Deformations and cohomologies of relative Rota-Baxter Lie algebras

Rong Tang
(Joint work with C. Bai, L. Guo, A. Lazarev and Y. Sheng)

Department of Mathematics, Jilin University, China

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Outline

- Background and motivation
- Deformation and cohomology of LieRep pairs
- Deformation and cohomology of relative Rota-Baxter operators
- Deformation of relative Rota-Baxter Lie algebras
- Cohomology of relative Rota-Baxter Lie algebras

History

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The deformation of algebraic structures began with the seminal work of Gerstenhaber for associative algebras:

M. Gerstenhaber, On the deformation of rings and algebras. *Ann. Math. (2)* **79** (1964), 59-103.

Then it is extended to Lie algebras by Nijenhuis and Richardson. Deformations of other algebraic structures such as pre-Lie algebras, Leibniz algebras, n-Lie algebras have also been well developed. More generally, deformation theory for algebras over quadratic operads was developed by Balavoine.

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Slogan

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There is a well known slogan, often attributed to Deligne, Drinfeld and Kontsevich: *every reasonable deformation theory is controlled by a differential graded Lie algebra (an L_∞-algebra), determined up to quasi-isomorphism*. This slogan has been made into a rigorous theorem by Lurie and Pridham.

J. Lurie, *DAG X: Formal moduli problems*, available at http://www.math.harvard.edu/ lurie/papers/DAG-X.pdf.

J. P. Pridham, Unifying derived deformation theories. *Adv. Math.* **224** (2010), 772-826.

Goal

• What do we want to do

- Idea: we try to extend the above deformation theories to the study of deformations of relative Rota-Baxter Lie algebras.
- Goal: we develop a deformation theory of relative Rota-Baxter Lie algebras which is remarkably consistent with the general principles of deformation theories.
  1. There is a suitable L_∞-algebra whose Maurer-Cartan elements characterize relative Rota-Baxter Lie algebras and their deformations.
  2. There is a cohomology theory which controls the infinitesimal and formal deformations of relative Rota-Baxter Lie algebras.

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The concept of Rota-Baxter operators on associative algebras was introduced by G. Baxter in his study of probability theory. It has found many applications, including Connes-Kreimer's algebraic approach to the renormalization in perturbative quantum field theory. Moreover, Rota-Baxter operators on associative algebras are closely related to symmetric functions and double Poisson algebras.

In the Lie algebra context, a Rota-Baxter operator was introduced independently as the operator form of the classical Yang-Baxter equation that plays important roles in integrable systems and quantum groups.

Rota-Baxter operators lead to the splitting of operads. For further details on Rota-Baxter operators, see Li Guo's book.

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Rota-Baxter type operators

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Definition

A relative Rota-Baxter Lie algebra is a triple $((\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}), \rho, T)$, *where* $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ *is a Lie algebra,* $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ *is a representation of* $\mathfrak{g}$ *on a vector space* V *and* $T : V \longrightarrow \mathfrak{g}$ *is a* relative Rota-Baxter operator, *i.e.*

$$[Tu, Tv]_\mathfrak{g} = T\big(\rho(Tu)(v) - \rho(Tv)(u)\big), \quad \forall u, v \in V.$$

When the representation is the adjoint representation, we obtain Rota-Baxter Lie algebras.

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Triangular Lie bialgebra

Definition-Example

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra. We also use the notation* $[\cdot,\cdot]_{\mathfrak{g}}$ *to denote the graded Lie bracket on the exterior algebra* $\wedge^{\bullet}\mathfrak{g}$*. An element* $r \in \wedge^2\mathfrak{g}$ *is called a skew-symmetric* r*-matrix if* r *satisfies the classical Yang-Baxter equation (CYBE):*

$$[r, r]_{\mathfrak{g}} = 0.$$

A skew-symmetric r*-matrix gives rise to a relative Rota-Baxter operator* $r^{\sharp} : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ *with respect to the coadjoint representation* ad^**, where* $r^{\sharp}$ *is defined by*

$$\langle r^{\sharp}(\xi), \eta\rangle = \langle r, \xi \wedge \eta\rangle.$$

A skew-symmetric r*-matrix will give rise to a triangular Lie bialgebra, which we denote by* $(\mathfrak{g}, r)$*.*

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Nijenhuis-Richardson bracket

