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# On abelian 2-ramification torsion modules of quadratic fields

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**Abstract** For a number field $F$ and a prime number $p$, the $\mathbb{Z}_p$-torsion module of the Galois group of the maximal abelian pro-$p$ extension of $F$ unramified outside $p$ over $F$, denoted as $\mathcal{T}_p(F)$, is an important subject in abelian $p$-ramification theory. In this paper we study the group $\mathcal{T}_2(F) = \mathcal{T}_2(m)$ of the quadratic field $F = \mathbb{Q}(\sqrt{m})$. Firstly, assuming $m > 0$, we prove an explicit 4-rank formula for quadratic fields that $\mathrm{rk}_4(\mathcal{T}_2(-m)) = \mathrm{rk}_2(\mathcal{T}_2(-m)) - \mathrm{rank}(R)$ where $R$ is a certain explicitly described Rédei matrix over $\mathbb{F}_2$. Furthermore, using this formula, we obtain the 4-rank density of $\mathcal{T}_2$-groups of imaginary quadratic fields. Secondly, for $l$ an odd prime, we obtain results about the 2-power divisibility of orders of $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$, both of which are cyclic 2-groups. In particular we find that $\#\mathcal{T}_2(l) \equiv 2\#\mathcal{T}_2(2l) \equiv h_2(-2l) \pmod{16}$ if $l \equiv 7 \pmod 8$ where $h_2(-2l)$ is the 2-class number of $\mathbb{Q}(\sqrt{-2l})$. We then obtain density results for $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$ when the orders are small. Finally, based on our density results and numerical data, we propose distribution conjectures about $\mathcal{T}_p(F)$ when $F$ varies over real or imaginary quadratic fields for any prime $p$, and about $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$ when $l$ varies, in the spirit of Cohen-Lenstra heuristics. Our conjecture in the $\mathcal{T}_2(l)$ case is closely connected to Shanks-Sime-Washington's speculation on the distributions of the zeros of 2-adic $L$-functions and to the distributions of the fundamental units.

**Keywords** quadratic fields, density theorems, abelian 2-ramification

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## 1 Introduction

Let $p$ be a prime number. For a number field $F$, let $M = M(F,p)$ be the maximal abelian pro-$p$ extension of $F$ unramified outside $p$. By class field theory, $\mathrm{Gal}(M/F)$ is a finitely generated $\mathbb{Z}_p$-module of rank $r_2(F)+\delta_p(F)+1$, where $r_2(F)$ is the number of complex places of $F$ and $\delta_p(F) \geqslant 0$ is the Leopoldt defect of $F$ at $p$. Leopoldt's Conjecture is that $\delta_p(F) = 0$ for all $p$ and $F$ and this has been proved when $F/\mathbb{Q}$ is abelian. We call the $\mathbb{Z}_p$-torsion subgroup of $\mathrm{Gal}(M/F)$, a finite abelian $p$-group, the $\mathcal{T}_p$-group of $F$ and denote it by $\mathcal{T}_p(F)$. The study of $\mathrm{Gal}(M/F)$ and $\mathcal{T}_p(F)$ which goes back to fundamental contributions

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of Serre, Shafarevich and Brumer, is the so-called abelian $p$-ramification theory. We refer the reader to the historical survey [8] by Gras for this theory, in which the $p$-rank formula for $\mathcal{T}_p(F)$ due to himself is stated. When $F$ is totally real, assuming $\delta_p(F)=0$, the work of Coates [2] and Colmez [3] shows that the order of $\mathcal{T}_p(F)$ is essentially the residue of the $p$-adic zeta function of $F$ up to a $p$-adic unit. This motivates us to study the group structure of $\mathcal{T}_p(F)$ in more detail. Like class groups, the study of $\mathcal{T}_p(F)$ can be much more explicit in the case that $F$ is a quadratic field and $p=2$. In this paper, we will mainly consider this case, and our main purpose is to study the distribution of $\mathcal{T}_2(F)$ when $F$ varies in a certain family of quadratic fields.

Note that the structure of a finite abelian $p$-group $A$ is completely determined by its $p^i$-rank $\mathrm{rk}_{p^i}(A) := \dim_{\mathbb{F}_p} p^{i-1}A/p^iA$ for all $i$. As a consequence, to study $\mathcal{T}_p(F)$, it is necessary and sufficient to study $\mathrm{rk}_{p^i}(\mathcal{T}_p(F))$ for all $i$.

The general $p$-rank formula for $\mathcal{T}_p(F)$ becomes very explicit for $p=2$ and $F$ quadratic, after a computation of genus class numbers; see Theorem 2.1. If $F$ is imaginary quadratic, we shall prove an explicit 4-rank formula of $\mathcal{T}_2(F)$, namely, $\mathrm{rk}_4(\mathcal{T}_2(F))$ is the difference of $\mathrm{rk}_2(\mathcal{T}_2(F))$ and the rank of a certain explicitly described Rédei matrix; see Theorem 2.4. This formula is new and is analogous to the classical 4-rank formula for narrow class groups of quadratic fields. Applying this result, we deduce the following 4-rank density formula for $\mathcal{T}_2$-groups of imaginary quadratic fields, which is the main result of this paper:

**Theorem 1.1** (4-rank density formula for $\mathcal{T}_2$ of imaginary quadratic fields)**.** *For integers $t \geqslant 1$ and $r \geqslant 0$, and a real number $x>0$, put*

$$N_x := \{m \in \mathbb{Z}_{>0} \mid m \leqslant x \ \textit{squarefree}\},$$
$$N_{t;x} := \{m \in N_x \mid \ \textit{exactly } t \textit{ prime numbers are ramified in } \mathbb{Q}(\sqrt{-m})\},$$
$$T^r_{t;x} := \{m \in N_{t;x} \mid \mathrm{rk}_4(\mathcal{T}_2(\mathbb{Q}(\sqrt{-m}))) = r\}.$$

*Then for any integer $r \geqslant 0$, the limit $d^T_{\infty,r}$, which is defined by*

$$d^T_{\infty,r} := \lim_{t\to\infty}\lim_{x\to\infty}\frac{\#T^r_{t;x}}{\#N_{t;x}} \tag{1.1}$$

*exists and*

$$d^T_{\infty,r} = \frac{\prod_{i=r+2}^{\infty}(1-2^{-i})}{2^{r(r+1)}\prod_{i=1}^{r}(1-2^{-i})} = \frac{\eta_\infty(2)}{2^{r(r+1)}\eta_r(2)\eta_{r+1}(2)} \tag{1.2}$$

*where $\eta_s(q) := \prod_{i=1}^{s}(1-q^{-i})$ for $s\in\mathbb{Z}_{>0}\cup\{\infty\}$ and $q\geqslant 2$ and $\eta_0(q):=1$.*

**Remark 1.2.** Theorem 1.1 is analogue to the density theorem of Gerth [5] on the 4-rank of narrow class groups of quadratic fields, and to the theorem of Yue-Yu [24] on the 4-rank of the tame kernel of quadratic fields.

We then turn to study the $\mathcal{T}_2$-groups of subfamilies of quadratic fields, namely $\mathbb{Q}(\sqrt{\pm l})$ and $\mathbb{Q}(\sqrt{\pm 2l})$ where $l$ is an odd prime. For simplicity, write $\mathcal{T}_2(m)$ for $\mathcal{T}_2(\mathbb{Q}(\sqrt{m}))$, $t_2(m)$ for its order, and $h_2(m)$ for the 2-class number of $\mathbb{Q}(\sqrt{m})$. Such questions for $\mathcal{T}_2(l)$ and $\mathcal{T}_2(2l)$ have been studied by many researchers before. For example, consider $\mathbb{Q}(\sqrt{l})$ and let $L_2(1,\chi_l)$ be the 2-adic $L$-function where $\chi_l$ is the quadratic character associated with $\mathbb{Q}(\sqrt{l})$. Recalling that $h_2(l)=1$, then Coates' order formula (see [2, Appendix 1] or Proposition 3.3) directly relates $\#\mathcal{T}_2(l)$ to the 2-adic regulator of $\mathbb{Q}(\sqrt{l})$ and therefore to the 2-adic valuation of $L_2(1,\chi_l)$ by the class number formula. The latter two objects and their relation to $h_2(-l)$ and $h_2(-2l)$ have been studied by Kaplan, Leonard, Williams (see [10], [13], [23]) and by Shanks-Sime-Washington [19]. However, it seems that there is no study for $\mathcal{T}_2(-l)$ and $\mathcal{T}_2(-2l)$ before.

