THE UNIVERSAL KOLYVAGIN RECURSION IMPLIES THE KOLYVAGIN RECURSION

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Abstract. Let $\mathcal{U}_z$ be the universal norm distribution and M a fixed power of prime p, by using the double complex method employed by Anderson, we study the universal Kolyvagin recursion occurred in the canonical basis in the cohomology group $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$. We furthermore show that the universal Kolyvagin recursion implies the Kolyvagin recursion in the theory of Euler systems. One certainly hopes this could lead a new way to find new Euler systems.

1. Introduction

Let $\mathbb{F}$ be a finite real abelian extension of $\mathbb{Q}$. Let M be an odd positive integer. For every squarefree positive integer r the prime factors of which are congruent to 1 modulo M and split completely in $\mathbb{F}$, the corresponding Kolyvagin class $\kappa_r \in \mathbb{F}^\times/\mathbb{F}^{\times M}$ satisfies a remarkable and crucial recursion which for each prime number ℓ dividing r determines the order of vanishing of κ_r at each place of $\mathbb{F}$ above ℓ in terms of $\kappa_{r/\ell}$. In the note [2], Anderson and Ouyang gave the recursion a new and universal interpretation with the help of the double complex method. Namely, the recursion satisfied by Kolyvagin classes is shown to be the specialization of a universal recursion independent of $\mathbb{F}$ satisfied by the universal Kolyvagin classes in the group cohomology of the universal ordinary distribution.

In the note [2], the question of whether such kind of Kolyvagin recursion holds or not for the universal Euler systems is raised. The goal of this paper to answer this question.

Let X be a totally ordered set(in application, elements in X are often prime ideals in a number field K). Let Z be the set of formal products of elements in X. Let $\mathcal{O}$ be the integer ring of a number field or a local field with characteristic 0 and $\mathcal{T}$ be an $\mathcal{O}$-algebra which is a finite free $\mathcal{O}$-module. For every element $z \in Z$, let $z(x) = x^n$ be the x-part of z for $x \in X$ and $x \mid z$. We associate z with a group $G_z = \prod_{x|z} G_{x^n}$ and a $\mathcal{T}[G_z]$-module $\mathcal{U}_z$, a universal model satisfying certain distribution relations. This module $\mathcal{U}_z$ is called the *universal norm distribution* of level z, a generalization of the *universal ordinary distribution* of Kubert [3] and the *universal Euler system* of Rubin [8]. A special case of the universal norm distribution was first introduced

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in Ouyang [5]. The more general case was defined and studied in Ouyang [6]. Suppose M is a nonzero element in $\mathcal{O}$ dividing the order of all G_{x^n}, then the double complex method(see for example Anderson [1] and Ouyang [4]) produces a canonical basis for the cohomology group $H^*(G_z,\mathcal{U}_z/M\mathcal{U}_z)$ as a free $\mathcal{T}/M\mathcal{T}$-module. In particular, the canonical basis of $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$ can be given as $\{c_y : y \mid z, y \text{ squarefree}\}$. Moreover, one can consider $H^0(G_{z/x^n},\mathcal{U}_{z/x^n}/M\mathcal{U}_{z/x^n})$ as a subgroup of $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$ with $\mathcal{T}/M\mathcal{T}$-basis $\{c_y : y \mid \frac{z}{x^n}, y \text{ squarefree}\}$.

In this paper, we first find a natural and well defined map Δ_x from $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$ to $H^0(G_{z/x^n},\mathcal{U}_{z/x^n}/M\mathcal{U}_{z/x^n})$. We then show that Δ_x maps c_{xy} to c_y and c_y to 0 for xy a squarefree factor of z by the double complex method. Acting Δ_{x_i} consequently, we thus obtain a sequence of elements in $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$. We thus say that the family consisting of classes $\{c_y\}$ satisfies the *universal Kolyvagin recursion*. In addition, we show that the family of the original Kolyvagin classes(i.e., the cocycles which map to $D_r\xi_r$ as in Rubin [7]) also satisfies the universal Kolyvagin recursion.

Now given a number field K and a p-adic representation T of G_K with coefficients in $\mathcal{O}$, the universal norm distribution $\mathcal{U}_z$ in this situation becomes the universal Euler system, for which Rubin introduced and studied extensively in Chapter 4 of [8], resulting the proof of the celebrated Theorem 4.5.4 there. If let ξ be a G_K-homomorphism from $\mathcal{U}_z$ to $H^1(F(z),T)$ for $F(z)$ a certain extension of K. Let $W_M = M^{-1}T/T$, then ξ induces a map from $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$ to $H^1(F,W_M)$. Theorem 4.5.4 then states that the family of the Kolyvagin classes satisfies an important recursive relation, which is essential to the effectiveness of Euler system. We call this recursion the *Kolyvagin recursion*. We show that the universal Kolyvagin recursion implies the Kolyvagin recursion in Theorem 4.7.

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2. Review about the universal norm distribution

We first give a brief review of the results in Ouyang [6].

2.1. **The universal norm distribution.** Let X be a given totally ordered set. Let Y be the set of all squarefree formal products of elements in X, Let Z be the set of all (finite or infinite) formal product of elements in X. Elements in X will be denoted by x, x' and be called *primes*, elements in Y will be denoted by y, y' etc and elements in Z will be denoted by z, z', w and $\mathbf{z}$. Moreover, y, y' and w will usually assumed to be *finite* (i.e., finite product of $x \in X$) and $\mathbf{z}$ will assume to be *infinite*.

For every $z \in Z$, the *support* of z is the unique element $\bar{z} \in Y$ such that if $x \mid z$ then $x \mid \bar{z}$. For every $z \in Z$, a stalk of z is a factor $z' \mid z$ satisfying $\gcd(z', z/z') = 1$ and is denoted by $z' \mid_s z$. Fix z, for each $y \mid \bar{z}$, let $z(y)$ be corresponding stalk of z whose support is y. Let $v_x(z)$ be the integer n such that $x^n = z(x)$.

For every $z \in Z$, G_z is an abelian group which is the direct product of $G_{z(x)}$ for all $x \mid z$. Furthermore, for every $z' \mid z$, $G_{z'}$ is a subgroup of G_z and hence also a quotient of G_z. For any $z' \mid z$ and $g \in G_z$, Let $g_{z'}$ denote the restriction of g to $G_{z'}$. For each pair $x \in X$ and $z \in Z$, the Frobenius element Fr_x is a given element in G_z whose restriction to G_x is the identity.

Let $\mathcal{O}$ be an integral domain and let Φ be its fractional field. Let $\mathcal{T}$ be a fixed $\mathcal{O}$-algebra which is torsion free and finitely generated as an $\mathcal{O}$-module. For each $x \in X$, a polynomial

$$p(x;t) \in \mathcal{T}[\,t\,]$$

is chosen corresponding to x.

For every finite $z \in Z$, $\mathcal{B}_z$ is the free $\mathcal{T}$-module $\mathcal{T}[G_z] = \langle B_z \rangle_{\mathcal{T}}$ generated by

$$B_z = \{[g\ z] : g \in G_z\}.$$

For every $z \in Z$, finite or not, let

$$A_z = \bigcup_{\substack{z' \text{ finite} \\ z' |_s z}} B_{z'}$$

and let

$$\mathcal{A}_z = \langle A_z \rangle_{\mathcal{T}} = \bigoplus_{z' |_s z} \mathcal{B}_{z'} = \langle [g\ z'] : z' \mid_s z, z' \text{ finite},\ g \in G_{z'} \rangle_{\mathcal{T}}.$$

For every pair $z' \mid_s z$, the group G_z acts on $\mathcal{A}_{z'}$ by

$$g \cdot [g'' z''] = [g_{z''} \cdot g'' z''],\ z'' \text{ finite}, z'' \mid_s z'$$

and by this way $\mathcal{A}_{z'}$ becomes a $\mathcal{T}[G_z]$-module.