Let $\mathfrak{g}$ be a vector space. We consider the graded vector space $C^*(\mathfrak{g},\mathfrak{g}) = \oplus_{n=0}^{+\infty} C^n(\mathfrak{g},\mathfrak{g}) = \oplus_{n=0}^{+\infty} \mathrm{Hom}\,(\wedge^{n+1}\mathfrak{g},\mathfrak{g})$. Then $C^*(\mathfrak{g},\mathfrak{g})$ equipped with the Nijenhuis-Richardson bracket

$$[P,Q]_{\mathsf{NR}} = P\bar{\circ}Q - (-1)^{pq} Q\bar{\circ}P, \quad \forall P \in C^p(\mathfrak{g},\mathfrak{g}), Q \in C^q(\mathfrak{g},\mathfrak{g}),$$

is a graded Lie algebra, where $P\bar{\circ}Q \in C^{p+q}(\mathfrak{g},\mathfrak{g})$ is defined by

$$\begin{aligned} & (P\bar{\circ}Q)(x_1,\cdots,x_{p+q+1}) \\ = \ & \sum_{\sigma} (-1)^{\sigma} P(Q(x_{\sigma(1)},\cdots,x_{\sigma(q+1)}),x_{\sigma(q+2)},\cdots,x_{\sigma(p+q+1)}). \end{aligned}$$

Lemma

For $\mu \in \mathrm{Hom}\,(\wedge^2\mathfrak{g},\mathfrak{g})$ and $\rho \in \mathrm{Hom}\,(\mathfrak{g}\otimes V, V)$. Then μ is a Lie algebra structure on $\mathfrak{g}$ and ρ is a representation of Lie algebra $\mathfrak{g}$ on V if and only if

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Maurer-Cartan elements characterizing LieRep pairs

Definition

A LieRep pair consists of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$ on a vector space V.

Denote by

$$\mathcal{L}_{LieRep} = \oplus_{k=0}^{+\infty}(\mathrm{Hom}\,(\wedge^{k+1}\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{k}\mathfrak{g} \otimes V, V)).$$

Proposition (Arnal)

Let $\mathfrak{g}$ and V be two vector spaces. Then $(\mathcal{L}_{LieRep}, [\cdot,\cdot]_{\mathsf{NR}})$ is a graded Lie algebra. Its MC elements are precisely LieRep pairs.

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Let $((\mathfrak{g}, \mu); \rho)$ be a LieRep pair. Define the set of n-cochains $\mathfrak{C}^n(\mathfrak{g}, \rho)$ to be

$$\mathfrak{C}^n(\mathfrak{g}, \rho) := \mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g} \otimes V, V).$$

Define the coboundary operator $\partial : \mathfrak{C}^n(\mathfrak{g}, \rho) \longrightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, \rho)$ by

$$\partial f := (-1)^{n-1}[\mu + \rho, f]_{\mathsf{NR}}.$$

Then $\partial \circ \partial = 0$. Thus we obtain the complex $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g}, \rho), \partial)$.

Definition

The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g}, \rho), \partial)$ is called the cohomology of the LieRep pair $((\mathfrak{g}, \mu); \rho)$. The resulting n-th cohomology group is denoted by $\mathcal{H}^n(\mathfrak{g}, \rho)$.

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Maurer-Cartan elements characterizing relative Rota-Baxter operators

Let $(V;\rho)$ be a representation of a Lie algebra $\mathfrak{g}$. Consider the graded vector space

$$\mathcal{C}^*(V,\mathfrak{g}) := \oplus_{k=0}^{+\infty}\mathrm{Hom}\,(\wedge^k V,\mathfrak{g}).$$

Define a skew-symmetric bracket operation

$$[[\cdot,\cdot]] : \mathrm{Hom}\,(\wedge^n V,\mathfrak{g}) \times \mathrm{Hom}\,(\wedge^m V,\mathfrak{g}) \longrightarrow \mathrm{Hom}\,(\wedge^{m+n} V,\mathfrak{g})$$

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Proposition

$(\mathcal{C}^*(V,\mathfrak{g}),[[\cdot,\cdot]])$ is a gLa. Its Maurer-Cartan elements are precisely the relative Rota-Baxter operators on $\mathfrak{g}$ with respect to $(V;\rho)$.

Proof. The Nijenhuis-Richardson bracket $[\cdot,\cdot]_{\mathsf{NR}}$ associated to $\mathfrak{g}\oplus V$ gives rise to a graded Lie algebra $(\oplus_{k\geq 0}\mathrm{Hom}\,(\wedge^k(\mathfrak{g}\oplus V),\mathfrak{g}\oplus V),[\cdot,\cdot]_{\mathsf{NR}})$. Obviously $\oplus_{k\geq 0}\mathrm{Hom}\,(\wedge^k V,\mathfrak{g})$ is an abelian subalgebra. A linear map $\mu:\wedge^2\mathfrak{g}\longrightarrow\mathfrak{g}$ is a Lie algebra structure and $\rho:\mathfrak{g}\otimes V\longrightarrow V$ is a representation of $\mathfrak{g}$ on V iff $\mu+\rho$ is a Maurer-Cartan element of the gLa $(\oplus_{k\geq 0}\mathrm{Hom}\,(\wedge^k(\mathfrak{g}\oplus V),\mathfrak{g}\oplus V),[\cdot,\cdot]_{\mathsf{NR}})$, defining a differential $d_{\mu+\rho}$ via $d_{\mu+\rho}=[\mu+\rho,\cdot]_{\mathsf{NR}}$. Further, the differential $d_{\mu+\rho}$ gives rise to a graded Lie algebra structure on the graded vector space $\oplus_{k\geq 0}\mathrm{Hom}\,(\wedge^k V,\mathfrak{g})$ via the derived bracket

