3.1 非粘性流体血流的波动速度

◆假定一： 理想化血管—无限长， 圆管， 弹性管， 独立于环境

理想化血流—均质， 不可压， 无粘流体 （homogeneous, incompressible, inviscid flow）

◆假定二：管径比波长小很多

◆假定三：管内流动是一维流动

只有一个速度成分u（x，t），x：流动方向，t：时间这样，血管内血液流动的连续方程和动量方程是

（3.1）

$$\frac{\partial A}{\partial t}+\frac{\partial (uA)}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial P}{\partial x}=0 \quad （3.2）$$

还需要一个管径a和内压p之间的关系，最简单的弹性模型为

$$\boxed{da=\frac{\gamma}{2}dp} \quad （3.3）$$

γ -- 弹性管的顺应性（compliance）2—只是为了以后推导方便

波动方程

由连续方程3.1可以得到

$$\frac{\partial a}{\partial t} = -\frac{a}{2}\frac{\partial u}{\partial x} \quad （3.4）$$

$$\frac{\partial u}{\partial x} = -\frac{\gamma}{a}\frac{\partial P}{\partial t} \quad （3.6）$$

由3.3弹性模型公式得到 $\frac{\partial a}{\partial t} = \frac{\gamma}{2}\frac{\partial P}{\partial t}$ （3.5）