Let $\lambda_{z(x)}$ be the $\mathcal{T}[G_z]$-homomorphism given by

$$\lambda_{z(x)} : [z'] \longmapsto \begin{cases} p(x; \mathrm{Fr}_x^{-1})[z'] - N_{z(x)}[z(x)z'], & \text{if } x \nmid z', \\ 0, & \text{if } x \mid z'. \end{cases}$$

Let $\mathcal{D}_z$ be the submodule of $\mathcal{A}_z$ generated by the images of $\lambda_{z(x)}(\mathcal{A}_{z/z(x)})$ for all $x \mid z$. The *universal norm distribution* $\mathcal{U}_z$ is then defined to be the quotient $\mathcal{T}[G_z]$-module $\mathcal{A}_z/\mathcal{D}_z$.

From Ouyang [6], for any $z' \mid_s z \in Z$, the map

$$\mathcal{U}_{z'} \to \mathcal{U}_z$$

induced by the inclusion $\mathcal{A}_{z'} \subseteq \mathcal{A}_z$ is an injective G_z-homomorphism of free $\mathcal{T}$-modules with free cokernel and hence the induced map

$$H^0(G, \mathcal{U}_{z'}/M\mathcal{U}_{z'}) \to H^0(G, \mathcal{U}_z/M\mathcal{U}_z)$$

is also injective. Thus we henceforth identify $\mathcal{U}_{z'}$ (resp. $H^0(G, \mathcal{U}_{z'}/M\mathcal{U}_{z'})$) with a subgroup of $\mathcal{U}$ (resp. $H^0(G, \mathcal{U}_z/M\mathcal{U}_z)$). Note that we have

$$\mathcal{U}_z = \bigcup_{\substack{z' \text{ finite} \\ z' |_s z}} \mathcal{U}_{z'}, \quad H^0(G, \mathcal{U}_z/M\mathcal{U}_z) = \bigcup_{\substack{z' \text{ finite} \\ z' |_s z}} H^0(G, \mathcal{U}_{z'}/M\mathcal{U}_{z'}).$$

2.2. **Anderson's resolution $\mathcal{L}_z$.** For every $z \in Z$, $\mathcal{L}_z$ is the free $\mathcal{T}$-module generated by

$$\{[a, y] : [a] \in A_{z/z(y)}, y \mid \bar{z}\}$$

We equip $\mathcal{L}_z$ with a grading by declaring that

$$\deg[a, y] = -\deg y = -(\text{number of primes } x \mid y).$$

For any $g \in G_z$ and $[g'z'] \in A_{z/z(y)}$, We equip $\mathcal{L}_z$ with a G_z-action by the rule

$$g[g'z', y] := [g_{z'} g'z', y],$$

then $\mathcal{L}_z$ becomes a graded $\mathcal{T}[G_z]$-module. $\mathcal{L}_z$ is bounded above and is bounded if and only if z is finite. We equip $\mathcal{L}_z$ with a G_z-equivariant differential d of degree 1 by the rule

$$d[a,y] = \sum_{x|y} \omega(x,y)(p(x;\mathrm{Fr}_x^{-1})[a,y/x] - N_{z(x)}[az(x),y/x])$$

where ω is as defined as

$$(x,y) \longmapsto \begin{cases} (-1)^{\#\{x'|y:x'<x\}}, & \text{if } x \mid y; \\ 0, & \text{if } x \nmid y. \end{cases}$$

Then the complex $(\mathcal{L}_z^\bullet, d)$ is acyclic for degree $n \neq 0$ and $H^0(\mathcal{L}_z^\bullet, d) \cong \mathcal{U}_z$ induced by

$$\mathbf{u} : [a,y] \longmapsto \begin{cases} [a], & \text{if } y = \mathbf{1}; \\ 0, & \text{if } y \neq \mathbf{1}. \end{cases}$$

We call the complex $(\mathcal{L}_z^\bullet, d)$ (or simply $\mathcal{L}_z^\bullet$) *Anderson's resolution* of the universal norm system $\mathcal{U}_z$.

2.3. **The double complex $\mathbf{K}_z^{\bullet,\bullet}$.** Assume that $G_{z(x)}$ is acyclic. Let $\sigma_{z(x)}$ be a generator of $G_{z(x)}$. Let $\mathbf{K}_z^{\bullet,\bullet}$ be the free graded $\mathcal{T}$-module with basis

$$\{[a,y,w] : y \mid \bar{z}, a \in A_{z/z(y)}, \bar{w} \mid \bar{z}\}$$

and with the double grading given by

$$\deg[a,y,w] = (-\deg y, \deg w).$$

where $\deg w = \sum_{x|w} v_x(w)$. We equip $\mathbf{K}_z^{\bullet,\bullet}$ with a $\mathcal{T}[G_z]$-module structure by the rule

$$g[a,y,w] = [ga,y,w], \forall\ g \in G_z$$

For every $x \mid z$, we equip $K_z^{\bullet,\bullet}$ with G_z-equivariant differentials d_x of bidegree $(1,0)$ by the rule

$$d_x[a,y,w] = \omega(x,y)(-1)^{\sum_{x'<x} v_{x'}(w)} \left(p(x;\mathrm{Fr}_x^{-1})[a,y/x,w] - N_{z(x)}[az(x),y/x,w]\right),$$

and with G_z-equivariant differentials δ_x of bidegree $(0,1)$ by the rule

$$\delta_x[a,y,w] = (-1)^{\sum_{x'\le x} v_{x'}(y)}(-1)^{\sum_{x'<x} v_{x'}(w)} a_{z(x)}[a,y,wx]$$

where $a_{z(x)}$ is equal to $1-\sigma_{z(x)}$ if $v_x(w)$ even and $N_{z(x)}$ if $v_x(w)$ odd. Any two distinct differentials in the family $\{d_x\} \cup \{\delta_x\}$ anticommute. If let

$$d = \sum_x d_x, \qquad \delta = \sum_x \delta_x,$$

then d is a differential of bidegree $(1,0)$ and δ is of bidegree $(0,1)$ and d and δ are anticommute. $(\mathbf{K}_z; d, \delta)$ is a double complex of G_z-modules. Let $\mathbf{K}_z^\bullet$ be the single total complex of this double complex.

