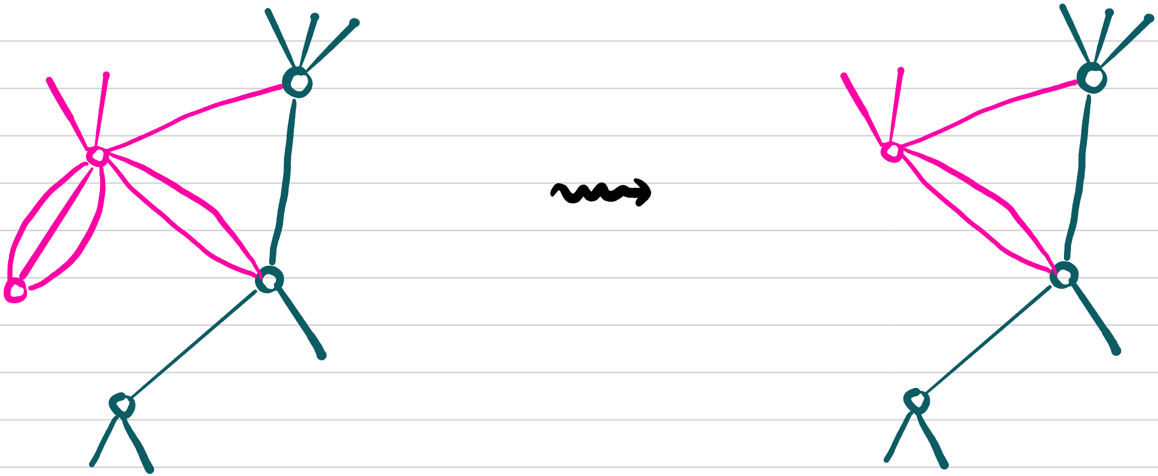




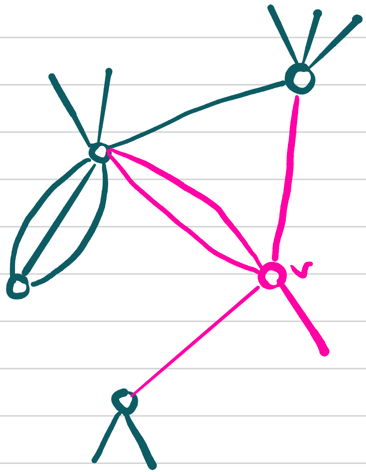


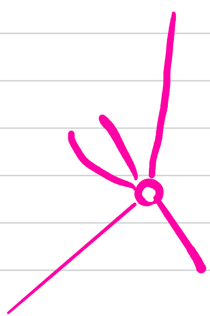


good subgraphs

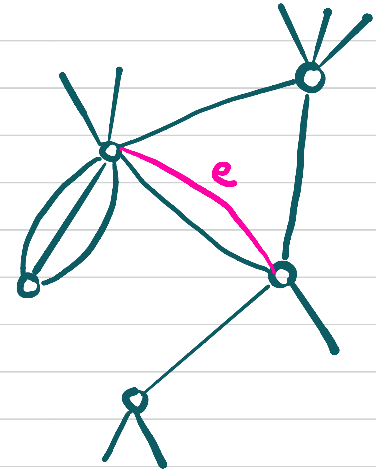


good subgraphs





c_v



η_e

structure of Sb(G)

the set of "good subgraphs" of G

$E(G) \to Sb(G)$

$V(G) \to Sb(G)$

"Elementary subgraphs"

 : η_e

 : C_v

in, out: $Sb(G) \to \wp(E(G))$

"boundary functions"

