

生物流体力学附录—
有限元法
在流体计算中的应用

非压缩流体的N—S方程，连续方程

连续方程

$$\nabla \cdot \mathbf{V} = 0$$

动量方程

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \Delta \mathbf{V}$$

能量方程

$$\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla) T = \alpha \Delta T$$

V:	velocity
p:	pressure
T:	temperature

对控制方程进行无量纲化，引
入如下无量纲参数

$$\mathbf{u} = \frac{\mathbf{V}}{U_{\infty}}$$

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}$$

$$p^* = \frac{p}{\rho U_{\infty}^2}$$

$$T^* = \frac{T - T_r}{T_{\infty} - T_r}$$

非压缩流体流动的N-S方程，连续方程和能量方程

无量纲形式

- Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u}$$

- Continuity equation

$$\nabla \cdot \mathbf{u} = 0$$

- Energy equation

$$\frac{\partial T^*}{\partial t} + (\mathbf{u} \cdot \nabla) T^* = \frac{1}{Pe} \Delta T^*$$

Re: Reynolds number

$$Re = \frac{U_\infty L}{\nu}$$

惯性力和粘性力大小的比

Pe: Peclet number

$$Pe = \frac{U_\infty L}{\alpha} = \frac{U_\infty L}{\nu} \cdot \frac{\nu}{\alpha} = Re Pr$$

求解压力的SMAC (Simplified MAC method) 法

MAC: Marker and Cell

对运动方程进行时间方向上的离散

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t \left\{ (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n - \frac{1}{\text{Re}} \Delta \mathbf{u}^n \right\} - \Delta t \nabla p^{n+1} \quad (1)$$

令中间流速定义为

$$\tilde{\mathbf{u}} = \mathbf{u}^n - \Delta t \left\{ (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n - \frac{1}{\text{Re}} \Delta \mathbf{u}^n \right\} \quad (2)$$

代入(1)式, 则有

$$\mathbf{u}^{n+1} = \tilde{\mathbf{u}} - \Delta t \nabla p^{n+1} \quad (3)$$

对两边取散度

$$\nabla \cdot \mathbf{u}^{n+1} = \nabla \cdot \tilde{\mathbf{u}} - \Delta t \nabla \cdot \nabla p^{n+1} \quad (4)$$

由连续方程可以得到

$$\nabla \cdot \mathbf{u}^{n+1} = 0$$

这样式(4)就可以变成

$$\Delta p^{n+1} = \frac{1}{\Delta t} \nabla \tilde{\mathbf{u}} \quad (5)$$

具体步骤:

1. 由公式(2)求出中间速度 $\tilde{\mathbf{u}}$
2. 由公式(5)求出压力 p^{n+1}
3. 由公式(3) 求出速度 \mathbf{u}^{n+1}

采用伽辽金法对中间速度, 压力, 速度以及温度分布公式进行离散化

离散化方法之一

权余法：用近似值代替严格值代入原微分方程后产生的余量与选择的加权函数在定义域内作内积，并要求所选用的加权函数能使内积为零；这时域内任意点的近似值便是离散化的数值解

伽辽金法：

伽辽金法是选用形状函数 (shape function) 作为加权函数的权余法，满足内积

$$(R, N_i) = 0$$

$i=1, 2$ 代表需要两个方向的速度, $\alpha=1-3$ 代表三角形元素的三个顶点

未知函数可以用下面的公式表示

$$\tilde{u}_i = N_{\alpha} \tilde{u}_{\alpha i}$$

$$u_i = N_{\alpha} u_{\alpha i}$$

$$P = N_{\alpha} P_{\alpha}$$

加权函数取未知函数的形状函数 N_{α} , 将未知函数, 加权函数代入中间速度, 压力和速度公式, 则, 权余法公式为

X方向的中间速度

$$\int_{\Omega_e} N_\alpha \tilde{u} d\Omega = \int_{\Omega_e} N_\alpha u^n d\Omega$$