Let $\bar{\mathbf{K}}_z^\bullet$ be the quotient of free $\mathcal{T}$-module generated by

$$\{[a,w], a \in A_z, \bar{w} \mid \bar{z}\}$$

modulo relations generated by

$$p(x;\mathrm{Fr}_x^{-1})[a,w] - N_{z(x)}[a\ z(x),w],\ a \in A_{z/z(x)},\ \bar{w} \mid \bar{z},\ x \mid z.$$

With the grading given by the rule

$$\deg[a,w] = \deg w$$

and the differential δ given by the rule

$$\delta[a, w] = \sum_{x \in z} (-1)^{\sum_{x' < x} v_{x'} w} a_{z(x)}[wx],$$

$\bar{\mathbf{K}}_z^\bullet$ is a complex of G_z-modules whose cohomology is nothing but the group $H^*(G_z, \mathcal{U}_z)$. The homomorphism

$$\mathbf{u} : \mathbf{K}_z^\bullet \longrightarrow \bar{\mathbf{K}}_z^\bullet, \ [a, y, w] \longmapsto \begin{cases} [a, w], & \text{if } y = \mathbf{1} \\ 0, & \text{if } y \neq \mathbf{1} \end{cases}$$

is a quasi-isomorphism. Thus it induces isomorphisms between $H^*(\mathbf{K}_z^\bullet, d + \delta)$ and $H^*(G_z, \mathcal{U}_z)$(resp. $H^*(\mathbf{K}_z^\bullet / M\mathbf{K}_z^\bullet)$ and $H^*(G_z, \mathcal{U}_z / M\mathcal{U}_z)$ for $0 \neq M \in \mathcal{O}$). In particular, for any 0-cocycle c in $\mathbf{K}_z^\bullet / M\mathbf{K}_z^\bullet$, the map $\mathbf{u}$ sends its bidegree $(0, 0)$-component $\sum [a, \mathbf{1}, \mathbf{1}]$ to $\sum [a, 1] \in \bar{\mathbf{K}}_z^0 / M\bar{\mathbf{K}}_z^0$ and then to $\sum [a] \in \mathcal{U}_z / M\mathcal{U}_z$, the resulting element is fixed by G_z and hence is a cocycle in $H^0(G_z, \mathcal{U}_z / \mathcal{U}_z)$.

For every $z' \mid_s z$, let $\mathbf{K}_{z'}$ be the submodule of $\mathbf{K}_z$ generated by

$$\{[a, y, w] : y \mid z', a \in B_{z'/z(y)}, \bar{w} \mid z'\}$$

and let $\mathbf{K}_z(z')$ be the submodule generated by

$$\{[a, y, w] : y \mid z', a \in B_{z'/z(y)}, \bar{w} \mid z\}$$

Then $\mathbf{K}_{z'}$ and $\mathbf{K}_z(z')$ are compatible with differentials d_x and δ_x. The $(d + \delta)$-cohomology of $\mathbf{K}_{z'}$(resp. $\mathbf{K}_{z'} / M\mathbf{K}_{z'}$) is just $H^*(G_{z'}, \mathcal{U}_{z'})$(resp. $H^*(G_{z'}, \mathcal{U}_{z'} / M\mathcal{U}_{z'})$) and the $(d + \delta)$-cohomology of $\mathbf{K}_z(z')$(resp. $\mathbf{K}_z(z') / M\mathbf{K}_z(z')$) is $H^*(G_z, \mathcal{U}_{z'})$(resp. $H^*(G_z, \mathcal{U}_{z'} / M\mathcal{U}_{z'})$).

2.4. **The canonical basis for $H^0(G_z, \mathcal{U}_z / M\mathcal{U}_z)$.** Suppose now that M is a common divisor of $|G_{z(x)}|$ and $p(x; 1)$ for every $x \mid z$. Let $\mathbf{S}_z$ be the $\mathcal{T}$-submodule of $\mathbf{K}_z$ generated by

$$\{[a, y, w] : a \in B_{z/z(y)}, \ y \mid z, \ \bar{w} \mid z, a \notin B_{\mathbf{1}} \text{ if } y \mid w\}.$$

Then $\mathbf{S}_z / M\mathbf{S}_z$ is a submodule of $\mathbf{K}_z / M\mathbf{K}_z$ which is also $d-$ and $\delta-$ stable and thus is a subcomplex of $\mathbf{K}_z / M\mathbf{K}_z$ with respect to the multi-complex structure of $\mathbf{K}_z / M\mathbf{K}_z$. Let $\mathbf{Q}_z / M\mathbf{Q}_z$ be the quotient of $\mathbf{K}_z / M\mathbf{K}_z$ by $\mathbf{S}_z / M\mathbf{S}_z$. Then $\mathbf{Q}_z / M\mathbf{Q}_z$ is a free $\mathcal{T} / M\mathcal{T}$-module generated by

$$\{[\mathbf{1}, y, w] : y \mid \bar{w} \mid z\}$$

with all induced differentials $d = \delta = 0$. Write the quotient map from $\mathbf{K}_z / M\mathbf{K}_z$ to $\mathbf{Q}_z / M\mathbf{Q}_z$ as ρ_M. The homomorphism ρ_M is a quasi-isomorphism as claimed in Ouyang [6]. Every element $[\mathbf{1}, y, y]$ for y finite and dividing z in $\mathbf{Q}_z / M\mathbf{Q}_z$ thus induces a 0-cocycle in $\mathbf{K}_z / M\mathbf{K}_z$ and henceforth induces a cocycle in $H^0(G_z, \mathcal{U}_z / M\mathcal{U}_z)$. Write this cocycle $\bar{c}_y$, then

$$\{\bar{c}_y : y \text{ finite}, y \mid z\}$$

forms a basis for $H^0(G_z, \mathcal{U}_z / M\mathcal{U}_z)$. We call it the *canonical basis.*

3. The universal Kolyvagin Recursions

3.1. **The Kolyvagin conditions.** In this section, we fix an infinite $\mathbf{z} \in Z$. we write $\mathcal{U}_{\mathbf{z}}$($\mathcal{L}_{\mathbf{z}}$, $G_{\mathbf{z}}$ and $\mathbf{K}_{\mathbf{z}}$ etc.) as $\mathcal{U}$($\mathcal{L}$, G and $\mathbf{K}$ etc.). For any $x \mid \mathbf{z}$, let $r_x(t) \in \mathcal{O}[t]$ and

$$\gamma_{\mathbf{z}(x)} = p(x; \mathrm{Fr}_x^{-1}) - r_x(\mathrm{Fr}_x^{-1})|G_{\mathbf{z}(x)}|.$$

Assume the following *Kolyvagin conditions* hold:

- There exists $M \in \mathcal{O}$ such that $M \mid |G_x|$ and $M \mid p(x; 1)$ for every $x \mid \mathbf{z}$. We fix M here after.
- The group $G_{\mathbf{z}(x)}$ is cyclic for every $x \mid \mathbf{z}$.
- The homomorphism $\gamma_{\mathbf{z}(x)} : \mathcal{U}_{\mathbf{z}/z(x)} \to \mathcal{U}_{\mathbf{z}/z(x)}$ has a trivial kernel for every $x \mid \mathbf{z}$.

Following the first two assumptions, we can and hence will apply the results in § 2. If we let $n_x = \mathrm{Norm}(x)$ and let

$$r_x(\mathrm{Fr}_x^{-1}) = \frac{p(x; \mathrm{Fr}_x^{-1}) - p(x; n_x \, \mathrm{Fr}_x^{-1})}{|G_{\mathbf{z}(x)}|},$$

then $\gamma_{\mathbf{z}(x)} = p(x; n_x \, \mathrm{Fr}_x^{-1})$. When $p(x; t)$ is coming from certain characteristic polynomial in a p-adic representation, then $\gamma_{\mathbf{z}(x)}$ can be shown to satisfy the last assumption. See Lemma 4.6 in § 4 for more details.