$$[[P,Q]]:=(-1)^n[[\mu+\rho,P]_{\mathsf{NR}},Q]_{\mathsf{NR}},\quad \forall P\in\mathrm{Hom}\,(\wedge^n V,\mathfrak{g}),Q\in\mathrm{Hom}\,(\wedge^m V,\mathfrak{g}),$$

which is exactly the above bracket.

For $T:V\longrightarrow\mathfrak{g}$, we have

$$[[T,T]](u_1,u_2)=2\big(T(\rho(Tu_1)u_2)-T(\rho(Tu_2)u_1)-[Tu_1,Tu_2]\big).$$

Thus, Maurer-Cartan elements are relative Rota-Baxter operators.

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Now we define the cohomology governing deformations of a relative Rota-Baxter operator $T : V \longrightarrow \mathfrak{g}$. Define the vector space of n-cochains $\mathfrak{C}^n(T)$ as $\mathfrak{C}^n(T) = \mathrm{Hom}\,(\wedge^{n-1}V, \mathfrak{g})$.

Define the coboundary operator $\delta : \mathfrak{C}^n(T) \longrightarrow \mathfrak{C}^{n+1}(T)$ by

$$\delta\theta = (-1)^{n-2}\,[[T, \theta]] = (-1)^{n-2}[[\mu + \rho, T]_{\mathsf{NR}}, \theta]_{\mathsf{NR}}. \qquad (1)$$

Then $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(T), \delta)$ is a cochain complex.

Definition

The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(T), \delta)$ is called the cohomology of the relative Rota-Baxter operator $T : V \longrightarrow \mathfrak{g}$. The corresponding n-th cohomology group is denoted by $\mathcal{H}^n(T)$.

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The relation with pre-Lie algebras

Definition

A pre-Lie algebra is a pair $(V, \cdot_V)$, where V is a vector space and $\cdot_V : V \otimes V \longrightarrow V$ is a bilinear multiplication satisfying that for all $x, y, z \in V$, the associator

$$(x, y, z) := (x \cdot_V y) \cdot_V z - x \cdot_V (y \cdot_V z)$$

is symmetric in x, y, that is, $(x, y, z) = (y, x, z)$, or equivalently,

$$(x \cdot_V y) \cdot_V z - x \cdot_V (y \cdot_V z) = (y \cdot_V x) \cdot_V z - y \cdot_V (x \cdot_V z).$$

Theorem

Let $T : V \to \mathfrak{g}$ be a relative Rota-Baxter operator on a Lie algebra $\mathfrak{g}$ with respect to a representation $(V; \rho)$. Then $(V, \cdot_T)$ is a pre-Lie algebra, where

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Let V be a vector space. For $\alpha \in \mathrm{Hom}(\wedge^n V \otimes V, V)$ and $\beta \in \mathrm{Hom}(\wedge^m V \otimes V, V)$, define $\alpha \circ \beta \in \mathrm{Hom}(\wedge^{n+m} V \otimes V, V)$ by

$$
\begin{aligned}
& (\alpha \circ \beta)(u_1, \cdots, u_{m+n+1}) \\
= & \sum_{\sigma \in \mathbb{S}_{(m,1,n-1)}} (-1)^{\sigma} \alpha(\beta(u_{\sigma(1)}, \cdots, u_{\sigma(m+1)}), u_{\sigma(m+2)}, \cdots, u_{\sigma(m+n)}, u_{m+n+1}) \\
& + (-1)^{mn} \sum_{\sigma \in \mathbb{S}_{(n,m)}} (-1)^{\sigma} \alpha(u_{\sigma(1)}, \cdots, u_{\sigma(n)}, \beta(u_{\sigma(n+1)}, \cdots, u_{\sigma(m+n)}, u_{m+n+1}))
\end{aligned}
$$

Then $C^*(V, V) := \oplus_{k \geq 0} \mathrm{Hom}(\wedge^k V \otimes V, V)$ equipped with the Matsushima-Nijenhuis bracket $[\cdot, \cdot]^C$ given by

$$
[\alpha, \beta]^C := \alpha \circ \beta - (-1)^{mn} \beta \circ \alpha,
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is a graded Lie algebra.