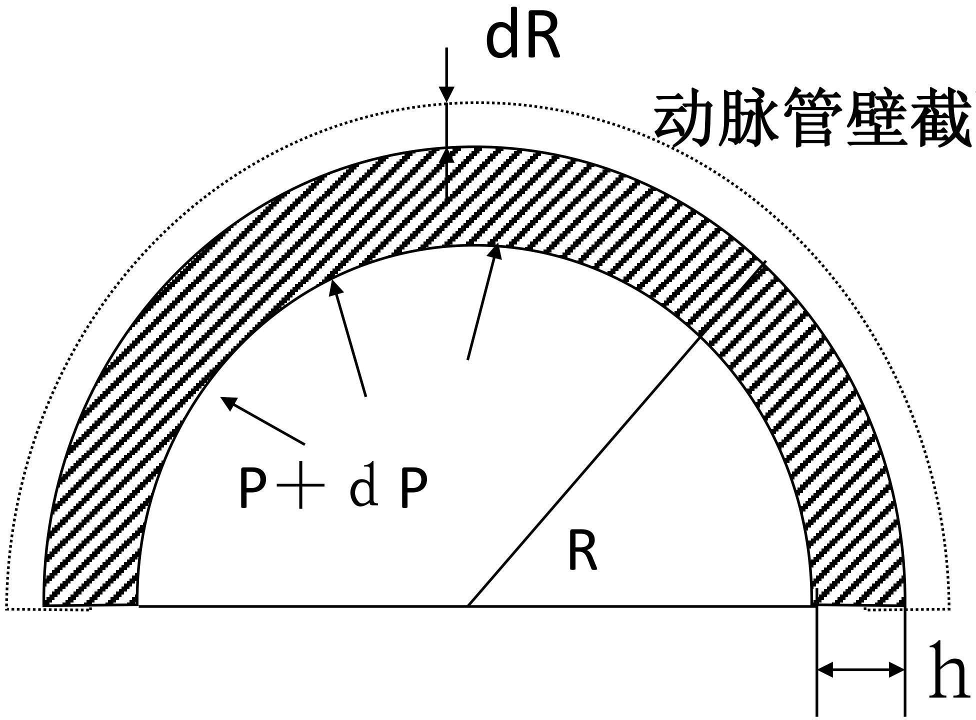
将3.6对t微分，将动量方程3.2对x微分

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}$$

？？

$$c^2 = \frac{R}{\rho\gamma}$$

波动速度



动脉管壁截面

顺应性的公式

$$\gamma = \frac{2R^2}{Eh}$$

代入3.3 弹性模型公式

$$dR = (\frac{R^2}{Eh})dP$$

根据虎克定律，周向应力 $= E\dfrac{dR}{R}$

$$RdP = (E\frac{dR}{R})h$$

Laplace Equation：壁面张力　　壁面张力

薄壁弹性管内不可压缩非粘性流体的波动速度

$$c = \sqrt{\frac{R}{\rho\gamma}} = \sqrt{\frac{Eh}{2\rho R}}$$

E： Young’s modulus

h: wall thickness

ρ: density

R: radius

Moens-Korteweg equation

波动速度测定方法

◆在大动脉弓相距**5 cm**处放入传感器（带有双传感器的导管）

◆记录两处的压力波

◆让两个压力波相互重叠，波动速度为记录点间距离与滞后时间之比

$$c=\frac{\Delta z}{\Delta t}=\sqrt{\frac{Eh}{2\rho R}}$$

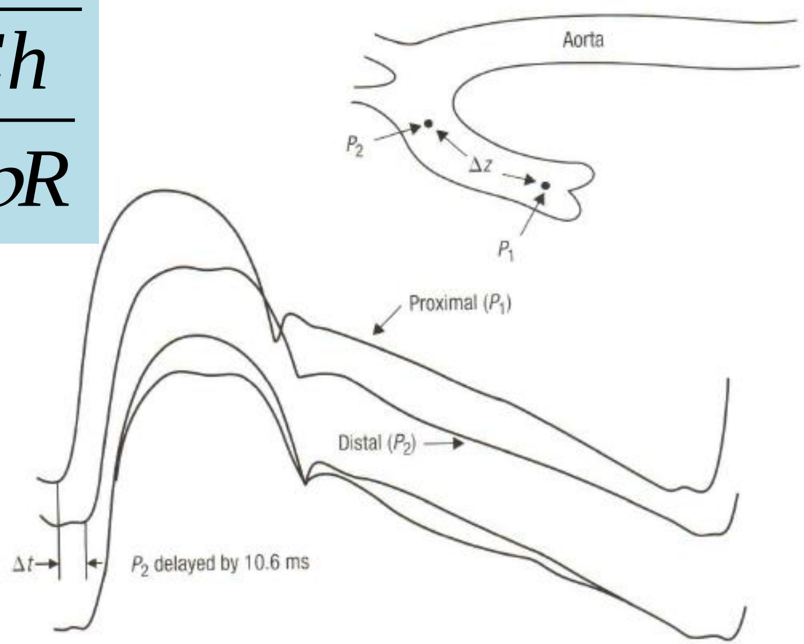


Figure 3.5 *Pressure (dual-sensor catheter) waves recorded in the ascending aorta of a dog. The pressure sensors were placed 5.0 cm apart on the catheter. The delay time (Δt) between the two pressures ($P_1 - P_2$) is the transmission time (= 10.6 ms) and when divided by distance (Δz) between the two pressure recording sites gives pulse wave velocity (c_0 = 472 cm/s)*

3.2粘性流体管内脉动流动的速度分布

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{\mu}\frac{\partial P}{\partial z}=\frac{\rho}{\mu}\frac{\partial w}{\partial t}=\frac{1}{\nu}\frac{\partial w}{\partial t}$$

$$\frac{\partial P}{\partial z}=A^* e^{i\omega t}$$

$$\omega = 2\pi f$$

简单的调和运动

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{\mu}A^* e^{i\omega t}=\frac{1}{\nu}\frac{\partial w}{\partial t} \rightarrow \frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{1}{\nu}\frac{\partial w}{\partial t}=-\frac{A^*}{\mu}e^{i\omega t}$$

$$\frac{\partial^2 (ue^{i\omega t})}{\partial r^2} + \frac{1}{r}\frac{\partial (ue^{i\omega t})}{\partial r} - \frac{1}{\nu}\frac{\partial (ue^{i\omega t})}{\partial t} = -\frac{A^*}{\mu}e^{i\omega t}$$

$$\frac{\partial^2 (u\cancel{e^{i\omega t}})}{\partial r^2} + \frac{1}{r}\frac{\partial (u\cancel{e^{i\omega t}})}{\partial r} - \boxed{\frac{1}{\nu} i\omega u\cancel{e^{i\omega t}}} = -\frac{A^*}{\mu}\cancel{e^{i\omega t}}$$

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{\nu} i\omega = -\frac{A^*}{\mu}$$

贝塞尔方程形式

$$u = \frac{A^*}{i\omega\rho}\left\{1 - \frac{J_0\left[r\sqrt{(\omega/\nu)}i^{3/2}\right]}{J_0\left[R\sqrt{(\omega/\nu)}i^{3/2}\right]}\right\}$$

Womersley 数

$$w=\frac{A^*}{i\omega\rho}\left\{1-\frac{J_0(\alpha y i^{3/2})}{J_0(\alpha i^{3/2})}\right\}e^{i\omega t}$$

$$J_0(xi^{3/2})$$

第一类零阶Bessel函数Dwight（1961）已列出具体数值

$$J_0(\alpha y i^{3/2})=M_0(y)e^{i\theta_0(y)}$$

$$J_0(\alpha i^{3/2})=M_0 e^{i\theta_0}$$

$$A^* e^{i\omega t}=M[\cos(\omega t-\phi)+i\sin(\omega t-\phi)]$$

$$e^{i\theta_0}=\cos\theta_0+i\sin\theta_0$$

取实部

$$w=\frac{M}{\omega\rho}\sin(\omega t-\phi)-\frac{M}{\omega\rho}\frac{M_0(y)}{M_0}\sin(\omega t-\phi-\delta_0)$$