3.2. **The submodule I_x of $\mathcal{U}$.** Let $x \mid \mathbf{z}$ be given. We define

$$I_x \subset \mathcal{U}$$

to be the $\mathcal{T}[G]$-submodule generated by all elements of $\mathcal{U}$ represented by the expressions of the form

$$[z\mathbf{z}(x)] - g[z\mathbf{z}(x)] \quad \text{or} \quad r_x(\mathrm{Fr}_x^{-1})[z] - g[z\mathbf{z}(x)] (g \in G_{\mathbf{z}(x)}, z \text{ finite } z \mid_s \mathbf{z}, x \nmid z).$$

When the dependence on $\mathbf{z}$ is needed to emphasize, we write I_x as $I_{x,\mathbf{z}}$. Note that

$$(\sigma_{\mathbf{z}(x)} - 1)\mathcal{U} \subset I_x,$$

the quotient $\mathcal{U}/I_x$ can be viewed as a $G/G_{\mathbf{z}(x)}$-module.

Proposition 3.1. *The sequence*

$$0 \longrightarrow \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \xrightarrow{\gamma_{\mathbf{z}(x)}} \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \longrightarrow \mathcal{U}/I_x \longrightarrow 0$$

is exact where the map $\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \to \mathcal{U}/I_x$ is that induced by the inclusion of $\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \subset \mathcal{U}$.

Proof. We consider the complex homomorphism

$$\gamma_{\mathbf{z}(x)} : \mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)} \longrightarrow \mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)}$$

The mapping cone of $\gamma_{\mathbf{z}(x)}$ is just the complex

$$Cone^{\bullet}(\gamma_{\mathbf{z}(x)}) = (\mathcal{L}^{n+1}_{\mathbf{z}/\mathbf{z}(x)}, \mathcal{L}^{n}_{\mathbf{z}/\mathbf{z}(x)})$$

with the differential given by

$$d(a, b) = (-da, \gamma_{\mathbf{z}(x)}(a) + db), \text{for } (a, b) \in Cone^n(\gamma_{\mathbf{z}(x)}).$$

From homological algebra, we know there exists the following exact sequence

$$\Sigma : 0 \longrightarrow \mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)} \longrightarrow Cone^{\bullet}(\gamma_{\mathbf{z}(x)}) \longrightarrow \mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)}[1] \longrightarrow 0.$$

Let

$$s_x : \mathcal{L} \to \mathcal{L}(\mathbf{z}/\mathbf{z}(x))$$

be the unique homomorphism such that

$$s_x[a, y] \equiv \begin{cases} \omega(x, y)[a, y/x] & \text{if } x \mid y \\ 0 & \text{otherwise} \end{cases}$$

for all symbols $[a, y]$ in the canonical basis of $\mathcal{L}$. The homomorphism s_x is of degree 1 and satisfies the relation

$$s_x d = -ds_x$$

as can be verified by a straightforward calculation. Now consider the sequence

$$\Sigma' : \ 0 \to \mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)} \longrightarrow \mathcal{L}^{\bullet}/\mathcal{L}' \xrightarrow{s_x} \mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)}[1] \to 0$$

where $\mathcal{L}'$ is the $\mathcal{T}[G]$-submodule of $\mathcal{L}$ generated by all elements of the form

$$[z\mathbf{z}(x), y] - [gz\mathbf{z}(x), y] \quad \text{or} \quad r_x(\mathrm{Fr}_x^{-1})[z, y] - [gz\mathbf{z}(x), y](\mathbf{z}(y)z\mathbf{z}(x) \mid \mathbf{z}, \ g \in G_{\mathbf{z}(x)}).$$

and the map $\mathcal{L}_{\mathbf{z}/\mathbf{z}(x)} \to \mathcal{L}/\mathcal{L}'$ is that induced by the inclusion $\mathcal{L}_{\mathbf{z}/\mathbf{z}(x)} \subset \mathcal{L}$. It is easy to verify that Σ' is short exact. The two complexes Σ and Σ' are actually isomorphic: just let the two side maps be the identities and let the middle map from $\mathcal{L}^{\bullet}/\mathcal{L}'$ to $Cone^{\bullet}$ be given by

$$[a, yx] \mapsto (\omega(x, xy)[a, y], 0), \qquad [a, y] \mapsto (0, [a, y]), \forall\ y \mid \mathbf{z}/\mathbf{z}(x).$$

Since $\mathcal{L}'$ is a graded d- and G-stable subgroup of $\mathcal{L}^{\bullet}$ and $(\sigma_{\mathbf{z}(x)} - 1)\mathcal{L} \subset \mathcal{L}'$, it follows that Σ' can be viewed as a short exact sequence of complexes of $G/G_{\mathbf{z}}$-modules. Because $H^*(\mathcal{L}^{\bullet}_{\mathbf{z}/\mathbf{z}(x)}, d)$ is concentrated in degree 0, the long exact sequence of $G/G_{\mathbf{z}(x)}$-modules deduced from Σ' by taking d-cohomology has at most four nonzero terms and after making the evident identifications takes the form

$$\cdots \to 0 \to H^{-1}(\mathcal{L}^{\bullet}/\mathcal{L}', d) \to \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \xrightarrow{\gamma_{\mathbf{z}(x)}} \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \longrightarrow \mathcal{U}/I_x \to 0 \to \dots$$

where the map $\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \to \mathcal{U}/I_x$ is that induced by the inclusion $\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \subset \mathcal{U}$. By the assumption of the Kolyvagin conditions, we have

$$H^{-1}(\mathcal{L}/\mathcal{L}', d) = \ker\left(\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \xrightarrow{\gamma_{\mathbf{z}(x)}} \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}\right) = 0,$$

whence the result. □

Theorem 3.2. *For every prime number x dividing $\mathbf{z}$ there exists a unique homomorphism*

$$\Delta_x : H^0(G, \mathcal{U}/M\mathcal{U}) \to H^0(G, \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}/M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)})$$

such that

$$\frac{(1 - \sigma_{\mathbf{z}(x)})a}{M} \equiv \frac{\gamma_x b}{M} \bmod I_x \Leftrightarrow \Delta_x(a \bmod M\mathcal{U}) = b \bmod M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$$

for all $a \in \mathcal{U}$ representing a class in $H^0(G, \mathcal{U}/M\mathcal{U})$ and $b \in \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$ representing a class in $H^0(G, \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}/M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)})$. Moreover one has

$$\Delta_x H^0(G, \mathcal{U}_z/M\mathcal{U}_z) \subset H^0(G, \mathcal{U}_{z/\mathbf{z}(x)}/M\mathcal{U}_{z/\mathbf{z}(x)})$$

for all finite $z \mid_s \mathbf{z}$ and divisible by x.

Proof. Put

$$\begin{array}{lcl} A & := & \left\{a \in \mathcal{U} \mid a \text{ represents a class in } H^0(G,\mathcal{U}/M\mathcal{U})\right\}, \\ B & := & \left\{b \in \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \mid \gamma_x(b) \in M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}\right\}, \\ C & := & \left\{(a,b) \in X \times Y \,\middle|\, \frac{(1-\sigma_{\mathbf{z}(x)})a}{M} \equiv \frac{\gamma_x(b)}{M} \bmod I_x\right\}. \end{array}$$

Fix a finite $z \mid_s \mathbf{z}$ and divisible by x. To prove the proposition it is enough to prove the following three claims:

(1) $C \cap (M\mathcal{U} \times B) = M\mathcal{U} \times M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$.
(2) $(\sigma - 1)C \subset M\mathcal{U} \times M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$ for all $\sigma \in G$.
(3) For all $a \in A \cap \mathcal{U}_z$ there exists $b \in B \cap \mathcal{U}_{z/\mathbf{z}(x)}$ such that $(a,b) \in C$.

