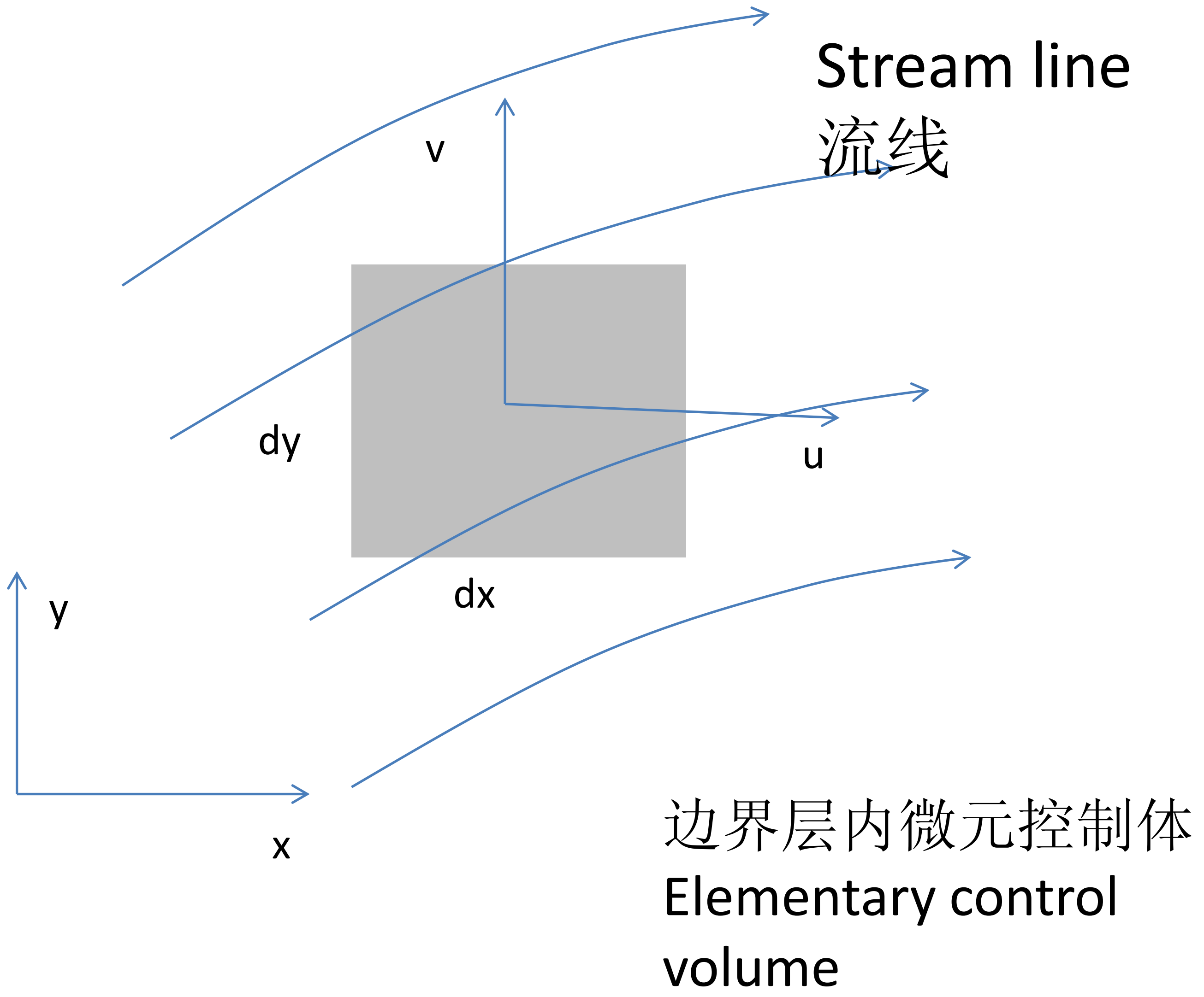
