The Theory and Practice of Cosmological Perturbations

Part III -- Nonlinear Fluctuations

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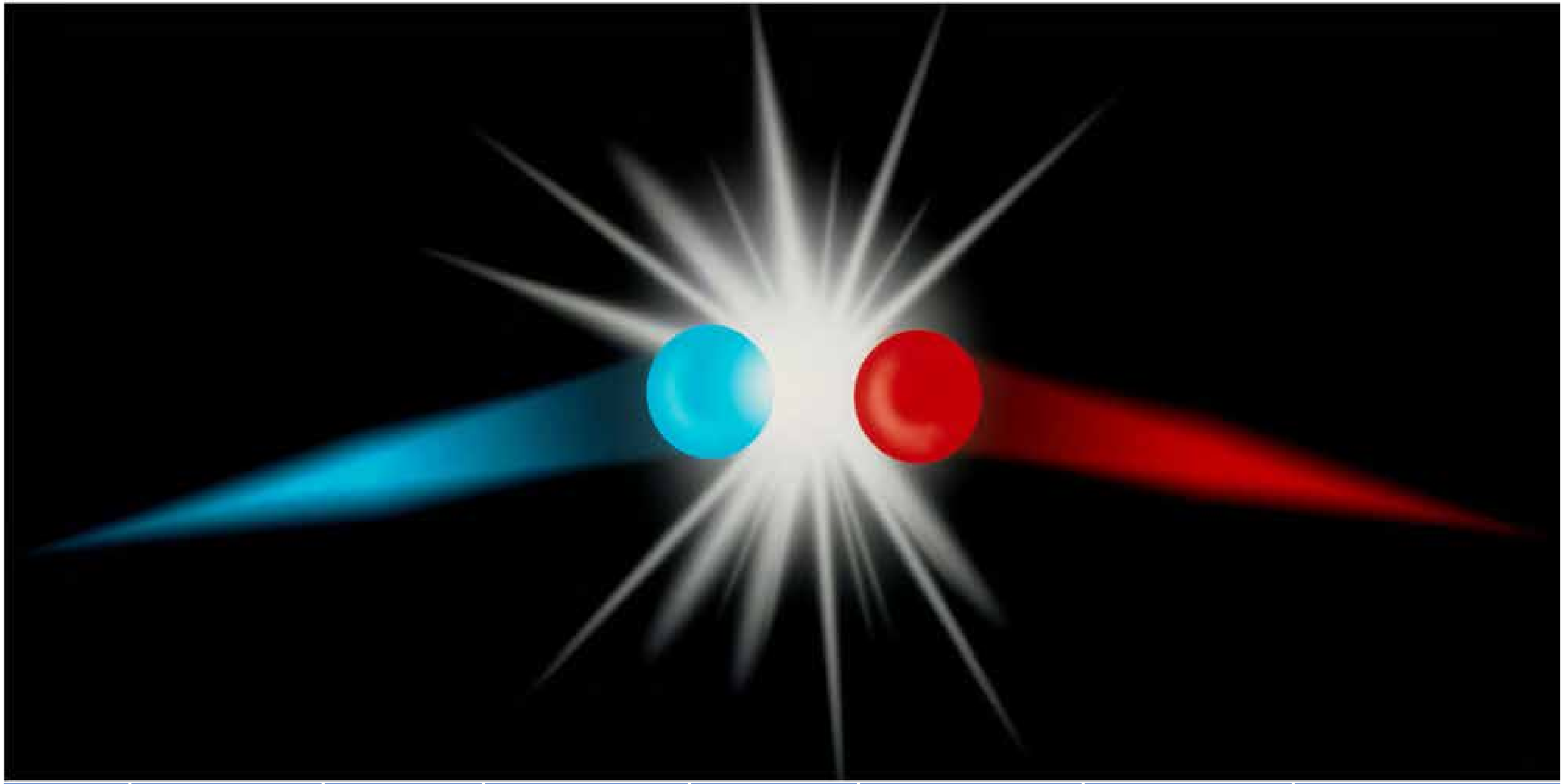
Why nonlinear fluctuations (non-Gaussianities)?

1. They exist (gravity is nonlinear)
2. They tell “which inflation model”
3. They tell “what additional” happened
4. They turn inflation into a particle collider
5. They tell evolution history of primordial universe

The in-in formalism

Particle collider physics: in-out formalism



The in-in formalism

Particle collider physics: in-out formalism

$$\langle\Omega|\, T\{\phi(x)\phi(y)\}\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\, T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle}{\langle 0|\, T\left\{\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\}|0\rangle}$$

The in-in formalism

Particle collider physics: in-out formalism

$$\langle\Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\left\{\phi_I(x)\phi_I(y) \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\} |0\rangle}{\langle 0| T\left\{\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right\} |0\rangle}$$

The case of cosmological correlation functions:

- Initial states not prepared
- Final states not infinite future

So, the quick answer:

$$\langle\Omega|Q(\tau)|\Omega\rangle = \langle 0| \left[\bar{T} e^{i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right] Q^I(\tau) \left[T e^{-i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right] |0\rangle$$

The in-in formalism

Derive the formalism: Step 1: Split the Hamilton into BG and perturbation parts.

Given Hamiltonian $H[\phi(\mathbf{x},\tau),\pi(\mathbf{x},\tau)]$

$$[\phi_a(\mathbf{x},\tau),\pi_b(\mathbf{y},\tau)] = i\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) \ , \quad [\phi_a(\mathbf{x},\tau),\phi_b(\mathbf{y},\tau)] = 0 \ , \quad [\pi_a(\mathbf{x},\tau),\pi_b(\mathbf{y},\tau)] = 0$$

$$\phi_a' = i\,[H,\phi_a] \ , \qquad \pi_a' = i\,[H,\pi_a]$$

$$\phi_a(\mathbf{x},\tau) = \bar\phi_a(\tau) + \delta\phi_a(\mathbf{x},\tau) \ , \qquad \pi_a(\mathbf{x},\tau) = \bar\pi_a(\tau) + \delta\pi_a(\mathbf{x},\tau)$$

$$[\delta\phi_a(\mathbf{x},\tau),\delta\pi_b(\mathbf{y},\tau)] = i\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) \ , \quad [\delta\phi_a(\mathbf{x},\tau),\delta\phi_b(\mathbf{y},\tau)] = 0 \ , \quad [\delta\pi_a(\mathbf{x},\tau),\delta\pi_b(\mathbf{y},\tau)] = 0$$

$$H[\phi,\pi] = H[\bar\phi,\bar\pi] + \sum_a \int d^3x \left.\frac{\delta H[\phi,\pi]}{\delta\phi(\mathbf{x},\tau)}\right|_{\bar\phi,\bar\pi} \delta\phi(\mathbf{x},\tau) + \sum_a \int d^3x \left.\frac{\delta H[\phi,\pi]}{\delta\pi(\mathbf{x},\tau)}\right|_{\bar\phi,\bar\pi} \delta\pi(\mathbf{x},\tau) + \widetilde H[\delta\phi,\delta\pi;\tau]$$

$$\delta\phi_a' = i\left[\widetilde H,\delta\phi_a\right] \quad \rightarrow \quad \delta\phi_a(\tau) = U^{-1}(\tau,\tau_0)\delta\phi_a(\tau_0)U(\tau,\tau_0)$$

$$\delta\pi_a' = i\left[\widetilde H,\delta\pi_a\right] \quad \rightarrow \quad \delta\pi_a(\tau) = U^{-1}(\tau,\tau_0)\delta\pi_a(\tau_0)U(\tau,\tau_0)$$

(Formally $U = e^{-i\,\widetilde H(\tau-\tau_0)}$)

$$\partial_\tau U(t,\tau_0) = -i\widetilde H[\delta\phi(\tau_0),\delta\pi(\tau_0);\tau]U(\tau,\tau_0) \ , \qquad U(\tau_0,\tau_0) = 1$$

The in-in formalism

Step 2: Further split the perturbation Hamiltonian into free and interacting.

$$\widetilde{H}[\delta\phi, \delta\pi; \tau] = H_0[\delta\phi, \delta\pi; \tau] + H_I[\delta\phi, \delta\pi; \tau]$$

Define interaction picture fields $\delta\phi_a^I$, such that

$$\begin{aligned} \delta\phi_a^{I\prime} &\equiv i\left[H_0, \delta\phi_a^I\right] \quad \rightarrow \quad \delta\phi_a^I(\tau) = U_0^{-1}(\tau, \tau_0)\delta\phi_a^I(\tau_0)U_0(\tau, \tau_0) \\ \delta\pi_a^{I\prime} &\equiv i\left[H_0, \delta\pi_a^I\right] \quad \rightarrow \quad \delta\pi_a^I(\tau) = U_0^{-1}(\tau, \tau_0)\delta\pi_a^I(\tau_0)U_0(\tau, \tau_0) \end{aligned}$$

where U_0 is related to H_0 (similar to U and $\widetilde{H}$) by

$$\partial_\tau U_0(\tau, \tau_0) = -iH_0[\delta\phi^I(\tau_0), \delta\pi^I(\tau_0); \tau]U_0(\tau, \tau_0)\ , \qquad U_0(\tau_0, \tau_0) = 1$$

Together with

$$\partial_\tau U(t, \tau_0) = -i\widetilde{H}[\delta\phi(\tau_0), \delta\pi(\tau_0); \tau]U(\tau, \tau_0)\ , \qquad U(\tau_0, \tau_0) = 1$$

we can get

$$\partial_\tau[U_0^{-1}(\tau, \tau_0)U(\tau, \tau_0)] = -iU_0^{-1}(\tau, \tau_0)H_I[\delta\phi(\tau_0), \delta\pi(\tau_0); \tau]U(\tau, \tau_0) = -iU_0^{-1}(\tau, \tau_0)U(\tau, \tau_0)H_I[\delta\phi(\tau), \delta\pi(\tau); \tau]$$

The in-in formalism

$$\partial_\tau[U_0^{-1}(\tau,\tau_0)U(\tau,\tau_0)] = -iU_0^{-1}(\tau,\tau_0)H_I[\delta\phi(\tau_0),\delta\pi(\tau_0);\tau]U(\tau,\tau_0) = -iU_0^{-1}(\tau,\tau_0)U(\tau,\tau_0)H_I[\delta\phi(\tau),\delta\pi(\tau);\tau]$$

Step 3: Features and solutions of $F \equiv U_0^{-1}U$:

- F relates Heisenberg picture fields and interaction picture fields:

$$Q(\tau) = F^{-1}(\tau,\tau_0)Q^I(\tau)F(\tau,\tau_0) = \left[\bar{T}e^{i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right] Q^I(\tau) \left[Te^{-i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right]$$