二次微分要
进行变换(A)

$$-\Delta t \left[\int_{\Omega_e} N_\alpha u^n \frac{\partial u^n}{\partial x} d\Omega + \int_{\Omega_e} N_\alpha v^n \frac{\partial u^n}{\partial y} d\Omega - \frac{1}{Re} \int_{\Omega_e} N_\alpha \left(\frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) d\Omega \right]$$

Y方向的中间速度

$$\int_{\Omega_e} N_\alpha \tilde{v} d\Omega = \int_{\Omega_e} N_\alpha v^n d\Omega$$

二次微分要
进行变换(B)

$$-\Delta t \left[\int_{\Omega_e} N_\alpha u^n \frac{\partial v^n}{\partial x} d\Omega + \int_{\Omega_e} N_\alpha v^n \frac{\partial v^n}{\partial y} d\Omega - \frac{1}{Re} \int_{\Omega_e} N_\alpha \left(\frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) d\Omega \right]$$

压力公式

$$\int_{\Omega_e} N_\alpha \left(\frac{\partial^2 P^{n+1}}{\partial x^2} + \frac{\partial^2 P^{n+1}}{\partial y^2} \right) d\Omega = \frac{1}{\Delta t} \int_{\Omega_e} N_\alpha \left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) d\Omega$$

↑
二次微分要
进行变换(C)

X方向速度公式

$$\int_{\Omega_e} N_\alpha u^{n+1} d\Omega = \int_{\Omega_e} N_\alpha \tilde{u} d\Omega - \Delta t \int_{\Omega_e} N_\alpha \frac{\partial P^{n+1}}{\partial x} d\Omega$$

Y方向速度公式

$$\int_{\Omega_e} N_\alpha v^{n+1} d\Omega = \int_{\Omega_e} N_\alpha \tilde{v} d\Omega - \Delta t \int_{\Omega_e} N_\alpha \frac{\partial P^{n+1}}{\partial y} d\Omega$$

运用格林公式，可以把二次微分项转换成一次微分项

(A)

$$(B) \int_{\Omega_e} N_\alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega = \int_{\Gamma_e} N_\alpha \frac{\partial u}{\partial n} d\Gamma - \int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega$$

(C)

$$\int_{\Omega_e} N_\alpha \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) d\Omega = \int_{\Gamma_e} N_\alpha \frac{\partial v}{\partial n} d\Gamma - \int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial v}{\partial y} \right) d\Omega$$

$$\int_{\Omega_e} N_\alpha \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) d\Omega = \int_{\Gamma_e} N_\alpha \frac{\partial p}{\partial n} d\Gamma - \int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial p}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial p}{\partial y} \right) d\Omega$$

对任意单元, 经变换后, 中间速度方程的权余法公式可写成

X方向

$$\int_{\Omega_e} N_\alpha N_\alpha \tilde{u} d\Omega = \int_{\Omega_e} N_\alpha N_\alpha u_\alpha^n d\Omega$$
$$-\Delta t \left[\int_{\Omega_e} N_\alpha N_\alpha u_\alpha^n \frac{\partial N_\alpha}{\partial x} u_\alpha^n d\Omega + \int_{\Omega_e} N_\alpha N_\alpha v_\alpha^n \frac{\partial N_\alpha}{\partial y} u_\alpha^n d\Omega + \frac{1}{Re} \int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial N_\alpha}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial N_\partial}{\partial y} \right) u_\alpha^n d\Omega \right]$$

Y方向

$$\int_{\Omega_e} N_\alpha N_\alpha \tilde{v} d\Omega = \int_{\Omega_e} N_\alpha N_\alpha v_\alpha^n d\Omega$$
$$-\Delta t \left[\int_{\Omega_e} N_\alpha N_\alpha u_\alpha^n \frac{\partial N_\alpha}{\partial x} v_\alpha^n d\Omega + \int_{\Omega_e} N_\alpha N_\alpha v_\alpha^n \frac{\partial N_\alpha}{\partial y} v_\alpha^n d\Omega + \frac{1}{Re} \int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial N_\alpha}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial N_\partial}{\partial y} \right) v_\alpha^n d\Omega \right]$$