By the 2-rank formula (2.7), $\mathcal{T}_2(\pm l)=\mathcal{T}_2(\pm 2l)=0$ if $l\equiv\pm 3 \pmod 8$ and $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$ are nontrivial 2-cyclic groups if $l\equiv\pm 1 \pmod 8$. Applying our 4-rank formula and Coates' order formula for totally real fields, we obtain the following results:

- (Theorem 3.1) Determine the congruent conditions for $l$ satisfying $t_2(-l)$ or $t_2(-2l)=2$, 4 and $\geqslant 8$, and hence find the respective densities;

• (Theorem 3.7) Determine the conditions for $l \equiv 7 \pmod 8$ satisfying $t_2(l) = 4, 8$ and $\geqslant 16$, and deduce the formula

$$t_2(l) \equiv 2t_2(2l) \equiv h_2(-2l) \pmod{16}. \tag{1.3}$$

• (Proposition 3.9) Determine the conditions for $l \equiv 1 \pmod 8$ satisfying $t_2(l)$ or $t_2(2l) = 2$ or $4$.

Here Theorem 3.1 is new, Theorem 3.7 is an improvement of the result in [13] and Proposition 3.9 is essentially a summary of the results in [10], [13] and [23] using the language of $\mathcal{T}_2$-groups.

For the real case, we then have the following density result which is inspired by the work on the distribution of 2-adic valuation of $L_2(1, \chi_l)$ in [19].

**Theorem 1.3.** *For $i \in \{0, 1\}$ and $e \in \{0, 1\}$,*

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv (-1)^e \pmod 8,\ t_2(l) = 2^{i+1+e}\}}{\#\{l \leqslant x : l \equiv (-1)^e \pmod 8\}} = \frac{1}{2^{i+1}}. \tag{1.4}$$

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv (-1)^e \pmod 8,\ t_2(2l) = 2^{i+1}\}}{\#\{l \leqslant x : l \equiv (-1)^e \pmod 8\}} = \frac{1}{2^{i+1}}. \tag{1.5}$$

In the last section, we shall present several conjectures in light of the density results we proved in the spirit of Cohen-Lenstra heuristics. We shall present computational evidence for our conjectures in the Appendix.

## 2 The rank and density formulas for quadratic imaginary fields

### 2.1 Notations

We shall use the following notations.

(1) For a general number field $F$, $\mathcal{O}_F$ is the ring of integers of $F$, $\mathcal{O}_F^\times$ is the group of units of $F$, $r_1$ and $r_2$ are the numbers of real and complex places of $F$, $n = r_1 + 2r_2 = [F : \mathbb{Q}]$. For a finite place $v$ of $F$, we let $U_v$ and $U_{1,v}$ be the groups of local units and principal local units. For $v$ infinite, let $U_v = F_v^\times$. Let $\mathbb{A}_F$ be the adèle ring of $F$. The idèlic group of $F$, as the units of $\mathbb{A}_F$, is denoted by $\mathbb{A}_F^\times$.

Let $F^+ = \{\alpha \in F \mid v(\alpha) > 0 \text{ for all real places } v \text{ of } F\}$ be the subgroup of $F^\times$ of totally real elements. Hence $F^\times/F^+$ is an $\mathbb{F}_2$-vector space of dimension $r_1$, by the approximation theorem.

Let $S = S_p$ be the set of primes of $F$ lying above $p$. Let $\mathcal{O}_S, E_S, \mathrm{Cl}_S$ and $\mathrm{Cl}_S^+$ denote the ring of $S$-integers, the group of $S$-units, the $S$-class group, and the narrow $S$-class group of $F$, respectively. Let $E_S^+ = E_S \cap F^+$. Let $U_{1,S} = \prod_{v\in S} U_{1,v}$.

(2) In the special case that $F$ is a quadratic field, write $F = \mathbb{Q}(\sqrt{m})$, then $(r_1, r_2) = (2, 0)$ if $m > 0$ and $(0, 1)$ if $m < 0$. Let $G = \mathrm{Gal}(F/\mathbb{Q}) = \{1, \sigma\}$. Let $\mathrm{Cl}(m)$, $\mathrm{Cl}_p(m)$, $h(m)$, $h_p(m)$ and $\mathcal{T}_p(m)$ be the class group, the $p$-class group, the class number, the $p$-class number and the $\mathcal{T}_p$-group of $F = \mathbb{Q}(\sqrt{m})$ respectively. Let $t_p(m) = \#\mathcal{T}_p(m)$.

If $p = 2$, the size of $S$ is 2 if 2 splits and 1 if 2 is not split in $F$. If $F$ is imaginary, $F^+ = F^\times$ and $\sigma$ is the restriction of complex conjugation on $F$.

(3) For any abelian group $A$, $A[n]$ is the $n$-torsion subgroup of $A$ and $A[p^\infty]$ is the $p$-primary part of $A$. For a finite abelian group $A$ and a positive integer $i$, the $p^i$-rank $\mathrm{rk}_{p^i}(A) := \dim_{\mathbb{F}_p} p^{i-1}A/p^iA$. If $A$ is an $\mathbb{F}_2$-vector space, $\dim A := \dim_{\mathbb{F}_2} A$ is its dimension.

(4) For Jacobi, 2-nd Hilbert and Artin -symbols with values in $\mu_2 = \{1, -1\}$, we use [ ] instead of ( ) to represent the corresponding additive symbols with values in $\mathbb{F}_2 = \{0, 1\}$.

### 2.2 The 2-rank and 4-rank formulas in general

For $F$ a general number field, we recall some facts about $\mathcal{T}_p(F)$. All are standard consequences of global class field theory; see, for example, [22, Theorem 13.4]. The closed subgroup $\overline{F^\times \prod_{v\notin S} U_v}$ of $\mathbb{A}_F^\times$ corresponds to the maximal abelian extension of $F$ unramified outside $S$. Set

$$\mathcal{A}_F := \mathbb{A}_F^\times \Big/ \overline{F^\times \prod_{v\notin S} U_v}. \tag{2.1}$$

As known in the proof of [22, Theorem 13.4], the induced Artin map $\mathcal{A}_F \twoheadrightarrow \mathrm{Gal}(M/F)$ is surjective and has finite kernel of prime-to-$p$ order, thus induces a canonical isomorphism

$$\mathcal{A}_F^{\mathrm{pro}-p} \cong \mathrm{Gal}(M/F),$$

where $\mathcal{A}_F^{\mathrm{pro}-p}$ is the pro-$p$-part of $\mathcal{A}_F$. Let $H$ be the $p$-Hilbert class field of $F$. Then $\mathrm{Gal}(H/F) \cong \mathrm{Cl}_p(F)$ canonically. Let $\phi$ be the canonical diagonal embedding $F \hookrightarrow \prod_{v\in S} F_v$ and $E_{1,S} = \phi^{-1}(U_{1,S}) \cap \mathcal{O}_F^{\times}$. By class field theory, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_{1,S}/\overline{\phi(E_{1,S})} & \longrightarrow & \mathcal{A}_F^{\mathrm{pro}-p} & \longrightarrow & \mathrm{Cl}_p(F) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathrm{Gal}(M/H) & \longrightarrow & \mathrm{Gal}(M/F) & \longrightarrow & \mathrm{Gal}(H/F) & \longrightarrow & 0 \end{array} \tag{2.2}$$

The group $U_{1,S}$ is a finitely generated $\mathbb{Z}_p$-module of rank $n = r_1 + 2r_2$ and the submodule $\overline{\phi(E_{1,S})}$ is of rank $r_1 + r_2 - 1 - \delta_p(F)$ for some integer $\delta_p(F) \geqslant 0$. It follows that $\mathrm{Gal}(M/F)$ is a finitely generated $\mathbb{Z}_p$-module of rank $r_2 + 1 + \delta_p(F)$. Leopoldt conjectured that $\delta_p(F)$ is always 0 and this has been proved when $F$ is abelian over $\mathbb{Q}$. Thus $\mathcal{T}_p(F)$, by definition the torsion subgroup of $\mathrm{Gal}(M/F)$, is finite and

$$\mathcal{T}_p(F) \cong \mathcal{A}_F[p^\infty], \tag{2.3}$$

and the $p$-rank of $\mathcal{T}_p(F)$ is given by

$$\mathrm{rk}_p(\mathcal{T}_p(F)) = \mathrm{rk}_p(\mathrm{Gal}(M/F)) - r_2 - 1 - \delta_p(F). \tag{2.4}$$

From now on, we identify $\mathcal{A}_F[p^\infty]$ with $\mathcal{T}_p(F)$. By abuse of notation, we write $\mathcal{A}_F$ and $\mathrm{Gal}(M/F)$ additively. Let $L$ be the maximal abelian extension of $F$ which is of exponent $p$ and unramified outside $p$. Then $L$ is the intermediate field of $M/F$ fixed by $p\mathrm{Gal}(M/F)$. The induced Artin map $\mathcal{A}_F \to \mathrm{Gal}(M/F)$ has kernel consisting of prime-to-$p$-torsion elements, hence is contained in $p\mathcal{A}_F$ and the induced map $\mathcal{A}_F/p\mathcal{A}_F \to \mathrm{Gal}(L/F)$ is an isomorphism. The kernel of the composite map

$$\varphi : \mathcal{T}_p(F)[p] \hookrightarrow \ \mathcal{A}_F \to \mathcal{A}_F/p\mathcal{A}_F = \mathrm{Gal}(L/F)$$

is $\mathcal{T}_p(F)[p] \cap p\mathcal{A}_F = p\mathcal{T}_p(F)[p^2]$, which is an $\mathbb{F}_p$-space of dimension $\mathrm{rk}_{p^2}(\mathcal{T}_p(F))$. This gives the identity

$$\mathrm{rk}_{p^2}(\mathcal{T}_p(F)) = \mathrm{rk}_p(\mathcal{T}_p(F)) - \dim_{\mathbb{F}_p} \mathrm{Im}(\varphi). \tag{2.5}$$

We first derive the 2 and 4-rank formulas of $\mathcal{T}_2$ for a general number field, and the 2-rank formula for a quadratic field. The general 2-rank formula (2.6) was proved in Gras [6, Théorème I 3], and the 4-rank formula is quite routine.