- F has mild time dependence, thus can be expanded & solved order-by-order

$$\begin{aligned}
F(\tau,\tau_0) \equiv& U_0^{-1}(\tau,\tau_0)U(\tau,\tau_0)\\
=&1 + (-i)\int_{\tau_0}^{\tau} d\tau_1\ H_I(\tau_1) + (-i)^2\int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2\ H_I(\tau_1)H_I(\tau_2)\\
&+ (-i)^3\int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \int_{\tau_0}^{\tau_2} d\tau_3\ H_I(\tau_1)H_I(\tau_2)H_I(\tau_3) + \cdots\\
=&Te^{-i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\ .
\end{aligned}$$

The in-in formalism

Step 4: the time evolution of the vacuum: Problem:

we need to work with the interacting vacuum $|\Omega\rangle$,

but only know the free vacuum $|0\rangle$

$$e^{-iH(\tau-\tau_0)}|0\rangle = \sum_n e^{-iH(\tau-\tau_0)}|n\rangle\langle n|0\rangle = \sum_n e^{-iE_n(\tau-\tau_0)}|n\rangle\langle n|0\rangle = e^{-iE_\Omega(\tau-\tau_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n'} e^{-iE_{n'}(\tau-\tau_0)}|n'\rangle\langle n'|0\rangle$$

$$\tau \to \tilde{\tau} = \tau(1-i\epsilon)$$

$$e^{-iH(\tilde{\tau}-\tilde{\tau}_0)}|\Omega\rangle = \frac{e^{-iH(\tilde{\tau}-\tilde{\tau}_0)}|0\rangle}{\langle\Omega|0\rangle} \qquad \Rightarrow \qquad F(\tilde{\tau},\tilde{\tau}_0)|\Omega\rangle = \frac{F(\tilde{\tau},\tilde{\tau}_0)|0\rangle}{\langle\Omega|0\rangle}$$

Issues: Do we have the true infinite past? (See non-BD section)

The in-in formalism

The result:

$$\langle\Omega|Q(\tau)|\Omega\rangle = \frac{\langle 0|\left[\bar{T}e^{i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right] Q^I(\tau)\left[Te^{-i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right]|0\rangle}{|\langle 0|\Omega\rangle|^2}$$

$$|\langle 0|\Omega\rangle|^2 = \frac{\langle 0|\left[\bar{T}e^{i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right]\left[Te^{-i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right]|0\rangle}{\langle\Omega|\Omega\rangle} = 1$$

(up to ϵ-dependent terms, which will eventually cancel)

$$\langle\Omega|Q(\tau)|\Omega\rangle = \langle 0|\left[\bar{T}e^{i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right] Q^I(\tau)\left[Te^{-i\int_{\tau_0}^{\tau} H_I(\tau')d\tau'}\right]|0\rangle$$

The in-in formalism

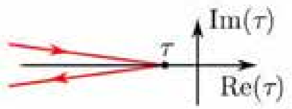
Faces of the in-in formalism

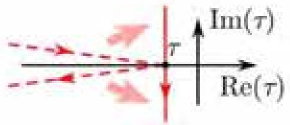
Factorized form: $\langle\Omega|Q(\tau)|\Omega\rangle = \langle 0|\left[\bar{T}e^{i\int_{\tau_0}^{\tau}H_I(\tau')d\tau'}\right]Q^I(\tau)\left[Te^{-i\int_{\tau_0}^{\tau}H_I(\tau')d\tau'}\right]|0\rangle$

1st order: $\langle\Omega|Q(\tau)|\Omega\rangle_1 = 2\mathrm{Im}\int_{\tau_0}^{\tau}d\tau_1\langle 0|Q^I(\tau)H_I(\tau_1)|0\rangle$

2nd order: $\langle\Omega|Q(\tau)|\Omega\rangle_2 = \int_{\tau_0}^{\tau}d\tau_1\int_{\tau_0}^{\tau}d\tau_2\langle 0|H_I(\tau_1)Q^I(\tau)H_I(\tau_2)|0\rangle - 2\mathrm{Re}\int_{\tau_0}^{\tau}d\tau_1\int_{\tau_0}^{\tau_1}d\tau_2\langle 0|Q^I(\tau)H_I(\tau_1)H_I(\tau_2)|0\rangle$

Euclidean factorized form: $\langle\Omega|Q(\tau)|\Omega\rangle = \langle 0|\bar{T}\left[Q^I(\tau)\exp\left(-\int_{-i\infty+\tau}^{i\infty+\tau}H_I(\tau_E)d\tau_E\right)\right]|0\rangle$





$= \langle 0|\bar{T}\left[Q^I(\tau)\exp\left(-\int_{-\infty}^{\infty}H_I(\tau+i\tau_E)d\tau_E\right)\right]|0\rangle$

Commutator form: $\langle\Omega|Q(\tau)|\Omega\rangle = \sum_{n=0}^{\infty}i^n\int_{\tau_0}^{\tau}d\tau_1\int_{\tau_1}^{\tau}d\tau_2\cdots\int_{\tau_{n-1}}^{\tau}d\tau_n\langle[H_I(\tau_1),[H_I(\tau_2),\cdots,[H_I(\tau_n),Q^I(\tau)]\cdots]]\rangle$

Mixed form: $\sum_{i=1}^{n}\int_{\tau_c}^{\tau}d\tau_1\cdots\int_{\tau_c}^{\tau_{i-1}}d\tau_i\ (\text{commutator form})\times\int_{-\infty}^{\tau_c}d\tau_{i+1}\cdots\int_{-\infty}^{\tau_{n-1}}d\tau_n\ (\text{factorized form})$

The in-in formalism

More faces: Two copies of fields (more helpful for formal problems)

(See, e.g. 0707.0842 by van der Meulen & Smit, 1010.4565 by Leblond & Pajer)

$$Z[J_+, J_-] = Z_0 \int \mathcal{D}\phi^+ \mathcal{D}\phi^- \exp\left[\mathrm{i} \int_{\mathrm{t_i}}^{\mathrm{t}} \mathrm{dt'} \int \mathrm{d^3x} \left(\mathcal{L}[\phi^+] - \mathcal{L}[\phi^-] + \mathrm{J}_+\phi^+ + \mathrm{J}_-\phi^- \right) \right]$$

Pairs of variations w.r.t. J_+ and J_- defines four Green's functions:

$$\begin{pmatrix} G^{++}(x,y) & G^{+-}(x,y) \\ G^{-+}(x,y) & G^{++}(x,y) \end{pmatrix} ,$$

$$G^{-+}(x,y) = i \langle \phi(x)\phi(y) \rangle \qquad G^{+-}(x,y) = i \langle \phi(y)\phi(x) \rangle ,$$

$$G^{++}(x,y) = i \langle \mathrm{T}\phi(\mathrm{x})\phi(\mathrm{y}) \rangle \qquad G^{--}(x,y) = i \langle \bar{\mathrm{T}}\phi(x)\phi(y) \rangle .$$

$$\langle \bar{\mathrm{T}} \left(\phi(x_1) \ldots \phi(x_n) \right) \mathrm{T} \left(\phi(x_{n+1}) \ldots \phi(x_{n+m}) \right) \rangle = \left. \frac{\delta^{n+m} Z[J_+, J_-, \rho(t_{\mathrm{in}})]}{\delta J_-(x_1) \cdots \delta J_-(x_n)\, \delta J_+(x_{n+1}) \cdots \delta J_+(x_{n+m})} \right|_{J_+, J_- = 0}$$

Single field slow roll

Minimal inflation: Minimal non-Gaussianity (Maldacena 2002)

Steps of calculation:

1. Calculate the gravitational 3rd order Lagrangian

   (Need 3pt of ζ because it is conserved nonlinearly)

   Method 1: $\delta\phi$-gauge + gauge transformation

   Method 2: ζ-gauge directly (lots of integration-by-parts to do)

2. Redefine fields (i.e. apply EoM) to simplify the Lagrangian
3. Transform to the 3rd order Hamiltonian
4. Use in-in formalism to calculate 3-point correlation function
5. Extract non-Gaussianity from 3pt

Single field slow roll

The 3rd order gravitational Lagrangian (ζ-gauge)

1. Expand the gravitational action
2. Solve the constraints N and N_i

   To calculate n-th order Lagrangian ($n \geq 3$),

   need to solve constraint to (n-2)-th order

   (last two orders vanish due to first two orders constraint eqs)

   Thus for 3rd order Lagrangian, we still only need linear constraints
3. Insert the constraints back
4. Integration by parts

Single field slow roll

Before inserting the constraints:

$$S_3 = \int dt d^3x \; a^3 \Big\{ 3H^2\alpha^3 - \epsilon H^2\alpha^3 - 9H^2\alpha^2\zeta + 3\epsilon H^2\alpha^2\zeta + \frac{27}{2}H^2\alpha\zeta^2 - \frac{9}{2}\epsilon H^2\alpha\zeta^2 - \frac{27}{2}H^2\zeta^3 + \frac{9}{2}\epsilon H^2\zeta^3 - 6H\alpha^2\dot{\zeta}$$
$$+ 18H\alpha\zeta\dot{\zeta} - 27H\zeta^2\dot{\zeta} + 3\alpha(\dot{\zeta})^2 - 9\zeta(\dot{\zeta})^2 - \frac{2}{a^2}H\alpha b_i\partial_i\zeta + \frac{2}{a^2}H\zeta b_i\partial_i\zeta - \frac{2}{a^2}H\alpha\partial_i\beta\partial_i\zeta + \frac{2}{a^2}H\zeta\partial_i\beta\partial_i\zeta + \frac{2}{a^2}b_i\dot{\zeta}\partial_i\zeta$$
$$+ \frac{2}{a^2}\partial_i\beta\dot{\zeta}\partial_i\zeta - \frac{1}{a^2}\alpha(\partial_i\zeta)^2 - \frac{1}{a^2}\zeta(\partial_i\zeta)^2 - \frac{1}{a^4}b_j\partial_i\zeta\partial_j b_i - \frac{1}{a^4}\partial_j\beta\partial_i\zeta\partial_j b_i - \frac{1}{a^4}b_i\partial_j\zeta\partial_j b_i - \frac{1}{a^4}\partial_i\beta\partial_j\zeta\partial_j b_i$$
$$- \frac{1}{4a^4}\alpha(\partial_j b_i)^2 - \frac{1}{4a^4}\zeta(\partial_j b_i)^2 - \frac{1}{4a^4}\alpha\partial_j b_i\partial_i b_j - \frac{1}{4a^4}\zeta\partial_j b_i\partial_i b_j + \frac{2}{a^2}H\alpha^2\partial^2\beta - \frac{2}{a^2}H\alpha\zeta\partial^2\beta + \frac{1}{a^2}H\zeta^2\partial^2\beta$$
$$- \frac{2}{a^2}\alpha\dot{\zeta}\partial^2\beta + \frac{2}{a^2}\zeta\dot{\zeta}\partial^2\beta - \frac{2}{a^4}b_j\partial_i\zeta\partial_i\partial_j\beta - \frac{2}{a^4}\partial_j\beta\partial_i\zeta\partial_i\partial_j\beta - \frac{1}{a^4}\alpha\partial_j b_i\partial_i\partial_j\beta - \frac{1}{a^4}\zeta\partial_j b_i\partial_i\partial_j\beta - \frac{1}{2a^4}\alpha(\partial_i\partial_j\beta)^2$$
$$- \frac{1}{2a^4}\zeta(\partial_i\partial_j\beta)^2 + \frac{1}{2a^4}\alpha(\partial^2\beta)^2 + \frac{1}{2a^4}\zeta(\partial^2\beta)^2 - \frac{2}{a^2}\alpha\zeta\partial^2\zeta - \frac{1}{a^2}\zeta^2\partial^2\zeta - \frac{9}{2}V\alpha\zeta^2 - \frac{9}{2}V\zeta^3\Big\} \,. \quad (167)$$