压力方程的离散公式

$$\int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial N_\alpha}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial N_\alpha}{\partial y} \right) d\Omega P_\alpha^{n+1} = -\frac{1}{\Delta t} \int_{\Omega_e} N_\alpha \left(\frac{\partial N_\alpha}{\partial x} \tilde{u}_\alpha + \frac{\partial N_\alpha}{\partial y} \tilde{v}_\alpha \right) d\Omega$$

速度方程的离散公式

X方向

$$\int_{\Omega_e} N_\alpha N_\alpha u_\alpha^{n+1} d\Omega = \int_{\Omega_e} N_\alpha N_\alpha \tilde{u}_\alpha d\Omega - \Delta t \int_{\Omega_e} N_\alpha \frac{\partial N_\alpha}{\partial x} P_\alpha^{n+1} d\Omega$$

Y方向

$$\int_{\Omega_e} N_\alpha N_\alpha v_\alpha^{n+1} d\Omega = \int_{\Omega_e} N_\alpha N_\alpha \tilde{v}_\alpha d\Omega - \Delta t \int_{\Omega_e} N_\alpha \frac{\partial N_\alpha}{\partial y} P_\alpha^{n+1} d\Omega$$

对于三角形单元的形状函数

$$N_1 = a_1 + b_1x + c_1y$$

$$N_2 = a_2 + b_2x + c_2y$$

$$N_3 = a_3 + b_3x + c_3y$$

对于形状函数的系数

$$a_i = \frac{1}{2\Delta} (x_j y_k - x_k y_j)$$

$$b_i = \frac{1}{2\Delta} (y_j - y_k) \quad (i, j, k = 1, 2, 3)$$

$$c_i = \frac{1}{2\Delta} (x_k - x_j)$$

三角形单元的面积

$$\Delta = \frac{1}{2} [x_i(y_j - y_k) + x_j(y_k - y_i) + x_k(y_i - y_j)]$$

将离散公式写成矩阵形式

$$M_{uv} \tilde{u}_\alpha = M_{uv} u_\alpha^n - \Delta t [(K_{u1} u_\alpha^n + K_{u2} v_\alpha^n) + \frac{1}{Re} A_{\alpha\beta} u_\alpha^n]$$

$$M_{uv} \tilde{v}_\alpha = M_{uv} v_\alpha^n - \Delta t [(K_{v1} u_\alpha^n + K_{v2} v_\alpha^n) + \frac{1}{Re} A_{\alpha\beta} v_\alpha^n]$$

$$A_p P_\alpha^{n+1} = -\frac{1}{\Delta t} (H_u \tilde{u}_\alpha + H_v \tilde{v}_\alpha)$$

$$M_u u_\alpha^{n+1} = (M_u \tilde{u}_\alpha - \Delta t H_u P_\alpha^{n+1})$$

$$M_u v_\alpha^{n+1} = (M_u \tilde{v}_\alpha - \Delta t H_v P_\alpha^{n+1})$$

以上各个离散公式中的系数是关于形状函数的积分

Mass matrix and convection term

$$M_{uv} = \int_{\Omega_e} N_\alpha N_\alpha d\Omega = \int_{\Omega_e} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} \langle N_1 \quad N_2 \quad N_3 \rangle d\Omega = \frac{\Delta}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$K_{u1} = \int_{\Omega_e} N_\alpha N_\alpha \frac{\partial N_\alpha}{\partial x} d\Omega u_\alpha^n = \frac{\Delta}{12} (b_1 u_1 + b_2 u_2 + b_3 u_3) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$K_{u2} = \int_{\Omega_e} N_\alpha N_\alpha \frac{\partial N_\alpha}{\partial y} d\Omega u_\alpha^n = \frac{\Delta}{12} (c_1 u_1 + c_2 u_2 + c_3 u_3) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$K_{v1} = \int_{\Omega_e} N_\alpha N_\alpha \frac{\partial N_\alpha}{\partial x} d\Omega v_\alpha^n = \frac{\Delta}{12} (b_1 v_1 + b_2 v_2 + b_3 v_3) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$K_{v2} = \int_{\Omega_e} N_\alpha N_\alpha \frac{\partial N_\alpha}{\partial x} d\Omega v_\alpha^n = \frac{\Delta}{12} (c_1 v_1 + c_2 v_2 + c_3 v_3) \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned}
A_{\alpha\beta} &= \int_{\Omega_e} \left(\frac{\partial N_\alpha}{\partial x} \frac{\partial N_\alpha}{\partial x} + \frac{\partial N_\alpha}{\partial y} \frac{\partial N_\alpha}{\partial y} \right) d\Omega \\
&= \Delta \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2^2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3^2 \end{bmatrix} + \Delta \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_2 c_1 & c_2^2 & c_2 c_3 \\ c_3 c_1 & c c & c_3^2 \end{bmatrix} \\
&= A_p
\end{aligned}$$

