The Theory and Practice of Cosmological Perturbations

Part I -- Linear Fluctuations

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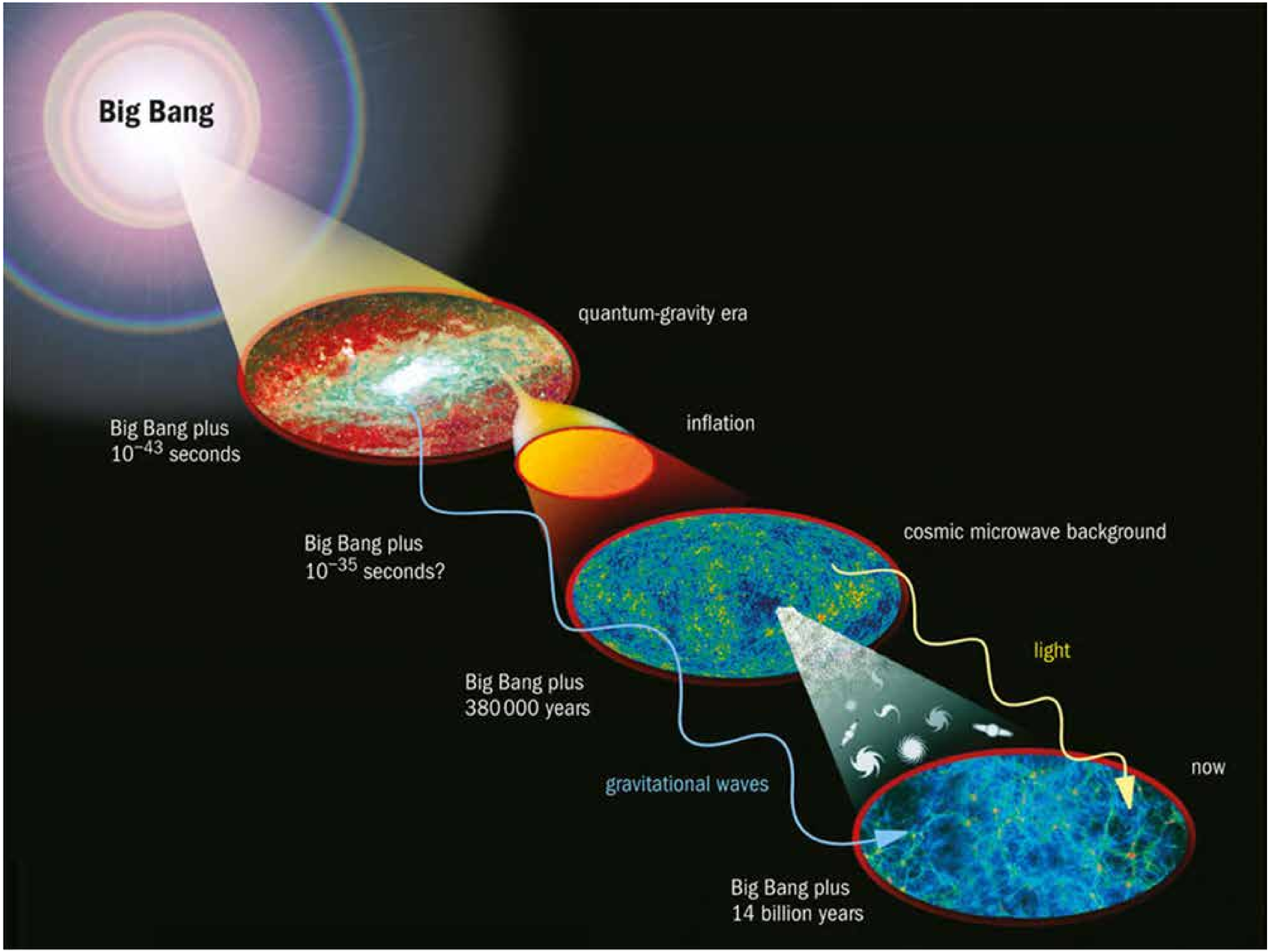
Where does structures in our universe come from?


Big Bang
quantum-gravity era
Big Bang plus
10^-43 seconds
inflation
Big Bang plus
10^-35 seconds?
cosmic microwave background
light
Big Bang plus
380 000 years
gravitational waves
now
Big Bang plus
14 billion years

Where does structures in our universe come from?

What can we learn from those structures?

- Better understanding of the late universe

Where does structures in our universe come from?

What can we learn from those structures?

- Better understanding of the late universe
- Understanding the primordial universe

Where does structures in our universe come from?

What can we learn from those structures?

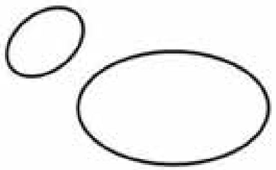
- Better understanding of the late universe
- Understanding the primordial universe
- Understanding the particle physics of the primordial universe

Plan:

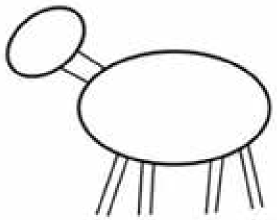
Part I – Linear Fluctuations

Part II – Computer Assisted Computation

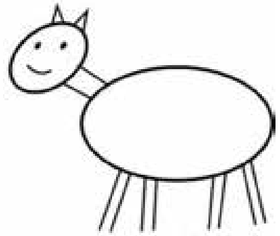
Lecture notes:
How to draw a horse?



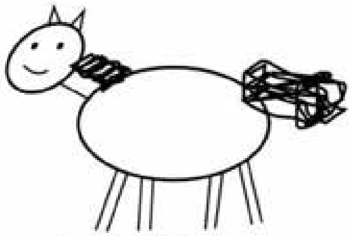
1. draw two circles



2. add neck and feet



3. add face



4. add hair

5. finally add some details,
That's it!

Plan:

Part I – Linear Fluctuations

Part II – Computer Assisted Computation

Part III – Nonlinear Fluctuations

Part IV – Inflationary Massive Fields

Part V – Computational Techniques

Apologize for the incompleteness of references.

Most references can be found at 1303.1523.