Single field slow roll

After inserting the constraints & integration by parts:

$$S_3 = \int dt d^3x \Big\{ a^3\epsilon^2\zeta\dot{\zeta}^2 + a\epsilon^2\zeta(\partial\zeta)^2 - 2a\epsilon\dot{\zeta}\partial\zeta\partial\chi + \frac{a^3\epsilon}{2}\dot{\eta}\zeta^2\dot{\zeta} + \frac{\epsilon}{2a}\partial\zeta\partial\chi\partial^2\chi + \frac{\epsilon}{4a}\partial^2\zeta(\partial\chi)^2 + 2f(\zeta)\left.\frac{\delta\mathcal{L}}{\delta\zeta}\right|_1 + \mathcal{L}_b \Big\},$$

$$\partial^2\chi \equiv a^2\epsilon\dot{\zeta}$$

$$f(\zeta) = \frac{\eta}{4}\zeta^2 + \frac{1}{H}\zeta\dot{\zeta} + \frac{-\partial\zeta\partial\zeta + \partial^{-2}[\partial_i\partial_j(\partial_i\zeta\partial_j\zeta)]}{4a^2H} + \frac{\partial\zeta\partial\chi - \partial^{-2}[\partial_i\partial_j(\partial_i\zeta\partial_j\chi)]}{2a^2H}$$

The terms proportional to EoM motivate us to redefine the fields:

$\zeta = \zeta_n - f(\zeta_n)$ Then the $f(\zeta)$ term is eliminated in S_3

Single field slow roll

After inserting the constraints & integration by parts:

$$S_3 = \int dt d^3x \Big\{ a^3\epsilon^2\zeta\dot{\zeta}^2 + a\epsilon^2\zeta(\partial\zeta)^2 - 2a\epsilon\dot{\zeta}\partial\zeta\partial\chi + \frac{a^3\epsilon}{2}\dot{\eta}\zeta^2\dot{\zeta} + \frac{\epsilon}{2a}\partial\zeta\partial\chi\partial^2\chi + \frac{\epsilon}{4a}\partial^2\zeta(\partial\chi)^2 + 2f(\zeta)\left.\frac{\delta\mathcal{L}}{\delta\zeta}\right|_1 + \mathcal{L}_b \Big\} ,$$

$O(\epsilon^2)$

The inflaton self-interaction:

$$V' \simeq -3H\dot{\phi} , \quad \partial_t(V') = V''\dot{\phi} = -3H\dot{\phi} \times \mathcal{O}(\epsilon) , \quad \partial_t(V'') = V'''\dot{\phi} = -3H \times \mathcal{O}(\epsilon^2)$$

$$V''' = \frac{H}{\dot{\phi}} \times \mathcal{O}(\epsilon^2) \sim \frac{H\mathcal{O}(\epsilon^2)}{\sqrt{P_\zeta}}$$

$$\mathcal{L}_3 \supset -V'''\frac{\dot{\phi}^3}{H^3}\zeta^3 = \mathcal{O}(\epsilon^2) \times \frac{\dot{\phi}^2}{H^2}\zeta^3 = \mathcal{O}(\epsilon^3)\zeta^3$$

About 100 times weaker!

Single field slow roll

From action to Hamiltonian:

Steps:

1. Write down full Lagrangian up to 3th order (or n-th needed)
2. Transform to the full Hamiltonian up to same order

   Note: $\Pi = \frac{\partial L}{\partial \dot{\phi}}$ is defined nonlinearly (precise up to n-1 order)

3. Split $\mathcal{H}(\phi, \Pi)$ into free and interaction parts
4. Change into interaction picture $\mathcal{H} = \mathcal{H}_0(\phi_I, \Pi_I) + \mathcal{H}_{\rm int}(\phi_I, \Pi_I)$

   Note: the time dependence of ϕ_I & Π_I follow from $\mathcal{H}_0$ instead of $\mathcal{H}$

5. Define $\dot{\phi}_I = \frac{\partial \mathcal{H}_0}{\partial \Pi_I}$ and use $\dot{\phi}_I$ to replace Π_I

Single field slow roll

From action to Hamiltonian:

Result:

For 3rd order: $\mathcal{H}_3 = -\mathcal{L}_3$

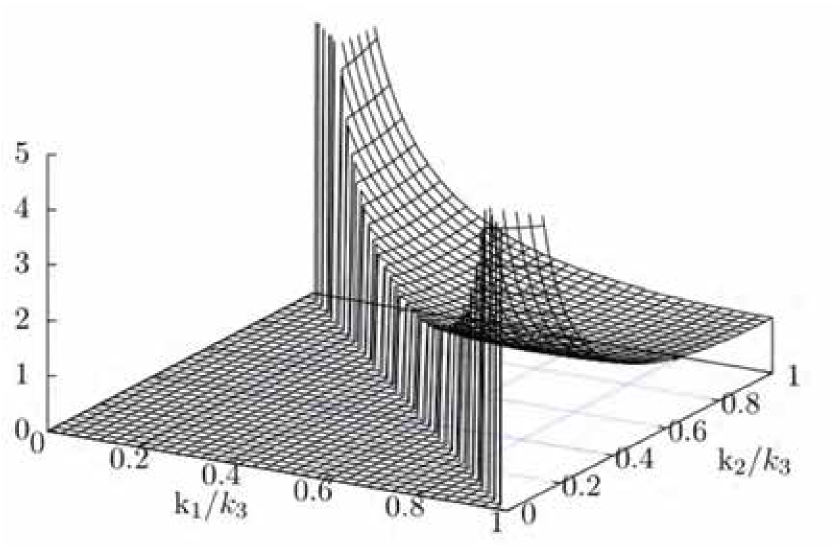
For 2nd order or 4th order, with derivative coupling: may be different

Single field slow roll

The in-in-calculated 3pt:

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^7\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{P_\zeta^2}{k_1^2k_2^2k_3^2}\mathcal{F}(k_1/k_3, k_2/k_3)$$

$$\mathcal{F}(k_1/k_3, k_2/k_3) \equiv \frac{3(\eta-\epsilon)}{8}\frac{(k_1^3+k_2^3+k_3^3)}{3k_1k_2k_3} + \frac{3\epsilon}{4}\frac{(k_1k_2^2+k_1k_3^2+k_2k_1^2+k_2k_3^2+k_3k_1^2+k_3k_2^2)}{6k_1k_2k_3} + \epsilon\frac{(k_1^2k_2^2+k_1^2k_3^2+k_2^2k_3^2)}{k_1k_2k_3(k_1+k_2+k_3)}$$



Local Non-Gaussianity

The dominate part of minimal non-G: local non-Gaussianity

$$\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + \frac{3}{5}f_{NL}\zeta_g^2(\mathbf{x}) + \frac{9}{25}g_{NL}\zeta_g^3(\mathbf{x}) + \frac{27}{125}h_{NL}\zeta_g^4(\mathbf{x}) + \frac{81}{625}i_{NL}\zeta_g^5(\mathbf{x}) + \frac{243}{3125}j_{NL}\zeta_g^6(\mathbf{x}) + \cdots$$

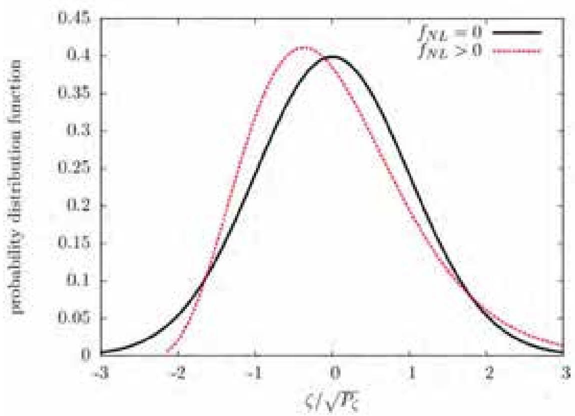
$$\zeta_{\mathbf{k}} = \zeta_{\mathbf{k}}^g + \frac{3}{5}f_{NL}\int \frac{d^3q}{(2\pi)^3}\zeta_{\mathbf{q}}^g\zeta_{\mathbf{q}-\mathbf{k}}^g$$

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = (2\pi)^7\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{3f_{NL}}{10k_1^3k_2^3}P_\zeta(k_1)P_\zeta(k_2) + (\mathbf{k}_2 \Leftrightarrow \mathbf{k}_3) + (\mathbf{k}_1 \Leftrightarrow \mathbf{k}_3)$$

Minimal non-G: $f_{NL} \sim O(\eta) \sim O(0.01)$

Planck (2015): $f_{NL} = 0.8 \pm 5.0$

Future: $\Delta f_{NL} \sim 0.5$ at LSS.