We turn to the proof of the first claim. Only the containment $\subset$ requires proof; the containment $\supset$ is trivial. Suppose we are given $(a,b) \in C \cap (M\mathcal{U} \times B)$. Then $\frac{\gamma_x(b)}{M} \in I_x \cap \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$ and hence by Proposition 3.1 there exists $c \in \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$ such that $\gamma_x(b) = M\gamma_x(c)$. It follows that $b = Mc$. Thus the first claim is proved. The second claim follows immediately from the first.

We turn finally to the proof of the third claim. Let

$$\beta_{\mathbf{z}(x)} : \mathcal{A}_z \to \mathcal{A}_{z/\mathbf{z}(x)}$$

be the unique homomorphism such that

$$\beta_{\mathbf{z}(x)}[z'] = r_x(\mathrm{Fr}_x^{-1})[z'/\mathbf{z}(x)]$$

for all $x \mid z' \mid_s z$. For each prime number x dividing z, recall that

$$\lambda_{\mathbf{z}(x)} : \mathcal{A}_{z/\mathbf{z}(x)} \to \mathcal{A}_z$$

is the unique homomorphism such that

$$\lambda_{\mathbf{z}(x)}[z'] := p(x; \mathrm{Fr}_x^{-1})[z'] - N_{\mathbf{z}(x)}[z'\mathbf{z}(x)]$$

for all $x \nmid z'$. Note that $\beta_{\mathbf{z}(x)}$ commutes with $\lambda_{\mathbf{z}(x')}$ for $x' \neq x$ and that the composite homomorphism $\beta_{\mathbf{z}(x)}\lambda_{\mathbf{z}(x)}$ induces the endomorphism $\gamma_{\mathbf{z}(x)}$ of $\mathcal{A}_{z/\mathbf{z}(x)}$. Choose a lifting $\mathbf{a} \in \mathcal{A}_z$ of a. By hypothesis there exists an identity

$$(\sigma_{\mathbf{z}(x)} - 1)\mathbf{a} = M\mathbf{b} + \sum_{x \mid z} \lambda_{\mathbf{z}(x)}\mathbf{b}_x \quad (\mathbf{b} \in \mathcal{A}_z, \;\; \mathbf{b}_x \in \mathcal{A}_{z/\mathbf{z}(x)}),$$

and hence also an identity

$$0 = M\beta_{\mathbf{z}(x)}\mathbf{b} + \gamma_{\mathbf{z}(x)}\mathbf{b}_x + \sum_{x' \mid \frac{z}{\mathbf{z}(x)}} \lambda_{\mathbf{z}(x')}\beta_{\mathbf{z}(x)}\mathbf{b}_{x'}.$$

Then the element $b \in \mathcal{U}_{z/\mathbf{z}(x)}$ represented by $\mathbf{b}_x$ has the desired property, namely that $(a,b) \in C$. Thus the third claim is proved and with it the result. $\square$

3.3. **The universal Kolyvagin recursion.** We say that a family of classes

$$\{c_y \in H^0(G,\mathcal{U}/M\mathcal{U})\}_{y \mid \bar{\mathbf{z}}}$$

indexed by finite $y \mid \bar{\mathbf{z}}$ satisfies the *universal Kolyvagin recursion* if the following conditions hold for all finite $y \mid \bar{\mathbf{z}}$ and primes $x \mid \mathbf{z}$:

- $c_y \in H^0(G_{\mathbf{z}(y)},\mathcal{U}_{\mathbf{z}(y)}/M\mathcal{U}_{\mathbf{z}(y)}) = H^0(G,\mathcal{U}_{\mathbf{z}(y)}/M\mathcal{U}_{\mathbf{z}(y)}) \subset H^0(G,\mathcal{U}/M\mathcal{U})$.
- $x \mid y \Rightarrow \Delta_x c_y = c_{y/x}$.

3.4. **The diagonal shift operator Δ_x.** For each x dividing $\mathbf{z}$, we define the corresponding *diagonal shift* operator Δ_x on $\mathbf{K}$ of bidegree $(1,-1)$ by the rule

$$\Delta_x[a,y,w] := \begin{cases} [a,y/x,w/x], & \text{if } x \mid y \text{ and } x \mid w, \\ 0, & \text{otherwise.} \end{cases}$$

One has

$$\Delta_x d_{x'} = d_{x'}\Delta_x, \quad \Delta_x\delta_{x'} = \delta_{x'}\Delta_x$$

for all primes $x' \mid \mathbf{z}$ distinct from x. One has

$$\Delta_x d_x = d_x\Delta_x = 0, \quad (\delta_x\Delta_x - \Delta_x\delta_x)\mathbf{K} \subset M\mathbf{K}.$$

For every finite z dividing $\mathbf{z}$ one has

$$\Delta_x\mathbf{K}_z \subset \begin{cases} \mathbf{K}_{z/\mathbf{z}(x)}, & \text{if } x \mid z, \\ \{0\}, & \text{otherwise.} \end{cases}$$

The action of Δ_x therefore passes to

$$H^0(\mathbf{K}_z/M\mathbf{K}_z, d+\delta) = H^0(G_z, \mathcal{U}_z/M\mathcal{U}_z)$$

and in the limit to

$$H^0(\mathbf{K}/M\mathbf{K}, d+\delta) = H^0(G, \mathcal{U}/M\mathcal{U}).$$

Proposition 3.3. *For each $x \mid \mathbf{z}$, the endomorphism of $H^0(G,\mathcal{U}/M\mathcal{U})$ induced by the diagonal shift operation Δ_x coincides with Δ_x defined in Theorem 3.2.*

Proof. Fix a finite $z \mid_s \mathbf{z}$ divisible by x. Fix a class

$$c \in H^0(G_z, \mathcal{U}_z/M\mathcal{U}_z).$$

It suffices to show that $\mathit{\Delta}_x$ and the endomorphism of $H^0(G,U/MU)$ induced by Δ_x applied to c give the same result. Let $\mathbf{c}$ be a 0-chain in $\mathbf{K}_z$ reducing modulo $M\mathbf{K}_z$ to a 0-cycle representing c. Write

$$0 = (d+\delta)\mathbf{c} + M\mathbf{b}$$

where $\mathbf{b}$ is a 1-chain of $\mathbf{K}_z$. For any finite $y \mid z$ and finite w such that $\bar{w} \mid z$, let

$$(\mathbf{a} \mapsto \mathbf{a} \otimes [y,w]) : \mathcal{A}_{z/\mathbf{z}(y)} \to \mathbf{K}_z$$

be the unique homomorphism such that

$$[a] \otimes [y,w] := [a,y,w]$$

for all $a \in A_{z/\mathbf{z}(y)}$. Write

$$\mathbf{c} = \sum \mathbf{c}_{y,w} \otimes [y,w], \quad \Delta_x\mathbf{c} = \sum \mathbf{c}_{yx,wx} \otimes [y,w] \quad (\mathbf{c}_{y,w} \in \mathcal{A}_{z/\mathbf{z}(y)})$$

and

$$\mathbf{b} = \sum \mathbf{b}_{y,w} \otimes [y,w] \quad (\mathbf{b}_{y,w} \in \mathcal{A}_{z/\mathbf{z}(y)})$$

where all the sums are extended over pairs (y,w) consisting of finite $y \mid z$ and w with $\bar{w} \mid z$. Let $\beta_{\mathbf{z}(x)} : \mathcal{A}_z \to \mathcal{A}_{z/\mathbf{z}(x)}$ be as in the proof of Proposition 3.2. By hypothesis one has an identity

$$0 = \sum_{\substack{x' \mid z \\ x' < x}} \lambda_{\mathbf{z}(x')}\mathbf{c}_{x',x} + \lambda_{\mathbf{z}(x)}\mathbf{c}_{x,x} - \sum_{\substack{x' \mid z \\ x' > x}} \lambda_{\mathbf{z}(x')}\mathbf{c}_{x',x} + (1-\sigma_{\mathbf{z}(x)})\mathbf{c}_{\mathbf{1},\mathbf{1}} + M\mathbf{b}_{\mathbf{1},x}$$

and hence also an identity

$$0=\sum_{\substack{x'|z\\x'<x}}\lambda_{\mathbf{z}(x')}\beta_{\mathbf{z}(x)}\mathbf{c}_{x',x}+\gamma_{\mathbf{z}(x)}\mathbf{c}_{x,x}-\sum_{\substack{x'|z\\x'>x}}\lambda_{\mathbf{z}(x')}\beta_{\mathbf{z}(x)}\mathbf{c}_{x',x}+M\beta_{\mathbf{z}(x)}\mathbf{b}_{\mathbf{1},x}.$$