在壁面处，速度为零，因为

$$\frac{M_0(y)}{M_0}=1$$

$$\delta_0=\theta_0-\theta_0(y)=0$$

$$w=\frac{M}{\omega\rho}\sin(\omega t-\phi)-\frac{M}{\omega\rho}\frac{M_0(y)}{M_0}\sin(\omega t-\phi-\delta_0)$$

$$h_0=\frac{M_0(y)}{M_0}$$

$$M_0^{'}=(1+h_0^2-2h_0\cos\delta_0)^{1/2}$$

$$\tan\varepsilon_0=\frac{h_0\sin\delta_0}{1-h_0\cos\delta_0}$$

$$w=\frac{M}{\omega\rho}M_0^{'}\sin(\omega t-\phi+\varepsilon_0)$$

$$\frac{M_0^{'}}{\alpha}=\frac{1-y^2}{4}$$

$$M\sin(\omega t-\phi+\varepsilon_0)=\frac{\Delta P}{L}$$

定常流动速度

$$\alpha^2=R^2\ \omega\rho/\mu)$$

$$w=\frac{MR^2}{\mu}\frac{M_0^{'}}{\alpha}\sin(\omega t-\phi+\varepsilon_0)$$

不同Womersley及角频率下管内速度分布

$$M = 1 \quad \phi = 0 \qquad w = \frac{R^2}{\mu} \frac{M_0^{'}}{\alpha} \sin(\omega t + \varepsilon_0)$$

✓频率f：2.8 Hz
✓从左至右：角频率依次为1，2，3，4倍
✓半个周期

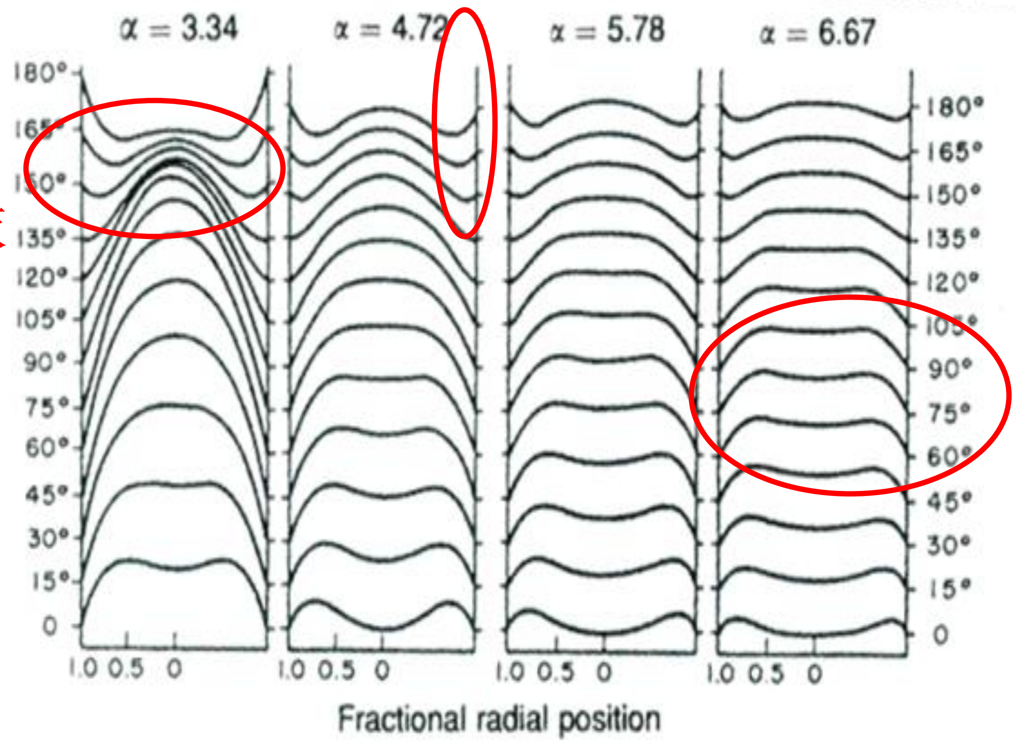


Fig. 3.9 Velocity profiles in a straight rigid tube at various points in the cycle of a sinusoidal pressure gradient for different values of α.