$in(\eta_e) = \{e\}$

$out(\eta_e) = \{e\}$

$in(C_v) = in(v)$

$out(C_v) = out(v)$

unions: $H, K \in Sb(G)$

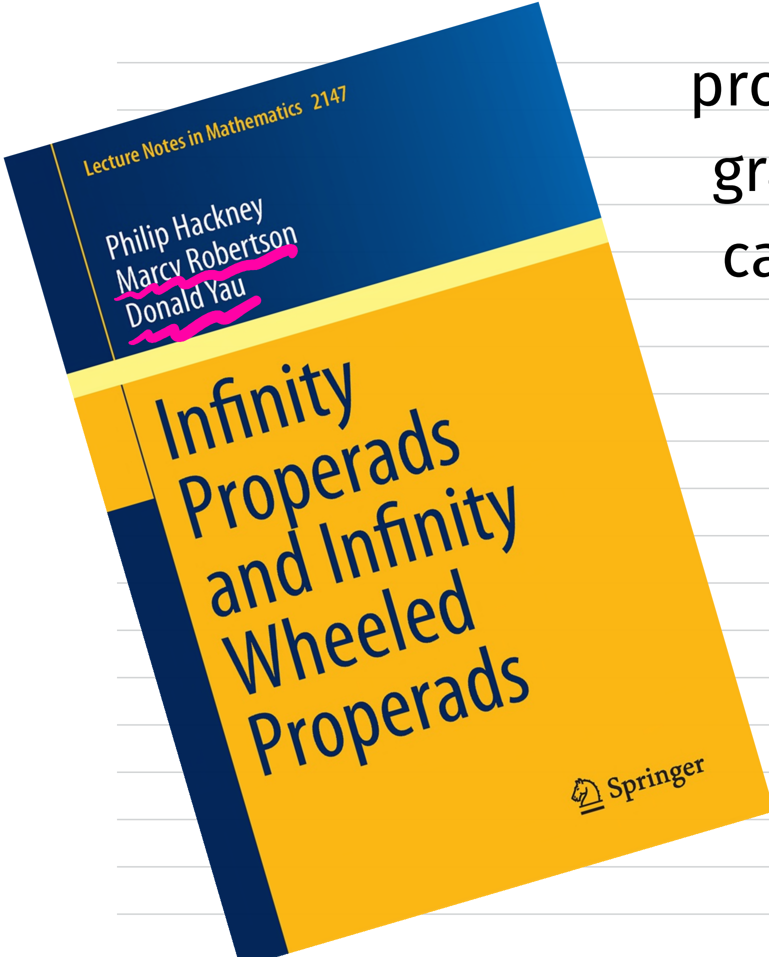
Do $E(H) \cup E(K) \subset E(G)$

&

$V(H) \cup V(K) \subset V(G)$

determine a good subgraph?

If so, this is $H \cup K \in Sb(G)$



properadic
graphical
category

Γ

Hongyi Chu

properadic graphical category Γ (Chu, H. 2020)

Objects: graphs

Morphisms: $f : G \to H$ consists of a pair of functions $f_0 : E(G) \to E(H)$
$f_1 : Sb(G) \to Sb(H)$

so that

a)
$$\begin{tikzcd}
\mathcal{P}(E(G)) \arrow[d, "\mathcal{P}(f_0)"] & Sb(G) \arrow[l, "in"'] \arrow[r, "out"] \arrow[d, "f_1"] & \mathcal{P}(E(G)) \arrow[d, "\mathcal{P}(f_0)"] \\
\mathcal{P}(E(H)) & Sb(H) \arrow[l, "in"'] \arrow[r, "out"] & \mathcal{P}(E(H))
\end{tikzcd}$$
commutes

b) If J, K, $J \cup K$ are good subgraphs of G, then

$$f_1(J \cup K) = f_1(J) \cup f_1(K)$$

First example: if G is a good subgraph of H, then we have $G \to H$

graphs can be built out of iterated unions of smaller graphs,
so by axiom (b)