$$H_u = \int_{\Omega_e} \left(N_\alpha \frac{\partial N_\alpha}{\partial x} \right) d\Omega = \frac{\Delta}{3} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$H_v = \int_{\Omega_e} \left(N_\alpha \frac{\partial N_\alpha}{\partial y} \right) d\Omega = \frac{\Delta}{3} \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

单个单元内的矩阵方程

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix} \begin{Bmatrix} u_i^e \\ u_j^e \\ u_k^e \end{Bmatrix} = \begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \end{Bmatrix}$$

最后的合成方程有如下形式

$$\begin{bmatrix} K_{11}^1 & K_{13}^1 & K_{12}^1 & K_{13}^2 \\ K_{31}^1 & K_{33}^1 + K_{11}^2 & K_{32}^1 + K_{12}^2 & K_{13}^2 \\ K_{21}^1 & K_{23}^1 + K_{21}^2 & K_{22}^1 + K_{22}^2 & K_{23}^2 \\ & K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

求解连立一次方程

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

可以采用直接法 (Gauss 法) 或迭代法对方程求解
共轭梯度法 (CG: Conjugate Gradient Method) 是常用的迭代法

$$\mathbf{x}^{k+1} = \mathbf{B}\mathbf{x}^k + \mathbf{f} \quad (k = 0, 1, 2, \dots)$$

如果 $\lim_{k \rightarrow \infty} \mathbf{x}^k$ 存在, 则称迭代法收敛, 方程的解为

$$\mathbf{x}^* = \lim_{k \rightarrow \infty} \mathbf{x}^k$$

程序实现

```
c....*
subroutine makebc(x,y,ie,nop,b,c,area,aa03,aa12)
implicit none
c
real*8 x,y
dimension x(NoNode),y(NoNode)
c
integer nop
dimension nop(3,NoElem)
dimension b(3),c(3)
c
integer i1,i2,i3
real*8 b,c,area,aa03,aa12,a2
real*8 x1,x2,x3,y1,y2,y3
c
integer ie

i1 = nop(1,ie)
i2 = nop(2,ie)
i3 = nop(3,ie)
```

c --- coordinate ----

x1 = x(i1)

x2 = x(i2)

x3 = x(i3)

y1 = y(i1)

y2 = y(i2)

y3 = y(i3)

c --- area of the triangle element ---

a2 = x1*(y2 - y3) + x2*(y3 - y1) + x3*(y1 - y2)