In principle & distant future: 10^{-4} for 21cm, CMB distortion

Beyond the minimal non-G

Examples of generalizations:

- Generalized Lagrangian (K, DBI, Galileons, … -> EFT)
- Generalized initial conditions (non-BD)
- Generalized slow roll conditions (ultra-slow-roll)
- Adding additional light fields
- Adding additional heavy fields

Effective Field Theory

Generalizing the Lagrangian:

rule of game:

ghost free: plus sign in front of $\left(\delta\dot{\phi}\right)^2$

no kinetic instability: $c_s^2 > 0$ (or slow roll suppressed instability)

no tachyonic instability: $m^2 > 0$ (or slow roll suppressed instability)

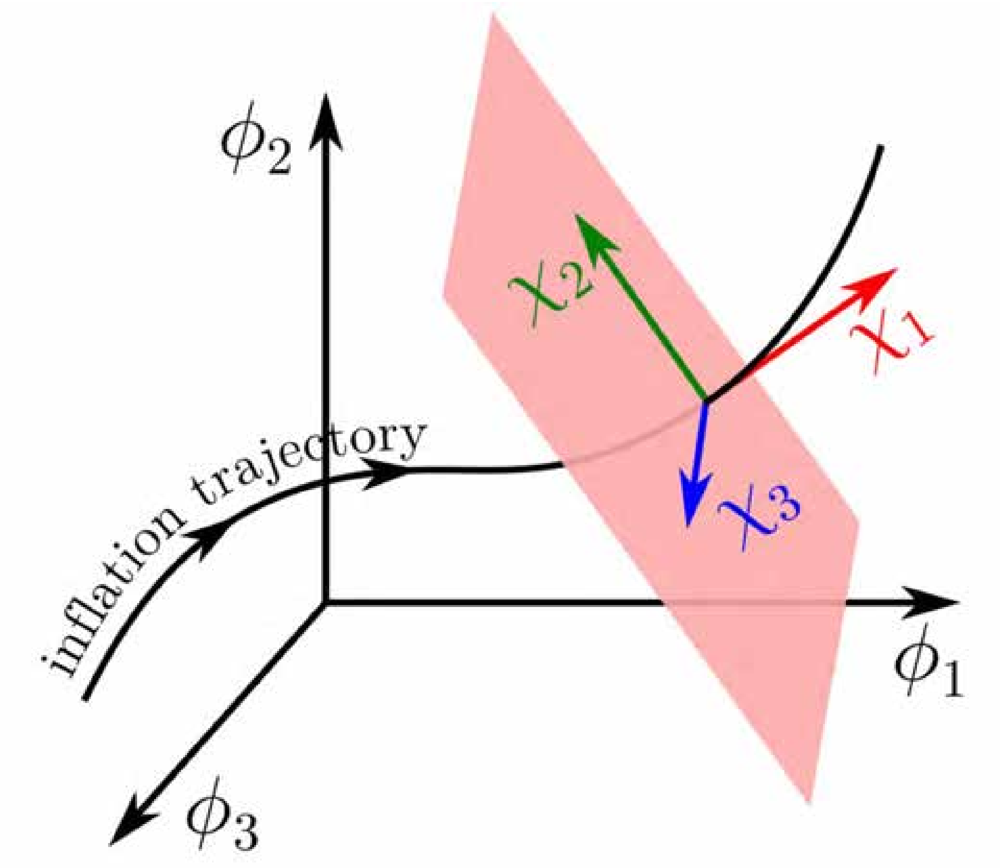
Example of generalized Lagrangian:

$\mathcal{L} = P(X, \phi)$, where $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

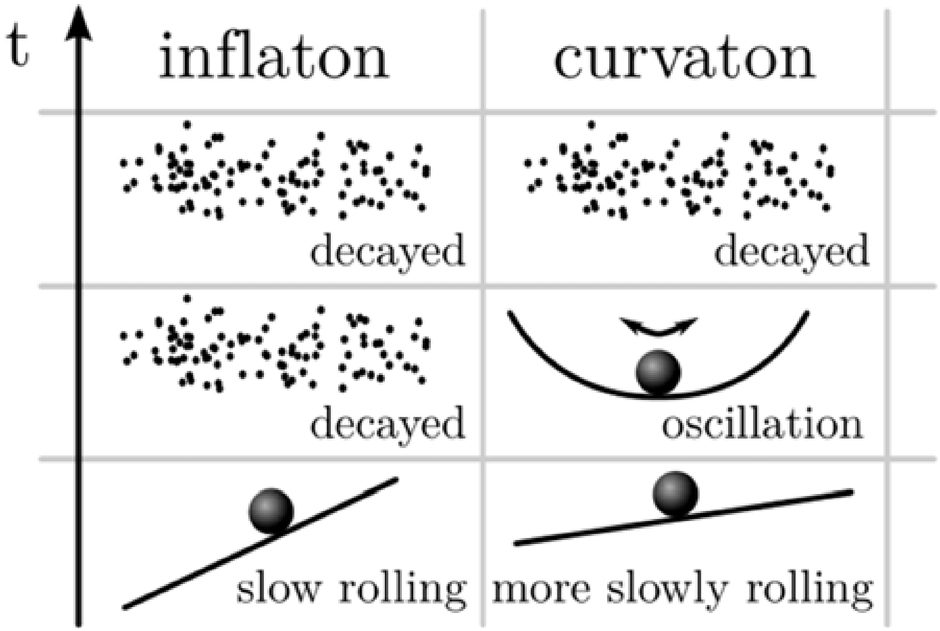
A more general form:

$$\begin{aligned}\mathcal{L} =& P(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ &+ G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]\end{aligned}$$

Multi-field inflation



Multi-field: Curvaton



assuming same decay product
entropy perturbation becomes adiabatic

curvaton density catches up
perturbation starts to gravitate

curvaton has entropy perturbation
inflaton perturbation assumed small

Multi-field: Curvaton

Non-Gaussianity of the curvaton scenario:

In general, non-G can be large because

- Curvature perturbation from subdominant component

  (larger fluctuations are more easily non-linear)

- No slow roll constraints at decay of curvaton (no ϵ suppression)

Multi-field: Curvaton

Putting in some numbers: Assuming only mass term for curvaton

$$r = \frac{3\rho_\sigma}{3\rho_\sigma + 4\rho_r}, \qquad f_{NL} = \frac{5}{4r} - \frac{5}{3} - \frac{6r}{5}$$ (typically >O(1))

Multi-field: Curvaton

Putting in some numbers: Assuming only mass term for curvaton

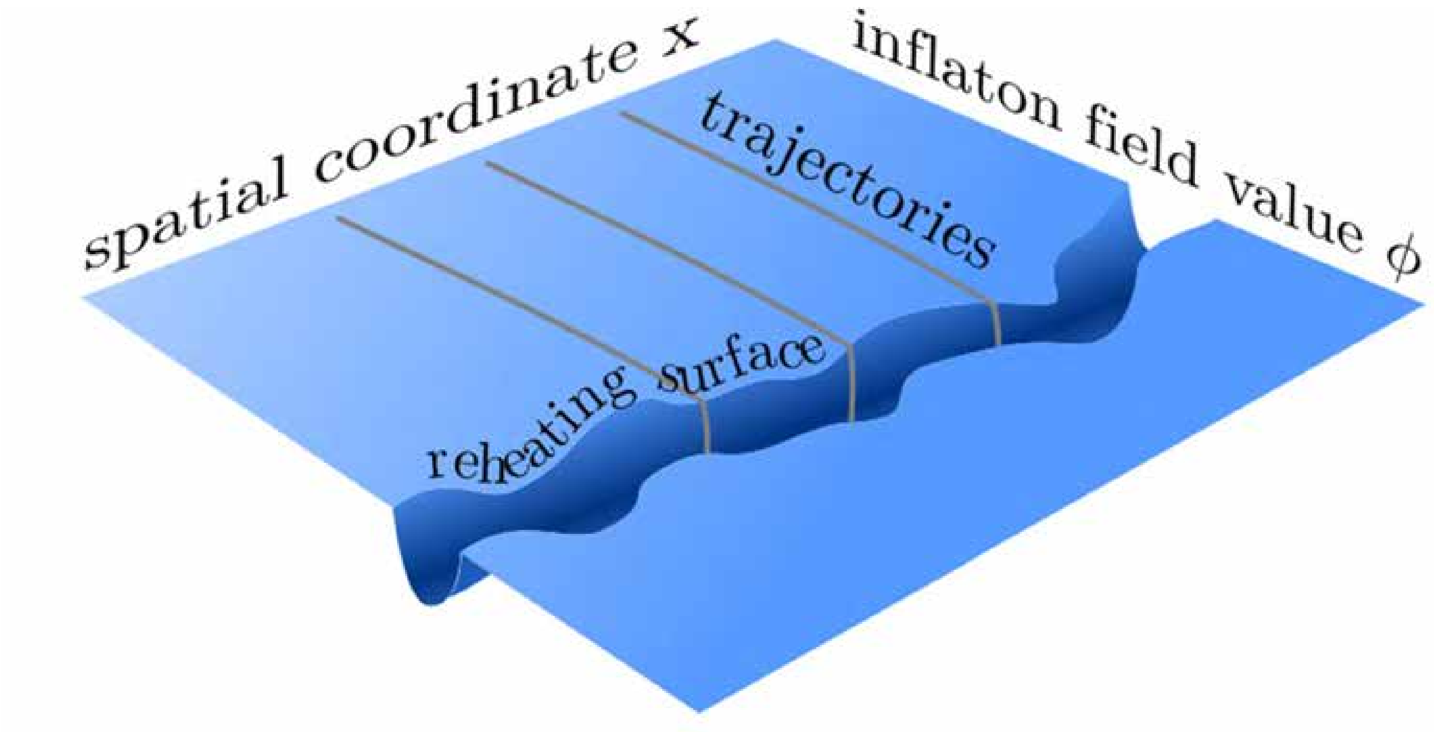
$$r = \frac{3\rho_\sigma}{3\rho_\sigma + 4\rho_r}, \qquad f_{NL} = \frac{5}{4r} - \frac{5}{3} - \frac{6r}{5} \qquad \text{(typically >O(1))}$$

⇩

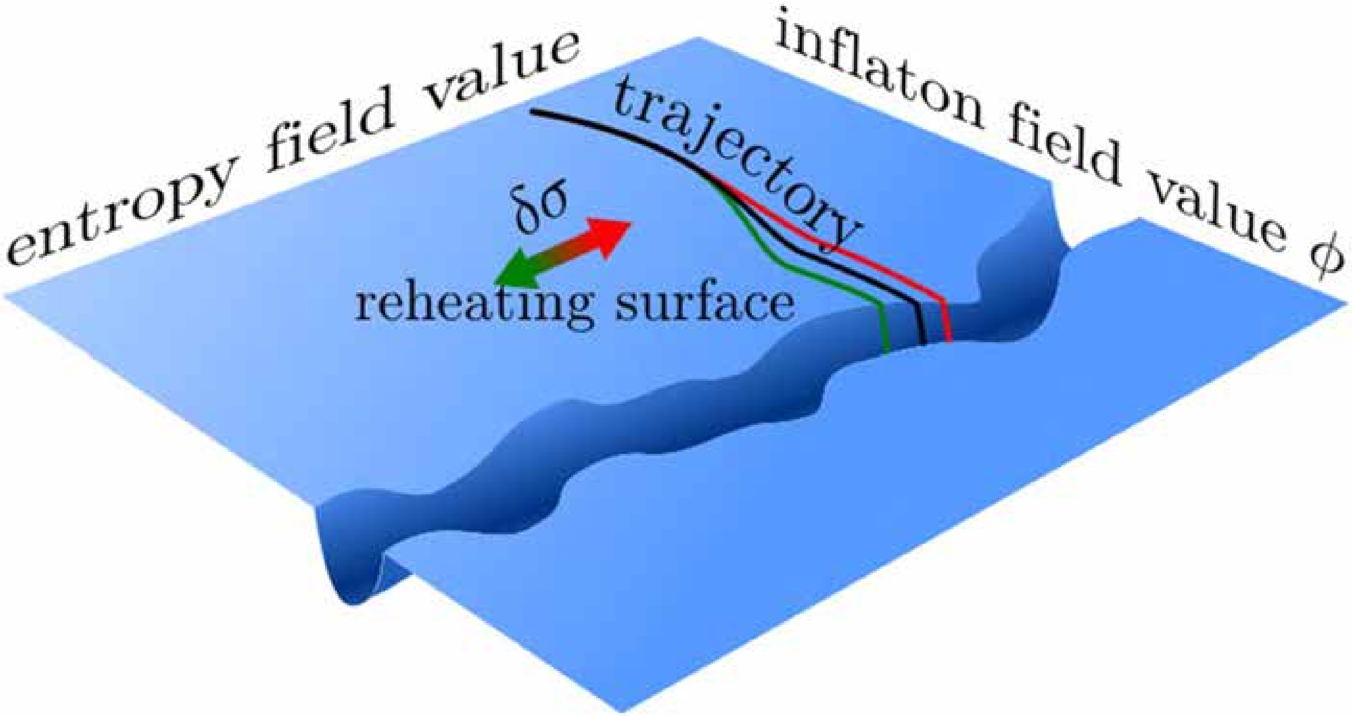
Super-horizon → local non-Gaussianity

(quite general for multi-field inflation)