$$f_1 : Sb(G) \longrightarrow Sb(H)$$

is determined by its value on elementary graphs

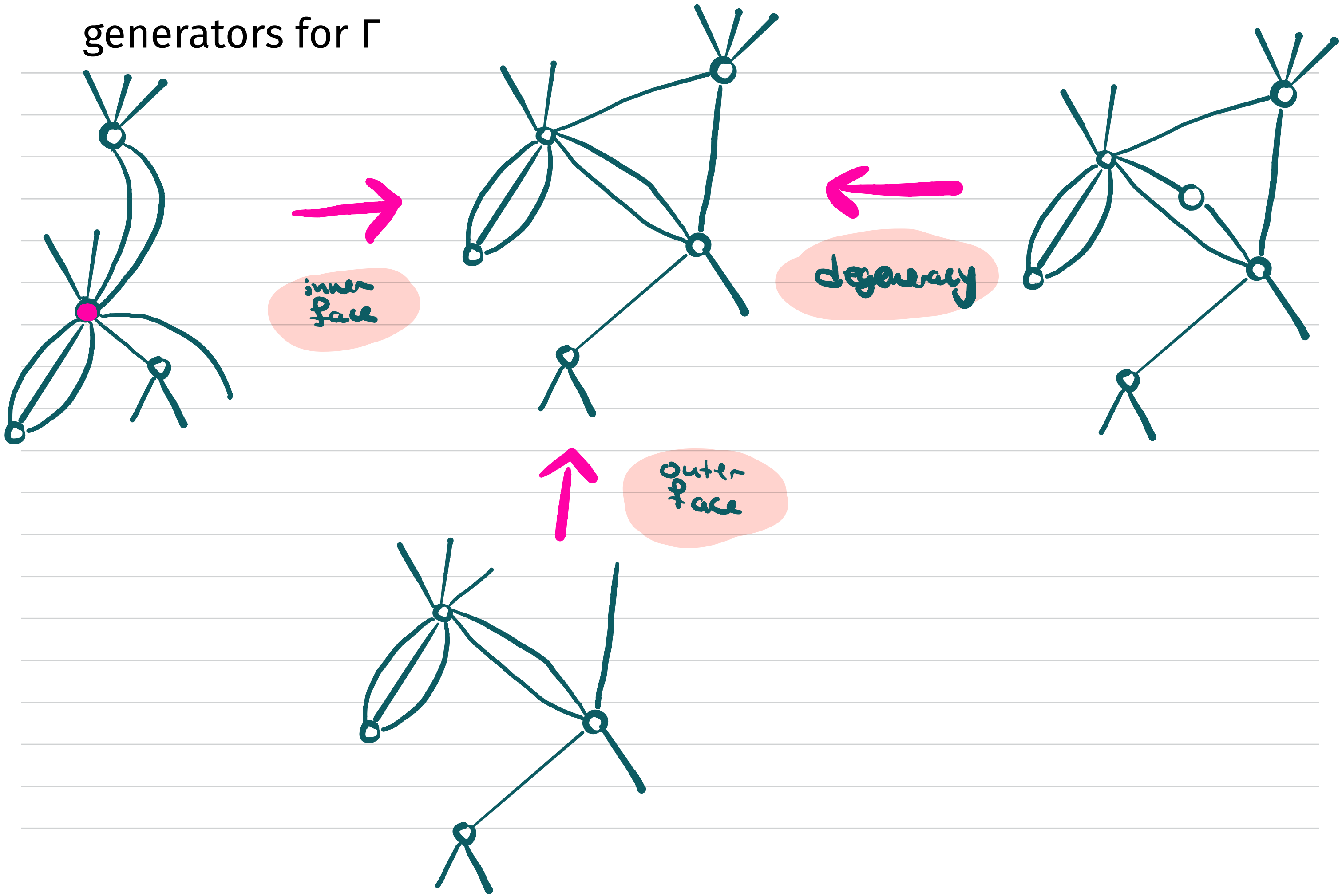
i.e. by the composite functions

$$\begin{tikzcd}
E(G) \arrow[r, "f_0"] \arrow[d] & E(H) \arrow[d] \\
Sb(G) \arrow[r, "f_1"] & Sb(H)
\end{tikzcd}$$

and

$$V(G) \longrightarrow Sb(G) \xrightarrow{f_1} Sb(H)$$

generators for Γ



Structure of the category Γ

$\rightsquigarrow$ like the simplicial category Δ

Theorem (H., Robertson, Yau 2015)