**Theorem 2.1.** *Let $F$ be a number field. Let $S$ be the set of primes in $F$ above 2 and $\mathrm{Cl}_S^+$ the narrow $S$-class group of $F$.*

(1) (*Gras*) *The 2-rank of $\mathcal{T}_2(F)$ is given by the formula*

$$\mathrm{rk}_2\mathcal{T}_2(F) = \#S + \mathrm{rk}_2(\mathrm{Cl}_S^+) - 1 - \delta_2(F). \tag{2.6}$$

*In particular, if $m$ is a squarefree integer with $t$ odd prime factors, then for $F = \mathbb{Q}(\sqrt{m})$,*

$$\mathrm{rk}_2(\mathcal{T}_2(F)) = \begin{cases} t & \text{if } q \equiv \pm 1 \pmod 8 \text{ for all odd prime } q \mid m, \\ t-1 & \text{if } q \equiv \pm 3 \pmod 8 \text{ for some odd prime } q \mid m. \end{cases} \tag{2.7}$$

(2) *Suppose $A$ is a finite set of idèles which generates $\mathcal{T}_2(F) \subset \mathcal{A}_F := \mathbb{A}_F^{\times}/\overline{F^{\times}\prod_{v\nmid 2} U_v}$. Suppose $B$ is a finite set of elements in $F^{\times}$ such that $F(\sqrt{B})$ is the maximal abelian extension of $F$ of exponent 2 unramified outside 2. For $a \in A$ and $b \in B$, let $[a,b] = \log_{-1}(a, F(\sqrt{b})) \in \mathbb{F}_2$ be the additive Artin symbol. Let $R = ([a,b])_{a\in A, b\in B}$. Then*

$$\mathrm{rk}_4(\mathcal{T}_2(F)) = \mathrm{rk}_2(\mathcal{T}_2(F)) - \mathrm{rank}(R). \tag{2.8}$$

**Remark 2.2.** (1) The minimal size of $A$ is $\mathrm{rk}_2(\mathcal{T}_2(F))$, and the minimal size of $B$ is $\mathrm{rk}_2(\mathrm{Gal}(M/F))) = \mathrm{rk}_2(\mathcal{T}_2(F)) + r_2(F) + 1 + \delta_2(F)$. Moreover, if $F(\sqrt{b})$ is contained in a $\mathbb{Z}_2$-extension of $F$, then $[a,b]=0$ for all $a \in A$ and we can delete the corresponding row in $R$.

(2) The $p^2$-rank formula for $\mathcal{T}_p(F)$ in the case $\mu_p \subseteq F$ can be proved similarly, as the kernel of the map $\mathcal{T}_p(F)[p] \hookrightarrow \mathcal{A}_F \to \mathcal{A}_F/p\mathcal{A}_F \cong \mathrm{Gal}(L/F)$ is $p\mathcal{T}_p[p^2]$. Moreover, one can similarly deduce the formula $\mathrm{rk}_{p^{i+1}}(\mathcal{T}_p(F)) = \mathrm{rk}_{p^i}(\mathcal{T}_p(F)) - \dim_{\mathbb{F}_p} \mathrm{Im}(\mathcal{T}_p(F)[p^i] \to \mathrm{Gal}(L/K))$.

*Proof.* We are in the case $p=2$. Then $L$ is the maximal abelian extension of $F$ of exponent 2 unramified outside $S$. By Kummer Theory, $L = F(\sqrt{J})$, where $J$ is the finite subgroup of $F^\times/F^{\times 2}$ given by

$$J := \{\beta \in F^+ \mid \beta\mathcal{O}_S = \mathfrak{b}^2 \text{ for some } \mathcal{O}_S\text{-fractional ideal } \mathfrak{b} \text{ of } F\}/(F^\times)^2. \tag{2.9}$$

(1) First suppose $F$ is general. The non-degeneracy of the Kummer pairing $J \times \mathrm{Gal}(L/F) \to \{\pm 1\}$ then implies

$$\mathrm{rk}_2\mathrm{Gal}(M/F) = \dim \mathrm{Gal}(L/F) = \dim J. \tag{2.10}$$

Let pr be the natural projection $\mathrm{Cl}_S^+ \to \mathrm{Cl}_S$ and $\mathrm{Cl}_{S,+} = \mathrm{pr}(\mathrm{Cl}_S^+[2]) \subset \mathrm{Cl}_S[2]$. For $[\beta] \in J$, $\beta\mathcal{O}_S = \mathfrak{b}^2$, then the class map $\mathrm{cl}_S(\mathfrak{b})$ lies in $\mathrm{Cl}_{S,+}$. This gives an exact sequence of $\mathbb{F}_2$-vector spaces:

$$1 \to E_S^+/E_S^2 \to J \xrightarrow{\beta \mapsto \mathrm{cl}_S(\mathfrak{b})} \mathrm{Cl}_{S,+} \to 1. \tag{2.11}$$

Let $F^\times\mathcal{O}_S = \{\alpha\mathcal{O}_S \mid \alpha \in F^\times\}$ and $F^+\mathcal{O}_S = \{\alpha\mathcal{O}_S \mid \alpha \in F^+\}$, then $\ker \mathrm{pr} = F^\times\mathcal{O}_S/F^+\mathcal{O}_S \subset \mathrm{Cl}_S^+[2]$. This gives an exact sequence of $\mathbb{F}_2$-vector spaces

$$1 \to F^\times\mathcal{O}_S/F^+\mathcal{O}_S \to \mathrm{Cl}_S^+[2] \to \mathrm{Cl}_{S,+} \to 1. \tag{2.12}$$

We also have the following natural exact sequence of $\mathbb{F}_2$-vector spaces:

$$1 \to E_S/E_S^+ \to F^\times/F^+ \to F^\times\mathcal{O}_S/F^+\mathcal{O}_S \to 1. \tag{2.13}$$

Combining the above results, we get

$$\begin{aligned}\mathrm{rk}_2\mathrm{Gal}(M/F) &= \dim E_S^+/E_S^2 + \dim \mathrm{Cl}_{S,+}\\ &= \dim E_S^+/E_S^2 + \dim \mathrm{Cl}_S^+[2] - r_1 + \dim E_S/E_S^+\\ &= \dim E_S/E_S^2 + \dim \mathrm{Cl}_S^+[2] - r_1\\ &= r_2 + \#S + \dim \mathrm{Cl}_S^+[2],\end{aligned}$$

where $\dim F^\times/F^+ = r_1$ by the approximation theorem, and $\dim E_S/E_S^2 = r_1 + r_2 + \#S$ by Dirichlet's unit theorem that $E_S \cong \mathbb{Z}^{r_1+r_2+\#S-1} \times \mathbb{Z}/d\mathbb{Z}$ with $d$ even. By (2.4), we then get the general 2-rank formula (2.6) for $\mathcal{T}_2$-group of a general base field (see [6] for a slightly different approach).

Now suppose $F = \mathbb{Q}(\sqrt{m})$ is a quadratic field. Then $\delta_2(F) = 0$. Write $G = \mathrm{Gal}(F/\mathbb{Q})$. Since $\mathbb{Q}$ has class number 1, we conclude that $\mathrm{Cl}_S^+[2] = (\mathrm{Cl}_S^+)^G$. Recall that $t$ is the number of odd prime factors of $m$. Applying the $S$-narrow version of the ambiguous class number formula (see, for example [16, Remark 4.5]) gives the following result:

$$\dim(\mathrm{Cl}_S^+)^G = \begin{cases} t-2, & \text{if 2 splits and } 2 \notin N(F);\\ t-1, & \text{if 2 splits and } 2 \in N(F) \text{ or 2 does not split and } 2 \notin N(F);\\ t, & \text{if 2 does not split and } 2 \in N(F).\end{cases} \tag{2.14}$$

By Lemma 2.3 below, $2 \in N(F)$ if and only if $q \equiv \pm 1 \pmod 8$ for all odd primes $q \mid m$, the 2-rank formula (2.7) fo $F = \mathbb{Q}(\sqrt{m})$ then follows.