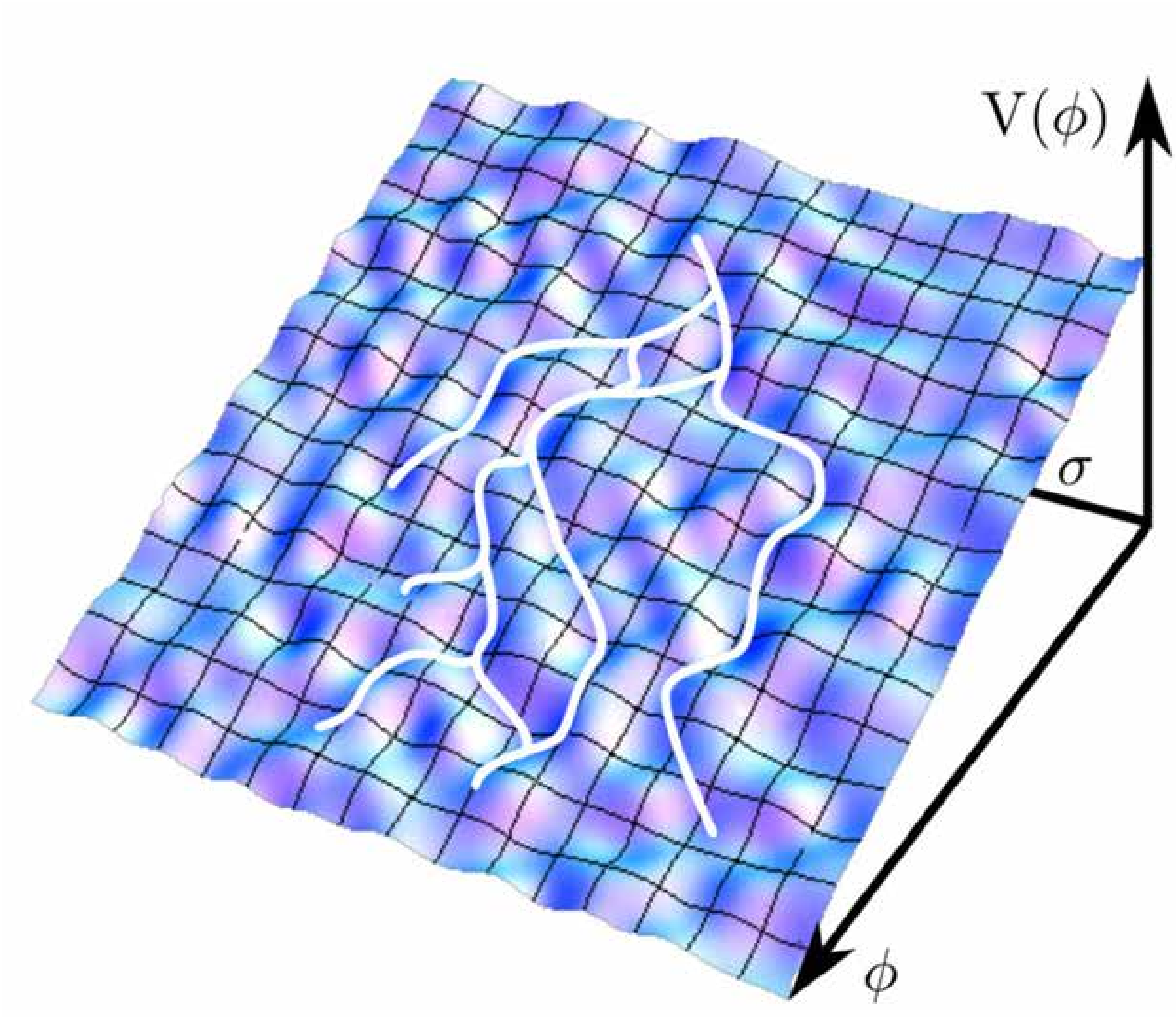
Multi-field: Modulated reheating



Multi-field: Multi-brid inflation

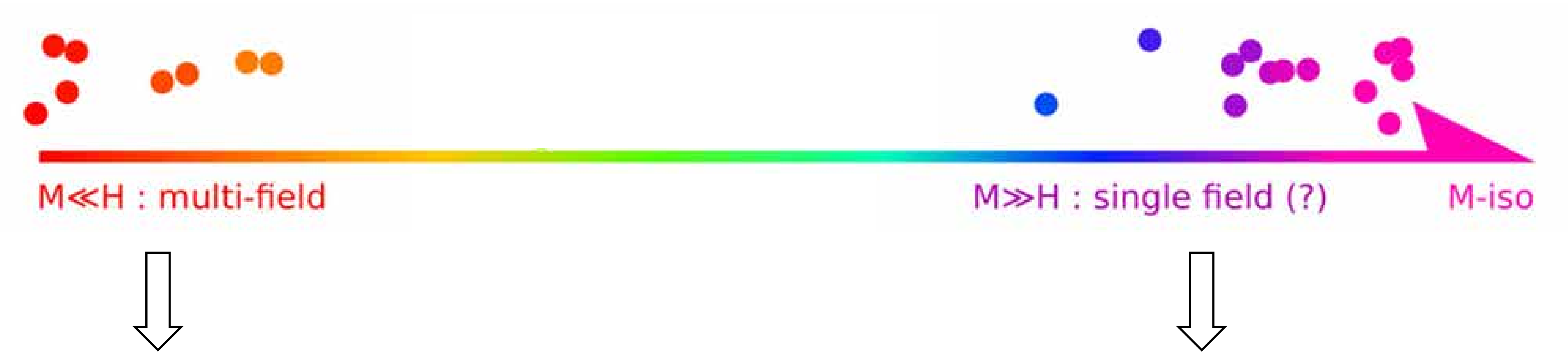


Multi-field: Multi-stream inflation



Quasi-Single Field Inflation (QSFI)

Single field vs multi-field: how to define them?

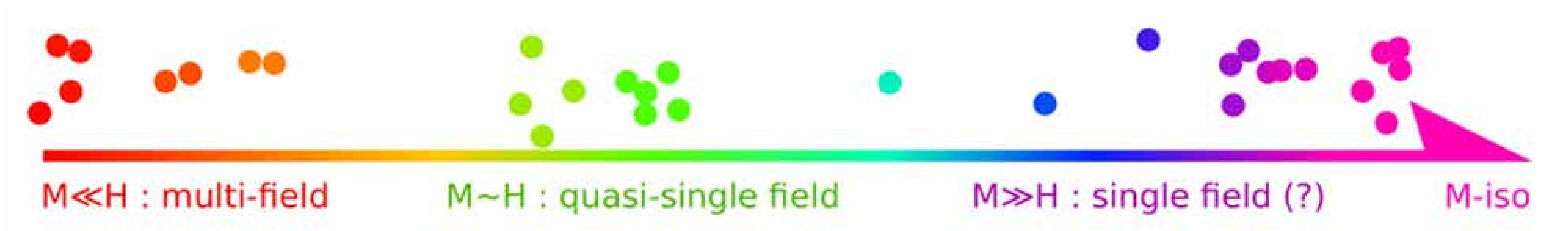


In general, $M < \sqrt{\eta}H$ (inflaton mass)

In general, $M \gg H$

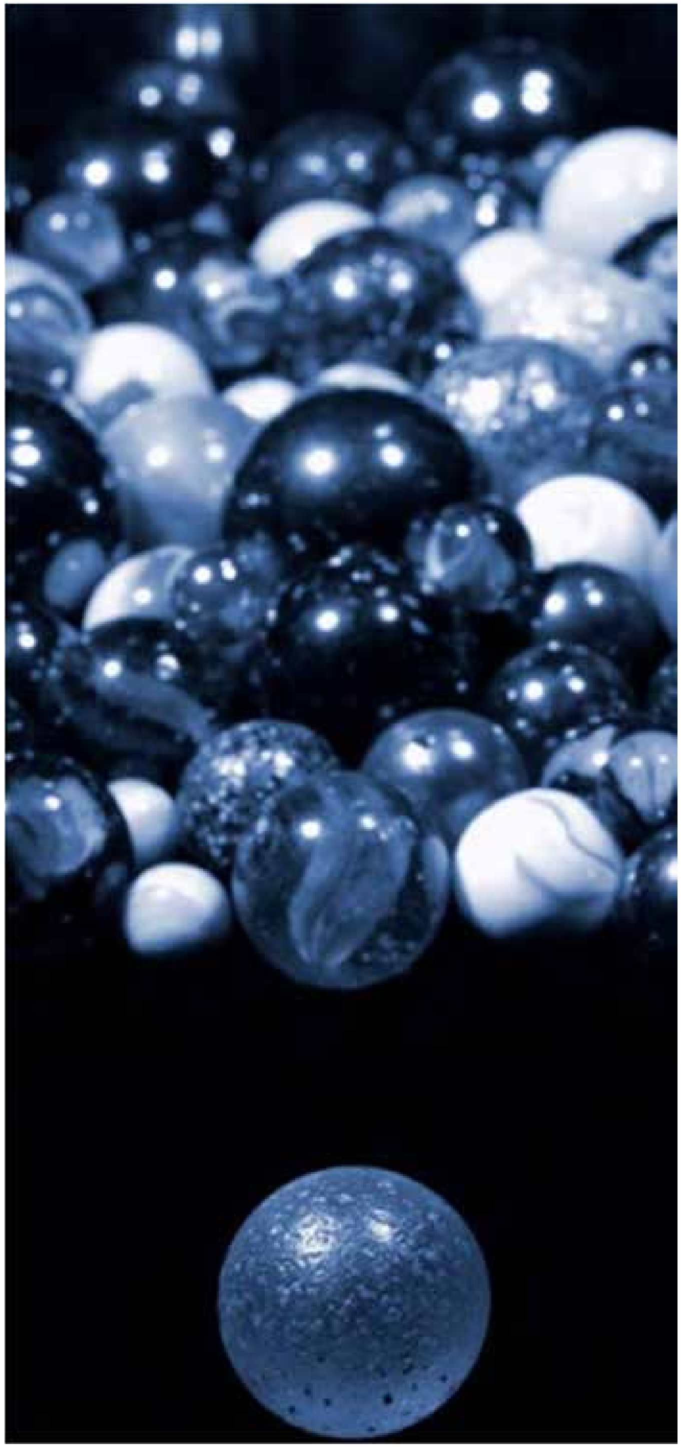
Can integrate out → single field EFT

Quasi-Single Field Inflation (QSFI)