Γ is a generalized Reedy category the sense of Berger & Moerdijk

$\deg : \mathrm{Ob}(\Gamma) \to \mathbb{N}$

Γ^- generated by degeneracy maps

Γ^+ generated by (inner & outer) face maps

Theorem (H., Robertson, Yau 2018)

With this structure, Γ is an Eilenberg–Zilber category

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "f^-"'] & & \cdot \\ & \cdot \arrow[ur, "f^+"'] & \end{tikzcd}$$

Γ^- = split epimorphisms

Γ^+ = monomorphisms

Theorem (Joachim Kock 2016)

Γ has an active-inert orthogonal factorization system

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\text{active} = \text{bdry preserving}"'] & & \cdot \\ & \cdot \arrow[ur, "\text{inert} \cong \text{subgraph inclusion}"'] & \end{tikzcd}$$

<u>active</u> $f : G \to H$

f_0 induces bijections

$\mathrm{in}(G) \cong \mathrm{in}(H)$

$\mathrm{out}(G) \cong \mathrm{out}(H)$

<u>inert</u> $f : G \to H$

f is isomorphic to a subgraph inclusion

Γ_{act} generated by degeneracy maps and inner face maps

Γ_{int} generated by outer face maps

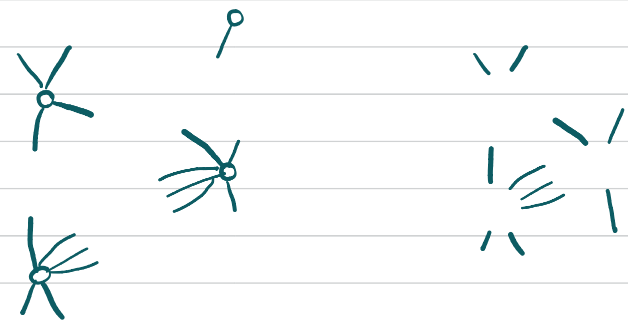
Γ presheaves = graphical sets

(like Δ-presheaves = simplicial sets)

functors $X : \Gamma^{op} \to \mathrm{Set}$

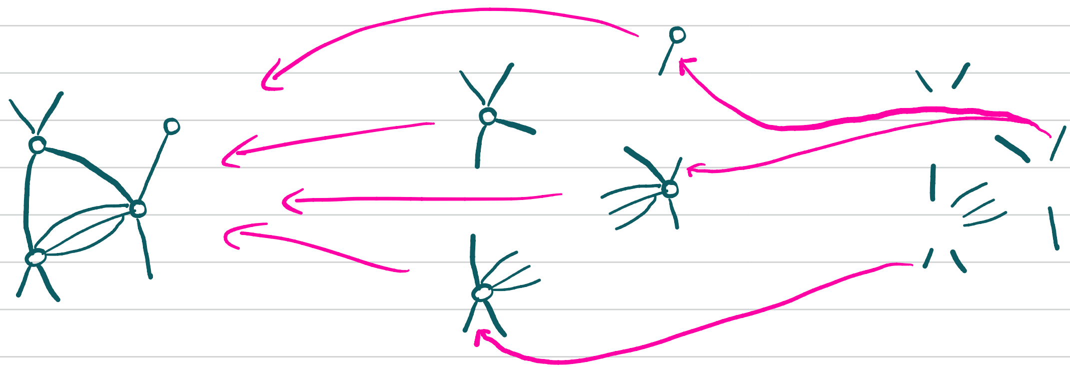
X satisfies the Segal condition just when X(G) is determined by the values of X on its elementary subgraphs