area = a2 * 0.5d0

aa03 = area / 3d0

aa12 = area / 12d0

a2 = 1d0 / a2

c --- coefficient of the shape function ---

b(1) = (y2 - y3) * a2

b(2) = (y3 - y1) * a2

b(3) = (y1 - y2) * a2

c(1) = (x3 - x2) * a2

c(2) = (x1 - x3) * a2

c(3) = (x2 - x1) * a2

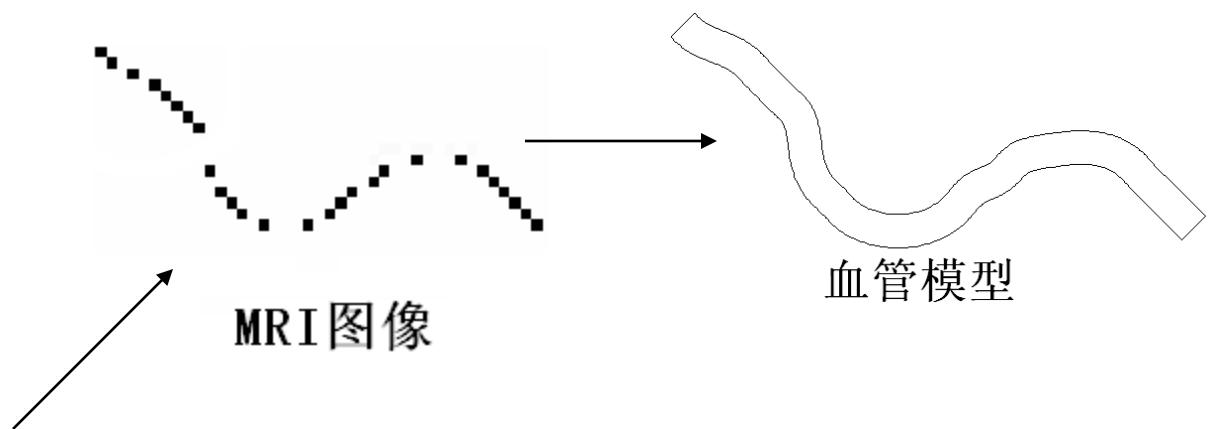
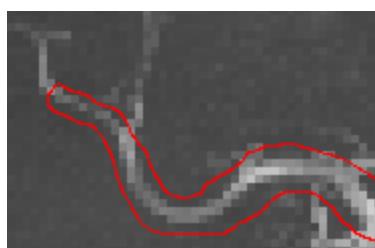
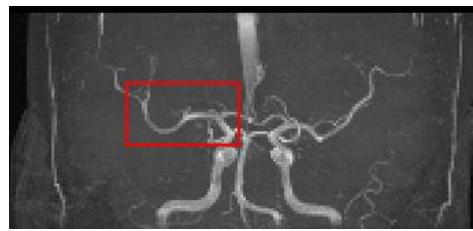
return