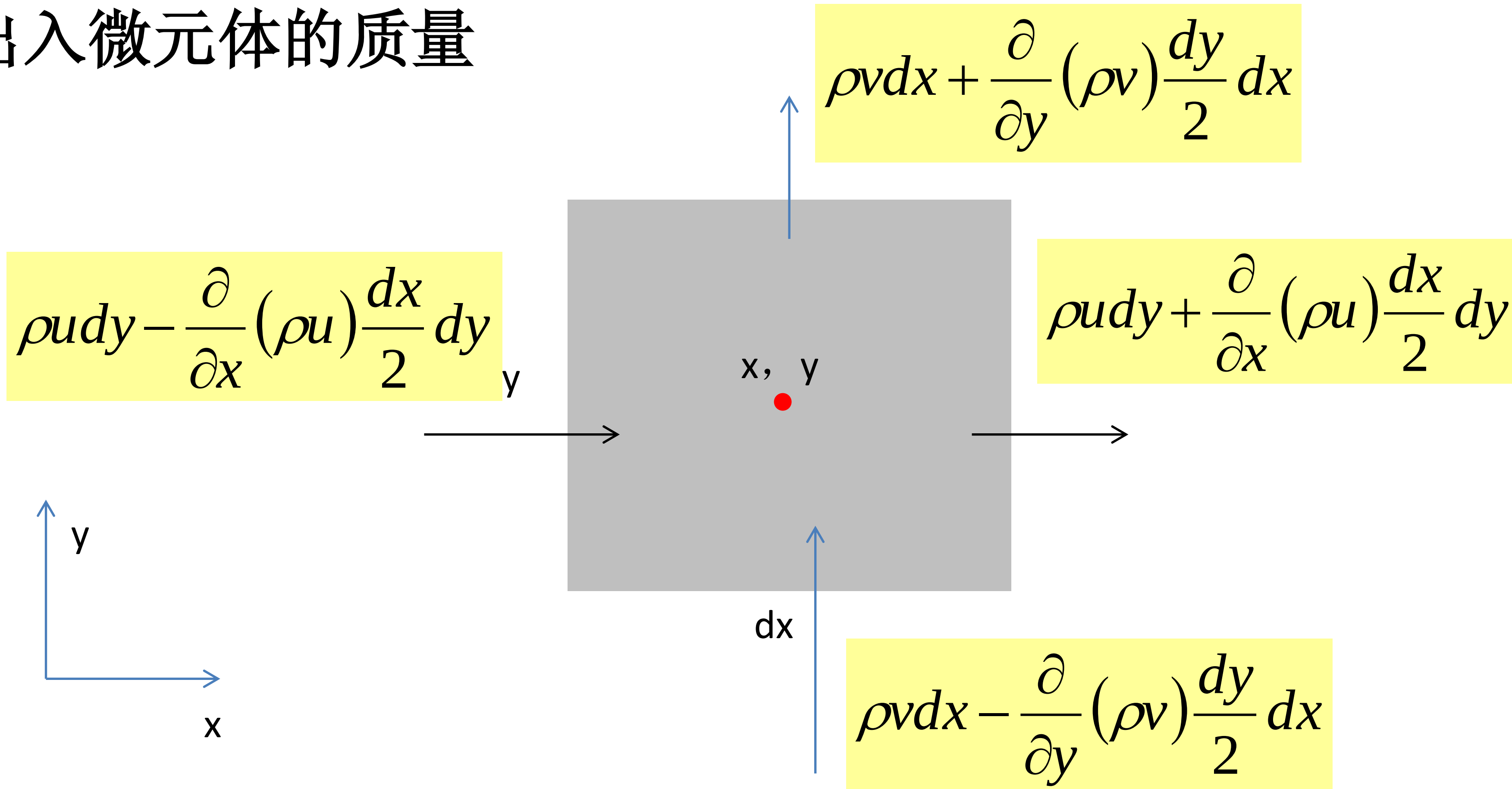