G



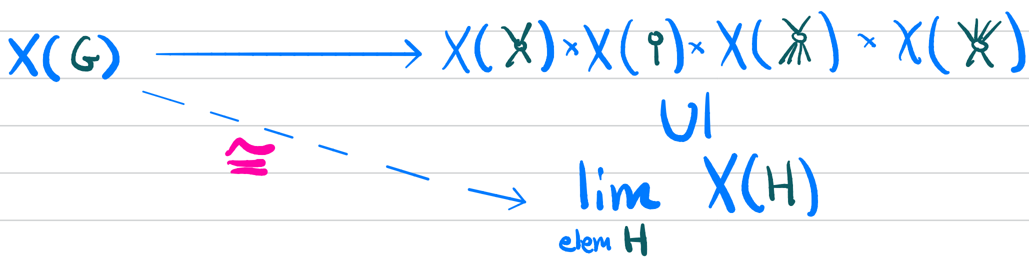
elementary subgraphs of G

Γ



G

elementary subgraphs of G

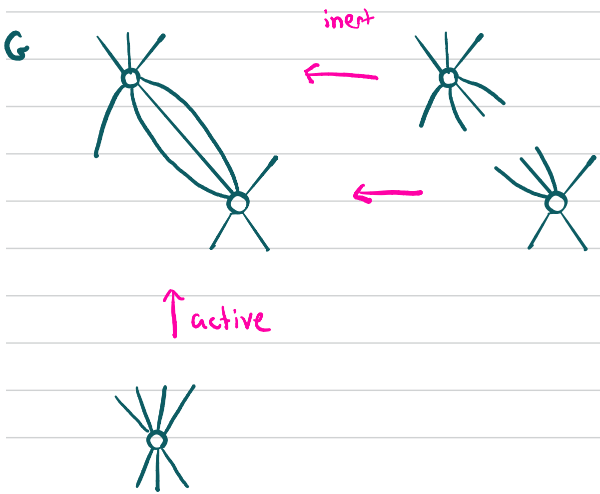


properads

A presheaf $X : \Gamma^{op} \to Set$ is **Segal** when the map

$$X(G) \xrightarrow{\cong} \lim_{\substack{\text{elem} \\ H \subseteq G}} X(H) \quad \text{is a bijection for all } G$$

Definition: (or Theorem (H, Robertson, Yau 2015))
A **properad** (in Set) is the same thing as a Segal presheaf.



$$\begin{tikzcd} X(G) \arrow[r, "\cong"] \arrow[d] & X(C^3_4) \times_{X(\uparrow)^{\times 3}} X(C^4_2) \arrow[dl, dashed] \\ X(C^4_3) & \end{tikzcd}$$

like composition from first slide

[colored properad]

