Gelfand-Kirillov Dimension of Nonsymmetric Operads

The 3rd Conference on Operad Theory and Related Topics

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Joint work

This talk is based on a joint work with Yongjun Xu, James J. Zhang and Xiangui Zhao.

Plan

- History
- Gelfand-Kirillov dimension of associative algebras
- Nonsymmetric operads
- Gelfand-Kirillov dimension of nonsymmetric operads
- Gap theorem of GKdim of nonsymmetric operads
- Another construction of NS operads with given GKdim

1. History

- 1966, Gelfand-Kirillov conjecture

I.M Gel'fand, A.A. Kirillov, *On fields connected with the enveloping algebras of Lie algebras*. (Russian) Dokl. Akad. Nauk SSSR **167** 1966 503-505.

I.M Gel'fand, A.A. Kirillov, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*. (French) Inst. Hautes Ètudes Sci. Publ. Math. No. **31** (1966), 5-19.

- 1968, Milnor, Growth of groups

J. Milnor, *A note on curvature and fundamental group.* J. Diff. Geom. **2** (1968), 1-7.

- 1955, A.S. Švarc

A.S. Švarc, *A volume invariant of coverings*. (Russian) Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 32-34.

1. History

- 1976, Borho and Kraft showed that GK dimension can be any real number bigger than 2.

W. Borho and H.Kraft, *Über die Gelfand-Kirillov Dimension*. Math. Ann. **220** (1976), 1-24.

- 1978, Bergman proved the Gap Theorem for GK dimension.

G.M. Bergman, *A note on growth functions of algebras and semigroups*. Research Note, University of California, Berkeley, (1978).

- 1984, Warfield gave another construction of algebras with GK dimension any real number bigger than 2.

R. B. Warfield, *The Gelfand-Kirillov dimension of a tensor product*. Math. Zeit. **185** (1984), no.4, 441-447.

1. History

- Boardman, Vogt
- May

J.M. Boardman and R.M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Math., vol. **347**, Springer-Verlag, Berlin · Heidelberg · New York, 1973.

J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. **271**.

- Ginzburg, Kapranov

V. Ginzburg and M. M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203-272.

- Kontsevich
- Tamarkin

M. Kontsevich, *Deformation quantization of Poisson manifolds.* Lett. Math. Phys. **66**(2003), 157-216.

D. Tamarkin, *Another proof of M. Kontsevich formality theorem*, preprint, arXiv:9803025.

1. History

- 2020, Bao, Ye and Zhang defined GK dimension of a finitely generated operad.

Y.-H. Bao, Y. Ye and J.J. Zhang, *Truncation of Unitary Operads*, Advances in Mathematics. **372** (2020): 107290.

Let $\mathbb{K}$ be a field. Let A be a $\mathbb{K}$-algebra and V be a finite dimensional subspace of A spanned by $a_1, \ldots, a_m$. For $n \geq 1$, let V^n denote the space spanned by all monomials in $a_1, \ldots, a_m$ of length n. Define

$$d_V(n) = dim(V_n), \text{ where } V_n := \mathbb{K} + V + V^2 + \cdots + V^n$$

Definition

The Gelfand-Kirillov dimension of a $\mathbb{K}$-algebra A is

$$\mathrm{GKdim}(A) = \sup_V \overline{\lim} \log_n d_V(n)$$

where the supremum is taken over all finite dimensional subspaces V of A

Remark

For a finitely generated $\mathbb{K}$-algebra A with finite dimensional generating space V,

$$\mathrm{GKdim}(A) = \overline{\lim} \log_n d_V(n),$$

which is independent of the choice of V.

2. GK-dimension of algebras

Proposition

Let A be a finitely generated commutative $\mathbb{K}$*-algbra and* $cl.Kdim(A)$ *be the classical Krull dimension of A, then*

$$GKdim(A) = cl.Kdim(A).$$

Proposition

$GKdim\,(A) = 0$ if and only if A is locally finite dimensional, meaning that every finitely generated subalgebra is finite dimensional.
$GKdim\,(A) \geq 1$ if algebra A is not locally finite dimensional.

Proposition

Let A be a $\mathbb{K}$*-algebra, and let* $B = A[x_1, \ldots, x_n]$*. Then* $GKdim(B) = GKdim(A) + n$*.*

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2. GK-dimension of algebras

Problem

Which real numbers occur as the Gelfand-Kirillov dimension of a $\mathbb{K}$-algebra?

2. GK-dimension of algebras

Theorem (Borho and Kraft 1976)

For any real number $r > 2$, there exists a $\mathbb{K}$-algebra such that $GKdim(A) = r$.