Womersley数

$$\alpha = R\sqrt{\frac{\omega}{\nu}} = R\sqrt{\frac{\omega\rho}{\mu}}$$

物理意义：表示流体脉动性的无量纲常数

当α<1时：流动为准定常，在任何时刻，截面上的速度都可以近似为Poiseuille 流；瞬时流量为

$$Q = Q_0 \sin \omega t$$

（Poiseuille流动流量+脉动）

当α值较大时，在截面上速度分布发生畸变，流量Q的相位落后于压力差的相位，

$$Q = Q_0 \sin[\omega(t - \phi)]$$

ϕ--滞后时间

即t时刻的流量由t-ϕ时刻的压力差给出

典型的几种动物和人的Womersley 数

	Aorta	Femoral Arteries
Mouse	1.5	
Rabbit	4	1.6
Dog	10	2.4
Human	15	3
Elephant	50	

距离大动脉弓7-10cm处的血管内流速分布- 测量值

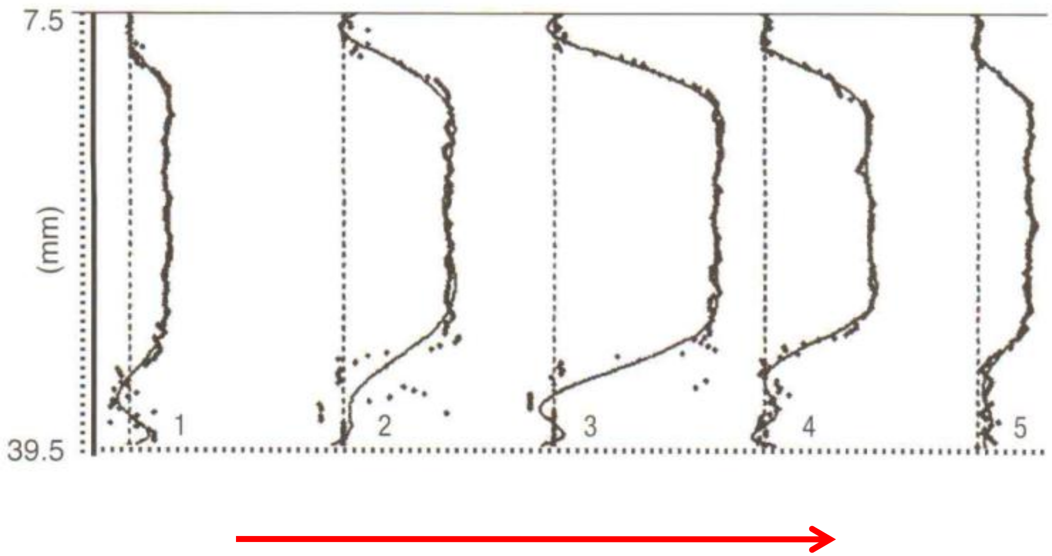


Figure 2.27 *Velocity profiles at various times in systole detected at a location 7–10 cm down from the aortic arch in a patient with no cardiovascular disease. From Tortoli* et al. *(2002)*

流动方向

脉动流动的流量

$$\frac{\partial P}{\partial z} = A^* e^{i\omega t}$$

简单的调和运动

$$w = \frac{A^*}{i\omega\rho}\left\{1 - \frac{J_0(\alpha y i^{3/2})}{J_0(\alpha i^{3/2})}\right\} e^{i\omega t}$$

速度变化

管内体积流量

$$Q = \frac{\pi R^2 A^*}{i\omega\rho}\left\{1 - \frac{2J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})}\right\} e^{i\omega t}$$

？ ？

$$A^* e^{i\omega t} = M[\cos(\omega t - \phi) + i\sin(\omega t - \phi)]$$

$$Q = \frac{R^2}{\omega\rho}[1 - F_{10}]\frac{A^*}{i}e^{i\omega t}$$

取实部

$$Q=\frac{\pi R^2}{\omega\rho}M\left[1-F_{10}\right]\sin(\omega t-\phi)$$

+

$$\left[1-F_{10}\right]=M_{10}^{'}e^{i\varepsilon_{10}}$$

写成三角函数型式

？？

$$Q=\frac{\pi R^2}{\omega\rho}MM_{10}^{'}\sin(\omega t-\phi+\varepsilon_{10})$$

$$\alpha=\frac{R^2\omega}{\nu}$$

$$Q=\frac{\pi R^4 M}{\mu}\frac{M_{10}^{'}}{\alpha^2}\sin(\omega t-\phi+\varepsilon_{10})$$

流量

$$\frac{M_{10}^{'}}{\alpha^2}\rightarrow 1/8 \qquad \alpha\rightarrow 0 \qquad \varepsilon_{10}\rightarrow 90^{o}$$