end

脑血管血流动力学计算算例

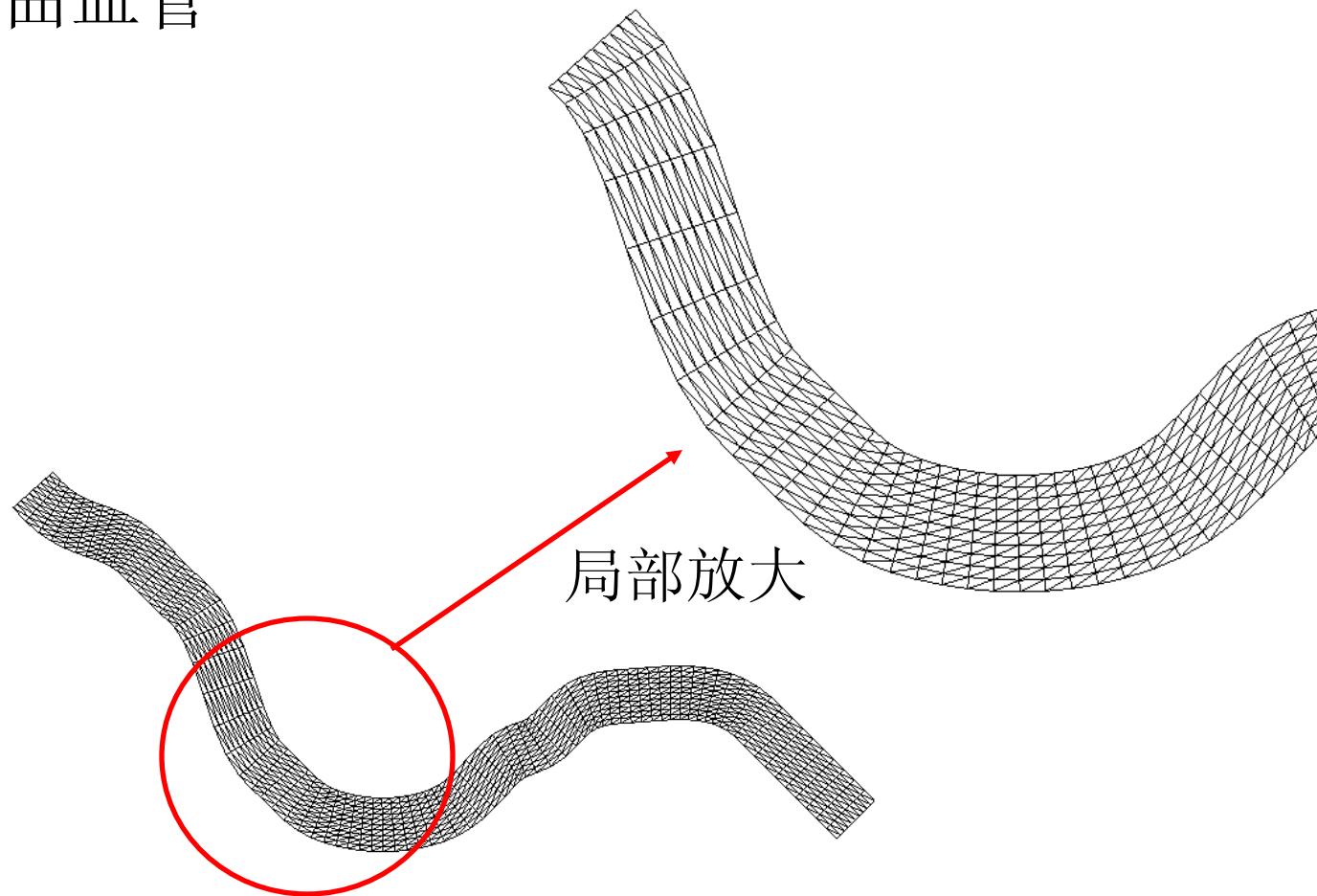
血管建模

- 弯曲血管



划分网格

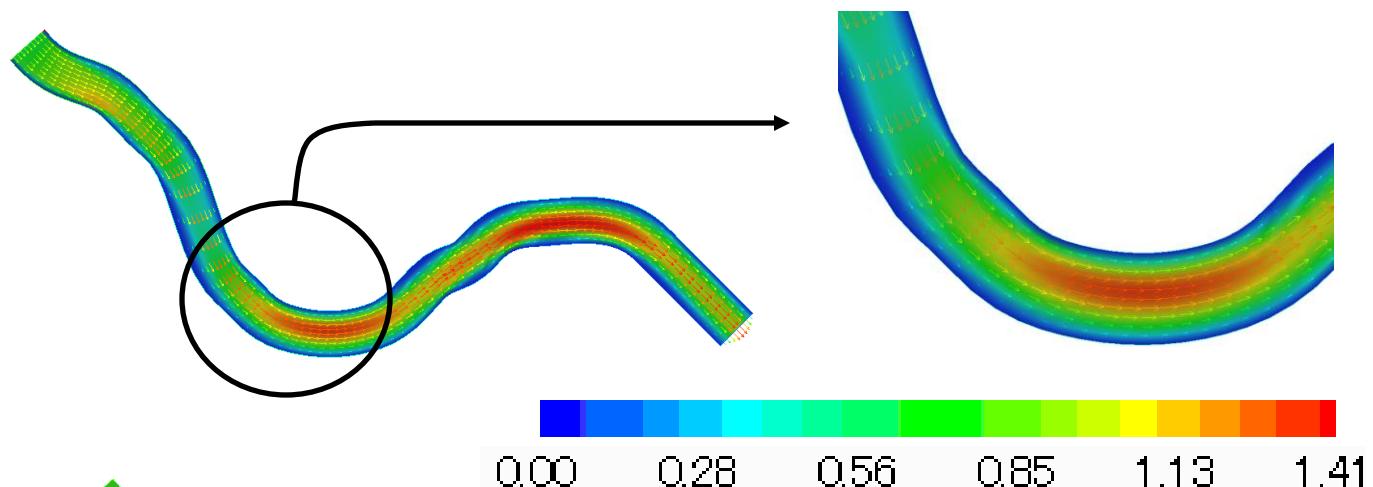
- 弯曲血管



计算结果及分析

- 弯曲血管

1.速度:



2.压力:

