Homotopy embedding tensors

Maosong Xiang

Huazhong University of Science and Technology

Sep 19, 2020

Joint work with Z.Chen and H. Ge
The 3rd conference on operad theory and related topics, Jilin University.

Embedding tensors

The embedding tensor formalism was introduced by Nicolai and Samtleben in the gauging procedure of 3D supergravity theories. An embedding tensor is a linear map $f : V \to \mathfrak{g}$, where

1. V is the space of fields;
2. $\mathfrak{g}$ is the Lie algebra of the rigid symmetry group G;

satisfying

$$f(\xi \triangleright x) = [\xi, f(x)]_{\mathfrak{g}}, \quad \forall \xi \in \mathfrak{h} = \mathrm{Im}(f), x \in V.$$

f is said to be strict, if this relation holds for all $\xi \in \mathfrak{g}$. In other words, f is an averaging operator of $\mathfrak{g}$. Any (strict) embedding tensor induces a Leibniz algebra structure $\circ$ on V

$$x \circ y := f(x) \triangleright y, \quad \forall x, y \in V.$$

Goal of this talk

Question

1. *What is the "correct" notion of homotopy embedding tensor?*
2. *What algebraic structures can we obtain from homotopy embedding tensors?*

For the first question, we first show that an embedding tensor is an algebra over a 2-colored operad. Then we consider its Boardman-Vogt resolution due to Berger and Moerdijk. We define a homotopy embedding tensors as an algebra over the cofibrant 2-colored operad obtained from BV resolution.
For the second question, we will show that homotopy Leibniz algebras arise from homotopy embedding tensors, whose structure maps can be realized as a sum over rooted forests.

Definition of a colored dg operad

A colored (dg) operad (a.k.a. multicategory) $\mathscr{C}$ consists of the following data:

- A finite set $\{X, Y, Z, \cdots\}$ of objects or colors of $\mathscr{C}$.
- For every finite set I, every I-indexed collection of colors $\{X_i\}_{i \in I}$, and every color Y, a cochain complex $\mathscr{C}(\{X_i\}_{i \in I}, Y)$.
- For each color $X \in \mathscr{C}$ a unit element $\mathrm{id}_X \in \mathscr{C}(X, X)$.
- For every map of finite sets $I \to J$ with fibers $\{I_j\}_{j \in J}$, every finite collections of colors $X_I = \{X_i\}_{i \in I}$ and $Y_J = \{Y_j\}_{j \in J}$, and every color Z, a composition map

$$\mu^Z_{Y_J} : \mathscr{C}(\{Y_j\}_{j \in J}, Z) \otimes \bigotimes_{j \in J} \mathscr{C}(\{X_i\}_{i \in I_j}, Y_j) \to \mathscr{C}(\{X_i\}_{i \in I}, Z).$$

Definition of a colored dg operad: continued

These data are supposed to satisfy the following conditions:

1. Each cochain complex $\mathscr{C}(\{X_i\}_{i\in I}, Y)$ admits a (right) representation of the permutation group S_n, where $n = |I|$ is the cardinal number of the finite set I, defined by

$$\sigma \in S_n \to \left(\mathscr{C}(\{X_i\}_{i\in I}, Y) \mapsto \mathscr{C}(\{X_{\sigma(i)}\}_{i\in I}, Y)\right).$$

2. (Equivariance) The composition map is equivariant under the symmetry group action.
3. (Unit axiom) For all colors $X, X_1, \cdots, X_n \in \mathscr{C}$ and each $f \in \mathscr{C}(\{X_i\}_{i=1}^n, X)$, we have

$$\mu_X^X(\mathrm{id}_X, f) = \mu_{X_1,\cdots,X_n}^X(f, \mathrm{id}_{X_1}, \cdots, \mathrm{id}_{X_n}) = f.$$

4. (Associativity) The composition is associative in the natural way (cf. Definition 2.1.1.1 in *Higher algebras* by Lurie).