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Let V be a vector space. For $\alpha \in \mathrm{Hom}(\wedge^n V \otimes V, V)$ and $\beta \in \mathrm{Hom}(\wedge^m V \otimes V, V)$, define $\alpha \circ \beta \in \mathrm{Hom}(\wedge^{n+m} V \otimes V, V)$ by

$$\begin{aligned}
&(\alpha \circ \beta)(u_1, \cdots, u_{m+n+1}) \\
= &\sum_{\sigma \in \mathbb{S}_{(m,1,n-1)}} (-1)^\sigma \alpha(\beta(u_{\sigma(1)}, \cdots, u_{\sigma(m+1)}), u_{\sigma(m+2)}, \cdots, u_{\sigma(m+n)}, u_{m+n+1}) \\
&+(-1)^{mn} \sum_{\sigma \in \mathbb{S}_{(n,m)}} (-1)^\sigma \alpha(u_{\sigma(1)}, \cdots, u_{\sigma(n)}, \beta(u_{\sigma(n+1)}, \cdots, u_{\sigma(m+n)}, u_{m+n+1}))
\end{aligned}$$

Then $C^*(V,V) := \oplus_{k \geq 0} \mathrm{Hom}(\wedge^k V \otimes V, V)$ equipped with the Matsushima-Nijenhuis bracket $[\cdot,\cdot]^C$ given by

$$[\alpha, \beta]^C := \alpha \circ \beta - (-1)^{mn} \beta \circ \alpha,$$

is a graded Lie algebra.

The relation with pre-Lie algebras

Remark

For $\alpha \in \mathrm{Hom}\,(V \otimes V, V)$*, we have*

$$[\alpha,\alpha]^C(u,v,w) \\ = 2\big(\alpha(\alpha(u,v),w) - \alpha(\alpha(v,u),w) - \alpha(u,\alpha(v,w)) + \alpha(v,\alpha(u,w))\big).$$

Thus, α *defines a pre-Lie algebra structure on* V *if and only if* $[\alpha,\alpha]^C = 0$*, that is,* α *is a Maurer-Cartan element of the graded Lie algebra* $(C^*(V,V),[\cdot,\cdot]^C)$*.*

The relation with pre-Lie algebras

Define a linear map
$\Phi : \mathrm{Hom}\,(\wedge^k V, \mathfrak{g}) \longrightarrow \mathrm{Hom}\,(\wedge^k V \otimes V, V), k \geq 0$, by

$$\Phi(f)(u_1, \cdots, u_k, u_{k+1}) = \rho(f(u_1, \cdots, u_k))(u_{k+1}).$$

Proposition

Let $(V;\rho)$ be a representation of a Lie algebra $\mathfrak{g}$. Then Φ is a homomorphism of graded Lie algebras from $(C^*(V,\mathfrak{g}), [[\cdot,\cdot]])$ to $(C^*(V,V), [\cdot,\cdot]^C)$.

R. Tang, C. Bai, L. Guo and Y. Sheng, Deformations and their controlling cohomologies of $\mathcal{O}$-operators, *Comm. Math. Phys.* 368 (2019), 665 – 700.

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L_∞-algebra

Definition

An L_∞-algebra *is a* $\mathbb{Z}$*-graded vector space* $\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k$ *equipped with a collection* $(k \geq 1)$ *of linear maps* $l_k : \otimes^k\mathfrak{g} \longrightarrow \mathfrak{g}$ *of degree* 1 *with the property that, for any homogeneous elements* $x_1, \cdots, x_n \in \mathfrak{g}$*, we have*

(i) (graded symmetry) *for every* $\sigma \in \mathbb{S}_n$,

$$l_n(x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma) l_n(x_1, \cdots, x_{n-1}, x_n),$$

(ii) (generalized Jacobi identity) *for all* $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma\in\mathbb{S}_{(i,n-i)}} \varepsilon(\sigma) l_{n-i+1}(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0.$$

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Maurer-Cartan elements

Definition

The set of MC elements, denoted by $\mathrm{MC}(\mathfrak{g})$*, of a filtered* L_∞*-algebra* $\mathfrak{g}$ *is the set of those* $\alpha \in \mathfrak{g}^0$ *satisfying the MC equation*

$$\sum_{k=1}^{+\infty} \frac{1}{k!} l_k(\alpha, \cdots, \alpha) = 0.$$

Remark

The condition of being filtered ensures convergence of the series figuring in the definition of MC elements and MC twistings.