(2) We may assume $\widetilde{B} = \{b \pmod{F^{\times 2}} \mid b \in B\}$ is an $\mathbb{F}_2$-basis of $J$. Then

$$\mathrm{Gal}(L/F) \hookrightarrow \prod_{b \in B} \mathrm{Gal}(F(\sqrt{b})/F)$$

is an isomorphism. Written additively, the map $\varphi$ sends $a \in \mathcal{T}_2(F)[2] \subset \mathcal{A}_F$ to $([a, F(\sqrt{b})])_{b \in B}$. Thus $\dim_{\mathbb{F}_2}(\mathrm{Im}(\varphi))$ is nothing but the rank of $([a,b])_{a \in A, b \in B}$. By (2.5), we get the 4-rank formula. □

We have the following easy lemma to transform the norm conditions into congruent conditions.

**Lemma 2.3.** *Let $m$ be a positive squarefree integer. Let $F=\mathbb{Q}(\sqrt{-m})$ and $\widetilde{F}=\mathbb{Q}(\sqrt{m})$. Then*

$$2\in N(F) \iff 2\in N(\widetilde{F}) \iff q\equiv \pm 1 \pmod 8 \text{ for all odd prime } q\mid m;$$
$$-2\in N(\widetilde{F}) \iff q\equiv 1,\ 3 \pmod 8 \text{ for all odd prime } q\mid m;$$
$$-1\in N(\widetilde{F}) \iff q\equiv 1 \pmod 4 \text{ for all odd prime } q\mid m.$$

*Proof.* By Hasse's norm theorem and the product formula, $2\in N(F)$ if and only if $2\in N(F_v)$ for all but one prime $v$ of $F$. If $v\nmid 2m$, then $v$ is always unramified and $2\in N(F_v)$ by local class field theory. For an odd prime $q\mid m$, $q$ is ramified in $F$. Let $\mathfrak{q}$ be the unique ramified prime of $F$ above $q$. Then $2\in N(F_{\mathfrak{q}})$ if and only if the Hilbert symbol $(2,-m)_q=1$, which is equivalent to that $q\equiv\pm1 \pmod 8$. If 2 splits in $F$, then $v\mid 2$ is unramified and $2\in N(F_v)$; in other cases, there is only one prime $v$ above 2 which can be excluded from consideration. Hence $2\in N(F)$ if and only if $q\equiv\pm1\pmod 8$ for every odd prime $q\mid m$. The other cases can be proved similarly. □

### 2.3 The explicit 4-rank formula for imaginary quadratic fields

We turn to work on the imaginary quadratic field case. We shall work out $A$ and $B$ explicitly for an imaginary quadratic field and hence obtain an explicit 4-rank formula in this case. This explicit formula will be used to deduce the 4-rank density formula of $\mathcal{T}_2$-groups of imaginary quadratic fields in next subsection.

We suppose $m>0$ and $F=\mathbb{Q}(\sqrt{-m})$. Let $\{q_1,\dots,q_t\}$ be the set of odd prime factors of $m$, arranged in such a way that $q_i\equiv\pm1\pmod 8$ if $1\leqslant i\leqslant k$ and $\pm3\pmod 8$ if $k<i\leqslant t$. Note that $k=0$ if $q\equiv\pm3 \pmod 8$ for all $q\mid m$. Let $\mathfrak{p}$ be a prime of $F$ above 2. Then $\mathfrak{p}$ is either the unique prime above 2 or $(2)=\mathfrak{p}\bar{\mathfrak{p}}$ splits in $F$ where $\bar{\mathfrak{p}}\neq\mathfrak{p}$ is the complex conjugate of $\mathfrak{p}$. Let $\mathfrak{q}_i$ be the unique prime of $F$ above $q_i$. For an odd prime $q$, let $q^*=(-1)^{(q-1)/2}q$. Then $q_i^*$ $(1\leqslant i\leqslant k)$ and $q_j^*q_{j'}^*$ $(k<j,j'\leqslant t)$ are squares in the 2-adic field $\mathbb{Q}_2$.

Our explicit 4-rank formula for $\mathcal{T}_2(\mathbb{Q}(\sqrt{-m}))$ is

**Theorem 2.4.** *Suppose $F=\mathbb{Q}(\sqrt{-m})$. For $0\leqslant i\leqslant t$, we define the idèles $a_i=(a_{i,v})\in\mathbb{A}_F^\times$ as follows:*
(1) *$a_{0,\mathfrak{p}}=\sqrt{-1}$ if $F_{\mathfrak{p}}\cong\mathbb{Q}_2(\sqrt{-1})$, and $a_{0,\mathfrak{p}}=-1$ if $2=\mathfrak{p}\bar{\mathfrak{p}}$ splits in $F$;*
(2) *if $1\leqslant i\leqslant k$, $a_{i,\mathfrak{q}_i}=\sqrt{-m}$ and $a_{i,v}=\sqrt{q_i^*}$ for $v\mid 2$;*
(3) *if $k<i<t$, $a_{i,\mathfrak{q}_i}=a_{i,\mathfrak{q}_t}=\sqrt{-m}$ and $a_{i,v}=\sqrt{q_i^*q_t^*}$ for $v\mid 2$;*
(4) *for all other places $v$, $a_{i,v}=1$. In particular, $a_t=1$ if $k<t$.*

*Let $\pi$ be a generator of $\mathfrak{p}^\lambda$ where $\lambda$ is the order of $\mathfrak{p}$ in the class group of $F$. If 2 is a norm of $F$, noting that $m$ is a norm of $\mathbb{Z}[\sqrt{2}]$, write $m=2g^2-h^2$ with $g,h\in\mathbb{Z}_{>0}$ and define*

$$\alpha=\begin{cases} h+\sqrt{-m}, & \text{if } 2\in N(F)\setminus N((\mathcal{O}_F\left[\tfrac12\right])^\times),\\ 1, & \text{otherwise.}\end{cases} \tag{2.15}$$

*Let*

$$A=\{a_0,\cdots,a_t\}\subset\mathbb{A}_F^\times,\qquad B=\{-1,q_1,\cdots,q_t,\pi,\alpha\}\subset F^\times. \tag{2.16}$$

*Then $A$ and $B\cup\{2\}$ satisfy the assumptions in Theorem 2.1(2), and $[a,2]=0$ for $a\in A$. Hence*

$$\mathrm{rk}_4(\mathcal{T}_2(F))=\mathrm{rk}_2(\mathcal{T}_2(F))-\mathrm{rank}(R)\ \text{ where } R=([a,b])_{a\in A,b\in B}\,. \tag{2.17}$$

**Remark 2.5.** For $F$ a general real quadratic field, it is still quite easy to find $B$, but the harder part is to find a set of generators $A$ for $\mathcal{T}_2(F)[2]$. One reason is that it is not known how to obtain a system of explicit generators $\mathrm{Cl}(F)[2]$ for an arbitrary real quadratic field $F$ by a general formula.

If $t=1$, then $F=\mathbb{Q}(\sqrt{-1})$ or $F=\mathbb{Q}(\sqrt{-2})$. In this case Theorem 2.4 can be verified directly. We shall assume $t>1$ in what follows. For an ideal $\mathfrak{a}$ of $F$, let $\mathrm{cl}(\mathfrak{a})$ be its ideal class in $\mathrm{Cl}(F)$, and $\mathrm{cl}_S(\mathfrak{a})$ be its class in the $S$-class group $\mathrm{Cl}_S$ of $F$.

Theorem 2.4 is then a consequence of the following three propositions.

**Proposition 2.6.** *Let $L$ be the maximal abelian extension of exponent 2 over $F$, unramified outside $S$. Then $L = F(\sqrt{B'})$ where $B' = B \cup \{2\} = \{-1, 2, q_1, \ldots, q_t, \pi, \alpha\}$.*

*Proof.* We include the proof, which is routine, for lack of exact references. Let $J'$ be the subgroup of $F^\times/(F^\times)^2$ generated by $B'$. It suffices to show $J' = J$ with $J$ defined in (2.9).

We note that for all $x \in B'$, $x \neq \alpha$, $F(\sqrt{x})/F$ is unramified outside $S$. Thus if one can show that $F(\sqrt{\alpha})/F$ is unramified outside $S$, then $J' \subseteq J$.