$$Q=\frac{\pi R^4\Delta P}{8\mu L}$$

Poiseuille Equation

Bessel函数的模量随Womersley变化的曲线

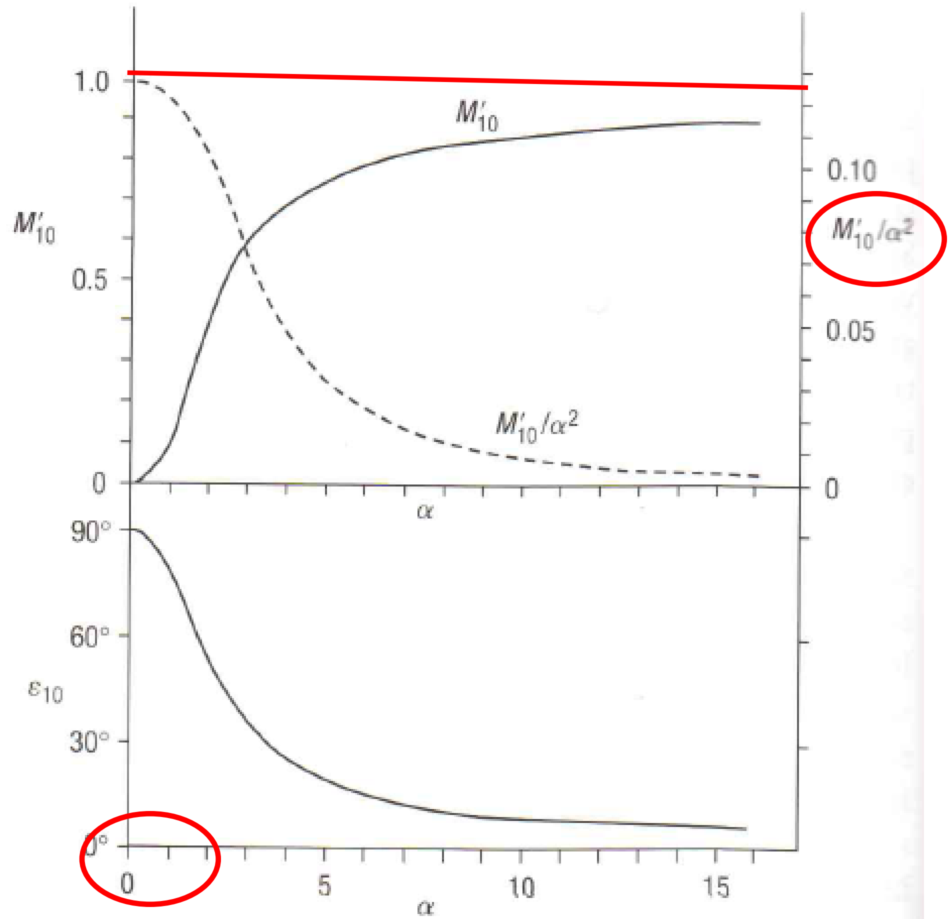


Figure 7.5 *The parameters* M'_{10}, M'_{10}/α^2 *and* ε_{10} *as functions of the non-dimensional constant* α. *As* α *approaches zero,* M'_{10}/α^2 *approaches 1/8, the numerical constant in the Poiseuille equation. The parameter* ε_{10} *approaches 90° as* α *approaches zero, whereas at high values of* α, ε_{10} *approaches 0°.* M'_{10} *approaches 1.0 at high values of* α. *Reproduced from McDonald (1974) as modified by Milnor (1989)*