Part I – Linear Fluctuations

Fluctuations:

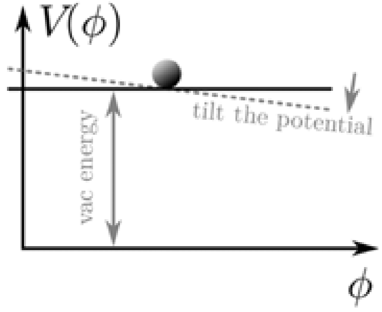
on top of homogeneous & isotropic background

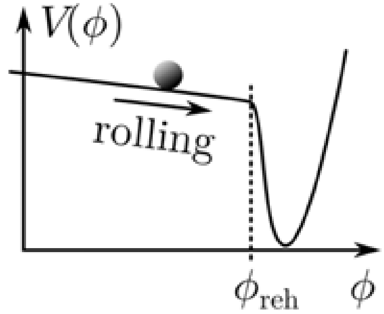
$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$, similarly for $g_{\mu\nu}(\mathbf{x}, t)$

Linear:

- Linear equation of motion (EoM)
- Variation of 2nd order action
- Results in a Gaussian random field
- The statistics is 2-point correlation function

A brief review of the inflationary background





Minimal model: $\mathcal{L}_\phi = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = a^3\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]$

Metric: $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2 = a(\tau)^2(-d\tau^2 + d\mathbf{x}^2)$

Scale factor: $a(t) \approx e^{Ht} \approx -1/(H\tau)$ Early: $\tau \to -\infty$, late: $\tau \to 0$

EoM: $3M_p^2H^2 = \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ $\qquad \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$

Slow roll: $\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \qquad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}$

Small
Higgs
Arctan
Mixed
First
Hilltop
Hybrid
Small
Higgs
Law
Field
Gauge
Valley
Brane
Brane
Inverse
String
Kahler
Witten
Pseudo
Law
Mixed
Kahler
Valley
Loop
Susy
Tip
Well
Large
Mass
First
Non
Horizon
Twisted
Quartic
Exponential
Hilltop
Mixed
Twisted
Arctan
Kahler
Brane
Canonical
Quartic
Double
Inflation
Orientifold
Massive
Inverse
Power
Mutated
Natural
Mixed
Well
Loop
Hilltop
Large
Kahler
Mssm
Quartic
Inflation
Massive
Spontaneous
Tachyonic
Potential
Non
Field
Witten
Order
Supergravity
Brane
Horizon
Constant
Open
Generalized
Renormalizable
Colemann
String
Corrected
Non
Tip
Radion
Susy
Large
Inverse
Twisted
Mass
Inflection
Mutated
Moduli
Large
Field
Law
Tip
Colemann
Spectrum
Power
Small
Dynamical
Radion
First
Gauge
Natural
Hybrid
Law
Flow
String
Oraifeartaigh
Potential
Moduli
Mssm
Inflection
Corrected
Susy
Flow
Loop
Breaking
Mssm
Symmetry
Radiatively
Moduli
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Well
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Gauge
Constant
First
Pseudo
Pseudo
Orientifold
Canonical
Renormalizable
Logarithmic

A Spectator Field in de Sitter

A Spectator Field in de Sitter

warm-up exercise

Two approximations:

- de Sitter: $a = e^{Ht}$
    - Ignore slow roll: $\epsilon \propto \dot{H} = 0$
    - Ignore the end of inflation
- Spectator:
    - No back-reaction to FRW
    - Massless $S = \int d^3x \, dt \, a^3 \left[\frac{1}{2}\dot{\sigma}^2 - \frac{1}{a^2}(\partial_i \sigma)^2 \right]$

A Spectator Field in de Sitter

$$S = \int d^3x \; dt \; a^3 \left[\frac{1}{2}\dot{\sigma}^2 - \frac{1}{a^2}(\partial_i \sigma)^2 \right]$$

Roughly: quantum fluctuations → classical fluctuations

May be understood in a few ways:

- Cosmological Schwinger effect
- Stretch of the vacuum wave function of σ
- Broken of WKB & particle production
- Quantum fluctuation in a “finite” box (outside: frozen)
- Explicit calculations (will do now)

A Spectator Field in de Sitter

$$S = \int d^3x \; dt \; a^3 \left[\frac{1}{2}\dot{\sigma}^2 - \frac{1}{a^2}(\partial_i \sigma)^2 \right]$$

Fourier transform

$$S = \int \frac{d^3k}{(2\pi)^3} \; dt \; a^3 \left[\frac{1}{2}\dot{\sigma}_{\mathbf{k}}\dot{\sigma}_{-\mathbf{k}} - \frac{k^2}{a^2}\sigma_{\mathbf{k}}\sigma_{-\mathbf{k}} \right]$$

Expand into "mode function" + creation/annihilation operators

$$\sigma_{\mathbf{k}}(t) = v_k(t) a_{\mathbf{k}} + v_k^*(t) a_{-\mathbf{k}}^\dagger$$

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$$

A Spectator Field in de Sitter

$$S = \int d^3x\ dt\ a^3 \left[\frac{1}{2}\dot{\sigma}^2 - \frac{1}{a^2}(\partial_i\sigma)^2\right]$$

Fourier transform

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Expand into "mode function" + creation/annihilation operators

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \qquad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$$

$$\sigma_{\mathbf{k}}(t) = v_k(t)a_{\mathbf{k}} + v_k^*(t)a^\dagger_{-\mathbf{k}}$$

$$(av_k)'' + \left(k^2 - \frac{a''}{a}\right)(av_k) = 0$$

$$v_k = c_1(1+ik\tau)e^{-ik\tau} + c_2(1-ik\tau)e^{ik\tau}$$

How to decide c_1 and c_2?

A Spectator Field in de Sitter

$$\sigma_{\mathbf{k}}(t) = v_k(t) a_{\mathbf{k}} + v_k^*(t) a_{-\mathbf{k}}^\dagger$$

$$v_k = c_1(1 + ik\tau)e^{-ik\tau} + c_2(1 - ik\tau)e^{ik\tau}$$

How to decide c_1 and c_2?

Method 1: Compare with flat QFT

Method 2:

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$$

$$\pi_{\mathbf{k}} \equiv \frac{\delta S}{\delta \dot{\sigma}_{\mathbf{k}}} = a^3 \dot{\sigma}_{-\mathrm{k}} \ , \qquad [\sigma_{\mathbf{k}}, \pi_{\mathbf{k}'}] = i(2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$$

consistent

$$c_1 c_1^* - c_2 c_2^* = \frac{H^2}{2k^3}$$

Vacuum has lowest energy

$\mathcal{H} \propto c_1 c_1^* + c_2 c_2^*$ at $\tau \to -\infty$

$$|c_1| = \frac{H}{\sqrt{2k^3}} \ , \qquad c_2 = 0$$

$$v_k(\tau) = \frac{H}{\sqrt{2k^3}}(1 + ik\tau)e^{-ik\tau}$$

A Spectator Field in de Sitter

$$\sigma_{\mathbf{k}}(t) = v_k(t) a_{\mathbf{k}} + v_k^*(t) a^\dagger_{-\mathbf{k}}$$

$$v_k(\tau) = \frac{H}{\sqrt{2k^3}}(1 + ik\tau)e^{-ik\tau}$$

The 1-point statistics vanishes: $\langle 0|\sigma_{\mathbf{k}}|0\rangle = 0$

The 2-point statistics is non-trivial:

$$\langle \sigma_{\mathbf{k}} \sigma_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P_\sigma(k) \ , \qquad P_\sigma(k) = \left(\frac{H}{2\pi}\right)^2$$