Suppose first that either $2 \in N(E_S)$ or $2 \notin N(F)$. In this case $\alpha = 1$ and hence $J' \subset J$. We shall use the exact sequence (2.11) to show that $J'$ is indeed equal to $J$. Since $F$ is imaginary, $F^+ = F^\times$. (2.11) becomes the following exact sequence:

$$1 \to E_S/E_S^2 \to J \xrightarrow{g} \mathrm{Cl}_S[2] \to 1. \tag{2.18}$$

Here we recall that the map $g$ sends $\beta$ to $\mathrm{cl}_S(\mathfrak{b})$, for $\beta \in J$ satisfying $\beta\mathcal{O}_S = \mathfrak{b}^2$ for some $\mathcal{O}_S$-fractional ideal $\mathfrak{b}$. Clearly $E_S/E_S^2 \subset J'$, as $E_S$ is generated by $-1, 2$ and $\pi$. Thus, in order to prove $J' = J$, it suffices to show that $g(J') = \mathrm{Cl}_S[2]$. Let $G = \mathrm{Gal}(F/\mathbb{Q})$. Then $\mathrm{Cl}_S^G = \mathrm{Cl}_S[2]$. Let $I_S$ be the subgroup of fractional ideals of $F$ which is generated by prime ideals not in $S$. There is an isomorphism (see [16, Section 4])

$$\mathrm{Coker}\left(I_S^G \to \mathrm{Cl}_S^G\right) \cong \left(\mathbb{Z}\left[\frac{1}{2}\right]\right)^\times \cap N(F^\times)/N(E_S). \tag{2.19}$$

Since $-1 \notin N(F)$ as $F$ is imaginary, the assumption that either $2 \in N(E_S)$ or $2 \notin N(F^\times)$ precisely implies that the group on the right hand of (2.19) is trivial. Thus $\mathrm{Cl}_S^G$ is generated by $I_S^G$. But $I_S^G$ is generated by the ramified primes (see [16, Lemma 4.4]), it follows that $\mathrm{Cl}_S^G = \langle \mathfrak{q}_1, \ldots, \mathfrak{q}_t \rangle$. Since $g(q_i) = \mathrm{cl}_S(\mathfrak{q}_i)$ for each $i$, this proves $g(J') = \mathrm{Cl}_S^G = \mathrm{Cl}_S[2]$. Therefore, we have $J' = J$ when either $2 \in N(E_S)$ or $2 \notin N(F)$.

Suppose next that $2 \in N(F)$ but $2 \notin N(E_S)$. By Lemma 2.3, $q_i \equiv \pm 1 \pmod 8$ for $1 \leqslant i \leqslant t$. Hence we can write $m = 2g^2 - h^2$ for some $g, h \in \mathbb{Z}_{>0}$. In this case, $\alpha = h + \sqrt{-m}$ (see (2.15)). Then $\alpha + \bar{\alpha} = 2h$ and $\alpha\bar{\alpha} = 2g^2$ where $\bar{\alpha}$ is the complex conjugate of $\alpha$. Clearly $\gcd(g, h) = 1$. It follows that $\gcd((\alpha), (\bar{\alpha})) \mid 2\mathcal{O}_F$.

(1) If $m \equiv 1 \pmod 8$ or $2 \mid m$; then $2\mathcal{O}_F = \mathfrak{p}^2$ is ramified in $F$ and $g$ is odd. In this case, $\mathfrak{p} \mid (\alpha)$ but $2 \nmid (\alpha)$, otherwise $4 \mid \alpha\bar{\alpha} = 2g^2$. Hence $\bar{\mathfrak{p}} = \mathfrak{p} \mid (\bar{\alpha})$ and $\gcd((\alpha), (\bar{\alpha})) = \mathfrak{p}$. Since the integral ideals $(\alpha)\mathfrak{p}^{-1}$ and $(\bar{\alpha})\mathfrak{p}^{-1}$ are coprime to each other and their product is a square, hence there exists an $\mathcal{O}_F$-integral ideal $\mathfrak{a}$ such that $(\alpha) = \mathfrak{p}\mathfrak{a}^2$.

(2) If $m \equiv 7 \pmod 8$, then $2\mathcal{O}_F = \mathfrak{p}\bar{\mathfrak{p}}$ splits and $g$ is even. Without loss of generality, we may assume $\mathfrak{p} \mid \alpha$. Then $\bar{\mathfrak{p}} \mid \bar{\alpha}$ and hence $\bar{\mathfrak{p}} \mid \alpha = 2h - \bar{\alpha}$. This means $2 \mid \alpha$ and $\gcd((\alpha), (\bar{\alpha})) = 2\mathcal{O}_F$. Now $\frac{\alpha}{2} \cdot \frac{\bar{\alpha}}{2} = 2 \cdot (\frac{g}{2})^2$, then one and only one of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ divides $\frac{\alpha}{2}$. Assume $\mathfrak{p} \mid \frac{\alpha}{2}$. Then the two integral ideals $(\alpha/2)\mathfrak{p}^{-1}$ and $(\bar{\alpha}/2)\bar{\mathfrak{p}}^{-1}$ are coprime and their product is a square, hence there exists an $\mathcal{O}_F$-integral ideal $\mathfrak{a}$ such that $(\alpha) = 2\mathfrak{p}\mathfrak{a}^2$.

Thus, in both cases, we have

$$\alpha\mathcal{O}_S = \mathfrak{a}^2\mathcal{O}_S. \tag{2.20}$$

This shows that $F(\sqrt{\alpha})/F$ is unramified outside $S$. Hence $J' \subset J$. Following the same argument in the previous case and applying (2.18), to show $J' = J$, we just need to show $g(J') = \mathrm{Cl}_S[2] = \mathrm{Cl}_S^G$. If we can prove $\mathrm{Cl}_S^G = \langle \mathrm{cl}_S(I_S^G), \mathrm{cl}_S(\mathfrak{a}) \rangle$, by the fact $\mathrm{cl}_S(I_S^G) \subset g(J')$ and $\mathrm{cl}_S(\mathfrak{a}) = g(\alpha) \in g(J')$, then we are done.

We are left to prove the claim $\mathrm{Cl}_S^G = \langle \mathrm{cl}_S(I_S^G), \mathrm{cl}_S(\mathfrak{a}) \rangle$. By the isomorphism (2.19) and by our assumption $2 \in N(F) \setminus N(E_S)$, we have $[\mathrm{Cl}_S^G : \mathrm{cl}_S(I_S^G)] = 2$. Thus we just need to show $\mathrm{cl}_S(\mathfrak{a}) \notin \mathrm{cl}_S(I_S^G)$. Suppose, on the contrary, $\mathrm{cl}_S(\mathfrak{a}) \in \mathrm{cl}_S(I_S^G)$. Then we would have $\mathrm{cl}(\mathfrak{a}) \in \langle \mathrm{cl}(I_S^G), \mathrm{cl}(\mathfrak{p}), \mathrm{cl}(\bar{\mathfrak{p}}) \rangle$, since by definition $\mathrm{Cl}_S = \mathrm{Cl}_F/\langle \mathrm{cl}(S) \rangle$. Also note that $\mathrm{cl}(\bar{\mathfrak{p}}) = \mathrm{cl}(\mathfrak{p})^{-1}$. So we can write $\mathrm{cl}(\mathfrak{a}) = \mathrm{cl}(\mathfrak{p})^{r_0} \prod_i \mathrm{cl}(\mathfrak{q}_i)^{r_i}$ for some integers $r_i \in \mathbb{Z}$. Then, $\mathrm{cl}(\mathfrak{a})^2 = \mathrm{cl}(\mathfrak{p})^{2r_0}$. But, we have shown that $\mathrm{cl}(\mathfrak{a})^2 = \mathrm{cl}(\mathfrak{p})^{-1}$. Hence $\mathfrak{p}^{2r_0+1}$ would be principal, say $\mathfrak{p}^{2r_0+1} = (\gamma)$. This would imply that $2 = N(\gamma/2^{r_0}) \in N(E_S)$ which contradicts to our assumption $2 \in N(F^\times) \setminus N(E_S)$. This proves the claim. $\square$

**Lemma 2.7.** *If* $m \equiv 3 \pmod 4$*, then* $\{\mathrm{cl}(\mathfrak{q}_1), \dots, \mathrm{cl}(\mathfrak{q}_{t-1})\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $\mathrm{Cl}(F)[2]$*. If* $m \equiv 1 \pmod 4$*, then* $\{\mathrm{cl}(\mathfrak{p}), \mathrm{cl}(\mathfrak{q}_1), \dots, \mathrm{cl}(\mathfrak{q}_{t-1})\}$ *is a basis of* $\mathrm{Cl}(F)[2]$*. If* $m \equiv 2 \pmod 4$*, then* $\{\mathrm{cl}(\mathfrak{q}_1), \dots, \mathrm{cl}(\mathfrak{q}_t)\}$ *is a basis of* $\mathrm{Cl}(F)[2]$*.*

*Proof.* The proof is the classical genus theory and we refer to [4, Theorem 6.1] for the details. □

**Proposition 2.8.** *Let* $\hat{A}$ *be the image of* $A$ *in* $\mathcal{A}_F$*. Then* $\mathcal{T}_2(F)[2] = \hat{A}$*.*

*Proof.* For each $i$, $a_i^2$ is clearly in $\overline{F^\times \prod_{v \notin S} U_v}$, hence $\hat{a}_i$, the image of $a_i$ in $\mathcal{A}_F$, is in $\mathcal{A}_F[2] = \mathcal{T}_2(F)[2]$, and $\hat{A} \subseteq \mathcal{T}_2(F)[2]$. We have the following exact sequence of $\mathbb{F}_2$-vector spaces induced from (2.2):

$$0 \longrightarrow U_{1,S}/\overline{\phi(E_{1,S})}[2] \longrightarrow \mathcal{T}_2(F)[2] \xrightarrow{f} \mathrm{Cl}(F)[2]. \tag{2.21}$$

Since $E_{1,S} = \{\pm 1\}$, the first term of (2.21) has order 2 and is generated by $\hat{a}_0$ if $F_\mathfrak{p} = \mathbb{Q}_2$ or $\mathbb{Q}_2(\sqrt{-1})$, and is trivial otherwise. Thus $\dim \mathrm{Ker}(f) = \dim \mathrm{Ker}(f|_{\hat{A}}) = 1$ if $F_\mathfrak{p} = \mathbb{Q}_2$ or $\mathbb{Q}_2(\sqrt{-1})$, and 0 if otherwise. By definition, $f(\hat{a}_i) = \mathrm{cl}(\mathfrak{q}_i)$ if $1 \leqslant i \leqslant k$ and $f(\hat{a}_j) = \mathrm{cl}(\mathfrak{q}_j)\mathrm{cl}(\mathfrak{q}_t)$ if $k < j < t$.