W. Borho and H.Kraft, *Über die Gelfand-Kirillov Dimension*. Math. Ann. **220** (1976), 1-24.

Theorem (Warfield 1984)

For any real number $r > 2$, there exists a two-generator algebra $A = \mathbb{K}\langle x, y\rangle / I$ with GKdim(A)=r.

R. B. Warfield, *The Gelfand-Kirillov dimension of a tensor product.* Math. Zeit. **185** (1984), no.4, 441-447.

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For $1 < r < 2$ the existence problem was open for some years until Bergman showed the following theorem.

Theorem (Bergman 1978, Gap Theorem)

No algebra has Gelfand-Kirillov dimension strictly between 1 and 2. So

$$GKdim \in R_{\mathrm{GKdim}} := \{0\} \cup \{1\} \cup [2, \infty) \cup \{\infty\}.$$

G.M. Bergman, *A note on growth functions of algebras and semigroups*. Research Note, University of California, Berkeley, (1978).

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G.M. Bergman, *A note on growth functions of algebras and semigroups*. Research Note, University of California, Berkeley, (1978).

Proposition

If $r \in R_{\mathrm{GKdim}}$, then there is a finitely generated monomial algebra A such that $\mathrm{GKdim}(A) = r$.

J.P. Bell, *Growth functions*, Commutative Algebra and Noncommutative Algebraic Geometry **1** (2015), 1.

Definition (partial definition)

A nonsymmetric operad is a collection of vector spaces $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ *(n is called the arity) equipped with an element* $\mathrm{id} \in \mathcal{P}(1)$ *and maps*

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), \alpha \otimes \beta \mapsto \alpha \circ_i \beta,\ 1 \leq i \leq m$$

which satisfy the following properties for all $\alpha \in \mathcal{P}(m)$, $\beta \in \mathcal{P}(n)$ *and* $\gamma \in \mathcal{P}(r)$*:*

(i) $(\alpha \circ_i \beta) \circ_{i+j-1} \gamma = \alpha \circ_i (\beta \circ_j \gamma)$ *for* $1 \leq i \leq m, 1 \leq j \leq n$*;*

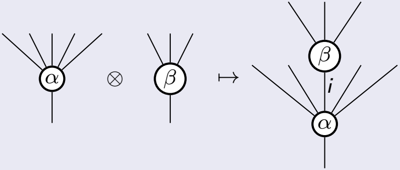
(ii) $(\alpha \circ_i \beta) \circ_{j+n-1} \gamma = (\alpha \circ_j \gamma) \circ_i \beta$ *for* $1 \leq i < j \leq m$*;*

(iii) $\mathrm{id} \circ_1 \alpha = \alpha,\ \alpha \circ_i \mathrm{id} = \alpha$ *for* $1 \leq i \leq n$*.*

Remark

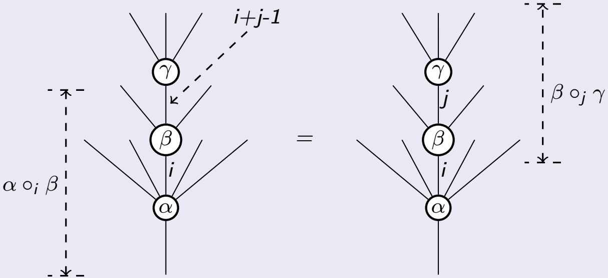
$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1)$$

$$\alpha \otimes \beta \mapsto \alpha \circ_i \beta$$



Remark

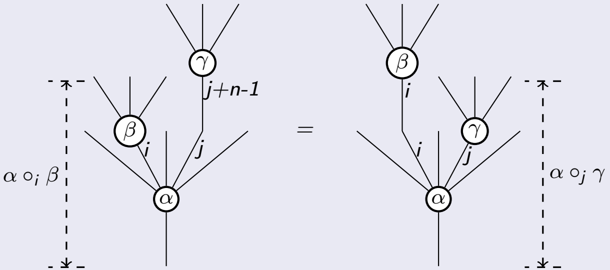
(i) $(\alpha \circ_i \beta) \circ_{i+j-1} \gamma = \alpha \circ_i (\beta \circ_j \gamma)$ *for* $1 \leq i \leq m, 1 \leq j \leq n$



3. Nonsymmetric operads

Remark

$$(ii)\ (\alpha \circ_i \beta) \circ_{j+n-1} \gamma = (\alpha \circ_j \gamma) \circ_i \beta \quad \textit{for } 1 \leq i < j \leq m$$



Example (operad of nonunital associative algebras)

Define $As = \{As(n)\}_{n \geq 1}$*, where* $As(1) = \mathbb{K}id$ *and* $As(n) = \mathbb{K}\mu_n$*.*

$$\mu_m \circ_i \mu_n := \mu_{m+n-1}, \; 1 \leq i \leq m.$$

Example

A unital associative algebra A *can be interpreted as an operad* $\mathcal{P}$ *with* $\mathcal{P}(1) = A$ *and* $\mathcal{P}(n) = 0$ *for all* $n \neq 1$*, and the compositions in* $\mathcal{P}$ *are given by the multiplication of* A*.*

Remark

An operad can be viewed as a generalization of an algebra.