The module operad introduced by Kapranov-Manin

Let $\mathscr{P}$ be a dg operad. A $\mathscr{P}$-module operad (or $\mathscr{P}$-moperad) $\mathscr{P}_1$ consists of

- a collection of right dg S_k-modules $\mathscr{P}_1(k), k \geq 0$;
- a unit element $\mathrm{id}_1 \in \mathscr{P}_1(0)$;
- composition morphisms

$$\mu_{1,k}\colon \mathscr{P}_1(k)\otimes\mathscr{P}_1(m_0)\otimes\mathscr{P}(m_1)\otimes\cdots\otimes\mathscr{P}(m_k) \to \mathscr{P}_1\left(\sum_{i=0}^{k} m_i\right).$$

satisfying

1. Equivariance: compositions are S-equivariant.
2. Unit axiom: For all $k \geq 0$ and all $a \in \mathscr{P}_1(k)$,

$$\mu_{1,0}(\mathrm{id}_1, a) = \mu_{1,k}(a, \mathrm{id}_1, \mathrm{id}, \cdots, \mathrm{id}) = a,$$

   where $\mathrm{id} \in \mathscr{P}(1)$ is the unit element for the dg operad $\mathscr{P}$.
3. Associativity.(cf. Definition 9 in *the homotopy braces formality morphism* by Willwacher.)

The 2-colored dg operad arising from a dg operad and its moperad

Let $\mathscr{P}$ be a dg operad and $\mathscr{P}_1$ its moperad. We have a 2-colored dg operad $\mathscr{C}(\mathscr{P}, \mathscr{P}_1)$ with the set of colors $\{1, 2\}$ as follows:

1. $\mathscr{P}(k)$ is the space of operations with k-inputs and the output of color 1;
2. $\mathscr{P}_1(k)$ is the space of operations with the first input and the output of color 2, and the last k-inputs of color 1.

Example

$\mathscr{P} = \mathrm{Lie}$, $\mathscr{P}_1 = \mathrm{LieMod}$. *Then the corresponding 2-colored dg operad* $\mathscr{C}(\mathrm{Lie}, \mathrm{LieMod})$ *governs dg Lie algebras and their representations.*

Definition of operadic embedding

$\mathscr{P}$: a dg operad. $\mathscr{P}_1$: a $\mathscr{P}$-moperad. An operadic $\mathscr{P}_1$-$\mathscr{P}$ embedding $\mathscr{E}$ is a morphism of $\mathscr{P}$-moperad from $\mathscr{P}_1$ to $\mathscr{P}$:

- a collection of right dg S_k-modules $\mathscr{E}(k)$ for $k = 0, 1, \cdots$, thought of as a space of operations with one input in color 2, k-input and the output in color 1;
- composition morphisms

$$e^{(l)}_{1,k}\colon \mathscr{P}(k+1) \otimes \mathscr{E}(m_0) \otimes \mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_k) \to \mathscr{E}(m_0 + \cdots + m_k),$$

$$e^{(r)}_{1,k,l}\colon \mathscr{E}(k) \otimes \mathscr{P}_1(l) \otimes \mathscr{P}(n_1) \otimes \cdots \mathscr{P}(n_l) \otimes \mathscr{P}(m_1) \otimes \cdots \mathscr{P}(m_k) \to \mathscr{E}(n_1 + \cdots + n_l + m_1 + \cdots + m_k),$$

satisfying axioms equivariance, unit axiom, and associativity.

Proposition (Chen-Ge-Xiang)

The triple $(\mathscr{P}, \mathscr{P}_1, \mathscr{E})$ determines a new 2-colored dg operad $\mathscr{C}(\mathscr{P}, \mathscr{P}_1, \mathscr{E})$.

Example

Consider the 2-colored dg operad $(\mathrm{Lie}, \mathrm{LieMod})$. We define an operadic $\mathrm{LieMod} - \mathrm{Lie}$ embedding $\mathscr{E}(k)$ as follows:

1. $\mathscr{E}(0)$ is the 1-dimensional space generated by the operation $2 \to 1$;
2. $\mathscr{E}(k), k \geq 1$ is obtained by grafting $\mathscr{E}(0)$ on the base elements in $\mathrm{Lie}(k)$. In other words, they are generated by Lie and $\mathscr{E}(0)$ via the composition $e^{(l)}_{1,k}$, i.e.,

$$\mathscr{E}(k) = e^{(l)}_{1,k}(\mathrm{Lie}(k+1) \otimes \mathscr{E}(0) \otimes \mathrm{Lie}(1) \otimes \cdots \otimes \mathrm{Lie}(1)).$$

Proposition (Chen-Ge-Xiang)

The algebra over the 2-colored dg operad $\mathscr{C}(\mathrm{Lie}, \mathrm{LieMod}, \mathscr{E})$ is a strict embedding tensor of dg Lie algebras.

Partial history on resolutions of operads

- Boardman and Vogt introduced an explicit resolution, called the W-construction, for topological operads;
- Ginzburg-Kapranov (cf. also Getzler-Jones, Kontsevich-Soibelman, and Dolgushev-Rogers) introduced the cobar-bar resolution for operads in (co)chain complexes (or dg operads);
- Berger and Moerdijk generalized the W-construction of Boardman-Vogt, also called the Boardman-Vogt (BV for short) resolution, for colored operads in monoidal model categories, which is isomorphic to the cobar-bar resolution when the monoidal model category is the one of (co)chain complexes.

An explicit description of the free colored operad

Let $\mathscr{C}$ be a colored operad. Intuitively speaking, elements in the free colored operad $F(\mathscr{C})$ are represented by rooted trees with

- inputs labelled by $1, \cdots, n$;
- edges labelled colors of $\mathscr{C}$;
- vertices labelled by an element in $\mathscr{C}(\{c_i\}_{i=1}^n, c)$ if its incoming edges are labelled by the colors $\{c_i\}_{i=1}^n$ and its outgoing edge is labelled by color c.

Furthermore, some identifications arising from tree-automorphisms are made. And compositions are given by grafting of trees.

An interval object in the category of cochain complexes

A *interval* in a cofibrantly generated monoidal model category E with cofibrant unit I is a factorization of the codiagonal

$$I \sqcup I \xrightarrow{(0,1)} H \xrightarrow{\epsilon} I,$$

where $(0,1)$ is a cofibration and the counit ϵ is a weak equivalence, equipped with an associative operation $\vee \colon H \otimes H \to H$, satisfying 0 is neutral and 1 is absorbing, i.e., $0 \vee x = x \vee 0 = x$ and $1 \vee x = x \vee 1 = 1$.

The projective monoidal model category of cochain complexes of vector spaces has an interval object $H = N^*(\Delta^1)$, where $N^0(\Delta^1) = \mathrm{span}\{\gamma_0, \gamma_1\}$, and $N^{-1}(\Delta^1) = \mathrm{span}\{\gamma\}$, satisfying

$$d\gamma = \gamma_1 - \gamma_0.$$

The binary relation $\vee \colon N^*(\Delta^1) \otimes N^*(\Delta^1) \to N^*(\Delta^1)$ is determined by requiring that γ_0 is neutral and γ_1 is absorbing.

The BV resolution $W(H, \mathscr{C})$ of $\mathscr{C}$

Elements in $W(H, \mathscr{C})$ are represented by the rooted planar trees as in $F(\mathscr{C})$, with an additional assignment of elements in H for each internal edge.
(1) edges of length γ_0 are contracted via the operation in $\mathscr{C}$;
(2) edges around a vertex labelled by a unit in $\mathscr{C}(c; c)$ are contracted into a single edge, deleting the vertex and assigning the operation $\vee$ of the corresponding lengths as new length.