不同病人的阻抗模量和相位随频率的变化

阻抗模量

$$|Z| = \frac{|P|}{|Q|} = \frac{\rho c'}{\pi R^2 M'_{10}}$$

表观相速

$$c' = \frac{\omega \Delta z}{\Delta \phi}$$

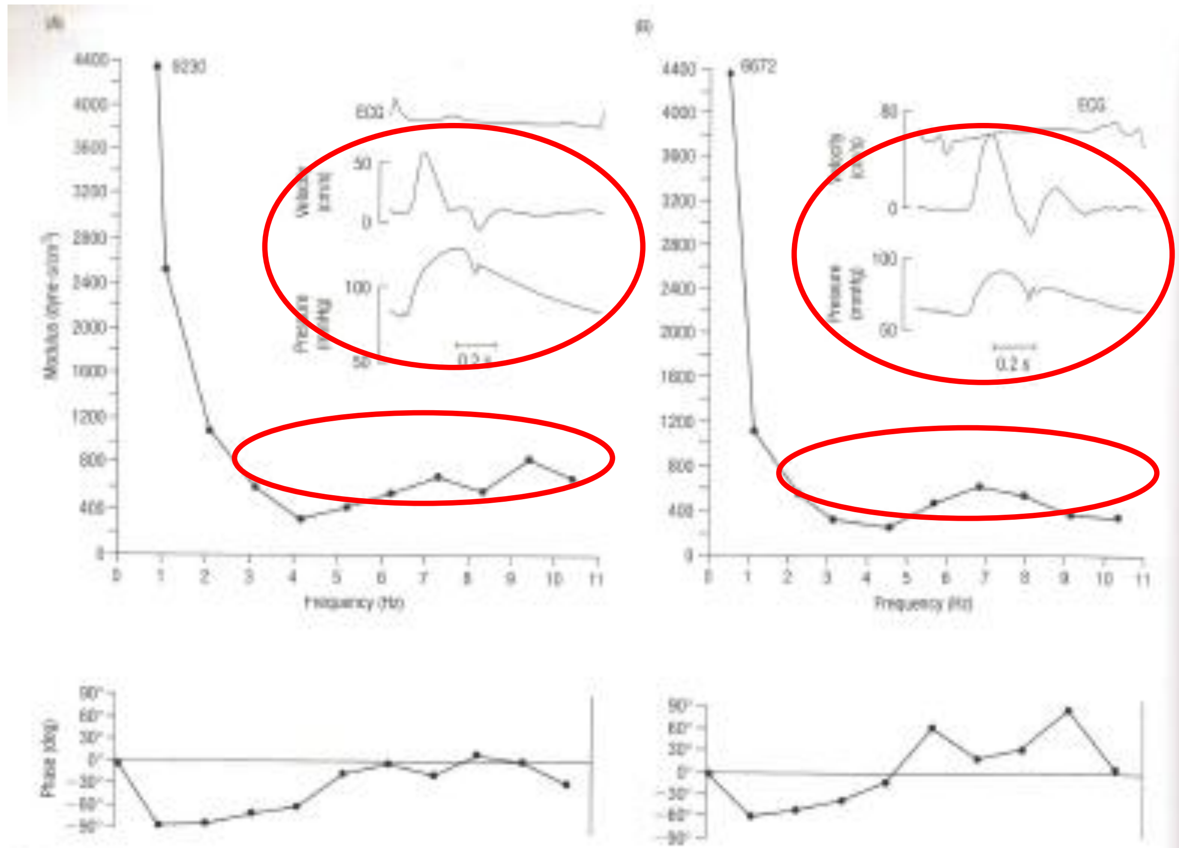


Figure 9.18 *Pressure and velocity waves* (**inset**) *in the brachiocephalic artery of two patients, together with the modulus* (**above**) *and phase* (**below**) *of impedance spectra determined from the waves. Similarity in the patterns of impedance indicates similarity of the physical properties of the vascular beds in the two patients, indicating that the different contours of flow and pressure waves are due to differences in duration of ventricular ejection, this being longer in the subject on the left with a type II wave and shorter in the patient on the right with a type III brachiocephalic flow. Reproduced from Mills et al. (1970)*

压力波的传播

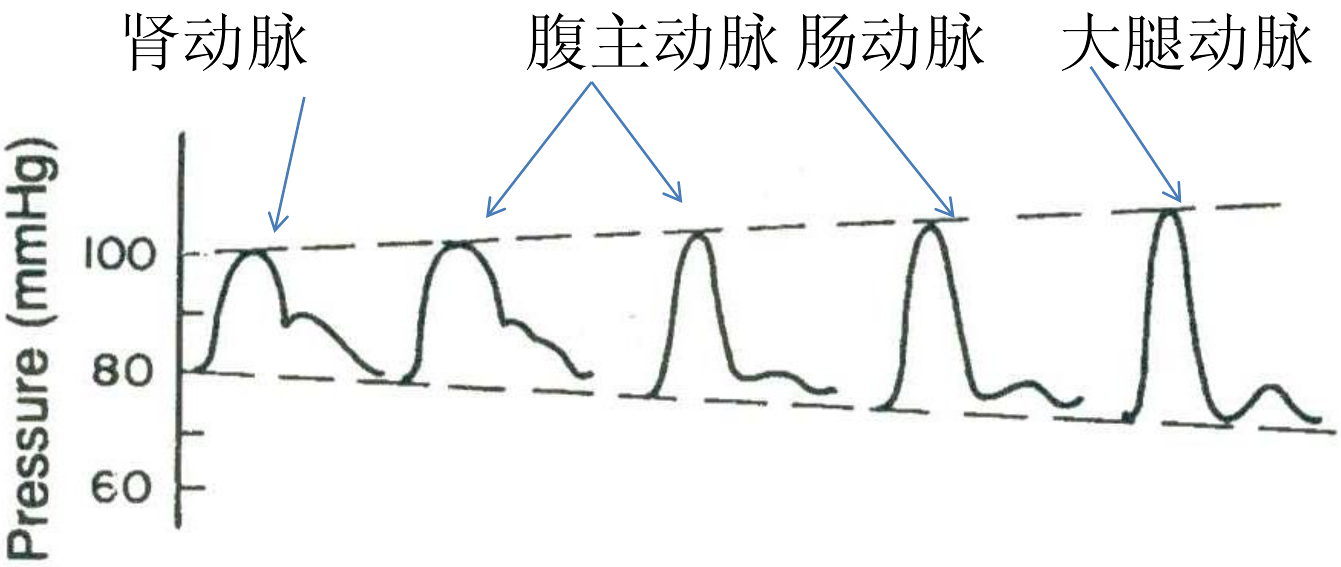
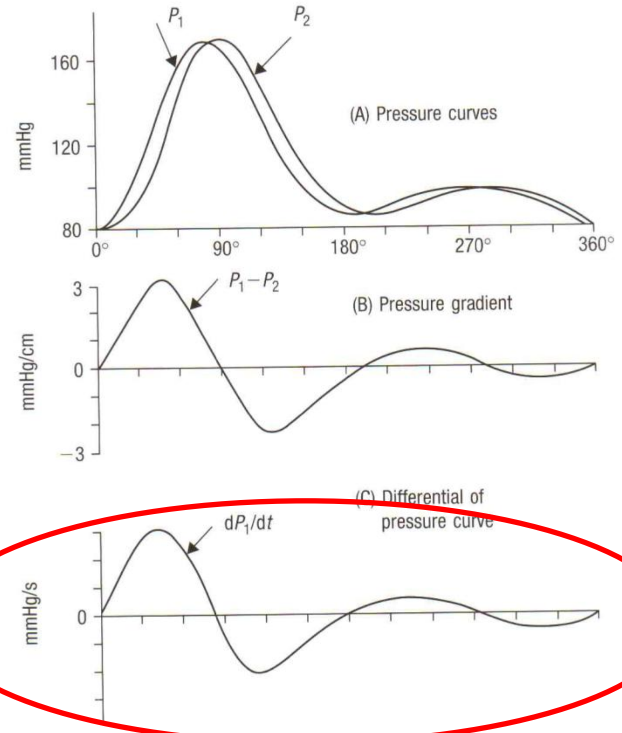