∞-properads

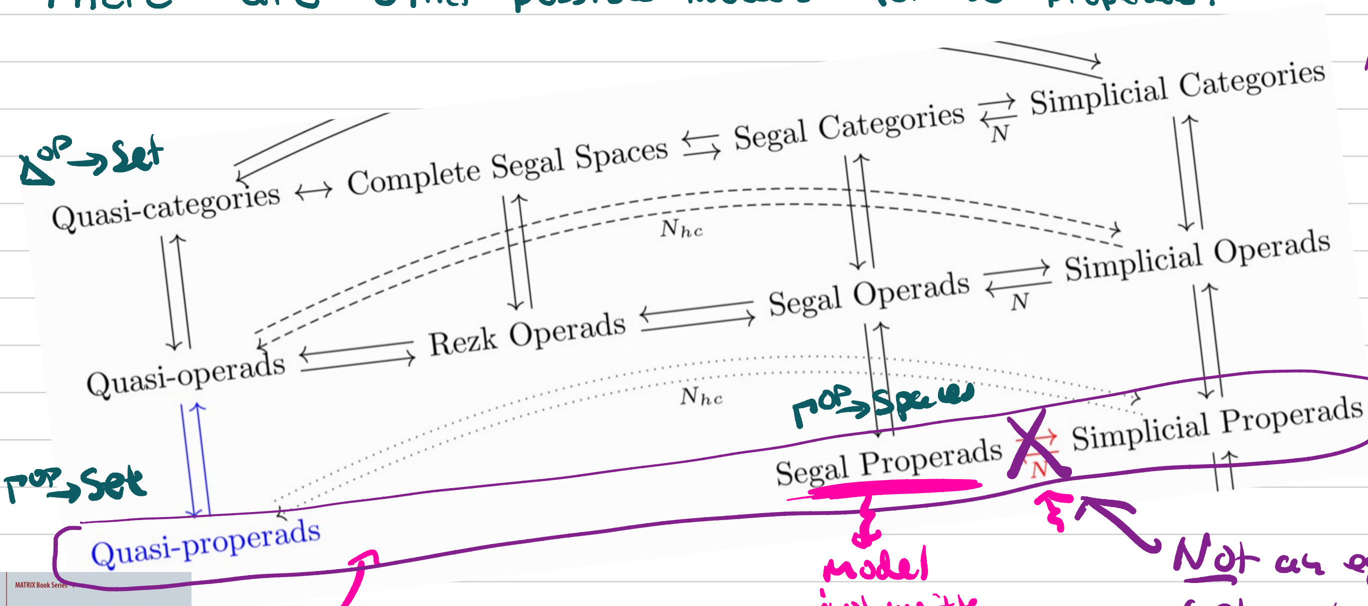
$X : \Gamma^{op} \longrightarrow \text{Spaces}$ is **Segal** just when

$X(G) \xrightarrow{\simeq} \lim\limits_{\substack{\text{elem} \\ H \subseteq G}} X(H)$ is a WEAK HOMOTOPY EQUIVALENCE for all G

An **∞-properad** is a presheaf $X : \Gamma^{op} \longrightarrow \text{Spaces}$ which is
- Reedy fibrant,
- has $X(\eta)$ homotopically discrete, and
- is Segal.

$$\begin{tikzcd} X(\quad) \arrow[r, "\simeq"] \arrow[d] & X(C_4^3) \times_{X(\eta)^{\times 3}} X(C_2^4) \\ X(C_3^4) & \end{tikzcd}$$

There are other possible models for ∞-properads:



∞-categories

∞-operads

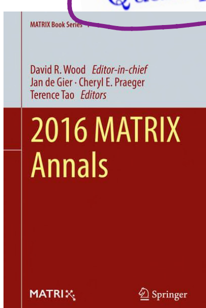
$\Delta^{op} \to Set$

$\Gamma^{op} \to Set$

$\Gamma^{op} \to Spaces$

model just writes down.

Not an equivalence (Chu, H. 2020)



(from H., Robertson 2016)

enriched ∞-properads

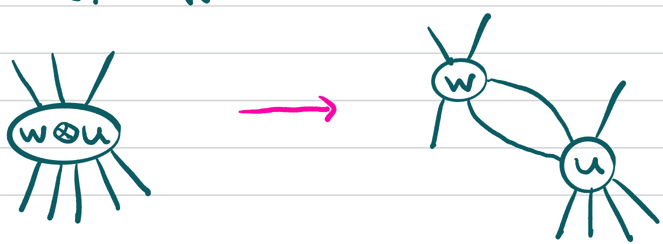
(based on ideas from Hongyi Chu thesis)