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V-structure

Definition (Voronov)

A *V-structure* *consists of a quadruple* $(L, \mathfrak{h}, P, \Delta)$ *where*

- $(L, [\cdot, \cdot])$ *is a graded Lie algebra,*
- $\mathfrak{h}$ *is an abelian graded Lie subalgebra of* $(L, [\cdot, \cdot])$*,*
- $P : L \longrightarrow L$ *is a projection, that is* $P \circ P = P$*, whose image is* $\mathfrak{h}$ *and kernel is a graded Lie subalgebra of* $(L, [\cdot, \cdot])$*,*
- Δ *is an element in* $\ker(P)^1$ *such that* $[\Delta, \Delta] = 0$*.*

Th. Voronov, Higher derived brackets and homotopy algebras. *J. Pure Appl. Algebra* **202** (2005), 133-153.

Higher derived brackets

Theorem (Voronov)

Let $(L, \mathfrak{h}, P, \Delta)$ *be a* V*-structure. Then the graded vector space* $L[1] \oplus \mathfrak{h}$ *is an* L_∞*-algebra, where* l_k *are given by*

$$\begin{aligned} l_1(x,a) &= (-[\Delta,x], P(x+[\Delta,a])), \\ l_2(x, s^{-1}y) &= (-1)^x[x,y], \\ l_k(x,a_1,\cdots,a_{k-1}) &= P[\cdots[[x,a_1],a_2]\cdots,a_{k-1}], \quad k \geq 2, \\ l_k(a_1,\cdots,a_{k-1},a_k) &= P[\cdots[[\Delta,a_1],a_2]\cdots,a_k], \quad k \geq 2. \end{aligned}$$

Here $a, a_1, \cdots, a_k \in \mathfrak{h}$ *and* $x, y \in L$.

Voronov's higher derived brackets, which is a useful tool to construct explicit L_∞-algebras.

Remark

Let L' *be a graded Lie subalgebra of* L *that satisfies* $[\Delta, L'] \subset L'$*. Then* $L'[1] \oplus \mathfrak{h}$ *is an* L_∞*-subalgebra of the above* L_∞*-algebra.*

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V-structure

Let $\mathfrak{g}$ and V be two vector spaces.

Proposition

We have a V-structure $(L, \mathfrak{h}, P, \Delta)$ as follows:

- *the graded Lie algebra $(L, [\cdot,\cdot])$ is given by $\big(\oplus_{n=0}^{+\infty} C^n(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V), [\cdot,\cdot]_{\mathsf{NR}}\big)$;*
- *the abelian graded Lie subalgebra $\mathfrak{h}$ is given by*

$$\mathfrak{h} := \oplus_{n=0}^{+\infty} \mathrm{Hom}\,(\wedge^{n+1} V, \mathfrak{g});$$

- *$P : L \longrightarrow L$ is the projection onto the subspace $\mathfrak{h}$;*
- *$\Delta = 0$.*

Consequently, we obtain an L_∞-algebra $(L[1] \oplus \mathfrak{h}, \{l_k\}_{k=1}^{+\infty})$, where l_k are given by

$$\begin{aligned} l_1(s^{-1}Q, \theta) &= P(Q), \\ l_2(s^{-1}Q, s^{-1}Q') &= (-1)^Q s^{-1}[Q, Q']_{\mathsf{NR}}, \\ l_k(s^{-1}Q, \theta_1, \cdots, \theta_{k-1}) &= P[\cdots[Q, \theta_1]_{\mathsf{NR}}, \cdots, \theta_{k-1}]_{\mathsf{NR}}, \end{aligned}$$

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for $\theta, \theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$, $Q, Q' \in L$.

Recall that $\mathcal{L}_{LieRep} = \oplus_{k=0}^{+\infty}(\mathrm{Hom}\,(\wedge^{k+1}\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{k}\mathfrak{g} \otimes V, V))$ is a subalgebra of L.

Corollary

With the above notation, $(\mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$ is an L_∞-algebra, where l_k are given by

$$\begin{aligned} l_2(Q, Q') &= (-1)^Q [Q, Q']_{\mathrm{NR}}, \\ l_k(Q, \theta_1, \cdots, \theta_{k-1}) &= P[\cdots[Q, \theta_1]_{\mathrm{NR}}, \cdots, \theta_{k-1}]_{\mathrm{NR}}, \end{aligned}$$

for $\theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$, $Q, Q' \in \mathcal{L}_{LieRep}$.

The L_∞-algebra governing relative RB Lie algebra

Recall that $\mathcal{L}_{LieRep} = \oplus_{k=0}^{+\infty}(\mathrm{Hom}\,(\wedge^{k+1}\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{k}\mathfrak{g} \otimes V, V))$ is a subalgebra of L.