Theorem (Berger-Moerdijk)

The counit $F(\mathscr{C}) \to \mathscr{C}$ of the free-forgetful adjunction has a factorization

$$F(\mathscr{C}) \rightarrowtail W(H, \mathscr{C}) \xrightarrow{\simeq} \mathscr{C},$$

where $F(\mathscr{C}) \rightarrowtail W(H, \mathscr{C})$ is defined by assigning length γ_1 for all internal edges, and $W(H, \mathscr{C}) \xrightarrow{\simeq} \mathscr{C}$ is defined by forgetting the length and applying compositions in $\mathscr{C}$.

The definition of homotopy embedding tensors

Consider the 2-colored dg operad $\mathscr{C}(\mathrm{Lie}, \mathrm{LieMod}, \mathscr{E})$ obtained from the operadic $\mathrm{LieMod} - \mathrm{Lie}$ embedding $\mathscr{E}$. Applying the BV resolution, we obtain a new 2-colored dg operad

$$W(H, \mathscr{C}(\mathrm{Lie}, \mathrm{LieMod}, \mathscr{E})).$$

Definiton

A homotopy embedding tensor is an algebra over the 2-colored dg operad $W(H, \mathscr{C}(\mathrm{Lie}, \mathrm{LieMod}, \mathscr{E}))$.

Unfolding the data, we obtain

Proposition (Chen-Ge-X)

A homotopy embedding tensor consists of a triple (L, V, f), where

1. *L is an L_∞-algebra, V is an L_∞ L-module;*
2. *$f: V \to L$ is an L_∞-morphism of L-modules.*

According to Buijs and Murillo, if adding certain locally finite constraints (a.k.a. mild conditions), we obtain two functors

$$C^\infty(-)\colon \mathrm{mLie}^\infty \to \mathrm{CDGA}, \quad C^\infty(L,-)\colon \mathrm{mMod}_L \to \mathrm{Mod}^{\mathrm{sf}}_{C^\infty(L)}.$$

Via the functor $C^\infty(L,-)$, a homotopy embedding tensor $f\colon V \to L$ (with certain local finite constraints) is identified as a morphism of dg $C^\infty(L)$-modules

$$F := C^\infty(L,f)\colon C^\infty(L,V) \to C^\infty(L,L).$$

Moreover, if we view L as a formal pointed dg manifold

$$L[1] \text{``}=\text{''} \operatorname{spec}(C^\infty(L)),$$

then the category of homotopy embedding tensors is identified as that of morphisms of dg vector bundles over $L[1]$ to its shifted tangent bundle $T[-1]L[1]$.

SH Leibniz algebras from homotopy embedding tensors

Definiton

An SH Leibniz algebra (or Leibniz$_\infty[1]$*-algebra) over a locally finite* L_∞*-algebra* L *is a semi-free dg* $C^\infty(L)$*-module* $C^\infty(L,V)$*, equipped with a sequence of* $C^\bullet(L)$*-linear maps*

$$\lambda_k : C^\bullet(L,V)^{\otimes k} \to C^\bullet(\mathfrak{g},V), k \geq 2,$$

such that $(C^\infty(L,V), \{\lambda_k\}_{k\geq 2})$ *is a Leibniz*$_\infty[1]$ *algebra.*

Theorem (Chen-Ge-X)

Let $f: V \rightsquigarrow L$ *be a homotopy embedding tensor with certain finiteness constraints. Then the dg* $C^\infty(L)$*-module* $C^\infty(L,V)$ *admits a Leibniz*$_\infty[1]$ *algebra structure* $\{\lambda_k\}_{k\geq 2}$ *over* L*. All those higher structure maps are given by a summation over rooted trees that we will discuss in the coming slides.*

A rooted tree is a directed tree T, whose set $V(T)$ of vertices admits a distinguished element $v_R \in V(T)$ of valency 1, called *root vertex*, such that the tree T is oriented toward the root vertex v_R. The orientation of T determines a map

$$N \colon V(T) \to V(T),$$

which maps v_R to itself, and assigns to each non-root vertex v the next vertex along the unique path from v to v_R. The map N defines a partial order on $V(T)$:

$$v_1 \prec v_2 \Leftrightarrow \exists k \geq 1, s.t. v_2 = N^k(v_1).$$

The height of any $v \in V(T) - \{v_R\}$ is the minimal integer n_v satisfying $N_T^k(v) = v_R$ for all $k \geq n_v$.
The height of T is $h(T) = \max\{n_v \mid v \in V(T)\}$.