space (but not FRW time) translation symmetry

power spectrum

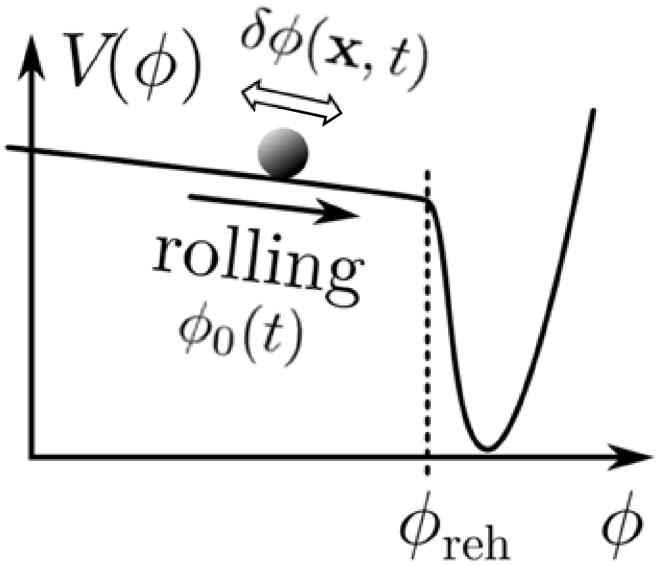
scale-invariant (of dS space)

From Spectator to Inflaton (Roughly)

$$\langle \sigma_{\mathbf{k}} \sigma_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P_\sigma(k) \,, \qquad P_\sigma(k) = \left(\frac{H}{2\pi} \right)^2$$

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t)$$

$$P_\zeta = \frac{H^2}{\dot\phi^2} P_\phi = \frac{H^4}{4\pi^2 \dot\phi^2} = \frac{H^2}{8\pi^2 \epsilon M_p^2}$$



Fluctuate down with $\delta\phi$

→ shorter e-fold $\zeta \equiv \delta N = -H\delta t = -H\delta\phi/\dot\phi_0$

→ earlier reheating → energy drop faster

→ lower energy density → hotter spot on CMB

Massive Spectator Fields

$$\ddot{v}_k + 3H\dot{v}_k + \frac{k^2}{a^2}v_k + m^2 v_k = 0$$

$$v_{\mathbf{k}} = -ie^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\frac{\sqrt{\pi}}{2}H(-\tau)^{3/2}H_\nu^{(1)}(-k\tau)$$

coefficient,

and choice of $H^{(1)}$:

match massless

at $\tau \to -\infty$

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

$$H_\nu^{(1)}(x) \equiv J_\nu(x) + iY_\nu(x)$$

Massive Spectator Fields

$$\ddot{v}_k + 3H\dot{v}_k + \frac{k^2}{a^2}v_k + m^2 v_k = 0$$

$$v_{\mathbf{k}} = -ie^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\frac{\sqrt{\pi}}{2}H(-\tau)^{3/2}H_\nu^{(1)}(-k\tau)$$

```
LogLinearPlot[
 Evaluate[Re[e^(i π ν/2) * Sqrt[π]/2 (x)^(3/2) HankelH1[ν, x]] /.
   {{ν → 3 / 2}, {ν → 1 / 2}, {ν → 0}, {ν → 3 i}}], {x, 0.001, 10},
 PlotLegends → {"ν=3/2  (m=0)", "ν=1/2  (m=√2 H)", "ν=0  (m=3H/2)",
   "ν=3i  (m=3.4H)"}]
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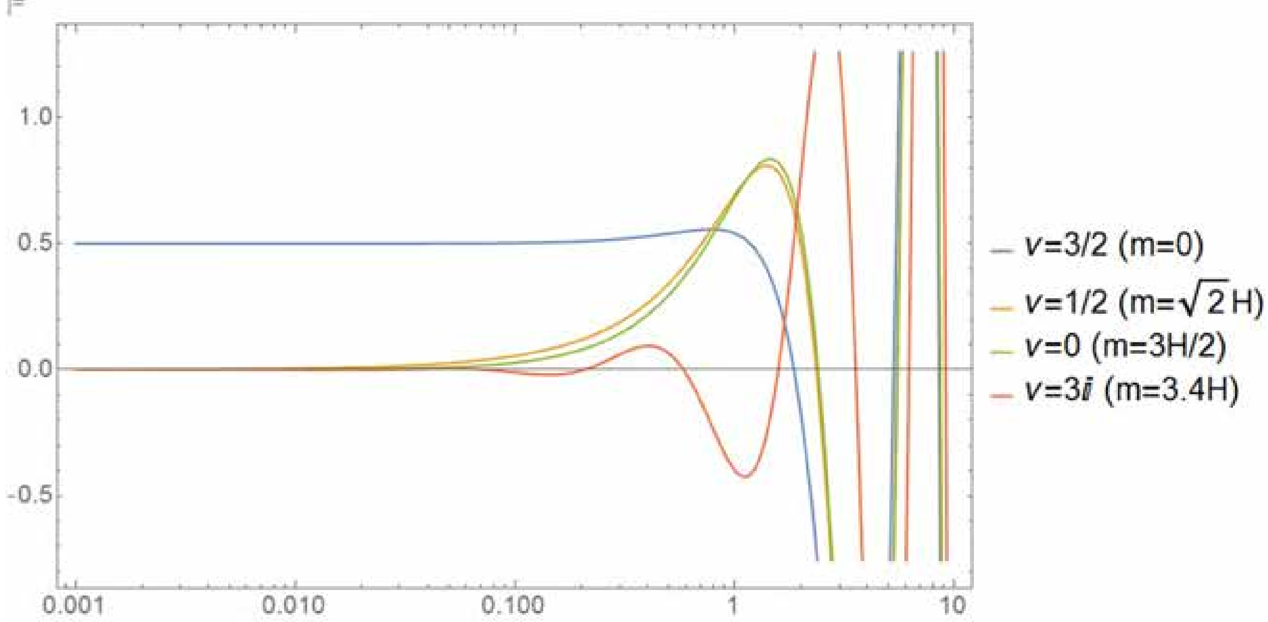
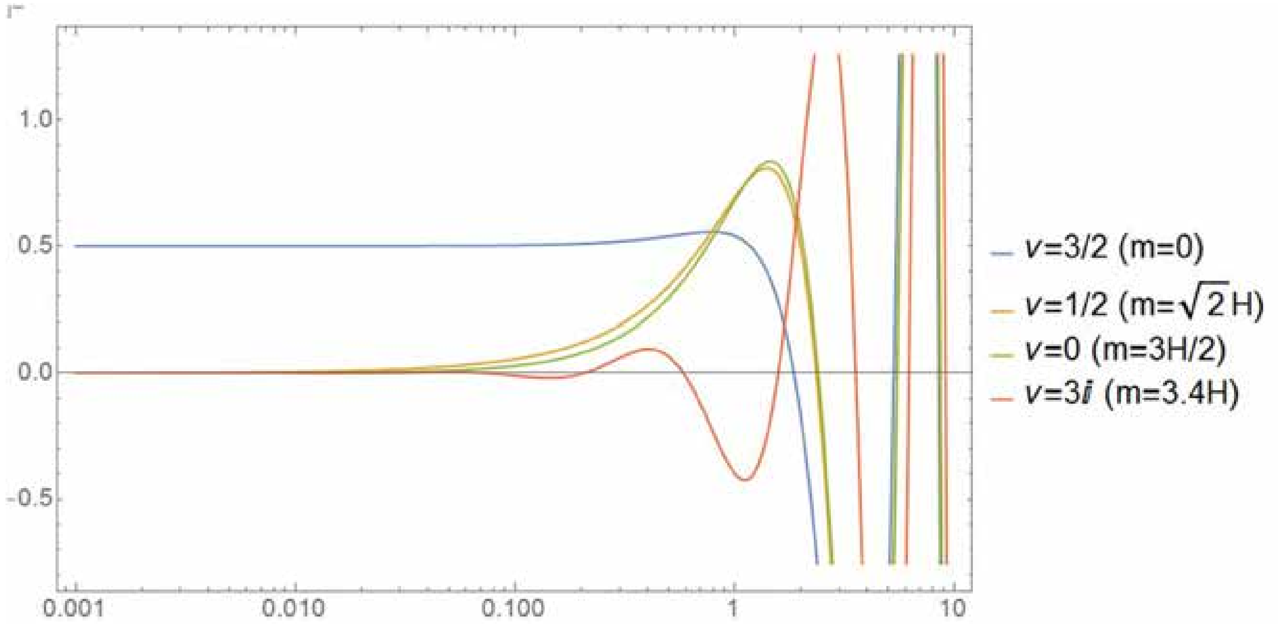
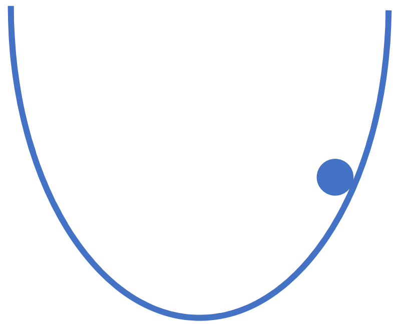


Massive Spectator Fields

$\frac{m^2}{H^2} < O(\epsilon, \eta)$: Field fluctuation exists until the end of inflation

$\frac{m^2}{H^2} > O(\epsilon, \eta)$: Field fluctuation decays away before the end of inflation

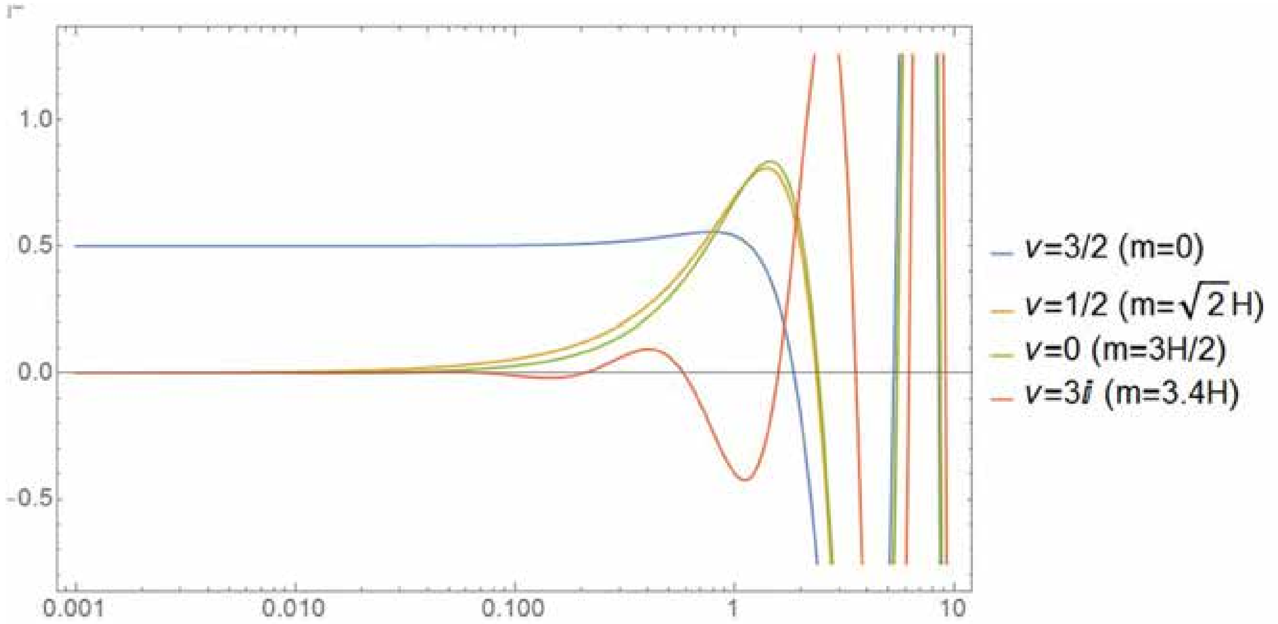


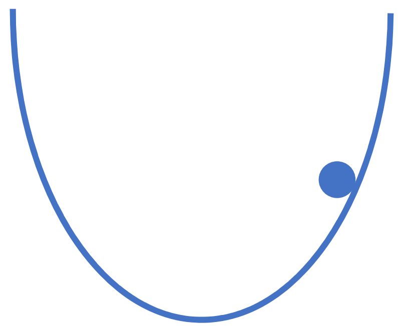


Massive Spectator Fields

$O(\epsilon,\eta) < \frac{m^2}{H^2} < \frac{9}{4}$: Over-damped oscillator

$\frac{m^2}{H^2} > \frac{9}{4}$: Under-damped oscillator





The Metric Fluctuations

Motivation:

Gravity tells matter how to move; Matter tells gravity how to curve.

FRW

$\sqrt{-g},\ g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

scalar fields, etc

$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

$T_{\mu\nu} \supset \partial_\mu\phi\partial_\nu\phi, V(\phi)$

$\phi = \phi_0(t) + \delta\phi(\mathbf{x}, t)$

$\phi_0 = 0 \rightarrow T_{\mu\nu} \sim O(\delta\phi^2) \rightarrow \delta g_{\mu\nu}$ is small

$\phi_0 \neq 0 \rightarrow T_{\mu\nu} \sim O(\delta\phi) \rightarrow \delta g_{\mu\nu}$ may be large

True for the inflaton!

How to Perturb the Metric?

Consider $g_{\mu\nu}(\mathbf{x}, t) = g_{\mu\nu}^{(0)}(t) + \delta g_{\mu\nu}(\mathbf{x}, t)$, where $g_{\mu\nu}^{(0)}$ is the FRW metric.

We need to further decompose $\delta g_{\mu\nu}(\mathbf{x}, t)$,

because the space-time symmetry is spontaneously broken by $\phi_0(t)$ and $a(t)$.

How to decompose $\delta g_{\mu\nu}(\mathbf{x}, t)$?

How to Perturb the Metric?

Consider $g_{\mu\nu}(\mathbf{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\mathbf{x},t)$, where $g^{(0)}_{\mu\nu}$ is the FRW metric.

We need to further decompose $\delta g_{\mu\nu}(\mathbf{x},t)$,

because the space-time symmetry is spontaneously broken by $\phi_0(t)$ and $a(t)$.

How to decompose $\delta g_{\mu\nu}(\mathbf{x},t)$?

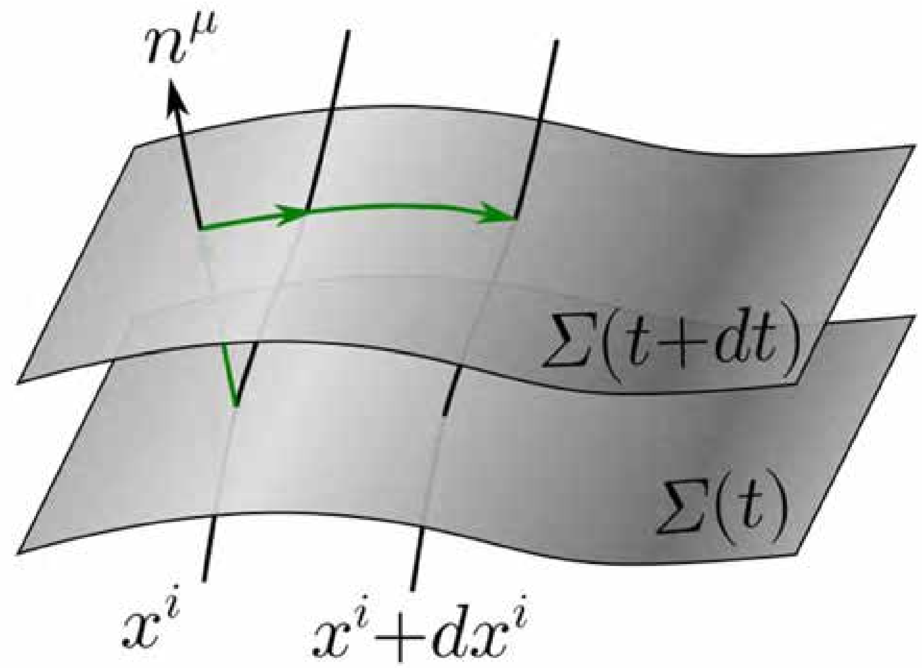
Remaining 3d symmetry.

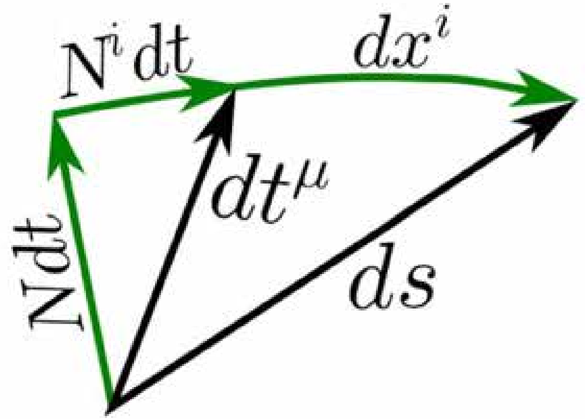
(4D tensor) → (3D scalar) × 4 + (3D div-free vector) × 2 + (3D tensor) × 1

Here we use the ADM decomposition.

ADM Decomposition of $g_{\mu\nu}$

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$





ADM Decomposition of $g_{\mu\nu}$

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

Some quantities can be calculated in closed form:

$$\sqrt{-g} = N\sqrt{h}$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^k N_k & N_i \\ N_j & h_{ij} \end{pmatrix} \quad \rightarrow \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^j/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix}$$

where $N_i \equiv h_{ij} N^j$, and h^{ij} is the inverse matrix of h_{ij} (note that in general $h^{ij} \neq g^{ij}$)

ADM Decomposition of $g_{\mu\nu}$

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

The Ricci scalar has a convenient form:

$$n_\mu = (-N, 0, 0, 0) = -\delta^0_\mu / \sqrt{-g^{00}} \, , \qquad n^\mu = (1/N, -N^i/N)$$

$$K_{\mu\nu} = (\delta^\sigma_\mu + n^\sigma n_\mu)\nabla_\sigma n_\nu = \frac{1}{2}\mathcal{L}_n (g_{\mu\nu} + n_\mu n_\nu)$$

$$K^{0\mu} = 0 \, , \qquad K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right]$$

$$R = \underline{{}^{(3)}R - K^2 + K^\mu_\nu K^\nu_\mu} - 2\nabla_\mu\left(-K n^\mu + n^\nu \nabla_\nu n^\mu\right)$$

Much easier to compute, if the boundary term can be dropped.

ADM Decomposition of $g_{\mu\nu}$

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

No time derivative on N and N_i
In the Einstein-Hilbert action.
Thus N and N_i can be solved as constraints.

$$K^{0\mu} = 0 \, , \qquad K_{ij} = \frac{1}{2N}\left[\dot{h}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right]$$

$$R = {}^{(3)}R - K^2 + K^{\mu}_{\nu} K^{\nu}_{\mu} - 2\nabla_{\mu}\left(-K n^{\mu} + n^{\nu}\nabla_{\nu} n^{\mu}\right)$$

ADM Decomposition of $g_{\mu\nu}$

$$ds^2 = -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

Further decomposing h_{ij}:

$$N_i = \partial_i \beta + b_i \ , \qquad h_{ij} = e^{2\zeta} a^2 \left[\delta_{ij} + \gamma_{ij} + \mathcal{D}_i \mathcal{D}_j E + \mathcal{D}_i F_j + \mathcal{D}_j F_i\right]$$

$$\mathcal{D}_i b^i = 0 \ , \qquad \mathcal{D}_i F^i = 0 \qquad\qquad \mathcal{D}_i \gamma^{ij} = 0 \ , \qquad \gamma^i_i = 0$$

Scalar sector: α, β, ζ, E

Vector sector: b_i, F_i

Tensor sector: γ_{ij}

Gauge Invariance and Fixing

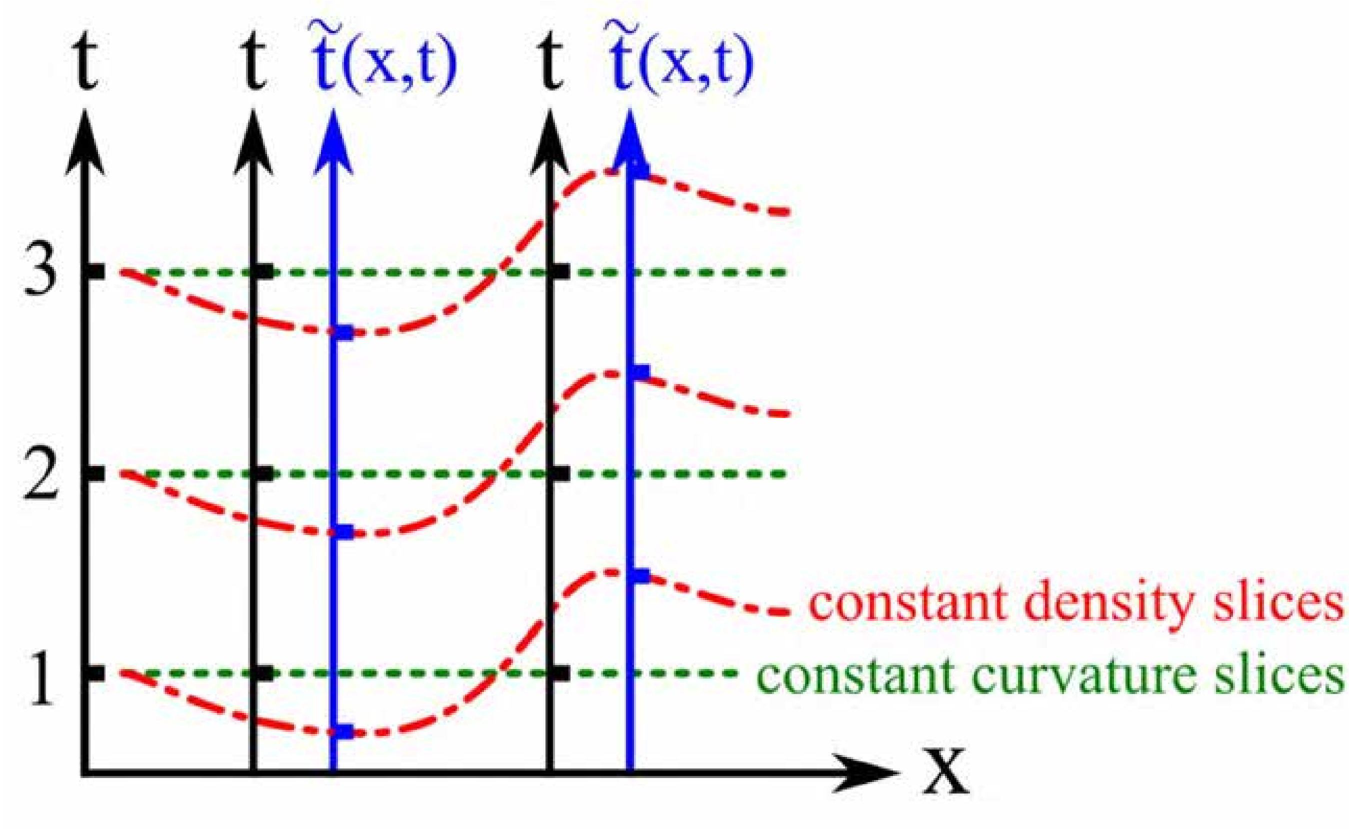
GR: we are free to choose coordinates ("gauge"): $\widetilde{x}^\mu = x^\mu + \xi^\mu(x)$

$$\delta_\xi g_{\alpha\beta}(x) \equiv \widetilde{g}_{\alpha\beta}(x) - g_{\alpha\beta}(x) = \widetilde{g}_{\alpha\beta}(x) - (\delta^\mu_\alpha + \partial_\alpha \xi^\mu)(\delta^\nu_\beta + \partial_\beta \xi^\nu)\widetilde{g}_{\mu\nu}(x+\xi)$$

$$\delta_\xi \phi(x) \equiv \widetilde{\phi}(x) - \phi(x) = \widetilde{\phi}(x) - \widetilde{\phi}(x+\xi)$$

Gauge Invariance and Fixing

An explicit example: freedom of choosing equal-time slices



Gauge Invariance and Fixing

How to get gauge (coordinate choice) independent predictions?

Method 1: Choose gauge invariant combinations.

For example, under $t \rightarrow \tilde{t} = t + \delta t(t, x)$, we have

$$\widetilde{\zeta} = H\delta t \ , \qquad \widetilde{\delta\phi} = -\dot{\phi}_0 \delta t$$

Thus the combination $\zeta^{\mathrm{gi}} \equiv \zeta + H\frac{\delta\phi}{\dot{\phi}}$

is gauge invariant at linear order.

Gauge Invariance and Fixing

How to get gauge (coordinate choice) independent predictions?

Method 2: To fix a gauge. Typically, we choose $E = 0$.

Many choices for the other gauge condition:

- Spatial flat gauge ($\delta\phi$-gauge): Choose $\zeta = 0$.

  Pros: intuitive and simple (minimize metric fluctuations)

- Uniform inflaton gauge (ζ-gauge): Choose $\delta\phi = 0$.

  Pros: ζ is conserved on super-Hubble (if no isocurvature)

- Newtonian gauge: scalar part of shift vector $\beta = 0$

  $$ds^2 = -(1+2\Phi)dt^2 + a^2(1-2\Psi)d\mathbf{x}^2$$

  Pros: Φ is Newtonian potential → connect to astrophysics

Gravitational Perturbations

The way of calculating linear cosmological perturbations with gravity:

1. Choose a gauge, identify the perturbation variables
2. Expand the action to second order in perturbations
3. Transform into Fourier space, with $\partial_i \to -ik_i$
4. Solve the constraints N and N_i
5. Insert the constraints into the second order action
6. Do IBP to bring the action into standard form
7. Quantize the fields using $u_k(\tau)a_{\mathbf{k}} + u_k^*(\tau)a_{\mathbf{k}}^\dagger$
8. Derive and solve the classical EoM for $u_k(\tau)$
9. $[a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger]$, $[\phi, \Pi]$, and vacuum → integration constants
10. Calculate the 2-point correlation function

} Done with a spectator field (steps 7–10)

Perturbations in the $\delta\phi$-Gauge

$$\delta\phi \equiv \phi(x) - \phi_0(t) \ , \qquad N \equiv 1 + \alpha \ , \qquad N_i \equiv \partial_i \beta$$

$$S_2 = \int dt \frac{d^3k}{(2\pi)^3} \Big[a^3(\epsilon - 3) M_p^2 H^2 \alpha^2 + 2k^2 M_p^2 a H \alpha\beta - a^3 V' \alpha \delta\phi - a^3 \dot{\phi}_0 \alpha \dot{\delta\phi} - k^2 a \dot{\phi}_0 \beta \delta\phi + \frac{1}{2} a^3 \dot{\delta\phi}^2 - \frac{1}{2} a (k^2 + a^2 V'') \delta\phi^2 \Big] \ .$$

$$\alpha = \frac{\dot{\phi}_0 \delta\phi}{2H} \ , \qquad \beta = \frac{a^2}{2Hk^2} (\dot{\phi}_0 \dot{\delta\phi} - H\epsilon \dot{\phi}_0 \delta\phi - a^2 \ddot{\phi}_0 \delta\phi)$$

$$S = \int dt \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2} a^3 \dot{\delta\phi}^2 - \frac{1}{2} k^2 a \delta\phi^2 - \frac{1}{2} a^3 \left[V'' - 2H^2 (3\epsilon - \epsilon^2 + \epsilon\eta) \right] \delta\phi^2 \right\}$$

Perturbations in the $\delta\phi$-Gauge

$$S = \int dt \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{2} a^3 \dot{\delta\phi}^2 - \frac{1}{2} k^2 a \delta\phi^2 - \frac{1}{2} a^3 \left[V'' - 2H^2(3\epsilon - \epsilon^2 + \epsilon\eta) \right] \delta\phi^2 \right\}$$

Effective mass: $m_{\text{eff}}^2 = V'' - 2H^2(3\epsilon - \epsilon^2 + \epsilon\eta)$

1. This is different from $V'' \sim \eta_V H^2$, if ϵ is not too small (wait for GW to tell!)
2. Still small compared to H^2 (energy scale of the perturbations).
3. Can we neglect this small mass?

   Should be careful on super-Hubble, because $N_e \times \eta \sim O(1)$

   Will return to this issue in ζ-gauge

Power spectrum (under slow roll approximation): $P_{\delta\phi} = \left(\frac{H}{2\pi} \right)^2$

Perturbations in the ζ-Gauge

$$S_2 = M_p^2 \int dt \frac{d^3k}{(2\pi)^3} \Big[-a^3(3-\epsilon)H^2\alpha^2 + (2aHk^2\beta + 2ak^2\zeta + 6a^3H\dot{\zeta})\alpha - 2k^2a\beta\dot{\zeta} - 3a^3\dot{\zeta}^2 - 18a^3H\zeta\dot{\zeta} + a(-9(3-\epsilon)a^2H^2 + k^2)\zeta^2 \Big] .$$