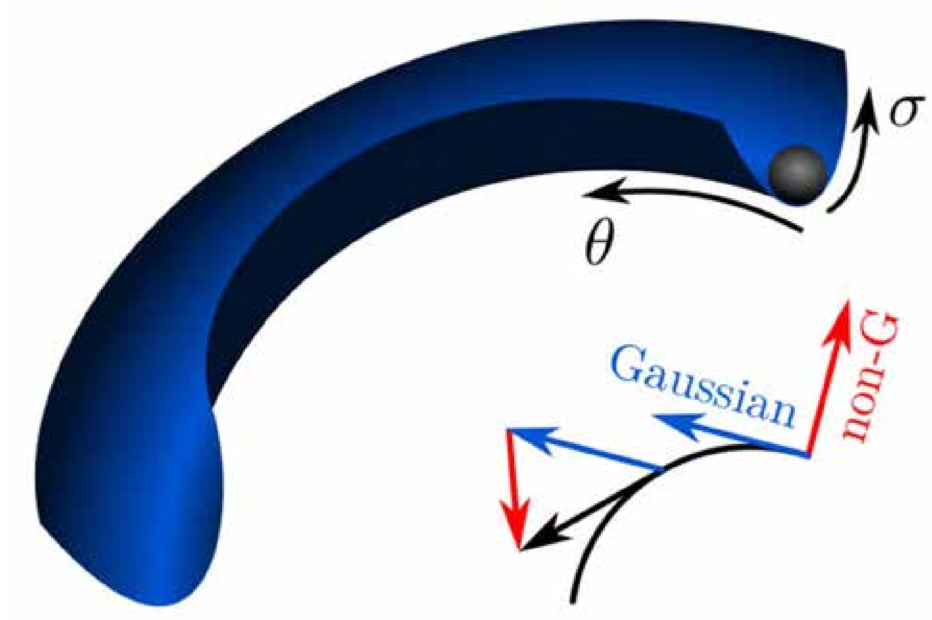
Gap between single field and multi-field:

BG: like single field; Pert: like multi-field



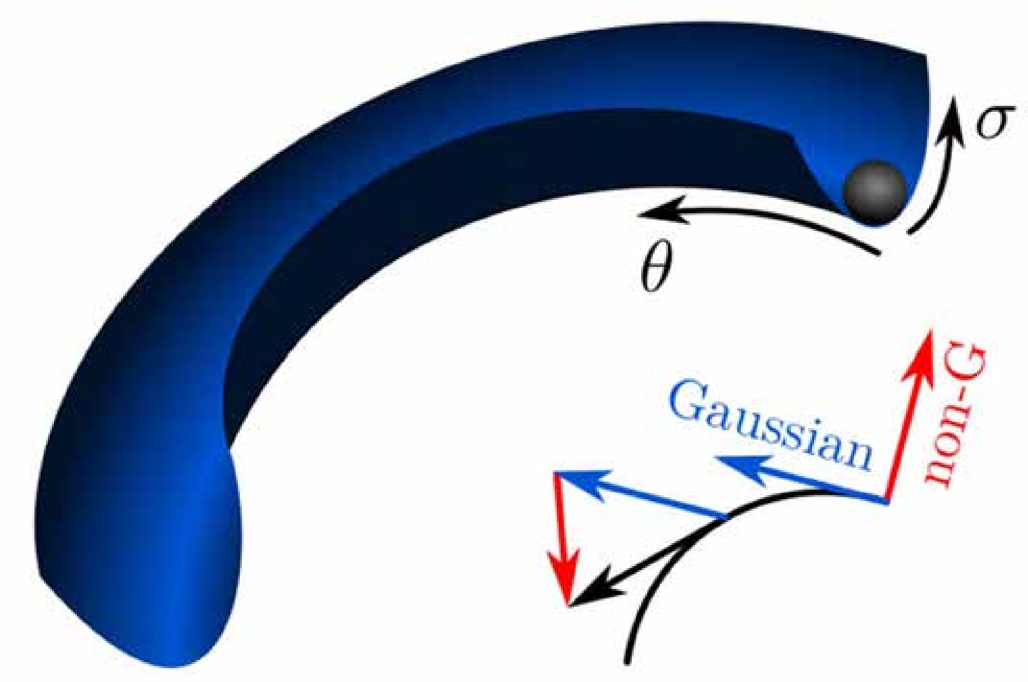
A toy model of QSFI

(more in Lecture 4)



Quasi-Single Field Inflation (QSFI)

A toy model and rough features of QSFI

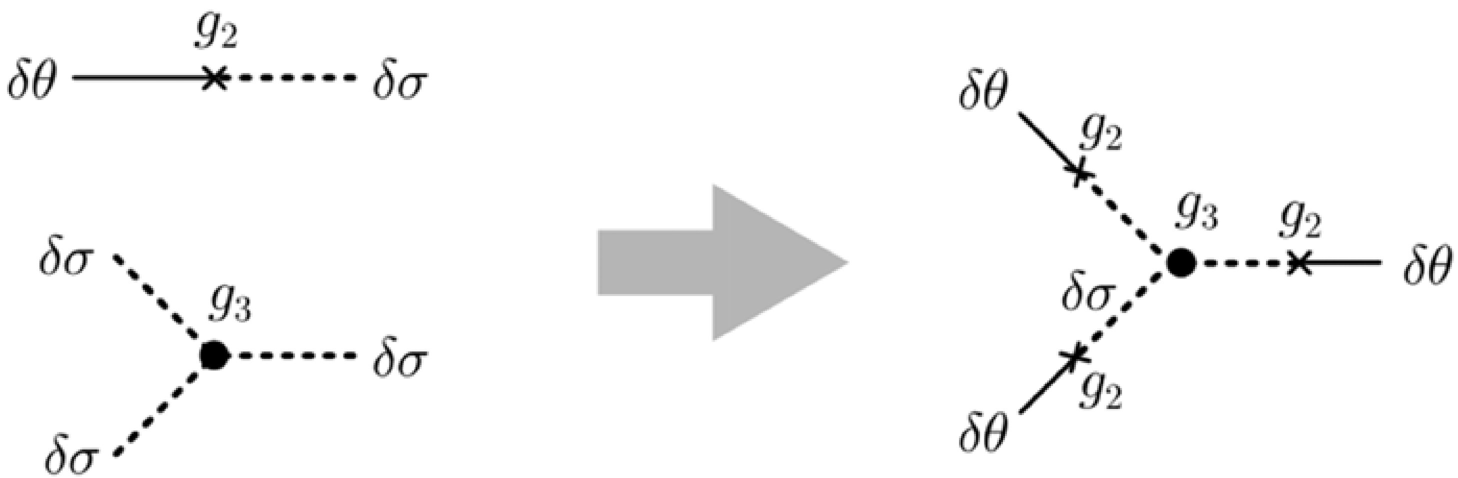


$$\Delta\mathcal{L}_2 = 2a^3\dot{\theta}_0 \times (\mathcal{R}\dot{\delta\theta}) \times \delta\sigma$$

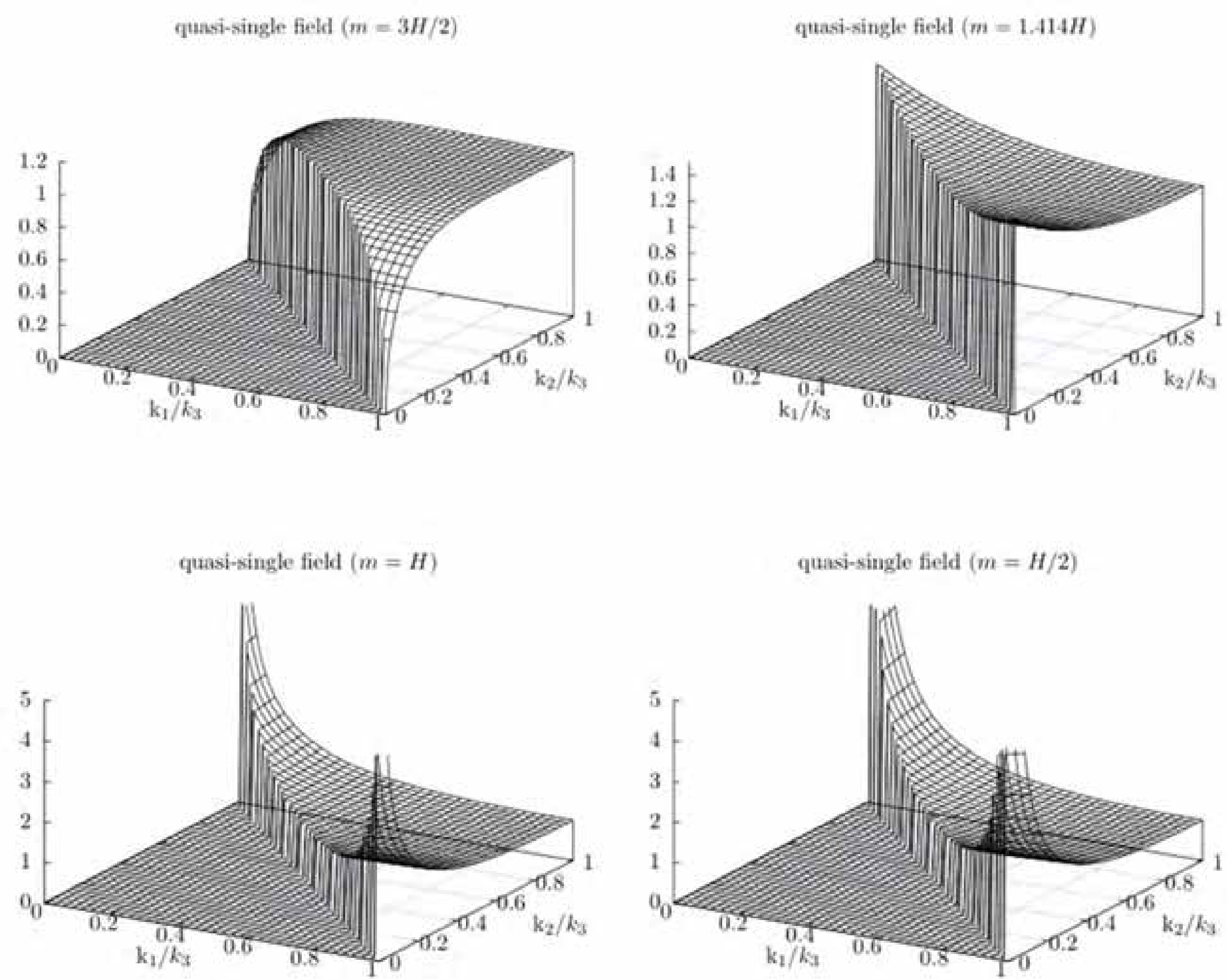
$$\Delta\mathcal{L}_3 = V'''\delta\sigma^3$$