Let $a\in\mathcal{U}_r$ be the element represented by $\mathbf{c_{1,1}}$ and let $b\in\mathcal{U}_{z/\mathbf{z}(x)}$ be the element represented by $\mathbf{c}_{x,x}$. One the one hand, the class of $H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)$ represented by the 0-cocycle $\mathbf{c}$ mod M of the complex $(\mathbf{K}_z/M\mathbf{K}_z,d+\delta)$ is a mod $M\mathcal{U}_z$ and the class of $H^0(G_{z/\mathbf{z}(x)},\mathcal{U}_{z/\mathbf{z}(x)}/M\mathcal{U}_{z/\mathbf{z}(x)})$ represented by the 0-cocycle $\Delta_x\mathbf{c}$ mod M of the complex $(K_{z/\mathbf{z}(x)}/MK_{z/\mathbf{z}(x)},d+\delta)$ is b mod $M\mathcal{U}_{z/\mathbf{z}(x)}$. But on the other hand, one has

$$\frac{(1-\sigma_{\mathbf{z}(x)})a}{M}\equiv\frac{\gamma_{\mathbf{z}(x)}b}{M}\bmod I_x$$

and hence

$$\mathit{\Delta}_x(a\bmod M\mathcal{U})\equiv b\bmod M\mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$$

by Theorem 3.2. Therefore the results of applying $\mathit{\Delta}_x$ and the endomorphism of $H^0(G,\mathcal{U}/M\mathcal{U})$ induced by Δ_x to the class a mod $M\mathcal{U}$ indeed coincide. □

Corollary 3.4. *The canonical basis $\{\bar{c}_y : y$ finite, $y\mid\mathbf{z}\}$ satisfies the universal Kolyvagin recursion.*

Proof. Clear. □

Corollary 3.5. *Any system of classes $\{b_y\}$ satisfying the universal Kolyvagin recursion and the normalization $b_{\mathbf{1}}=\bar{c}_{\mathbf{1}}$ is a $\mathcal{T}/M\mathcal{T}$-basis of $H^0(G,\mathcal{U}/M\mathcal{U})$.*

Proof. Fix a finite $y\mid\mathbf{z}$, let $z=\mathbf{z}(y)$. Let

$$y=x_1\cdots x_n$$

be the prime factorization of y. One then has

$$\mathit{\Delta}_{x_1}\cdots\mathit{\Delta}_{x_n}b_y=b_{\mathbf{1}}=\bar{c}_{\mathbf{1}}=\mathit{\Delta}_{x_1}\cdots\mathit{\Delta}_{x_n}\bar{c}_y$$

and hence

$$b_y-\bar{c}_y\in\ker\left(H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)\xrightarrow{\mathit{\Delta}_{x_1}\cdots\mathit{\Delta}_{x_n}}H^0(G_z,\mathcal{U}_z/M\mathcal{U}_z)\right)=\bigoplus_{\substack{y'|y\\y'\neq y}}\mathcal{T}/M\mathcal{T}\cdot\bar{c}_{y'},$$

whence the result. □

3.5. **The original Kolyvagin classes.** For any $x\mid\mathbf{z}$, we let

$$D_{\mathbf{z}(x)}=\sum_{k=0}^{|G_{\mathbf{z}(x)}|-1}k\cdot\sigma_{\mathbf{z}(x)}^k.$$

Then one has

$$(1-\sigma_{\mathbf{z}(x)})D_{\mathbf{z}(x)}=N_{\mathbf{z}(x)}-|G_{\mathbf{z}(x)}|.$$

For any finite $z\mid_s\mathbf{z}$, let

$$D_y=\prod_{x|y}D_{\mathbf{z}(x)}.$$

In particular, $D_{\mathbf{1}} = 1$. Let $c'_y = D_y[\mathbf{z}(y)] \in \mathcal{U}$. One sees that for $x \mid y$,

$$\begin{aligned}(1-\sigma_{\mathbf{z}(x)})c'_y =&(N_{\mathbf{z}(x)} - |G_{\mathbf{z}(x)}|)D_{y/x}[\mathbf{z}(y)] \\ =&p(x;\mathrm{Fr}_x^{-1})D_{y/x}[\mathbf{z}(y/x)] - |G_{\mathbf{z}(x)}|D_{y/x}[\mathbf{z}(y)] \\ =&\gamma_{\mathbf{z}(x)}c'_{y/x} + |G_{\mathbf{z}(x)}|D_{y/x}\left(r_x(\mathrm{Fr}_x^{-1})[\mathbf{z}(y/x)] - [\mathbf{z}(y)]\right).\end{aligned}$$

The above identity tells us two things. First, by induction, we see that $(1 - \sigma_{\mathbf{z}(x)})c'_y \subset M\mathcal{U}_{\mathbf{z}(y)}$ for every $x \mid y$, thus the image $\bar{c}'_y$ of c'_y in $\mathcal{U}_{\mathbf{z}(y)}/M\mathcal{U}_{\mathbf{z}(y)}$ is fixed by $G_{\mathbf{z}(y)}$ and hence is a 0-cocycle. Secondly, we see that

$$\frac{(1-\sigma_{\mathbf{z}(x)})c'_y}{M} = \frac{\gamma_{\mathbf{z}(x)}c'_{y/x}}{M} \pmod{I_x},$$

thus

$$\Delta_x \bar{c}'_y = \bar{c}'_{y/x}.$$

Hence $\{\bar{c}'_y : y \text{ finite}, y \mid \mathbf{z}\}$ satisfies the universal Kolyvagin recursion. In particular, one sees that $\bar{c}'_{\mathbf{1}} = \bar{c}_{\mathbf{1}}$. By Corollary 3.5, we thus have

Theorem 3.6. *The set of classes*

$$\{\bar{c}'_y = D_y[\mathbf{z}(y)] \mod M\mathcal{U} : y \textit{ finite}, y \mid \mathbf{z}\}$$

constitutes a $\mathcal{T}/M\mathcal{T}$-basis of $H^0(G,\mathcal{U}/M\mathcal{U})$.

Remark 3.7. 1. The classes $\bar{c}'_y$ are the original classes used in the study of Euler system. The above Theorem 3.6 is a generalization of Theorem B in Ouyang [4]. The proof here is following the proof of Theorem B given at the end of Anderson-Ouyang [2].

2. Apparently the definitions of D_x, D_y and c'_y depend on the choice of $\mathbf{z}$. We shall use $D_{\mathbf{z}(x)}$, $D_{\mathbf{z}(y)}$ and $c'_{\mathbf{z}(y)}$ when emphasis of the dependence is needed.