Stream line
流线
v
dy
u
dx
y
x
边界层内微元控制体
Elementary control volume

A 连续方程的诱导

假定： 流动为二维流动；定常层流；不受传热影响

出入微元体的质量



微元体体积 **dx×dy×1**

质量守恒原理：

定常状态下，**dt**时间内进入**dx×dy×1**微元体的质量等于流出微元体的质量

$$\rho u dy-\frac{\partial}{\partial x}(\rho u)\frac{dx}{2}dy+\rho v dx-\frac{\partial}{\partial y}(\rho v)\frac{dy}{2}dx$$
$$=\rho u dy+\frac{\partial}{\partial x}(\rho u)\frac{dx}{2}dy+\rho v dx+\frac{\partial}{\partial y}(\rho v)\frac{dy}{2}dx$$

⇓

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

适用于任何介质的连续方程

（**continuity equation**）

⇓

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

密度一定时的不可压缩流体

B. 动量方程的推导

- 牛顿定律 力的总和等于一定时间内动量的变化

$$\sum F = \frac{d(mv)}{dt}$$

- 在边界层内速度几乎与壁面平行，y方向的速度成分v很小，可以忽略不计
- 作用于y方向的剪切力可以忽略

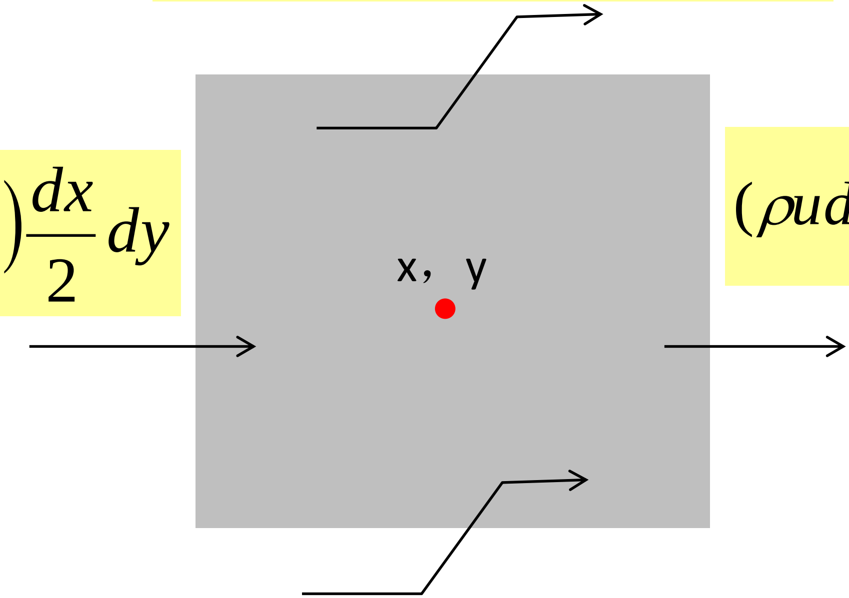
↓

对于边界层内的流动，**只考虑主流x方向的力和动量变化**

单位时间进出微元体的动量变化

$$(\rho v dx)u + \frac{\partial}{\partial y}\left(\rho uv\right)\frac{dy}{2}dx$$

$$(\rho u dy)u - \frac{\partial}{\partial x}\left(\rho u^2\right)\frac{dx}{2}dy$$



$$(\rho u dy)u + \frac{\partial}{\partial x}\left(\rho u^2\right)\frac{dx}{2}dy$$

$$(\rho v dx)u - \frac{\partial}{\partial y}\left(\rho uv\right)\frac{dy}{2}dx$$

微元体内总的动量变化

$$\frac{\partial}{\partial x}\left(\rho u^{2}\right)dxdy+\frac{\partial}{\partial y}\left(\rho uv\right)dxdy$$

$$[\rho u\frac{\partial u}{\partial x}+u\frac{\partial(\rho u)}{\partial x}+\rho v\frac{\partial u}{\partial y}+u\frac{\partial(\rho v)}{\partial y}]dxdy$$