Corollary

With the above notation, $(\mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$ *is an* L_∞*-algebra, where* l_k *are given by*

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\begin{aligned}
l_2(Q, Q') &= (-1)^{Q}[Q, Q']_{\mathsf{NR}}, \\
l_k(Q, \theta_1, \cdots, \theta_{k-1}) &= P[\cdots[Q, \theta_1]_{\mathsf{NR}}, \cdots, \theta_{k-1}]_{\mathsf{NR}},
\end{aligned}
$$

for $\theta_1, \cdots, \theta_{k-1} \in \mathfrak{h}$*,* $Q, Q' \in \mathcal{L}_{LieRep}$*.*

Theorem

Let $\mathfrak{g}$ *and* V *be two vector spaces,* $\mu \in \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$, $\rho \in \mathrm{Hom}(\mathfrak{g} \otimes V, V)$ *and* $T \in \mathrm{Hom}(V, \mathfrak{g})$. *Then* $((\mathfrak{g}, \mu), \rho, T)$ *is a relative Rota-Baxter Lie algebra if and only if* $(\mu + \rho, T)$ *is an MC element of the* L_∞*-algebra* $(\mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$.

Define $l_k^{(\mu+\rho,T)} : \otimes^k(\mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}) \longrightarrow \mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}$ by

$$l_k^{(\mu+\rho,T)}(x_1,\cdots,x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{(\mu+\rho,T),\cdots,(\mu+\rho,T)}_{n}, x_1,\cdots,x_k).$$

Theorem

With the above notation, we have the twisted L_∞-algebra

$$\left(\mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}, \{l_k^{(\mu+\rho,T)}\}_{k=1}^{+\infty}\right).$$

Moreover, for linear maps $T' \in \mathrm{Hom}(V,\mathfrak{g})$, $\mu' \in \mathrm{Hom}(\wedge^2\mathfrak{g},\mathfrak{g})$ and $\rho' \in \mathrm{Hom}(\mathfrak{g},\mathfrak{gl}(V))$, the triple $((\mathfrak{g},\mu+\mu'),\rho+\rho',T+T')$ is again a relative Rota-Baxter Lie algebra if and only if $\left((\mu'+\rho'),T'\right)$ is an MC element of the above twisted L_∞-algebra.

Deformations of relative RB Lie algebra

Define $l_k^{(\mu+\rho,T)} : \otimes^k(\mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}) \longrightarrow \mathcal{L}_{LieRep}[1] \oplus \mathfrak{h}$ by

$$l_k^{(\mu+\rho,T)}(x_1,\cdots,x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{(\mu+\rho,T),\cdots,(\mu+\rho,T)}_{n}, x_1,\cdots,x_k).$$

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Theorem

Let $((\mathfrak{g},\mu),\rho,T)$ be a relative Rota-Baxter Lie algebra. Then the L_∞-algebra $(\mathcal{L}_{LieRep}[1]\oplus\mathfrak{h},\{l_k^{(\mu+\rho,T)}\}_{k=1}^{+\infty})$ is a strict extension of the L_∞-algebra (dgla) $\mathcal{L}_{LieRep}[1]$ by the L_∞-algebra (dgla) $\oplus_{k=1}^{+\infty}\mathrm{Hom}\,(\wedge^k V,\mathfrak{g})$, that is, we have the following short exact sequence of L_∞-algebras:

$$0\longrightarrow \oplus_{k=1}^{+\infty}\mathrm{Hom}\,(\wedge^k V,\mathfrak{g})\stackrel{\iota}{\longrightarrow}\mathcal{L}_{LieRep}[1]\oplus\mathfrak{h}\stackrel{p}{\longrightarrow}\mathcal{L}_{LieRep}[1]\longrightarrow 0,$$

where $\iota(\theta)=(0,\theta)$ and $p(f,\theta)=f$.

Cohomology of relative Rota-Baxter Lie algebras

Define the space of n-cochains $\mathfrak{C}^n(\mathfrak{g},\rho,T)$ by

$$\begin{aligned}\mathfrak{C}^n(\mathfrak{g},\rho,T) &:= \mathfrak{C}^n(\mathfrak{g},\rho)\oplus\mathfrak{C}^n(T)\\ &= \Big(\mathrm{Hom}\,(\wedge^n\mathfrak{g},\mathfrak{g})\oplus\mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}\otimes V,V)\Big)\oplus\mathrm{Hom}\,(\wedge^{n-1}V,\mathfrak{g}).\end{aligned}$$

Define the coboundary operator $\mathcal{D}:\mathfrak{C}^n(\mathfrak{g},\rho,T)\longrightarrow\mathfrak{C}^{n+1}(\mathfrak{g},\rho,T)$ by

$$\mathcal{D}(f,\theta)=(-1)^{n-2}\Big(-[\pi,f]_{\mathsf{NR}},[[\pi,T]_{\mathsf{NR}},\theta]_{\mathsf{NR}}+\frac{1}{n!}\underbrace{[\cdots[[}_{n}f,T]_{\mathsf{NR}},T]_{\mathsf{NR}},\cdots,T]_{\mathsf{NR}}\Big),$$

where $f\in\mathrm{Hom}\,(\wedge^n\mathfrak{g},\mathfrak{g})\oplus\mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}\otimes V,V)$ and $\theta\in\mathrm{Hom}\,(\wedge^{n-1}V,\mathfrak{g})$.