Suppose first that $m \equiv 2 \pmod 4$. In this case $F_\mathfrak{p}$ can not be $\mathbb{Q}_2$ or $\mathbb{Q}_2(\sqrt{-1})$, so $\mathrm{Ker}(f) = 0$ and $\dim(\hat{A}) = \dim f(\hat{A})$. If $t = k$, then $\dim f(\hat{A}) = t$ by Lemma 2.7. Then $\mathcal{T}_2(F)[2] = \hat{A}$ by the 2-rank formula (2.7) for $\mathcal{T}_2(F)$. If $t > k$, one can write

$$(f(\hat{a}_1), \cdots, f(\hat{a}_k), f(\hat{a}_{k+1}\hat{a}_t), \dots, f(\hat{a}_{t-1}\hat{a}_t)) = (\mathrm{cl}(\mathfrak{q}_1), \cdots, \mathrm{cl}(\mathfrak{q}_t))M,$$

where $M$ is a matrix of rank $t-1$. Note that $\{\mathrm{cl}(\mathfrak{q}_1), \cdots, \mathrm{cl}(\mathfrak{q}_t)\}$ is an $\mathbb{F}_2$-basis of $\mathrm{Cl}(F)[2]$ by Lemma 2.7, then $\dim \hat{A} = \dim f(\hat{A}) = \mathrm{rank}(M) = t-1$. However, $\dim \mathcal{T}_2(F)[2] = t-1$ by (2.7) if $t > k$, hence $\mathcal{T}_2(F)[2] = \hat{A}$.

Suppose next that $m \equiv \pm 1 \pmod 8$. Then $t-k$ is even and $F_\mathfrak{p} = \mathbb{Q}_2$ or $\mathbb{Q}_2(\sqrt{-1})$. If $t = k$, it follows from Lemma 2.7 that $\dim f(\hat{A}) = t-1$ and hence $\dim \hat{A} = t$ which coincides with $\dim \mathcal{T}_2(F)[2]$ by (2.7). If $t-k$ is positive and even, this time we can write

$$(f(\hat{a}_1), \cdots, f(\hat{a}_k), f(\hat{a}_{k+1}\hat{a}_t), \dots, f(\hat{a}_{t-1}\hat{a}_t)) = (\mathrm{cl}(\mathfrak{q}_1), \cdots, \mathrm{cl}(\mathfrak{q}_{t-1}))M,$$

where $M$ is a matrix of rank $t-2$. Note that $\{\mathrm{cl}(\mathfrak{q}_1), \cdots, \mathrm{cl}(\mathfrak{q}_{t-1})\}$ is linearly independent by Lemma 2.7, then $\dim \hat{A} = \dim f(\hat{A}) + 1 = \mathrm{rank}(M) + 1 = t-1$, which coincides with $\dim \mathcal{T}_2(F)[2]$ by the 2-rank formula (2.7). This proves $\mathcal{T}_2(F)[2] = \hat{A}$ when $m \equiv \pm 1 \pmod 8$.

Finally, suppose that $m \equiv \pm 3 \pmod 8$. It follows that $t-k$ is an odd integer and the local field $F_\mathfrak{p}$ can not be $\mathbb{Q}_2$ or $\mathbb{Q}_2(\sqrt{-1})$. Then

$$(f(\hat{a}_1), \cdots, f(\hat{a}_k), f(\hat{a}_{k+1}\hat{a}_t), \dots, f(\hat{a}_{t-1}\hat{a}_t)) = (\mathrm{cl}(\mathfrak{q}_1), \cdots, \mathrm{cl}(\mathfrak{q}_{t-1}))M,$$

where $M$ is a matrix of rank $t-1$. Thus $\dim \hat{A} = \dim f(\hat{A}) = t-1$, which coincides with $\dim \mathcal{T}_2(F)[2]$ by the 2-rank formula (2.7). This proves $\mathcal{T}_2(F)[2] = \hat{A}$ when $m \equiv \pm 3 \pmod 8$. □

**Proposition 2.9.** $[a, 2] = 0$ *for all* $a \in A$*.*

*Proof.* Since $F(\sqrt{2})$ is the first layer of the cyclotomic $\mathbb{Z}_2$-extension of $F$, the proposition then follows from Remark 2.2(1). □

### 2.4 4-rank density formula

The aim of this subsection is to prove Theorem 1.1. We first give a simplification of the matrix $R$ in Theorem 2.4 when 2 is not a norm of $F = \mathbb{Q}(\sqrt{-m})$. Although only the result in the case $m \equiv 3 \pmod 4$ will be used in the proof of Theorem 1.1, we also present the simplification in the case $m \equiv 1, 2 \pmod 4$, for completeness.

**Theorem 2.10.** *Let $F=\mathbb{Q}(\sqrt{-m})$, where $m$ is a positive squarefree integer. Let $q_1, q_2, \cdots q_t$ be all the ramified prime numbers in $F$ and assume $q_1 = 2$ if 2 is ramified in $F$. Set*

$$R^C := \left(\left[\frac{q_i, -m}{q_j}\right]\right)_{2\leqslant i,j\leqslant t} \in M_{t-1}(\mathbb{F}_2)$$

*and*

$$\tau := \begin{cases} \left(\left[\frac{-2}{q_2}\right], \dots, \left[\frac{-2}{q_t}\right]\right)^T, & \text{if } m\equiv 3 \pmod 8 \\ \left(\left[\frac{2}{q_2}\right], \dots, \left[\frac{2}{q_t}\right]\right)^T, & \text{otherwise.} \end{cases}$$

*If $2\notin N(F)$, then*

$$\mathrm{rk}_4\mathcal{T}_2(F) = t-1-\mathrm{rank}\ (\tau, R^C). \tag{2.22}$$

**Remark 2.11.** A word on the notation: Note that in the above theorem, $q_1, \cdots, q_t$ denote the ramified primes in $F$ rather than the odd prime factors of $m$ as used in Theorem 2.4 and in last subsection. Clearly, this makes no difference when $m\equiv 3 \pmod 4$.

**Remark 2.12.** Recall that (see [16, §2] for example) the classical Rédei matrix for $\mathrm{Cl}_F$ is

$$R^{\mathrm{Cl}} := \left(\left[\frac{q_i, -m}{q_j}\right]\right)_{1\leqslant i,j\leqslant t} \quad \text{and} \quad \mathrm{rk}_4(\mathrm{Cl}_F) = t-1-\mathrm{rank}\ R^{\mathrm{Cl}}.$$

The matrix $R^C$ defined above is obtained from $R^{\mathrm{Cl}}$ by deleting its first row and first column. When $m\equiv 2,3 \pmod 4$, using the quadratic reciprocity law, one sees that the sums of each row and of each column of $R^{\mathrm{Cl}}$ are zero, hence rank $R^C$ = rank $R^{\mathrm{Cl}}$. Therefore

$$\mathrm{rk}_4\mathrm{Cl}_F = t-1-\mathrm{rank}\ R^C \quad \text{if } m\equiv 2,3 \pmod 4.$$

*Proof.* Firstly, we consider the case that 2 is unramified, i.e., $m\equiv 3 \pmod 4$. Then $\mathrm{rk}_2(\mathcal{T}_2(F)) = t-1$ by Theorem 2.1.

(1) Assume first $m\equiv 3 \pmod 8$. Then 2 is inert in $F$. Note that

$$\sum_{i=1}^{t}\left[\frac{-2}{q_i}\right] = \left[\frac{-2}{m}\right] = 0. \tag{2.23}$$

Hence we can rearrange $\{q_1, \dots, q_t\}$ without change the rank of $(\tau, R^C)$. In this case, the sets $A$ and $B$ of Theorem 2.4 are as follows: $a_0 = a_t = 1$, $\alpha = 1$ and $\pi = 2$. But by Proposition 2.9, $[a, 2] = 0$ for each $a\in A$. So we may assume that $A=\{a_1, \cdots, a_{t-1}\}$, $B=\{-1 := q_0, q_1, \dots, q_t\}$. Clearly, $B$ can be replaced by $\{-1 := q_0^*, q_1^*, \dots, q_t^*\}$ as they generate the same group.