Fig. 3.11 Change in arterial pressure pulse with increasing distance from the heart.

◆从大动脉到大腿部动脉，压力波振幅逐渐增大，年轻人可达60%
◆老年人的压力波振幅增大不明显，且观察不到二次波动
◆原因：波速增加，血管扩张性减弱？？

脉动压力-流量关系



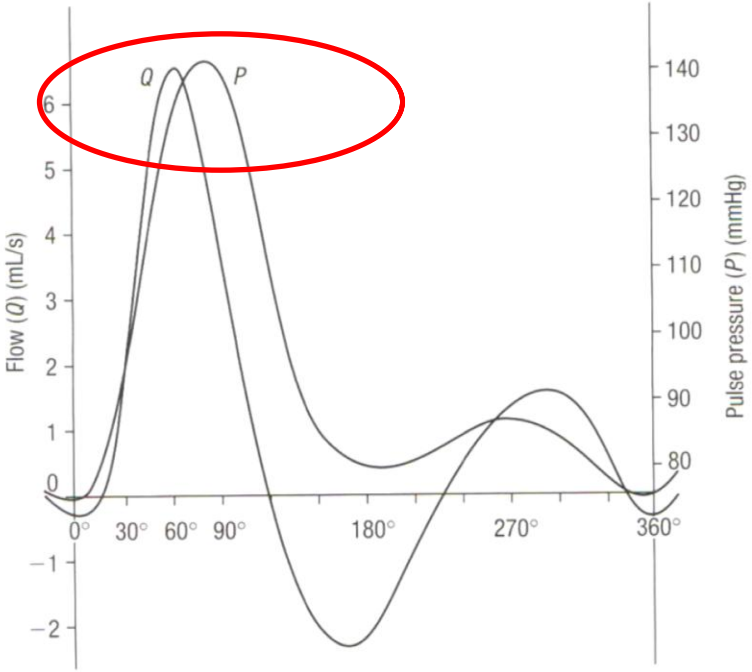


Figure 7.1 *A flow velocity pulse (Q) and the arterial pressure pulse (P) recorded simultaneously in the femoral artery of a dog. Flow velocity was recorded by high-speed cinematography, but the corresponding volume flow (mL/s) is plotted as the ordinate (left). Although superficially similar in shape when plotted on a comparable scale, the fact that the peak flow occurs before the pressure peak shows that there is no simple relation between these two curves. The flow is, in fact, determined by the pressure gradient (see text). The pulse frequency was 2.75 Hz in this experiment. The abscissa representing time is plotted as fractions of the cycle length, i.e. as degrees of arc. With pulse frequencies of this order one degree of arc is about 1.0 ms*

流量变化直接与压力梯度相关

- ✓压力波和流量波相位不相同
- ✓最大流量出现在最大压力之前

阻抗（impedance）

$$Q = \frac{\pi R^4 M}{\mu} \frac{M_{10}^{'}}{\alpha^2} \sin(\omega t - \phi + \varepsilon_{10})$$

$$\frac{\partial P}{\partial z} = A^* e^{i\omega t}$$

$$\bar{V} = \frac{Q}{\pi R^2} = \frac{A^* R^2}{i\mu} \frac{M_{10}^{'}}{\alpha^2} e^{i\varepsilon_{10}} e^{i\omega t}$$

$$Z_L = \frac{A^* e^{i\omega t}}{\bar{V}} = \frac{i\mu}{R^2} \frac{\alpha^2}{M_{10}^{'}} e^{-i\varepsilon_{10}}$$