Example ($\star$)

Suppose $A = \oplus_{i \geq 0} A_i$ *is a graded algebra with unit* 1_A. *Let* $\mathcal{P}(0) = 0$ *and* $\mathcal{P}(n) = A_{n-1}$ *for all* $n \geq 1$. *Define compositions as follows*

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(n+m-1),$$

$$a_{m-1} \otimes a_{n-1} \mapsto \begin{cases} c a_{m-1} & a_{n-1} = c 1_A, \\ a_{m-1} a_{n-1} & a_{n-1} \notin \mathbb{K} 1_A, i = 1, \\ 0 & a_{n-1} \notin \mathbb{K} 1_A, i \neq 1. \end{cases}$$

Then $\mathcal{P}$ *is an operad with* $\mathrm{id} = 1_A$.

3. Nonsymmetric operads

Definition

A collection $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ *of spaces (especially, an operad) is called finite dimensional if* $\dim \mathcal{P} := \dim\left(\oplus_{n\geq 0}\mathcal{P}(n)\right) < \infty$*;*
It is called locally finite if $\mathcal{P}(n)$ *is finite dimensional for all* $n \in \mathbb{N}$*.*

3. Nonsymmetric operads

Given a subcollection $\mathcal{V}$ of operad $\mathcal{P}$, let $\mathcal{V}^0 = (0, \mathbb{K}id, 0, 0, \dots)$ and $\mathcal{V}^m = \{\mathcal{V}^m(n)\}_{n\geq 0}$ for $m \geq 1$, where $\mathcal{V}^m(n)$ denotes the subspace of $\mathcal{P}(n)$ spanned by all elements that have the following form

$$((\cdots((a_1 \circ_{j_1} a_2) \circ_{j_2} a_3) \circ_{j_3} \cdots) \circ_{j_{m-1}} a_m), \text{ each } a_i \in \mathcal{V}. \tag{1}$$

We call $\mathcal{V}$ a *generating subcollection* of $\mathcal{P}$ if

$$\mathcal{P} = \sum_{m\geq 0} \mathcal{V}^m := \left\{ \sum_{m\geq 0} \mathcal{V}^m(n) \right\}_{n\geq 0}.$$

Definition

An operad $\mathcal{P}$ is called *finitely generated* if it has a finite dimensional generating subcollection $\mathcal{V} = \{\mathcal{V}(n)\}_{n\geq 0}$.

4. GK-dimension of NS operads

Definition (Bao-Ye-Zhang 2020)

Let $\mathcal{P}$ be a locally finite operad. The *Gelfand-Kirillov dimension* (*GK-dimension* for short) of $\mathcal{P}$ is defined to be

$$\mathrm{GKdim}(\mathcal{P}) := \overline{\lim} \log_n \left(\sum_{i=0}^{n} \dim \mathcal{P}(i) \right).$$

When we talk about the GK-dimension of an operad $\mathcal{P}$, we usually implicitly assume that $\mathcal{P}$ is locally finite.

Y.-H. Bao, Y. Ye and J.J. Zhang, *Truncation of Unitary Operads*, Advances in Mathematics. **372** (2020): 107290.

Example

Since $dim(As(n)) = 1$ *for all* $n \geq 1$,

$$GKdim(As) = \overline{\lim} \log_n \left(\sum_{i=0}^{n} dim(As(n)) \right) = \overline{\lim} \log_n(n) = 1.$$

Proposition

$\mathrm{GKdim}(\mathcal{P}) = 0$ *if and only if* $\mathcal{P}$ *is finite dimensional.*

Proposition

For any $r \in R_{\mathrm{GKdim}}$*, there exists a finitely generated operad* $\mathcal{P}$ *such that* $\mathrm{GKdim}(\mathcal{P}) = r$*.*

Idea of proof:
As in Example ($\star$), we can construct a finitely generated operad $\mathcal{P} := (0, \mathbb{K}, A_1, A_2, \dots)$ from a monomial algebra A which is naturally graded, such that $\mathrm{GKdim}(\mathcal{P}) = \mathrm{GKdim}(A)$.