For $1\leqslant i\leqslant t$, note that $\sqrt{q_i^*}\in F_{\mathfrak{p}} = \mathbb{Q}_2(\sqrt{5})$ and define

$$a_i' := (\cdots, \underset{\mathfrak{p}}{\sqrt{q_i^*}}, \dots, \underset{\mathfrak{q}_i}{\sqrt{-m}}, \cdots) \in \mathbb{A}_F^\times.$$

Then we have $a_i = a_i'$ for $1\leqslant i\leqslant k$ and $a_j = a_j' a_t'$ for $k<j\leqslant t-1$. Since $m\equiv 3 \pmod 8$, $t-k$ must be odd. Then a direct computation shows

$$a_1\cdots a_{t-1}\equiv a_t' \left(\mathrm{mod}\ \left(\mathbb{A}_F^{\times 2}, \overline{F^\times\prod_{v\notin S}U_v}\right)\right).$$

It follows that we may replace $A$ by $\{a_1', \cdots, a_t'\}$ as they generate the same group in $\mathcal{T}_2(F)$. Therefore, by Theorem 2.4, we have

$$\mathrm{rk}_4\mathcal{T}_2(F) = t-1-\mathrm{rank}\ ([a_i', q_j^*])_{1\leqslant i\leqslant t, 0\leqslant j\leqslant t}.$$

Using the quadratic reciprocity law, for $i,j\geqslant 1$, one checks that

$$[a_i', -1] = \left[\frac{-2}{q_i}\right] \quad \text{and} \quad [a_i',\ q_j^*] = \left[\frac{m, q_j^*}{q_i}\right] = \left[\frac{q_i, -m}{q_j}\right].$$

By the row-sum-zero and column-sum-zero property of the matrix mentioned in Remark 2.12 and the equation (2.23), we conclude that

$$\mathrm{rk}_4\mathcal{T}_2(F) = t - 1 - \mathrm{rank}\ (\tau, R^C).$$

(2) Assume next $m \equiv 7 \pmod 8$. The prime 2 splits in $F$. Note that $t > k$ since $2 \notin N(F)$, hence the element $a_t \in A$ is trivial. We still replace $B$ by $\{-1 := q_0, \pi, q_1^*, \dots, q_t^*\}$. Also note that both $a_0 \in A$ and $\pi \in B$ are nontrivial. We may choose the sign of $\pi$ such that $\left[\frac{\pi, -1}{\mathfrak{p}}\right] = 0$. Then $[a_0, \pi] = 0$. The matrix $R$ for $\mathcal{T}_2(F)$ in Theorem 2.4 is

$$R = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots \\ \left[\frac{-1}{q_1}\right] & \left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_1}\right] & \cdots & \left[\frac{m, q_j^*}{q_1}\right] & \cdots \\ \vdots & \vdots & & \vdots & \\ \left[\frac{-1}{q_t q_{k+1}}\right] & \left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_{k+1}}\right] + \left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_t}\right] & \cdots & \left[\frac{m, q_j^*}{q_{k+1}}\right] + \left[\frac{m, q_j^*}{q_t}\right] & \cdots \\ \vdots & \vdots & & \vdots & \\ \left[\frac{-1}{q_t q_{t-1}}\right] & \left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_{t-1}}\right] + \left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_t}\right] & \cdots & \left[\frac{m, q_j^*}{q_{t-1}}\right] + \left[\frac{m, q_j^*}{q_t}\right] & \cdots \end{pmatrix}. \tag{2.24}$$

We make the following elementary operations on the matrix $R$: Firstly, replace the first column by $(1, 0, \dots, 0)^T$, and replace the first row $(\cdots)$ by $\left(1, \left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_t}\right], \cdots, \left[\frac{m, q_j^*}{q_t}\right], \cdots\right)$. Secondly, add the first row to the $k+2, \cdots, t$-th row. Thirdly, move the first row to the bottom. Finally delete the first row. It follows that the matrix $R$ in (2.24) is equivalent to

$$(\tau, \beta, R^C). \tag{2.25}$$

where

$$\beta := \left(\left[\frac{\sqrt{-m}, \pi}{\mathfrak{q}_i}\right]\right)^T_{2 \leqslant i \leqslant t}.$$

By Lemma 2.13 below, $\mathrm{rank}(R) = \mathrm{rank}(\tau, R^C)$. This proves the case $m \equiv 7 \pmod 8$ by Theorem 2.4.

Now we consider the case that 2 is ramified whence $q_1 = 2$. Then $m = q_2 \cdots q_t \equiv 1 \pmod 4$ or $m = 2q_2 \cdots q_t \equiv 2 \pmod 4$. Write $2\mathcal{O}_F = \mathfrak{p}^2$. By our condition $2 \notin N(F)$, the $B$ in Theorem 2.4 is $B = \{q_0^* := -1, q_2^*, \dots, q_t^*\}$.

(3) Suppose $m \equiv 1 \pmod 8$. Then $A = \{a_0, a_2, \dots, a_{t-1}\}$. It is clear that the matrix $R$ for $\mathcal{T}_2(F)$ is

$$R = \begin{pmatrix} 0 & \cdots & 0 & \cdots \\ 0 & \cdots & \left[\frac{m, q_j^*}{q_2}\right] & \cdots \\ \vdots & & \vdots & \\ 0 & \cdots & \left[\frac{m, q_j^*}{q_{k+1}}\right] + \left[\frac{m, q_j^*}{q_t}\right] & \cdots \\ \vdots & & \vdots & \\ 0 & \cdots & \left[\frac{m, q_j^*}{q_{t-1}}\right] + \left[\frac{m, q_j^*}{q_t}\right] & \cdots \end{pmatrix}$$

Firstly, replace the first row $(\cdots)$ by $\left(1, \cdots, \left[\frac{m, q_j^*}{q_t}\right], \cdots\right)$. Then we get a matrix whose rank equals $1 + \mathrm{rank}\ R$. Secondly, add the first row to the $k+2, k+3, \cdots$ and the $t$-th row. Finally move the first row to the bottom. Now we get $(\tau, R^C)$. We have $\mathrm{rk}_2\ \mathcal{T}_2(F) = t - 2$ by Theorem 2.1. Thus

$$\mathrm{rk}_4\mathcal{T}_2(-m) = t - 2 - \mathrm{rank}\ R = t - 1 - \mathrm{rank}\ (\tau, R^C).$$

This proves the case $m \equiv 1 \pmod 8$. The arguments for the other cases are similar and we leave the details to the reader. $\square$

**Lemma 2.13.** *Assume that* $m \equiv 7 \pmod 8$ *having a prime factor* $q \equiv \pm 3 \pmod 8$. *Then* $\beta$ *is a sum of column vectors of* $R^C$.

*Proof.* Let $\lambda$ be the order of $\mathfrak{p}$ in $\mathrm{Cl}(F)$. Suppose $\pi = \frac{c+d\sqrt{-m}}{2}$ with $c, d \in \mathbb{Z}$ such that $\pi\mathcal{O}_F = \mathfrak{p}^\lambda$ and $\left(\frac{-1,\pi}{\mathfrak{p}}\right) = 1$. Note that $\lambda$ must be even; otherwise, $2 = N(\pi 2^{-\frac{\lambda-1}{2}}) \in N(F)$ which contradicts to the assumption. Write $\lambda = 2\lambda'$. Then we have a decomposition in $\mathbb{Z}$

$$(2^{\lambda'+1} - c)(2^{\lambda'+1} + c) = md^2. \tag{2.26}$$

Since $\left(\frac{-1,\pi}{\mathfrak{p}}\right) = 1$, it follows from the product formula that $\left(\frac{-1,\pi}{\bar{\mathfrak{p}}}\right) = 1$. We obtain $\pi \equiv 1 \pmod{\bar{\mathfrak{p}}^2}$ and $\bar{\pi} \equiv 1 \pmod{\mathfrak{p}^2}$. But $\mathfrak{p}^\lambda \mid \pi$ and $\lambda$ is even, we have $\pi \equiv 0 \pmod{\mathfrak{p}^2}$. Thus

$$c = \pi + \bar{\pi} \equiv 1 \pmod{\mathfrak{p})^2} \implies c \equiv 1 \pmod 4.$$

Then $2^{\lambda'+1} - c$ and $2^{\lambda'+1} + c$ are coprime, by (2.26), there exist positive integers $m_+, m_-, d_+, d_-$ such that $m = m_+m_-$, $d = d_+d_-$, $2^{\lambda'+1} + c = m_+d_+^2$, and $2^{\lambda'+1} - c = m_-d_-^2$. In particular, $m_+ \equiv c \equiv 1 \pmod 4$ and $m_- \equiv -1 \pmod 4$. We obtain

$$2c = m_+d_+^2 - m_-d_-^2.$$

Now the vector

$$\beta = \left(\left[\frac{q_i, 2c}{q_i}\right]\right)^T_{1\leqslant i\leqslant t}.$$