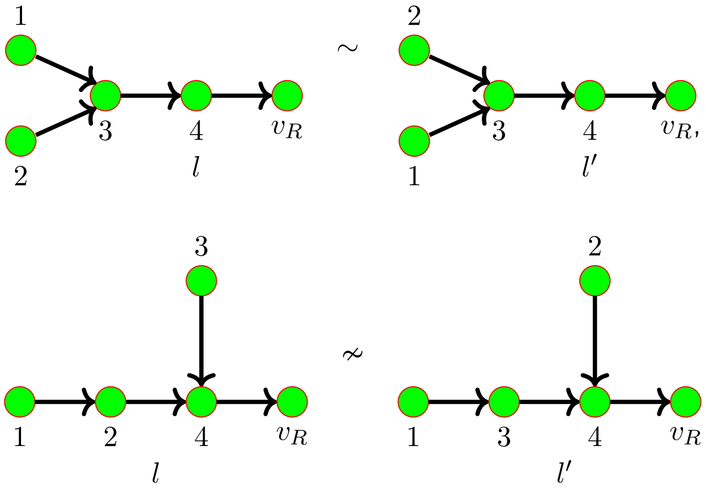
Monotonic orderings on rooted trees

A monotonic ordering l on a rooted tree $T \in \mathrm{RT}(n)$ is given by an order-preserving bijection

$$l\colon V(T) - \{v_R\} \to [n] := \{1, 2, \cdots, n\}.$$

Two monotonic orderings l and l' on T are said to be equivalent, if there exists an automorphism $\sigma : V(T) \to V(T)$ satisfying
(1) $l'(v) = l(\sigma(v))$ for all $v \in V(T) - \{v_R\}$;
(2) $\sigma(N(v)) = N(\sigma(v))$ for all $v \in V(T)$.
Denote by $[O(T)]$ the equivalent classes of monotonic orderings on T.

Examples of monotonic orderings



Decoration by homotopy embedding tensor

Given a homotopy embedding tensor $f\colon V \rightsquigarrow L$, let

$$F\colon C^\bullet(L,V) \to C^\bullet(L,L[1])[-1].$$

For each $x \in V^\bullet$, $F(x)$ is a finite sum of "trees" by local finiteness constraint. We now explain how to associate a multilinear map

$$\Theta^l_T\colon (V^\bullet)^{\otimes n} \to C^\bullet(L,L)[1-n]$$

to a rooted tree $T \in \mathrm{RT}(n)$ of height $h(T) = k+1$ with a monotonic ordering l. For all $x_1, \cdots, x_n \in V^\bullet$, we define $\Theta^l_T(x_1, \cdots, x_n)$ as follows:
(1) Label each non-root vertex $v \in V(T) - \{v_R\}$ by the element $x_{l(v)}$;
(2) Replace labels on tails $v_t \in V_t(T)$ by

$$L(v_t) = F(x_{l(v_t)}) \in C^\bullet(L,L[1]);$$

(3) Replace labels on internal vertices inductively as follows: Assume that each internal vertex $v_j \in V_i^j(T)$ of height j for $3 \leq j \leq k$ has been relabelled by $L(v_j) \in C^\bullet(L, L[1])$. For each internal vertex $v_{j-1} \in V_i^{j-1}(T)$ of height $j-1$ such that $N^{-1}(v_{j-1}) = \{v_j^1, \cdots, v_j^{|v_{j-1}|}\} \subset V^j(T)$, we relabel the vertex v_{j-1} by

$$L(v_{j-1}) := F(x_{l(v_{j-1})}) \bullet_{|v_{j-1}|} \Big(L(v_j^1), \cdots, L(v_j^{|v_{j-1}|}) \Big) \in C^\bullet(L, L[1]).$$