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

$$[\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}]dxdy$$

作用于控制流体的X方向的力：

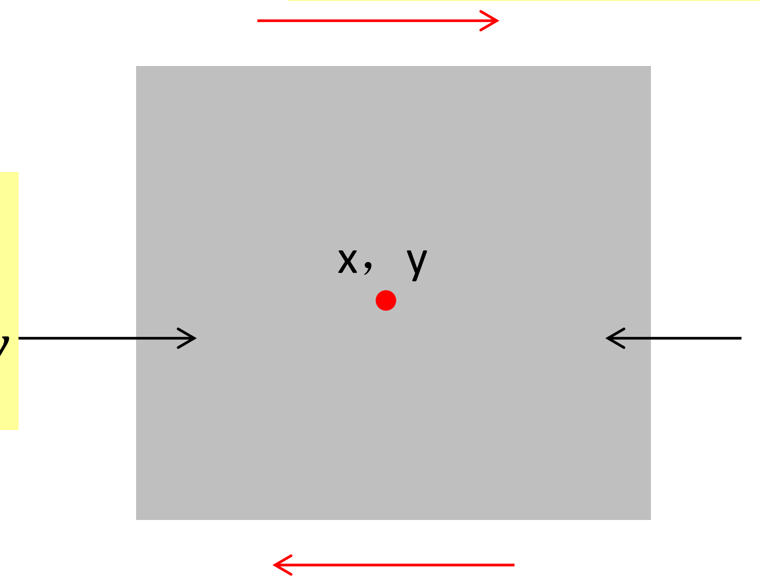
1.重力等体积力（强制对流时可忽略不计）

2. 压力（表面力）

3. 剪切力（表面力）

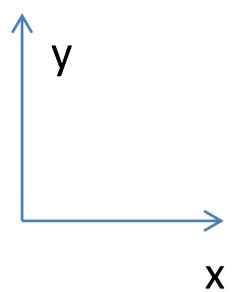
$$\tau_{yx}dx+\frac{\partial}{\partial y}(\tau_{yx})\frac{dy}{2}dx$$

$$p(x-dx/2,y)=pdy-\frac{\partial p}{\partial x}\frac{dx}{2}dy$$



$$pdy+\frac{\partial p}{\partial x}\frac{dx}{2}dy$$

$$\tau_{yx}dx-\frac{\partial}{\partial y}(\tau_{yx})\frac{dy}{2}dx$$



$$\tau_{yx}=\mu\frac{\partial u}{\partial y}$$

表面摩擦

边界层内X方向上的力量

$$[\tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx})\frac{dy}{2} - \tau_{yx} + \frac{\partial}{\partial y}(\tau_{yx})\frac{dy}{2}]dx$$

$$+ pdy - \frac{\partial p}{\partial x}\frac{dx}{2}dy - pdy - \frac{\partial p}{\partial x}\frac{dx}{2}dy$$

↓

$$\frac{\partial}{\partial y}(\tau_{yx})dxdy - \frac{\partial p}{\partial x}dxdy$$

$$= \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)dxdy - \frac{\partial p}{\partial x}dxdy$$

↓ 边界层的运动方程

教科书P208， 5-12

X方向 $\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x}$ **y方向** $\frac{\partial p}{\partial y} = 0$