$$= \frac{\mu\alpha^2}{R^2 M_{10}^{'}} \sin\varepsilon_{10} + \frac{i\mu\alpha^2}{R^2 M_{10}^{'}} \cos\varepsilon_{10}$$

$$= R + i\omega L$$

$$= \frac{\omega\rho}{M_{10}^{'}} \sin\varepsilon_{10} + \frac{i\omega\rho}{M_{10}^{'}} \cos\varepsilon_{10}$$

$$Z_L = \frac{\omega\rho}{M'_{10}}\sin\varepsilon_{10} + \frac{i\omega\rho}{M'_{10}}\cos\varepsilon_{10}$$

$$Z_0 = \frac{cZ_L}{i\omega}$$

$$Z_0 = \frac{\rho c}{M'_{10}} e^{-i\varepsilon_{10}}$$

$$\frac{c_0}{c} = \frac{\sqrt{(1-\sigma^2)}}{(M'_{10})^{1/2}} \exp\left(-i\tfrac{1}{2}\epsilon'_{10}\right)$$

$$Z_0 = \frac{\rho c_0}{\sqrt{1-\sigma^2}\,(M'_{10})^{1/2}} e^{-i\varepsilon/2}$$

？？

泊松比

Z_0

◆特征阻抗：无反射波时脉动压力对流量的比

◆输入阻抗：管进口处被反射波修正后的脉动压力与流量之比

刚性管内流阻和感应项随Womersley数的变化

Womersley数小于3时，与Poiseuille流动有相同阻抗

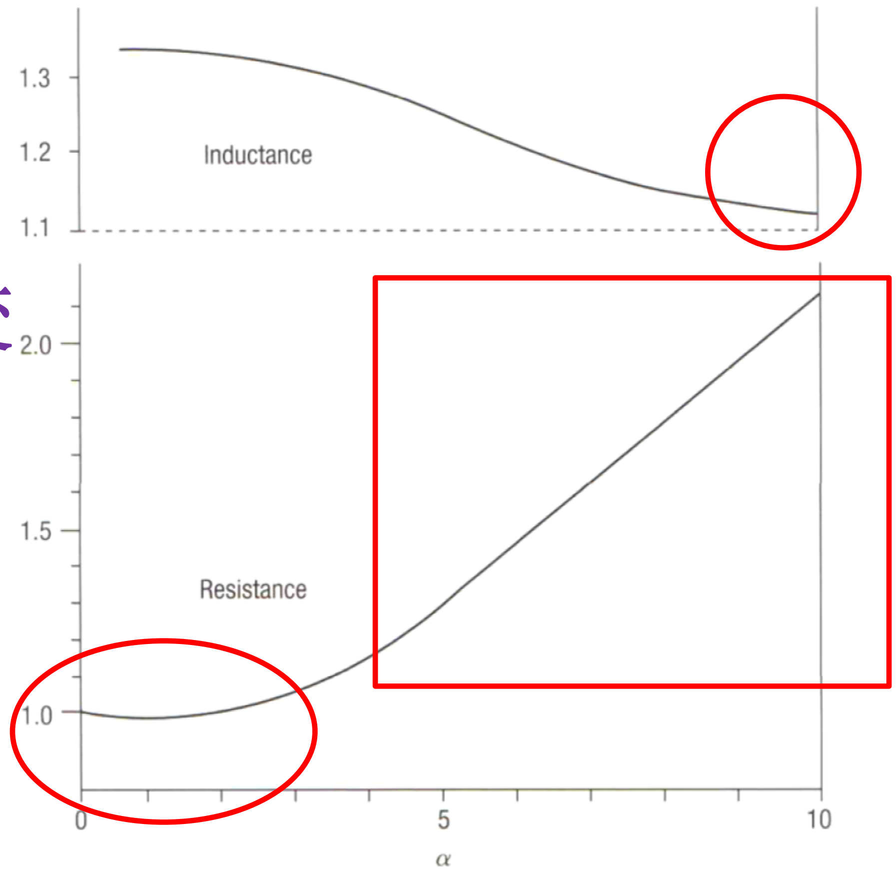


Figure 7.9 *Inductance (eqn 7.25) and resistance for oscillatory flow in a rigid tube plotted as a function of α. The values for resistance are given relative to the steady-flow, or Poiseuille, resistance and are 1/8 times that given by eqn 7.21. As α is a function of the square root of the frequency, it can be seen that both these components of the impedance are frequency-dependent, in contrast to the analogous electrical quantities. The application of the resistance in oscillatory flow to the manometer equations is discussed in Chapter 6*

思考题

1. 根据动量方程，弹性模型推导波动方程
2. 管内粘性流体脉动速度分布
3. 管内脉动流动阻抗的定义及公式