$$\alpha = \frac{\dot{\zeta}}{H} , \qquad \beta = -\frac{\zeta}{H} - \frac{a^2\epsilon\dot{\zeta}}{k^2}$$

$$S = M_p^2 \int dt \frac{d^3k}{(2\pi)^3} \epsilon (a^3\dot{\zeta}^2 - k^2 a\zeta^2)$$

Exactly massless (Goldstone, see Prof. Senatore's lectures)

Perturbations in the ζ-Gauge

$$S = M_p^2 \int dt \frac{d^3k}{(2\pi)^3} \epsilon (a^3 \dot{\zeta}^2 - k^2 a \zeta^2)$$

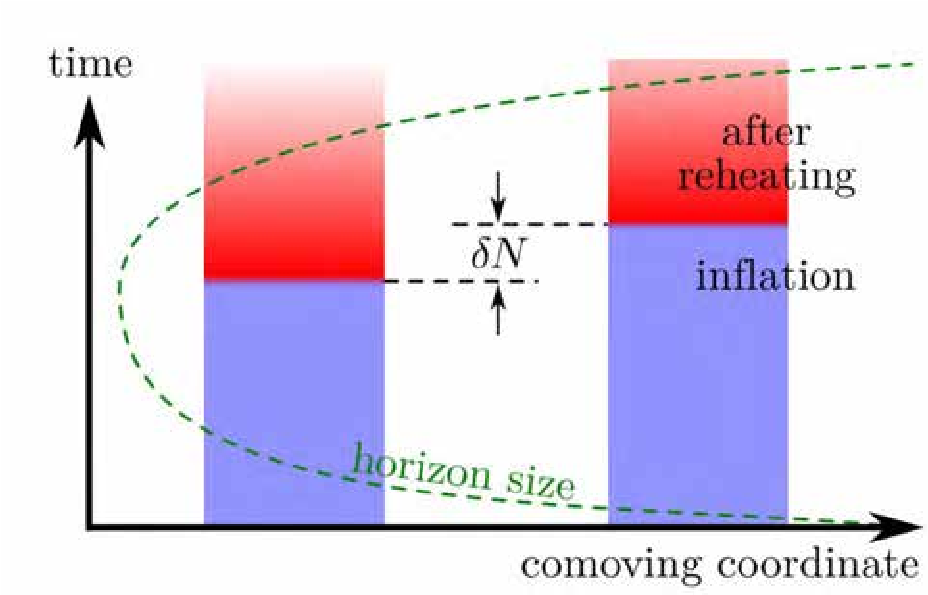
$$\ddot{\zeta} + (3+\eta) H \dot{\zeta} + \frac{k^2}{a^2} \zeta = 0$$

$$\zeta = \frac{H}{M_p \sqrt{4\epsilon k^3}} (1 + ik\tau) e^{-ik\tau}$$

$$\zeta_{\mathbf{k}} = u_k a_{\mathbf{k}} + u_k^* a_{-\mathbf{k}}^\dagger \,, \qquad u_k = \frac{H}{M_p \sqrt{4\epsilon k^3}} (1 + ik\tau) e^{-ik\tau}$$

From ζ to δN-formalism

The separate universe assumption (for long wavelength modes)



$$\widetilde{a}(t,\mathbf{x}) = a(t)e^{\zeta(t,\mathbf{x})}$$

$$\dot{\rho} + 3\widetilde{H}(\rho + p) = 0$$

$$\frac{\dot{a}}{a} + \dot{\zeta} = -\frac{1}{3}\frac{\dot{\rho}}{\rho + p}$$

$\dot{\zeta} = 0$ If $\rho = \rho(t)$

$p = p(\rho)$

$$\widetilde{N}(t_2, t_1, \mathbf{x}) = \int_{t_1}^{t_2} dt \widetilde{H} = \int_{t_1}^{t_2} dt \left(\frac{\dot{a}}{a} + \dot{\zeta}\right) = \ln\left[\frac{a(t_2)}{a(t_1)}\right] + \zeta(t_2, \mathbf{x}) - \zeta(t_1, \mathbf{x})$$

$$\zeta(t_2, \mathbf{x}) = \widetilde{N}(t_2, t_1, \mathbf{x}) - N(t_2, t_1) \equiv \delta N$$

$$\zeta^{(\mathrm{gi})}(t) = \delta N(t) + \frac{1}{3}\int_{\bar{\rho}(t)}^{\rho(t,\mathbf{x})} \frac{d\rho}{\rho + p}$$

The Power Spectrum

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P_\zeta(k) \ , \qquad P_\zeta(k) = \frac{H^4}{4\pi^2 \dot{\phi}^2} = \frac{H^2}{8\pi^2 \epsilon M_p^2}$$

The power spectrum is well measured: $P_\zeta \simeq 2 \times 10^{-9}$ (COBE).

Sometimes $\Delta_{\mathcal{R}} \equiv \sqrt{P_\zeta}$ is used.

Spectral index: $$n_s - 1 \equiv \frac{d \ln P_\zeta(k)}{d \ln k}$$

Observed value: $n_s - 1 = -0.032 \pm 0.006$ (non-zero at $> 5\sigma$ CL)

Running (not yet observed): $$\alpha_s \equiv \frac{d \ln n_s}{d \ln k} = -2\eta\epsilon - \frac{\dot{\eta}}{H\eta}$$

Observed bound: $\alpha_s = -0.0065 \pm 0.0076$

Beyond the Curvature Perturbation

Can we directly observe other primordial fluctuations (other than ζ)?

There *may* be additional light scalars (isocurvature fluctuations).

There *must* be gravitational waves (tensor mode of the metric).

Primordial Gravitational Waves

$$N_i = \partial_i \beta + b_i \ , \qquad h_{ij} = e^{2\zeta} a^2 \left[\delta_{ij} + \gamma_{ij} + \mathcal{D}_i \mathcal{D}_j E + \mathcal{D}_i F_j + \mathcal{D}_j F_i \right]$$

$$\mathcal{D}_i \gamma^{ij} = 0 \ , \qquad \gamma^i_i = 0$$

$$S = \frac{M_p^2}{8} \int \frac{d^3k}{(2\pi)^3} d\tau \ a^2 \left(\gamma^{i\prime}_j \gamma^{j\prime}_i - k^2 \gamma^i_j \gamma^j_i \right)$$

$$\gamma_{ij}(\mathbf{k}) = \frac{\sqrt{2}}{M_p} \left[\gamma_+(\mathbf{k}) e^+_{ij}(\mathbf{k}) + \gamma_\times(\mathbf{k}) e^\times_{ij}(\mathbf{k}) \right]$$

$$e^{+,\times}_{ij} = e^{+,\times}_{ji} \ , \qquad k^i e^{+,\times}_{ij} = 0, \qquad e^{+,\times}_{ii} = 0$$