C. 能量方程的导出

基于能量守恒和傅里叶定律

对外做的净功

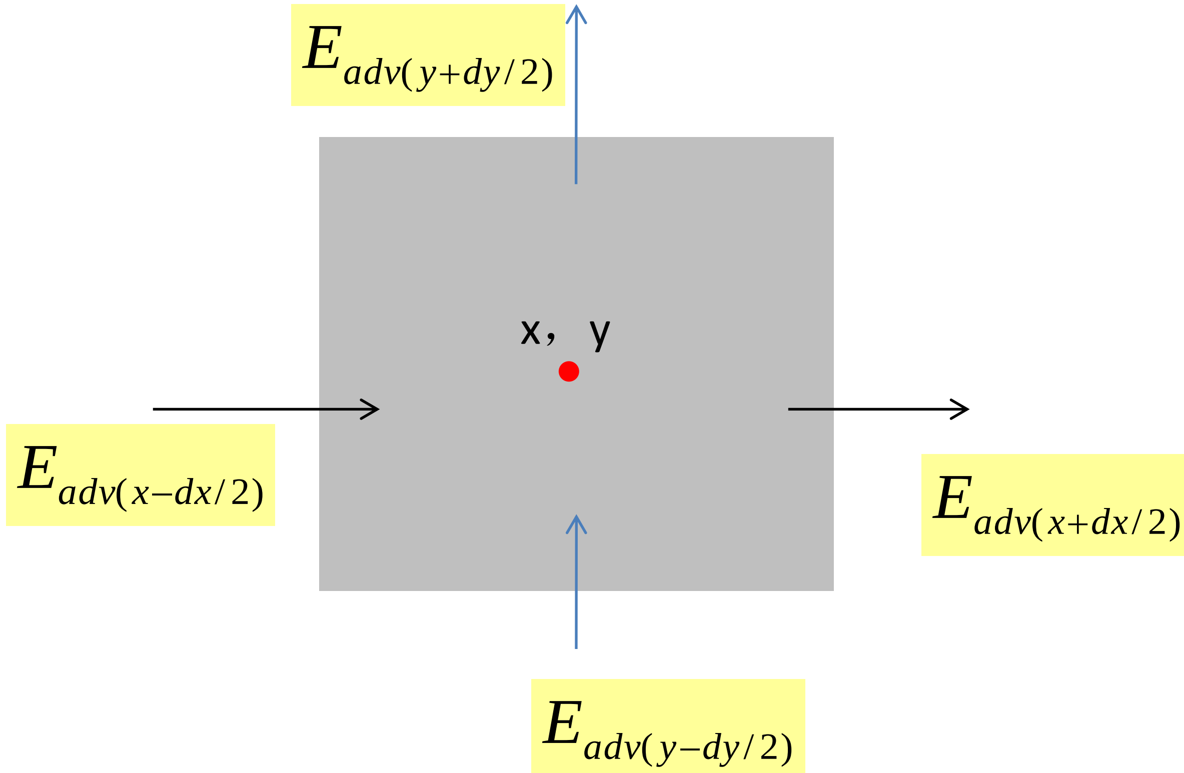
$$\frac{\partial E_{total}}{\partial t} = \Delta E_{conduction} + \Delta E_{convection} - \dot{W}_{net}$$

流体流动引起的能量变化

流体导热引起的能量变化

$$\Delta E_{conduction} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right)dxdy + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right)dxdy$$

流动引起的能量变化



$$\Delta E_{adv(x)} = E_{adv(x-dx/2)} - E_{adv(x+dx/2)}$$

$$= q_{in}(h + \cdots)_{in} - q_{out}(h + \cdots)_{out}$$

$$q_{in}h_{in} = \left[\rho uh - \frac{\partial}{\partial x}(\rho uh)\frac{dx}{2}\right]dy$$

$$q_{\text{out}}h_{\text{out}} = \left[\rho uh + \frac{\partial}{\partial x}(\rho uh)\frac{dx}{2}\right]dy$$

$$\Delta E_{adv(x)} = -\frac{\partial}{\partial x}(\rho uh)dxdy$$

$$\Delta E_{adv(x)} = -\frac{\partial}{\partial x}(\rho uh)dxdy$$

若流体不对外作功

$$\dot{W}_{net} = 0$$

$$\frac{\partial E_{total}}{\partial t} = \frac{\partial}{\partial t}(\rho h)dxdy$$

代入能量守恒方程

$$\frac{\partial}{\partial t}(\rho h)dxdy = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)dxdy + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)dxdy - \frac{\partial}{\partial x}(\rho uh)dxdy - \frac{\partial}{\partial y}(\rho vh)dxdy$$

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial}{\partial x}(\rho uh) + \frac{\partial}{\partial y}(\rho vh) = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)$$

$h = c_p T$ 比焓

$$\frac{\partial}{\partial x}(\rho uh) + \frac{\partial}{\partial y}(\rho vh) = \rho u\frac{\partial h}{\partial x} + \rho h\frac{\partial u}{\partial x} + \rho h\frac{\partial v}{\partial y} + \rho v\frac{\partial h}{\partial y}$$

$= 0$

$$\rho c_p\frac{\partial T}{\partial t} + \rho c_p u\frac{\partial T}{\partial x} + \rho c_p v\frac{\partial T}{\partial y} = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)$$