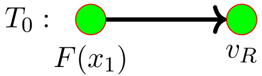
(4) We define $\Phi_T^l(x_1, \cdots, x_n)$ by

$$\Theta_T^l(x_1, \cdots, x_n) = F(x_{l(v_1)}) \bullet_{|v_1|} \Big(L(v_2^1), \cdots, L(v_2^{|v_1|}) \Big)$$

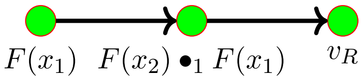
where v_1 is the unique vertex of height 1 that is adjacent to the root vertex v_R, and $\{v_2^1, \cdots, v_2^{|v_1|}\} = N^{-1}(v_1) \subset V^2(T)$.

Examples

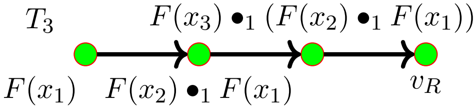
$T_0 \in \mathrm{RT}(1)$ of height 1:

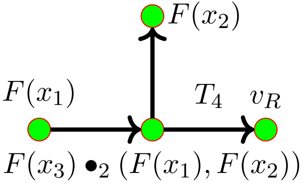


$T_1 \in \mathrm{RT}(2)$ of height 2:



$T_3, T_4 \in \mathrm{RT}(3)$ with height $h(T_3) = 3$ and $h(T_4) = 2$:





Linear maps from decorated trees

Note that the map $\Theta_T^l : \otimes^n V^\bullet \to C^\bullet(L, L[1])[-n]$ only depends on the equivalence classes of the monotonic ordering. Let

$$[\mathrm{ORT}(n)] = \{(T, l) \mid T \in \mathrm{RT}(n), l \in [O(T)]\}$$

be the set of equivalent monotonic ordered rooted trees with n non-root vertices. We define a multi-$C^\bullet(L)$-linear map

$$\Theta_n \colon \otimes^n_{C^\bullet(L)} C^\bullet(L, V) \to C^\bullet(L, L)[1-n]$$

by

$$\Theta_n(x_1, \cdots, x_n) := \sum_{(T,l)\in[\mathrm{ORT}(n)]} \Theta_T^l(x_1, \cdots, x_n),$$

for all $x_1, \cdots, x_n \in V^\bullet$.

Homotopy Leibniz algebra structure by summation over rooted trees

Theorem (Chen-Ge-X)

Let $f\colon V \rightsquigarrow L$ be a finite homotopy embedding tensor. Then the higher structure maps $\{\mu_{n+1}\}_{n\geq 1}$ of the Leibniz$_\infty$ $C(L)$-algebra structure on $C^\infty(L,V)$ has the form

$$\mu_{n+1}(x_1,\cdots,x_{n+1}) = \sum_{k=1}^{n} \sum_{n_1+\cdots n_k=n} \sum_{\sigma\in \mathrm{sh}(n_1,\cdots,n_k)} \frac{\epsilon(\sigma)}{k!}$$
$$\mu^V_{k+1}(\Theta_{n_1}(x_{\sigma(1)},\cdots,x_{\sigma(n_1)}), \Theta_{n_2}(x_{\sigma(n_1+1)},\cdots,x_{\sigma(n_1+n_2)}), \cdots, \Theta_{n_k}(x_{n-n_k+1},\cdots,x_n), x_{n+1}),$$

for all $x_1,\cdots,x_n\in V^\bullet$. Here $\mu^V_\bullet$ is the $C^\bullet(L)$-linear extension of the structure maps of the mild L_∞-module V.

Thank you for your attention.