4. The Kolyvagin recursion in Euler systems

In this section, we apply the results in the previous section to show that a family satisfying the universal Kolyvagin recursion maps to a family satisfying the Kolyvagin recursion in an Euler system. We shall follow heavily Chapter 4 of Rubin's book [8], which is actually the main motivation for this paper. Many results there will be introduced here without proof. We first give a brief review of the definition of the universal Euler system according to Rubin.

4.1. **The universal Euler system and the Euler system.** Let K be a fixed number field. Let p be a fixed rational prime number. Let Φ be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be the ring of integer of Φ. Let T be a p-adic representation of G_K with coefficients in $\mathcal{O}$. Assume that T is unramified outside a finite set of primes of K. Let $W = (T \otimes_{\mathcal{O}} \Phi)/T$. Fix a nonzero $M \in \mathcal{O}$ and let W_M be the M-torsion in W. Let $\mathbb{W}_M = Ind_{\{1\}}^{G_K} W_M$. The exact sequence

$$0 \to W_M \to \mathbb{W}_M \to \mathbb{W}_M/W_M \to 0$$

thus induce a canonical (surjective) map

$$\delta_L : (\mathbb{W}_M/W_M)^{G_L} \to H^1(G_L, W_M)$$

for every finite extension L of K.

Fix an ideal $\mathfrak{N}$ of K divisible by p and by all primes where T is ramified. Let X be the set of all primes x of K which is prime to $\mathfrak{N}$ and $K(x) \neq K(\mathbf{1})$, where

$K(x)$ is the ray class field of K modulo x and $K(1)$ is the Hilbert class field of K. Then Y is the set of squarefree products of primes in X and Z is the set of formal products of primes in X. Let $K(x^n)$ be the ray class field of K modulo x^n. Class field theory tells us that $G_{x^n} = \mathrm{Gal}(K(x^n)/K(\mathbf{1}))$ is a cyclic group and $K(x_1^{n_1}) \bigcap K(x_2^{n_2}) = K(\mathbf{1})$ for $x_1 \neq x_2$. Let σ_{x^n} be a generator of G_{x^n}. For every finite $z = x_1^{n_1} \cdots x_k^{n_k} \in Z$, let $K(z)$ be the composite

$$K(z) = K(x_1^{n_1}) \cdots K(x_k^{n_k}).$$

Fix a $\mathbb{Z}_p^d$-extension K_∞/K which no finite prime splits completely. We write $K \subset_f F \subset K_\infty$ to indicate F/K a finite subextension of K_∞/K. For $K \subset_f F \subset K_\infty$, we let $F(z) = FK(z)$. Let $G_z = \mathrm{Gal}(F(z)/F(\mathbf{1})) \cong \mathrm{Gal}(K(z)/K(\mathbf{1}))$ since in K_∞/K as a $\mathbb{Z}_p^d$-extension is unramified outside primes dividing p and z is prime to p by assumption. We see that for any $z' \mid_s z$, $G_z = G_{z'} \times G_{z/z'}$.

Let Fr_x denote a Frobenius of x in G_K, and let

$$p(x;t) = \det(1 - \mathrm{Fr}_x^{-1} t | T^*) \in \mathcal{O}[t].$$

Let $\mathcal{T} = \mathcal{T}(F) = \mathcal{O}[\mathrm{Gal}(F(\mathbf{1})/K)]$. With the above X, Y, Z, $\mathcal{O}$, Φ and $p(x;t)$, the corresponding universal norm distribution $\mathcal{U}_{F,z}$ (related to F) is called the *universal Euler system* of level (F,z). Since for all $x \in X$, x is unramified in the extensions F/K and $K(\mathbf{1})/K$, the Frobenius action of x in $\mathcal{U}_{F,z}$ is independent of the choice of Fr_x.

In application, we don't need the whole set X above but only a certain infinite subset $X_{F,M}$ of X consisting of elements x satisfying the following conditions:

- $M \mid |G_x|$ (hence $M \mid |G_{x^n}|$);
- $M \mid p(x;1)$;
- x splits completely in $F(\mathbf{1})/K$.

Hereafter, the pair (F,z) will always mean that z is finite and every $x \mid z$ is inside $X_{F,M}$. An *Euler system* is essentially a G_K-homomorphism

$$\xi : \bigcup_{F,z} \mathcal{U}_{F,z} = \varinjlim_{F,z} \mathcal{U}_{F,z} \longrightarrow \varinjlim_{F,z} H^1(F(z),T).$$

The following Proposition is crucial to the definition of Kolyvagin classes:

Proposition 4.1. *Suppose ξ is an Euler system. Then there exists a family of $\mathcal{O}[G_K]$-module maps $\{d_F\}$ such that the following diagrams are commutative:*

$$\begin{array}{ccc} \mathcal{U}_{F,z} & \xrightarrow{d_F} & (\mathbb{W}_M/W_M)^{G_{F(z)}} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \delta_{F(z)}} \\ H^1(F(z),T) & \longrightarrow & H^1(F(z),W_M) \end{array} \qquad \begin{array}{ccc} \mathcal{U}_{F',z} & \xrightarrow{d_{F'}} & (\mathbb{W}_M/W_M)^{G_{F'(z)}} \\ \downarrow{\scriptstyle N_{F'(z)/F(z)}} & & \downarrow{\scriptstyle N_{F'(z)/F(z)}} \\ \mathcal{U}_{F,z} & \xrightarrow{d_F} & (\mathbb{W}_M/W_M)^{G_{F(z)}} \end{array}$$

and d_F is unique up to $\mathrm{Hom}_{\mathcal{O}[G_K]}(\mathcal{U}_{F,z}, \mathbb{W}_M)$.

Proof. See Rubin [8], Proposition 4.4.8, page 87. □

From Proposition 4.1, for any element $c \in H^0(G_z, \mathcal{U}_{F,z}/M\mathcal{U}_{F,z})$, let $\tilde{c}$ be a lifting of c in $\mathcal{U}_{F,z}$. Then $d_F(N_{F(\mathbf{1})/F})\tilde{c}$ is an element in $(\mathbb{W}_M/W_M)^{G_F}$ and $\delta_F d_F(N_{F(\mathbf{1})/F})\tilde{c}$ is a well defined element in $H^1(F,W_M)$, independent of the choice of $\tilde{c}$. We denote by κ the map $c \mapsto \delta_F d_F(N_{F(\mathbf{1})/F})\tilde{c}$. Let $\hat{d}_F$ be a lifting of d_F in $\mathbb{W}_M$, then one can see immediately $\kappa(c)(\gamma) \in W_M$ is exactly $(\gamma - 1)\hat{\delta}_F d_F(N_{F(\mathbf{1})/F}\tilde{c})$.

4.2. **The Kolyvagin recursion.** Fix a number field F of K inside K_∞. Fix a infinite $\mathbf{z} \in Z$ with $x \in X_{F.M}$ for $x \mid \mathbf{z}$. For every $x \mid \mathbf{z}$, one see that

$$p(x;t) - p(x;1) = Q_x(t)(t-1),$$

for some polynomial $Q_x(t) \in \mathcal{O}[t]$, thus $p(x;t) \equiv Q_x(t)(t-1) \pmod{M}$, and $Q_x(t)$ is uniquely determined by the congruence relation since $t-1$ is not a zero divisor in $\mathcal{O}/M\mathcal{O}[t]$. Fix $Q_x(t)$.