二维稳态，无内热源的边界层内的流动与温度控制方程

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\partial p}{\partial x}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

与y方向热传导相比，X方向的热传导可以忽略

对流传热问题完整的数学描述

不可压缩，常物性，无内热源的二维问题

质量守恒方程

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

动量方程（Navier-Stokes方程）

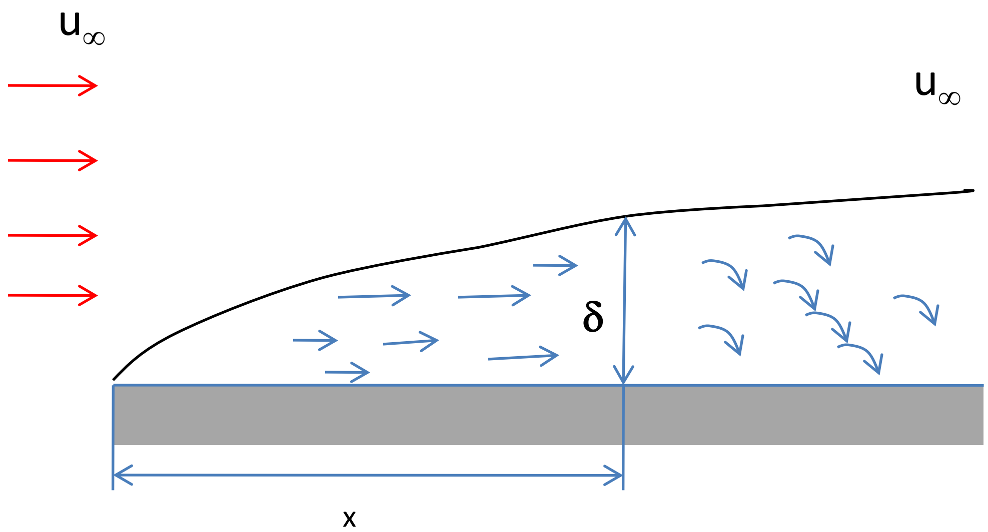
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=F_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=F_x-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

能量方程

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}+\alpha\frac{\partial^2 T}{\partial y^2}$$

5.4 流体外掠平板传热层流分析解



控制方程

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\partial p}{\partial x}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

求解方法：相似变换 ⟶ 将偏微分方程变为常微分方程

用流函数定义速度分量

$$u \equiv \frac{\partial \psi}{\partial y} \qquad v \equiv -\frac{\partial \psi}{\partial x}$$

代入连方程

$$\frac{\partial \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x \partial y} = 0$$

引入新的自变量和因变量

$$\eta \equiv y\sqrt{\frac{u_\infty}{\nu x}} \qquad f(\eta) \equiv \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}}$$

主流速度

将u，v写成η，f（η）的函数

$$u = u_\infty \frac{df}{d\eta}$$

$$v = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}\left[\eta \frac{df(\eta)}{d\eta} - f\right]$$

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\eta \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial u}{\partial y} = u_\infty \sqrt{\frac{u_\infty}{\nu x}} \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_\infty^2}{\nu x} \frac{d^3 f}{d\eta^3}$$

将各个参数代入动量方程 $u \; v \quad \frac{\partial u}{\partial x} \; \frac{\partial u}{\partial y} \; \frac{\partial^2 u}{\partial y^2}$

变换后的常微分方程

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

边界条件

$$u(x,0) = v(x,0) = 0 \qquad u(x,\infty) = u_\infty$$

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = 0$$

$$\left.\frac{df}{d\eta}\right|_{\eta=\infty} = 1$$

通过级数展开和数值计算可以得到常微分方程满足边界条件的解

平板边界层函数

$\eta \equiv y\sqrt{\dfrac{u_\infty}{\nu x}}$	f	$\dfrac{df}{d\eta}=\dfrac{u}{u_\infty}$	$\dfrac{d^2 f}{d\eta^2}$
0	0	0	0.332
4.8	3.085	0.988	0.022
5.2	3.485	0.994	0.011