$$e^{+,\times}_{ij}(-\mathbf{k}) = (e^{+,\times}_{ij}(\mathbf{k}))^* \qquad e^+_{ij}(e^{ij}_+)^* + e^\times_{ij}(e^{ij}_\times)^* = 4$$

Primordial Gravitational Waves

$$S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} d\tau \; a^2 \left[(\gamma'_+ {\gamma_+}' - k^2 \gamma_+ \gamma_+) + (\gamma'_\times {\gamma_\times}' - k^2 \gamma_\times \gamma_\times) \right]$$

$$P_\gamma^{+,\times} = \left(\frac{H}{2\pi}\right)^2$$

$$\bar{h}^{ik}\bar{h}^{jl}\langle \gamma_{ij}(\mathbf{k})\gamma_{kl}(\mathbf{k}')\rangle = (2\pi)^3 \delta^3(\mathbf{k}+\mathbf{k}') \frac{2\pi^2}{k^3} P_\gamma$$

$$P_\gamma = \frac{2H^2}{\pi^2 M_p^2}$$

Primordial Gravitational Waves

$$P_\gamma = \frac{2H^2}{\pi^2 M_p^2}$$

Assuming:

- GW from vacuum fluctuation
- Expansion (constant mode dominate)

⇨

GW only "cares" the energy scale of inflation in Planck unit

Primordial Gravitational Waves

c.f. $P_\zeta(k) = \frac{H^4}{4\pi^2\dot{\phi}^2}$: "cares" about H and $\dot{\phi}$.

Thus see GW (unfortunately not yet)

→ know energy scale of inflation

$$P_\gamma = \frac{2H^2}{\pi^2 M_p^2}$$

Assuming:

- GW from vacuum fluctuation
- Expansion (constant mode dominate)

⇨

GW only "cares" the energy scale of inflation in Planck unit

Primordial Gravitational Waves

$$S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} d\tau\ a^2 \left[(\gamma'_+ {\gamma_+}' - k^2 \gamma_+ \gamma_+) + (\gamma'_\times {\gamma_\times}' - k^2 \gamma_\times \gamma_\times) \right]$$

$$P_\gamma^{+,\times} = \left(\frac{H}{2\pi}\right)^2$$

$$\bar{h}^{ik}\bar{h}^{jl}\langle \gamma_{ij}(\mathbf{k})\gamma_{kl}(\mathbf{k}')\rangle = (2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}P_\gamma \qquad P_\gamma = \frac{2H^2}{\pi^2 M_p^2}$$

$$r \equiv \frac{P_\gamma}{P_\zeta} = 16\epsilon = 16\epsilon_V$$

$r < 0.07$ （Planck + BICEP2 + Keck）

Future: $\Delta r \sim 10^{-3}$ (LiteBIRD, CMB stage 4, see also Ali)

Primordial Gravitational Waves

$$S = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} d\tau \; a^2 \left[(\gamma_+' {\gamma_+}' - k^2 \gamma_+ \gamma_+) + (\gamma_\times' {\gamma_\times}' - k^2 \gamma_\times \gamma_\times) \right]$$

$$P_\gamma^{+,\times} = \left(\frac{H}{2\pi} \right)^2$$

$$\bar{h}^{ik} \bar{h}^{jl} \langle \gamma_{ij}(\mathbf{k}) \gamma_{kl}(\mathbf{k}') \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P_\gamma \qquad P_\gamma = \frac{2H^2}{\pi^2 M_p^2}$$

$$r \equiv \frac{P_\gamma}{P_\zeta} = 16\epsilon$$

$r < 0.07$ （Planck + BICEP2 + Keck）

$$n_T \equiv \frac{d \ln P_\gamma}{d \ln k} = -2\epsilon = -r/8$$

consistency relation for single field inflation

Primordial Gravitational Waves

Recall:

P_γ tells energy scale

P_ζ tells (energy scale) / (rolling speed of the inflaton)

Thus r tells rolling speed of the inflaton

$$\frac{d\phi}{dN} = \frac{\dot{\phi}}{H} = \frac{M_p}{4}\sqrt{2r}$$

For 60 e-folds of inflation: $\Delta\phi \simeq 15 M_p^2 \sqrt{2r}$

Detectable $r \rightarrow$ challenge for the EFT of inflationary background

Summary of This Lecture

Fluctuations, from quantum to classical

Spectator: Lagrangian, quantization, EoM, solution, power spectrum

Curvature ($\delta\phi, \zeta$)

Gravitational waves

} similar to spectator field