Define a linear operator $h_T:\mathfrak{C}^n(\mathfrak{g},\rho)\longrightarrow\mathfrak{C}^{n+1}(T)$ by

$$h_Tf:=(-1)^{n-2}\frac{1}{n!}\underbrace{[\cdots[[}_{n}f,T]_{\mathsf{NR}},T]_{\mathsf{NR}},\cdots,T]_{\mathsf{NR}}.$$

Then the coboundary operator can be written as

$$\mathcal{D}(f,\theta)=(\partial f,\delta\theta+h_Tf).$$

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Theorem

With the above notation, $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g},\rho,T),\mathcal{D})$ *is a cochain complex, i.e.* $\mathcal{D}\circ\mathcal{D}=0$.

Proof.

It follow from

$$\mathcal{D}(f,\theta)=(-1)^{n-2}l_1^{(\mu+\rho,T)}(f,\theta).$$

□

Definition

The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g},\rho,T),\mathcal{D})$ *is called the* ***cohomology of the relative Rota-Baxter Lie algebra*** $((\mathfrak{g},\mu),\rho,T)$*. Denote its* n*-th cohomology group by* $\mathcal{H}^n(\mathfrak{g},\rho,T)$*.*

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Now we give the formulas for h_T in terms of multilinear maps.

Lemma

The operator $h_T : \mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g} \otimes V, V) \longrightarrow \mathrm{Hom}\,(\wedge^n V, \mathfrak{g})$ *is given by*

$$\begin{aligned}(h_T f)(v_1, \cdots, v_n) &= (-1)^n f_{\mathfrak{g}}(Tv_1, \cdots, Tv_n) \\ &\quad + \sum_{i=1}^{n} (-1)^{i+1} T f_V\big(Tv_1, \cdots, Tv_{i-1}, Tv_{i+1}, \cdots, Tv_n, v_i\big),\end{aligned}$$

where $f = (f_{\mathfrak{g}}, f_V)$, *and* $f_{\mathfrak{g}} \in \mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g})$, $f_V \in \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g} \otimes V, V)$ *and* $v_1, \cdots, v_n \in V$.

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Relations among cohomologies

$\mathcal{D}$ can be well-explained by the following diagram:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathfrak{C}^n(\mathfrak{g},\rho) & \xrightarrow{\partial} & \mathfrak{C}^{n+1}(\mathfrak{g},\rho) & \xrightarrow{\partial} & \mathfrak{C}^{n+2}(\mathfrak{g},\rho) & \longrightarrow \cdots \\ & & \searrow^{h_T} & & \searrow^{h_T} & & \\ \cdots \longrightarrow & \mathfrak{C}^n(T) & \xrightarrow{\delta} & \mathfrak{C}^{n+1}(T) & \xrightarrow{\delta} & \mathfrak{C}^{n+2}(T) & \longrightarrow \cdots . \end{array}$$

Theorem

There is a short exact sequence of the cochain complexes:

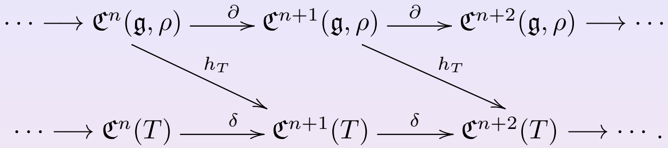
$$0 \longrightarrow (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(T),\delta) \xrightarrow{\iota} (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g},\rho,T),\mathcal{D}) \xrightarrow{p} (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g},\rho),\partial) \longrightarrow 0,$$

and there is a long exact sequence of the cohomology groups:

$$\cdots \longrightarrow \mathcal{H}^n(T) \xrightarrow{\mathcal{H}^n(\iota)} \mathcal{H}^n(\mathfrak{g},\rho,T) \xrightarrow{\mathcal{H}^n(p)} \mathcal{H}^n(\mathfrak{g},\rho) \xrightarrow{c^n} \mathcal{H}^{n+1}(T) \longrightarrow \cdots ,$$

where the connecting map c^n is defined by $c^n([\alpha]) = [h_T\alpha]$, for all $[\alpha] \in \mathcal{H}^n(\mathfrak{g},\rho)$.