$W^{\otimes} \longrightarrow \mathrm{FinSet}_*$ symmetric monoidal ∞-category

∞-cat ⇝
$$\begin{tikzcd} \Gamma^{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (W^{op,\otimes})^{op} \arrow[d, "\leftarrow \text{cartesian fibration}"] \\ \Gamma \arrow[r, "V(-)_+"'] & \mathrm{FinSet}_*^{op} \end{tikzcd}$$

$$G \longmapsto V(G)_+$$

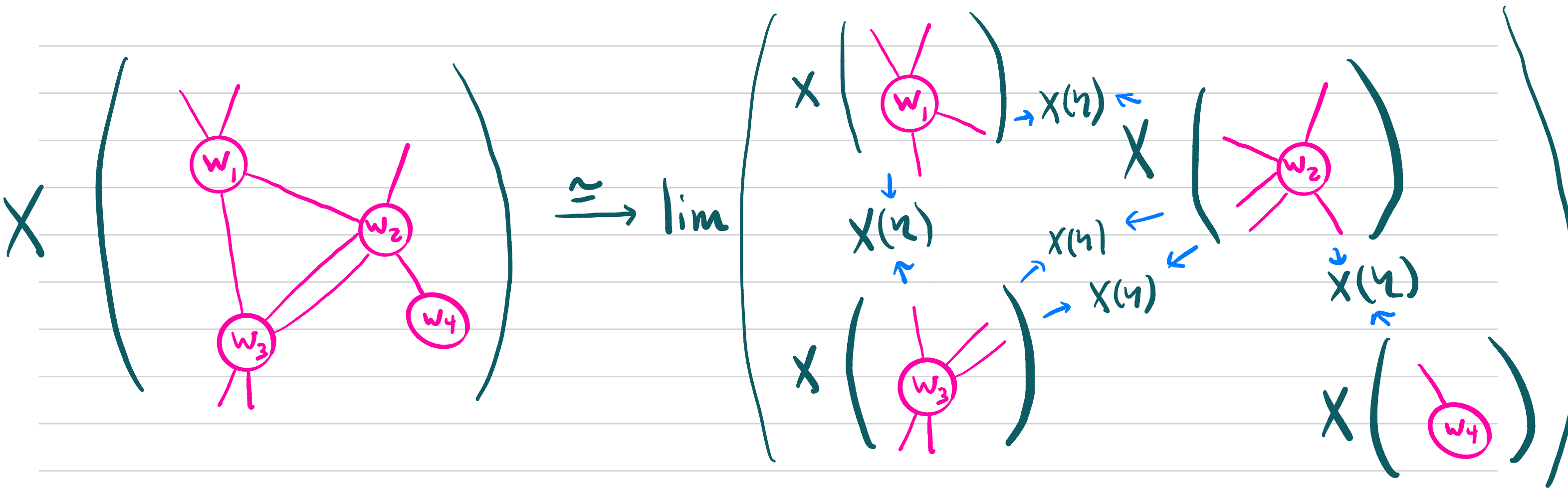
Objects of Γ^W are graphs whose vertices are decorated by objects of W



$(\Gamma^W)^{op}$ has an algebraic pattern structure (a notion due to Chu-Haugseng)

so we can make sense of the Segal condition for a presheaf

$$X : (\Gamma^W)^{op} \longrightarrow \text{Spaces}$$



Fibrewise representability:

$X(C_n^m(-)) : W^{op} \to \text{Spaces}$

$$\begin{tikzcd}
\mathrm{Map}_W(w, \bullet)_{\perp} \arrow[r] \arrow[d] & X(C_n^m(w)) \arrow[d, "\text{picks out colors}"] \\
\{a_1,\dots,a_m; b_1,\dots,b_n\} \arrow[r] & X(\eta)^{\times n+m}
\end{tikzcd}$$

$$\mathrm{Map}_X(a_1,\dots,a_m; b_1,\dots,b_n) = \bullet \in W$$

W-enriched ∞-properad $\quad X : (\Gamma^W)^{op} \longrightarrow \text{Spaces}$

which is

- Segal
- Fibrewise representable

ON RECTIFICATION AND ENRICHMENT OF INFINITY PROPERADS

HONGYI CHU AND PHILIP HACKNEY

Thank
You!