Quasi-Single Field Inflation (QSFI)

A toy model and rough features of QSFI

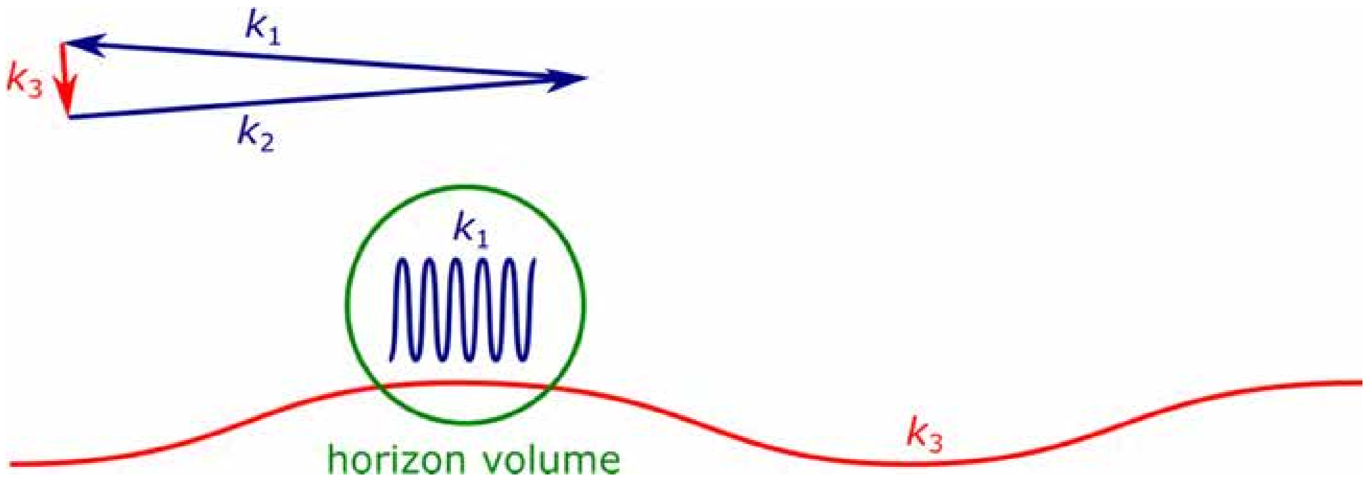


Quasi-Single Field Inflation (QSFI)



Soft limits of non-G

Soft limit of external momenta: Maldacena's consistency relation



3pt with one soft external momentum = scale dependence of 2pt

(assuming single field single mode: change of field = change of background)

Soft limits of non-G

Soft limit of external momenta: Maldacena's consistency relation

To make it explicit:

$$\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3} = (\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3})_0 + \zeta_{\mathbf{k}_1}\frac{\partial}{\partial\zeta_{\mathbf{k}_1}}(\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}) + \cdots$$

$$\zeta_{\mathbf{k}_1} \to \zeta_{\mathbf{k}_1} + \lambda(2\pi)^3\delta^3(\mathbf{k}_1)\ , \qquad \mathbf{k} \to \mathbf{k}e^{-\lambda}\ , \qquad (2\pi)^3\delta^3(\mathbf{k}_1)\frac{\partial}{\partial\zeta_{\mathbf{k}_1}} = \frac{\partial}{\partial\lambda} = -\frac{\partial}{\partial\ln k} = -\frac{\partial}{H\partial t}$$

$$\zeta_{\mathbf{k}_1}\frac{\partial}{\partial\zeta_{\mathbf{k}_1}}(\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}) = -\frac{\zeta_{\mathbf{k}_1}}{(2\pi)^3\delta^3(\mathbf{k}_1)}\frac{\partial}{H\partial t}(\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3})$$

$$\lim_{k_1/k_3\to 0}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = -(2\pi)^7\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{P_\zeta^2}{4k_1^3k_3^3}(n_s-1)$$

This is indeed satisfied by single field slow roll inflation.

What about curvaton?

Soft limits of non-G

Soft limit of external momenta: Maldacena's consistency relation

$$\lim_{k_1/k_3\to 0}\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = -(2\pi)^7\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{P_\zeta^2}{4k_1^3k_3^3}(n_s-1)$$

Examples when it breaks down:

- Multi-field inflation
- Ultra-slow-roll (kinetic term drive inflation for a few e-folds)
- Initial correlation in the UV (?)

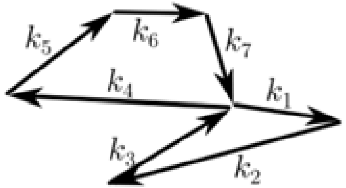
Soft limits of non-G

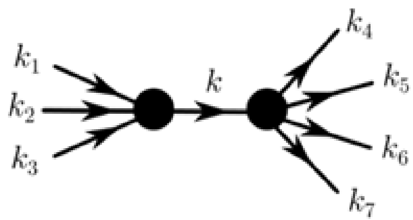
Soft limit of internal momenta: Suyama-Yamaguchi's relation

$$F \equiv F(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t)dt\right)$$

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = \langle F^\dagger\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}F\rangle$$

$$\langle\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{q}_1}\zeta_{\mathbf{q}_2}\rangle = \langle F^\dagger\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{q}_1}\zeta_{\mathbf{q}_2}F\rangle$$





At an initial time (sub-Hubble):

$$\mathbf{1} = \sum_i \int \frac{d^3p}{(2\pi)^3}\,|\phi_{\mathbf{p}}^{(i)}\rangle\langle\phi_{\mathbf{p}}^{(i)}| = \sum_i \int \frac{d^3p}{(2\pi)^3}\left(\frac{\phi_{\mathbf{p}}^{(i)}}{u_p^{(i)}}\right)^* |0\rangle\langle 0| \left(\frac{\phi_{\mathbf{p}}^{(i)}}{u_p^{(i)}}\right)$$

$$\phi_{\mathbf{p}}^{(i)} = u_p^{(i)}a_{\mathbf{p}}^{(i)} + (u_p^{(i)})^*(a_{-\mathbf{p}}^{(i)})^\dagger$$

For general time: $\mathbf{1} = \sum_i \int \frac{d^3p}{(2\pi)^3}\frac{1}{|u_p^{(i)}|^2}\phi_{\mathbf{p}}^{(i)}F|0\rangle\langle 0|F^\dagger(\phi_{\mathbf{p}}^{(i)})^\dagger$

$$\tau_{NL}(k,t) = \sum_i \frac{P^{(i)}(k,t)}{P(k,t)}\left(\frac{6}{5}f_{NL}^{(i)}(k,t)\right)^2 + \cdots$$

Soft limits of non-G

Soft limit of internal momenta: Suyama-Yamaguchi's relation

For single field: $\tau_{NL} = \frac{36}{25} f_{NL}^2$

In general: $\tau_{NL} \geq \frac{36}{25} f_{NL}^2$

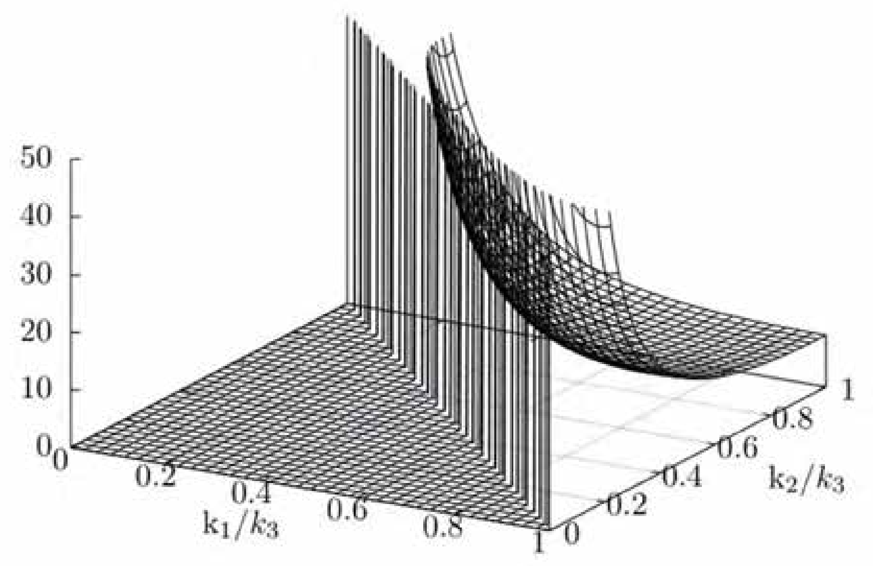
One can also use δN-formalism (when it is applicable) to derive this relation from Cauchy Schwarz inequality.

Non-G and Non-Non-G from Non-BD

What if the fluctuations do not start from a vacuum state?