Proposition

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4. GK-dimension of NS operads

Remark (Algebra Case)

For a finitely generated $\mathbb{K}$-algebra A with finite dimensional generating space V,

$$\mathrm{GKdim}(A) = \overline{\lim} \log_n d_V(n).$$

Proposition

Suppose $\mathcal{P}$ is a locally finite operad generated by a finite dimensional subcollection $\mathcal{V}$. Let $d_{\mathcal{V}}(n) = dim(\sum_{i=0}^{n} \mathcal{V}^i)$. Then

$$\mathrm{GKdim}(\mathcal{P}) = \overline{\lim} \log_n d_{\mathcal{V}}(n).$$

5. Gap theorem of GKdim of NS operads

Problem

Which real numbers occur as the Gelfand-Kirillov dimension of a nonsymmetric operad?

Theorem (Qi-Xu-Zhang-Zhao)

The range of GK-dimension of nonsymmetric operads is

$$R_{\mathrm{GKdim}} := \{0\} \cup \{1\} \cup [2, \infty) \cup \{\infty\}.$$

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5. Gap theorem of GKdim of NS operads

Proposition

No finitely generated nonsymmetric operad has GK-dimension strictly between 0 *and* 1*.*

Idea of proof:
Suppose $\dim(\mathcal{P}) = \infty$. We claim that $\mathcal{V}^{m+1} \neq \mathcal{V}^m$ for every m. Suppose to the contrary that $\mathcal{V}^{m+1} = \mathcal{V}^m$ for some m. Then by induction, one sees that $\mathcal{V}^n = \mathcal{V}^m$ for every $n > m$. So $\mathcal{P} = \cup_{n>m}\mathcal{V}^n = \mathcal{V}^m$, which is finite dimensional. Therefore $\dim \mathcal{V}^m \geq m+1$ for every m, and consequently,

$$\operatorname{GKdim}(\mathcal{P}) = \overline{\lim} \log_n \left(\sum_{i=0}^{n} \dim \mathcal{V}^i \right) \geq \overline{\lim} \log_n (n+1) = 1.$$

5. Gap theorem of GKdim of NS operads

Theorem (Qi-Xu-Zhang-Zhao 2020, Gap Theorem)

No finitely generated nonsymmetric operad has GK-dimension strictly between 1 *and* 2*.*

5. Gap theorem of GKdim of NS operads

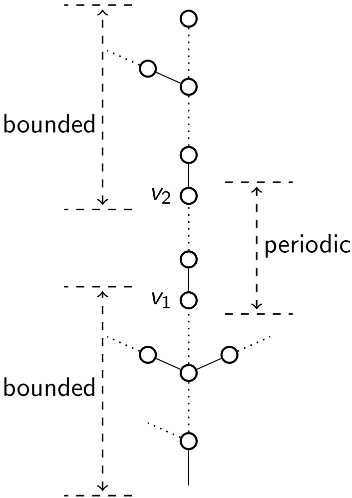
Idea of proof:

- If $\mathrm{GKdim}(\mathcal{P}) < 2$, then there exists a positive integer d such that $\dim \mathcal{V}^i \leq d$ for all i.
- So we have that

$$d_{\mathcal{V}}(n) = dim(\sum_{i=0}^{n} \mathcal{V}^i) \leq dn.$$

Consequently,

$$\mathrm{GKdim}(\mathcal{P}) = \overline{\lim} log_n d_{\mathcal{V}}(n) \leq 1.$$



Definition

An operad is called single-branched if it has a $\mathbb{K}$*-basis that consists of elements of the form*

$$x_1 \circ_{i_1} (x_2 \circ_{i_2} (\cdots (x_{n-2} \circ_{i_{n-2}} (x_{n-1} \circ_{i_{n-1}} x_n)) \cdots)).$$

Definition

An operad is called single-generated if it is generated by a single element.

6. Another construction of NS operads with given GKdim

Theorem (Qi-Xu-Zhang-Zhao)

If $r \in R_{\mathrm{GKdim}}$, then there is a single-generated single-branched locally finite nonsymmetric operad $\mathcal{P}$ such that $\mathrm{GKdim}(\mathcal{P}) = r$.

Idea of proof:
If $r \in R_{\mathrm{GKdim}}$, then there is a finitely generated monomial algebra A such that $\mathrm{GKdim}(A) = r$.
For any finitely generated graded monomial algebra A, construct a single-generated single-branched nonsymmetric operad $\mathcal{P}$ such that

$$\mathrm{GKdim}(\mathcal{P}) = \mathrm{GKdim}(A).$$

Thank you!