We say that a family of classes

$$\{\kappa_y \in H^1(F, W_M) : y \text{ finite}, y \mid \bar{\mathbf{z}}\}$$

satisfies the *Kolyvagin recursion* if for every finite $y \mid \bar{\mathbf{z}}$ and $x \mid y$, the following formula

$$Q_x(\mathrm{Fr}_x^{-1})\kappa_{y/x}(\mathrm{Fr}_x) = \kappa_y(\sigma_{\mathbf{z}(x)}) \in W_M$$

holds. We see that Theorem 4.5.4 in Rubin [8] essentially showed that the family $\{\kappa(D_y[\mathbf{z}(y)])\}$ satisfies the Kolyvagin recursion.

4.3. **The universal Kolyvagin recursion implies the Kolyvagin recursion.** We first gather a few lemmas from Rubin [8]:

Lemma 4.2. *Let $x \in X_{F,M}$, then $p(x; \mathrm{Fr}_x^{-1})$ annihilates W_M.*

Proof. See Rubin [8], Lemma 4.1.2(iv), page 77. □

Lemma 4.3. *Let $\hat{d}_F([z]) \in \mathbb{W}_M$ be a lifting of $d_F([z]) \in \mathbb{W}_M/W_M$. Then for any $x \mid \mathbf{z}$, ω a prime in $\bar{K}$ above x, g, g' elements in the decomposition group $\mathcal{D}$ of ω, and $\gamma \in G_K$, then*

$$gg'\gamma\hat{d}_F([z]) = g'g\gamma\hat{d}_F([z]).$$

Proof. See Rubin [8], Lemma 4.7.1, page 98. □

Lemma 4.4. *Let $\hat{d}_F$ be a lifting of d_F in $\mathbb{W}_M$, then for every $\gamma \in G_K$ and $z \mid_s \mathbf{z}$ and $x \mid z$,*

$$N_{G_{\mathbf{z}(x)}}\gamma\hat{d}_F([z]) = p(x; \mathrm{Fr}_x^{-1})\gamma\hat{d}_F([z/\mathbf{z}(x)]).$$

Proof. See Rubin [8], Lemma 4.7.3, page 99. □

Lemma 4.5. *Let n_x be the number of elements in $\mathcal{O}_K/x$. Let $r_x(t) = \frac{p(x;t)-p(x;n_x t)}{|G_{\mathbf{z}(x)}|}$. Then*

$$\xi(I_{x,z}) \in H^1(F(z)_\omega, W_M)$$

for every prime ω in $F(z)$ above x.

Proof. See Rubin [8], Corollary 4.8.1, page 102. □

With the choice of $r_x(t)$ in Lemma 4.5, one then has $\gamma_{\mathbf{z}(x)} = p(x; n_x \mathrm{Fr}_x^{-1})$ and

Lemma 4.6. *The map $\gamma_{\mathbf{z}(x)} : \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)} \to \mathcal{U}_{\mathbf{z}/\mathbf{z}(x)}$ is injective, thus the Kolyvagin conditions are satisfied.*

Proof. We only need to show that for any finite $z \mid_s \mathbf{z}$, $\gamma_{\mathbf{z}(x)} : \mathcal{U}_{z/\mathbf{z}(x)} \to \mathcal{U}_{z/\mathbf{z}(x)}$ is an injection. In this case, Fr_x^{-1} induces a linear transformation from $\mathcal{U}_{z/\mathbf{z}(x)} \otimes_{\mathcal{O}} \Phi$ to itself, whose eigenvalues λ has (logarithmic) discrete value 0 at every places above x. Since $p(x;t) \in \mathcal{O}[t]$, then except the constant term 1, other terms in $p(x; n_x \mathrm{Fr}_x^{-1})$ have discrete value no less than the discrete value of n_x, which is bigger than 0. Thus any eigenvalue of $\gamma_{\mathbf{z}(x)}$ can't be zero and the linear transformation $\gamma_{\mathbf{z}(x)}$ is injective. □

Finally we have

Theorem 4.7 (Kolyvagin recursion)**.** *Let r_x be given as Lemma 4.5. Let $\{c_y : y \in Y,\ y \mid \mathbf{z}\}$ be a family of classes in $H^0(G_{\mathbf{z}}, \mathcal{U}_{\mathbf{z}}/M\mathcal{U}_{\mathbf{z}})$ satisfying the universal Kolyvagin recursion related to $\gamma_{\mathbf{z}(x)}$. Then the family $\{\kappa(c_y) : y \in Y,\ y \mid \mathbf{z}\}$ satisfies the Kolyvagin recursion, i.e.,*

$$Q_x(\mathrm{Fr}_x^{-1})\kappa(c_{y/x})(\mathrm{Fr}_x) = \kappa(c_y)(\sigma_{\mathbf{z}(x)}) \in W_M.$$

Proof. Let $\hat{d}$ be a lifting of d_F in $\mathbb{W}_M$. By the definition of the connecting homomorphism $\delta_{F(z)}$, one has

$$\kappa(c_{y/x})(\mathrm{Fr}_x) = (\mathrm{Fr}_x - 1)N_{F(\mathbf{1})/F}\hat{d}(c_{y/x}) \in W_M,$$
$$\kappa(c_y)(\sigma_{\mathbf{z}(x)}) = (\sigma_{\mathbf{z}(x)} - 1)N_{F(\mathbf{1})/F}\hat{d}(c_y) \in W_M.$$

Then by Lemmas 4.2— 4.6, with the universal Kolyvagin recursion satisfied by c_y and $c_{y/x}$, one has

$$\begin{aligned}
&Q_x(\mathrm{Fr}_x^{-1})\kappa(c_{y/x})(\mathrm{Fr}_x) - \kappa(c_y)(\sigma_{\mathbf{z}(x)}) \\
&\quad = Q_x(\mathrm{Fr}_x^{-1})\,\mathrm{Fr}_x^{-1}\,\kappa(c_{y/x})(\mathrm{Fr}_x) - \kappa(c_y)(\sigma_{\mathbf{z}(x)}) \\
&\quad = Q_x(\mathrm{Fr}_x^{-1})(1 - \mathrm{Fr}_x^{-1})N_{F(\mathbf{1})/F}\hat{d}(c_{y/x}) - (\sigma_{\mathbf{z}(x)} - 1)N_{F(\mathbf{1})/F}\hat{d}(c_y) \\
&\quad = -P(x; \mathrm{Fr}_x^{-1})N_{F(\mathbf{1})/F}\hat{d}(c_{y/x}) + \gamma_{\mathbf{z}(x)}N_{F(\mathbf{1})/F}\hat{d}(c_{y/x}) \\
&\quad = -|G_{\mathbf{z}(x)}| \cdot r_x(\mathrm{Fr}_x^{-1})N_{F(\mathbf{1})/F}\hat{d}(c_{y/x}) \\
&\quad = 0,
\end{aligned}$$

which finishes the proof. □

Remark 4.8. The proofs of the above Lemmas don't require the use of the original Kolyvagin classes. Hence we indeed succeed to generalize Theorem 4.5.4 in [8]. We sincerely hope that our more abstract construction could lead to pursue new Euler systems.

Remark 4.9. In the special case $T = \mathbb{Z}_p(1)$, we see first that $p(x;t) = 1-t$ for every prime x. The elements $\kappa(c) \in H^1(F, W_M)$ for $c \in H^0(G, \mathcal{U}_{\mathbf{z}}/M\mathcal{U}_{\mathbf{z}})$ are elements inside $F^\times/F^{\times M}$. Anderson and Ouyang have studied this case thoroughly in the note [2].

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