$\mathcal{D}$ can be well-explained by the following diagram:



Theorem

There is a short exact sequence of the cochain complexes:

$$0 \longrightarrow (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(T),\delta) \stackrel{\iota}{\longrightarrow} (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g},\rho,T),\mathcal{D}) \stackrel{p}{\longrightarrow} (\oplus_{n=0}^{+\infty}\mathfrak{C}^n(\mathfrak{g},\rho),\partial) \longrightarrow 0,$$

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Cohomology of RB Lie algebras

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, T)$ be a Rota-Baxter Lie algebra. Define the space of n-cochains $\mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T)$ by

$$\mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T) := \mathfrak{C}^n_{\mathrm{Lie}}(\mathfrak{g};\mathfrak{g}) \oplus \mathfrak{C}^n(T) = \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g}) \oplus \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}).$$

Define $\mathcal{D}_{\mathrm{RB}} : \mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T) \longrightarrow \mathfrak{C}^{n+1}_{\mathrm{RB}}(\mathfrak{g}, T)$ by

$$\mathcal{D}_{\mathrm{RB}}(f,\theta) = \Big(\mathrm{d}_{\mathsf{CE}} f, \delta\theta + \Omega f\Big), \quad \forall f \in \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g}),\ \theta \in \mathrm{Hom}\,(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}),$$

where $\Omega : \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g}) \longrightarrow \mathrm{Hom}\,(\wedge^n\mathfrak{g}, \mathfrak{g})$ is defined by

$$\begin{aligned}(\Omega f)(x_1, \cdots, x_n) &= (-1)^n\Big(f(Tx_1, \cdots, Tx_n) \\ &\quad - \sum_{i=1}^{n} Tf(Tx_1, \cdots, Tx_{i-1}, x_i, Tx_{i+1}, \cdots, Tx_n)\Big).\end{aligned}$$

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Theorem

The map $\mathcal{D}_{\mathrm{RB}}$ is a coboundary operator, i.e. $\mathcal{D}_{\mathrm{RB}} \circ \mathcal{D}_{\mathrm{RB}} = 0$.

Definition

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, T)$ be a Rota-Baxter Lie algebra. The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n_{\mathrm{RB}}(\mathfrak{g}, T), \mathcal{D}_{\mathrm{RB}})$ is taken to be the cohomology of the Rota-Baxter Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, T)$*. Denote the n-th cohomology group by $\mathcal{H}^n_{\mathrm{RB}}(\mathfrak{g}, T)$.*

Cohomology of RB Lie algebras

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Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, r)$ be a triangular Lie bialgebra. Define the space of n-cochains $\mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g}, r)$ by

$$\mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g}, r) := \mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g}) \oplus \wedge^n \mathfrak{g}.$$

Define the *coboundary operator* $\mathcal{D}_{\mathrm{TLB}} : \mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g}, r) \longrightarrow \mathfrak{C}^{n+1}_{\mathrm{TLB}}(\mathfrak{g}, r)$ by

$$\mathcal{D}_{\mathrm{TLB}}(f, \chi) = \Big(\mathrm{d}_{\mathsf{CE}} f, \Theta f + \mathrm{d}_r \chi\Big), \quad \forall f \in \mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g}),\ \chi \in \wedge^n \mathfrak{g},$$

where $\mathrm{d}_r : \wedge^n \mathfrak{g} \longrightarrow \wedge^{n+1} \mathfrak{g}$ is given by $\mathrm{d}_r \chi = [r, \chi]_{\mathfrak{g}}$ and $\Theta : \mathrm{Hom}\,(\wedge^n \mathfrak{g}, \mathfrak{g}) \longrightarrow \wedge^{n+1} \mathfrak{g}$ is defined by

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Theorem

The map $\mathcal{D}_{\mathrm{TLB}}$ is a coboundary operator, i.e. $\mathcal{D}_{\mathrm{TLB}} \circ \mathcal{D}_{\mathrm{TLB}} = 0$.

Definition

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, r)$ be a triangular Lie bialgebra. The cohomology of the cochain complex $(\oplus_{n=0}^{+\infty} \mathfrak{C}^n_{\mathrm{TLB}}(\mathfrak{g}, r), \mathcal{D}_{\mathrm{TLB}})$ is called the cohomology of the triangular Lie bialgebra *$(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, r)$. Denote the n-th cohomology group by $\mathcal{H}^n_{\mathrm{TLB}}(\mathfrak{g}, r)$.*

A. Lazarev, Y. Sheng and R. Tang, *Deformations and homotopy theory of relative Rota-Baxter Lie algebras*. The MPIM preprint series, 2020.

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Working in progress

- Explain the cohomology of Rota-Baxter Lie algebras by Ext-functor over the enveloping algebras of Rota-Baxter Lie algebras;
- Construct cofibrant resolution of the operad of Rota-Baxter associative algebras.

Thanks for your attention!