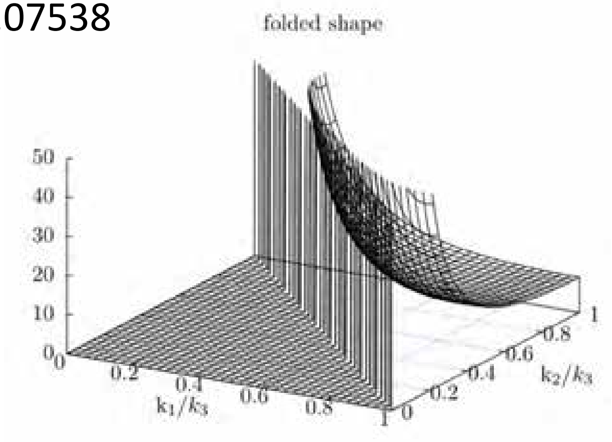
$$u_k = \frac{H}{\sqrt{4\epsilon k^3}} \left[C_k^{(+)}(1+ik\tau)e^{-ik\tau} + C_k^{(-)}(1-ik\tau)e^{ik\tau} \right]$$

Shape: $\mathrm{Re}[C_k^{(-)}] \frac{k_1^2k_2^2 + k_1^2k_3^2 + k_2^2k_3^2}{k_1k_2k_3} \left[\frac{1}{-k_1+k_2+k_3} + \frac{1}{k_1-k_2+k_3} + \frac{1}{k_1+k_2-k_3} \right]$



Non-G and Non-Non-G from Non-BD

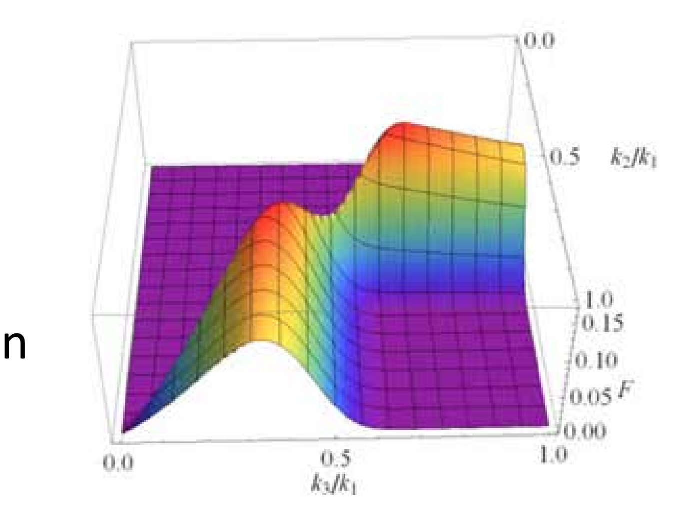
Hongliang Jiang, YW 1507.05193; Hongliang Jiang, YW, Siyi Zhou 1512.07538



Why is there a "folded divergence"?

Non-vacuum → unstable → decay via interaction

Unitarity:



$C_{\text{eff}} \sim \exp[\,-\Gamma\, f_{NL}^2\, (k\tau)^n\,]$,

where n depends on dimensionality of interaction

e.g. $n = 5$ for $\dot{\zeta}^3$ interaction

Non-G and Non-Non-G from Non-BD

Small non-G is not always "bad":

At least it enables us to probe a few more e-folds of inflation.

The Precision Era of non-G

(The following is my personal view point, may be biased)

The Precision Era of non-G

Non-G is becoming not-so-popular since 2013

Planck 2013 results. XXIV. Constraints on primordial non-Gaussianity

Planck Collaboration

P. A. R. Ade[87], N. Aghanim[60], C. Armitage-Caplan[93], M. Arnaud[73], M. Ashdown[70,6], F. Atrio-Barandela[18], J. Aumont[60], C. Baccigalupi[86], A. J. Banday[96,9], R. B. Barreiro[67], J. G. Bartlett[1,68], N. Bartolo[34,*], E. Battaner[97], K. Benabed[61,95], A. Benoit[58], A. Benoit-Lévy[25,61,95], J.-P. Bernard[96,9], M. Bersanelli[37,51], P. Bielewicz[96,9,86], J. Bobin[73], J. J. Bock[68,10], A.

Received: 22 March 2013

Accepted: 16 December 2013

Abstract

The *Planck* nominal mission cosmic microwave background (CMB) maps yield unprecedented constraints on primordial non-Gaussianity (NG). Using three optimal bispe estimators, separable template-fitting (KSW), binned, and modal, we obtain consistent values for the primordial local, equilateral, and orthogonal bispectrum ampli quoting as our final result $f_{NL}^{local} = 2.7 \pm 5.8$, $f_{NL}^{equil} = -42 \pm 75$, and $f_{NL}^{orth} = -25 \pm 39$ (68% CL statistical). Non-Gaussianity is detected in the data; using skew-C_ℓ sta we find a nonzero bispectrum from residual point sources, and the integrated-Sachs-Wolfe-lensing bispectrum at a level expected in the ΛCDM scenario. The results are on comprehensive cross-validation of these estimators on Gaussian and non-Gaussian simulations, are stable across component separation techniques, pass an ext suite of tests, and are confirmed by skew-C_ℓ, wavelet bispectrum and Minkowski functional estimators. Beyond estimates of individual shape amplitudes, we present r independent, three-dimensional reconstructions of the *Planck* CMB bispectrum and thus derive constraints on early-Universe scenarios that generate primordial NG, inc general single-field models of inflation, excited initial states (non-Bunch-Davies vacua), and directionally-dependent vector models. We provide an initial survey of dependent feature and resonance models. These results bound both general single-field and multi-field model parameter ranges, such as the speed of sound, $c_s \geq 0.02$ CL), in an effective field theory parametrization, and the curvaton decay fraction $r_D \geq 0.15$ (95% CL). The *Planck* data significantly limit the viable parameter space ekpyrotic/cyclic scenarios. The amplitude of the four-point function in the local model $\tau_{NL} < 2800$ (95% CL). Taken together, these constraints represent the highest pre tests to date of physical mechanisms for the origin of cosmic structure.

$$f_{NL}^{local} = 2.7 \pm 5.8,\ f_{NL}^{equil} = -42 \pm 75,\ \text{and}\ f_{NL}^{orth} = -25 \pm 39\ (68\%\ \text{CL statistical})$$

Planck 2015 results

XVII. Constraints on primordial non-Gaussianity

Planck Collaboration

P. A. R. Ade[97], N. Aghanim[63], M. Arnaud[79], F. Arroja[71,85], M. Ashdown[75,6], J. Aumont[63], C. Baccigalupi[95], M. Ballardini[51,53,34], A. J. Banday[109,10], R. B. Barreiro[70], N. Bartolo[33,71*], S. Basak[95], E. Battaner[110,111], K. Benabed[64,108], A. Benoit[61], A. Benoit-Lévy[26,64,108], J.-P. Bernard[109,10], M. Bersanelli[37,52], P. Bielewicz[89,10,95], J. J. Bock[72,12], A.

Received: 6 February 2015

Accepted: 27 January 2016

Abstract

The *Planck* full mission cosmic microwave background (CMB) temperature and *E*-mode polarization maps are analysed to obtain constraints on primordial non-Gaus: (NG). Using three classes of optimal bispectrum estimators - separable template-fitting (KSW), binned, and modal - we obtain consistent values for the primordial equilateral, and orthogonal bispectrum amplitudes, quoting as our final result from temperature alone $f^{\rm local}{}_{\rm NL}$ = 2.5 ± 5.7, $f^{\rm equil}{}_{\rm NL}$= -16 ± 70, , and $f^{\rm ortho}{}_{\rm NL}$ = -34 ± 32 CL, statistical). Combining temperature and polarization data we obtain $f^{\rm local}{}_{\rm NL}$ = 0.8 ± 5.0, $f^{\rm equil}{}_{\rm NL}$= -4 ± 43, and $f^{\rm ortho}{}_{\rm NL}$ = -26 ± 21 (68% CL, statistical). The resul based on comprehensive cross-validation of these estimators on Gaussian and non-Gaussian simulations, are stable across component separation techniques, pa extensive suite of tests, and are consistent with estimators based on measuring the Minkowski functionals of the CMB. The effect of time-domain de-glitching systemat the bispectrum is negligible. In spite of these test outcomes we conservatively label the results including polarization data as preliminary, owing to a known mismatch noise model in simulations and the data. Beyond estimates of individual shape amplitudes, we present model-independent, three-dimensional reconstructions of the CMB bispectrum and derive constraints on early universe scenarios that generate primordial NG, including general single-field models of inflation, axion inflation, initia modifications, models producing parity-violating tensor bispectra, and directionally dependent vector models. We present a wide survey of scale-dependent featur resonance models, accounting for the "look elsewhere" effect in estimating the statistical significance of features. We also look for isocurvature NG, and find no signal, b obtain constraints that improve significantly with the inclusion of polarization. The primordial trispectrum amplitude in the local model is constrained to be $g^{\rm local}{}_{\rm NL}$ = (- 7.7) X 10^4(68% CL statistical), and we perform an analysis of trispectrum shapes beyond the local case. The global picture that emerges is one of consistency wi premises of the ΛCDM cosmology, namely that the structure we observe today was sourced by adiabatic, passive, Gaussian, and primordial seed perturbations.

$f^{\rm local}{}_{\rm NL}$ = 0.8 ± 5.0, $f^{\rm equil}{}_{\rm NL}$= -4 ± 43, and $f^{\rm ortho}{}_{\rm NL}$ = -26 ± 21 (68% CL, statistical)

The Precision Era of non-G

COBE
(1990s)
$\Delta f_{NL} \sim 2000$

WMAP
1yr (2003)
$\Delta f_{NL} \sim 100$
7yr (2010)
$\Delta f_{NL} \sim 20$

Planck
(2013)
$\Delta f_{NL} \sim 5$

LSS
$\Delta f_{NL} \sim 0.5$

21cm:
$\Delta f_{NL} \sim 10^{-3}$
CMB distortion:
$\Delta f_{NL} \sim 10^{-3}$

The Precision Era of non-G

In $\sim$ 5-10 years Δf_{NL}~0.5 (e.g. SPHEREx)

(And Δf_{NL}~10^{-3} in the very distant future.)

What is the implication if $|f_{NL}| < 1$?

The Precision Era of non-G

In $\sim$ 5-10 years $\Delta f_{NL} \sim 0.5$ (e.g. SPHEREx)

(And $\Delta f_{NL} \sim 10^{-3}$ in the very distant future.)

What is the implication if $|f_{NL}| < 1$?

- Local: Curvaton will be very unlikely.
- Equilateral: $c_s \sim 1$, up to small corrections.

What is the motivation for future study?

The Precision Era of non-G

History of particle physics experiments:

- Early stage: studying external particle
  - α particle scattering
  - μ from cosmic rays
  - deep inelastic scattering
  - …
- Nowadays: study internal particle
  - Higgs - BSM - …

The Precision Era of non-G

Era	Pre-Planck	Post-Planck
Observable	CMB	LSS
NonG size	f_{NL} > O(1)	f_{NL} < O(1)
Physics	Curvaton, DBI, …	Massive states
Interest	External particles	Internal particles
Goal	Which inflation model	What particle physics
Toolkit	In-in formalism	+ EdS, O_{12}, nEFT, …

Summary of This Lecture

General methods:

in-in (see also δN)

Many kinds of non-G:

- Minimal
- EFT
- Multi-F
- QSFI
- Non-BD

Features:

- Maldacena's consistency relation
- Suyama-Yamaguchi relation

A precision era of non-G ahead!

Thank you ☺