If $q_i \mid m_-$, noting that $m_+ \equiv 1 \pmod 4$, we have

$$\left[\frac{q_i, 2c}{q_i}\right] = \left[\frac{q_i, m_+}{q_i}\right] = \sum_{q\mid 2m_+}\left[\frac{q_i, m_+}{q}\right] = \sum_{q\mid m_+}\left[\frac{q_i, m_+}{q}\right] = \sum_{q\mid m_+}\left[\frac{q_i, -m}{q}\right].$$

If $q_i \mid m_+$, noting that $-m_- \equiv 1 \pmod 4$, we also have

$$\left[\frac{q_i, 2c}{q_i}\right] = \left[\frac{q_i, -m_-}{q_i}\right] = \sum_{q\mid 2m_-}\left[\frac{q_i, -m_-}{q}\right] = \sum_{q\mid m_-}\left[\frac{q_i, -m_-}{q}\right] = \sum_{q\mid m_-}\left[\frac{q_i, -m}{q}\right] = \sum_{q\mid m_+}\left[\frac{q_i, -m}{q}\right].$$

This means that $\beta$ is the sum of the column vectors $\left(\left[\frac{q_i, -m}{q_j}\right]\right)^T_i$ for $q_j \mid m_+$ of $R^C$. $\square$

The rest of this subsection is dedicated to proving Theorem 1.1, which is based on the work of Gerth [5] and Yue-Yu [24]. As in the statement of Theorem 1.1, $x$ will always denote a positive real number and $t$ will denote a positive integer.

The set $N_{t,x}$ is the disjoint union of subsets $N_{t,x}^{(i)}$ $(i = 1, 2, 3)$ defined by (all $p_i$ are odd distinct primes)

$$N_{t,x}^{(1)} := \{m \in N_{t,x} \mid m = p_1 \cdots p_t \equiv 3 \pmod 4\};$$
$$N_{t,x}^{(2)} := \{m \in N_{t,x} \mid m = p_1 \cdots p_{t-1} \equiv 1 \pmod 4\};$$
$$N_{t,x}^{(3)} := \{m \in N_{t,x} \mid m = 2p_1 \cdots p_{t-1} \equiv 2 \pmod 4\}.$$

Following [5], we know that when $x \to \infty$,

$$\#N_{t;x}^{(1)} \sim \frac{1}{2}\frac{1}{(t-1)!}\frac{x(\log\log x)^{t-1}}{\log x},$$
$$\#N_{t;x}^{(2)} \sim \frac{1}{2}\frac{1}{(t-2)!}\frac{x(\log\log x)^{t-2}}{\log x} = o\left(\#N_{t;x}^{(1)}\right),$$
$$\#N_{t;x}^{(3)} \sim \frac{1}{(t-2)!}\frac{x(\log\log(x/2))^{t-2}}{2\log(x/2)} = o\left(\#N_{t;x}^{(1)}\right).$$

Here and after we denote $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. Then

$$\#N_{t;x} \sim \#N_{t;x}^{(1)} \sim \frac{1}{2}\frac{1}{(t-1)!}\frac{x(\log\log x)^{t-1}}{\log x}. \tag{2.27}$$

We define two equivalent relations in $N_{t,x}^{(1)}$.

**Definition 2.14.** For $m = p_1 \cdots p_t, n = q_1 \cdots q_t \in N_{t,x}^{(1)}$ arranging such that $p_1 < p_2 < \cdots < p_t$ and $q_1 < q_2 < \cdots < q_t$, we say that $m$ and $n$ have the same Rédei type if $q_i \equiv p_i \pmod 4$ for $i \leqslant t$ and $\left[\frac{q_j}{q_i}\right] = \left[\frac{p_j}{p_i}\right]$ for $1 \leqslant j < i \leqslant t$; we say that $m$ and $n$ have the same Rédei type modulo 8, if furthermore $q_i \equiv p_i \pmod 8$ for $i \leqslant t$. Denote by $[m]$ (resp. $[[m]]$) the equivalence class of $m$ with the same Rédei type (resp. modulo 8) respectively.

**Lemma 2.15.** *For any $m \in N_{t,x}^{(1)}$, we define*

$$R(m;t,x) := [m] \cap N_{t,x}^{(1)} = \{m' \in N_{t,x}^{(1)} \mid m' \text{ and } m \text{ have the same Rédei type}\},$$
$$S(m;t,x) := [[m]] \cap N_{t,x}^{(1)} = \{m' \in N_{t,x}^{(1)} \mid m' \text{ and } m \text{ have the same Rédei type modulo } 8\}.$$

*Then when $x \to \infty$, we have*

$$\#R(m;t,x) \sim 2^{1-\frac{t^2+t}{2}} \cdot \#N_{t,x}^{(1)},$$

*and*

$$\#S(m;t,x) \sim \frac{\#R(m;t,x)}{2^t}.$$

*Proof.* See [24, Lemma 2.1 and Corollary 2.2]. □

**Remark 2.16.** As mentioned in Gerth [5, Page 493], an intuitive explanation of the above lemma might proceed as follows. To decide the equivalence class $[m]$, we need to fix the conditions $p_i \pmod 4$ for $l \leqslant i \leqslant t-1$ since $m = \prod_{i=1}^t p_i \equiv 3 \pmod 4$, and the conditions $\left[\frac{p_j}{p_i}\right]$ for $1 \leqslant j < i \leqslant t$. Hence, there are $2^{\frac{t^2+t}{2}-1}$ equivalence classes and the proportion of each equivalence class in $N_{t,x}^{(1)}$ is the same by the above lemma. Furthermore, given a class $[m]$, then $\{p_1 \pmod 8, \dots, p_t \pmod 8\}$ have $2^t$ choice. Hence there are $2^t$ modulo 8 equivalence classes in $[m]$ and the proportion of each modulo 8 equivalence class in $[m]$ is the same by the above lemma again.

**Lemma 2.17.** *Let $W(t,x) = \{m \in N_{t,x}^{(1)} \mid 2 \in N(\mathbb{Q}(\sqrt{-m}))\}$. Then*

$$\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#W(t,x)}{\#N_{t,x}^{(1)}} = 0.$$

*Proof.* Put

$$f(m) = \begin{cases} 1, & \text{if } 2 \in N(\mathbb{Q}(\sqrt{-m})) \\ 0, & \text{otherwise.} \end{cases}$$

Given an equivalence class $[m]$, we claim that there is exactly one class $[[n]]$ in $[m]$ such that $f(n) = 1$. Indeed, $q_i \pmod 4$ is determined as $n = q_1 \cdots q_t \in [m]$. Then by Hasse's norm theorem, $f(n) = 1$ implies that $q_i$ must be $1 \pmod 8$ (resp. $7 \pmod 8$) in $1 \pmod 4$ (resp. $3 \pmod 4$). Hence follows the claim.

Now we have

$$\begin{aligned}\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#W(t,x)}{\#N_{t,x}^{(1)}} &= \lim_{t\to\infty}\lim_{x\to\infty} \frac{\sum\limits_{[m]} \sum\limits_{[[n]],n\in[m]} f(n)\cdot \#S(n;t,x)}{\sum\limits_{[m]} \sum\limits_{[[n]],n\in[m]} \#S(n;t,x)} \\ &= \lim_{t\to\infty}\lim_{x\to\infty} \frac{\sum\limits_{[m]} \#R(m;t,x)/2^t}{\sum\limits_{[m]} \#R(m;t,x)}\end{aligned}$$

$$= \lim_{t\to\infty} \frac{1}{2^t} = 0,$$

where the second equality is by lemma 2.15. □

*Proof of Theorem 1.1.* By Theorem 2.10, Lemma 2.17, and the estimate (2.27), it suffices to prove that for $r \geqslant 0$,

$$\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#\{m \in N_{t,x}^{(1)} \mid \text{rank } (\tau, R^C) = t-1-r\}}{\# N_{t;x}^{(1)}} = \frac{\eta_\infty(2)}{2^{r(r+1)}\eta_r(2)\eta_{r+1}(2)}.$$

For any matrix $A \in M_{t-1}(\mathbb{F}_2)$, write $\mathrm{Im}A := \{Ax \mid x \in \mathbb{F}_2^{t-1}\}$. Then we only need to prove that

$$\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#\{m \in N_{t,x}^{(1)} \mid \text{rank } R^C = t-1-r, \tau \in \mathrm{Im}R^C\}}{\# N_{t;x}^{(1)}} = \frac{1}{2^r} \cdot \frac{\eta_\infty(2)}{2^{r^2}\eta_r(2)^2} \tag{2.28}$$

and

$$\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#\{m \in N_{t,x}^{(1)} \mid \text{rank } R^{\mathrm{Cl}} = t-2-r, \tau \notin \mathrm{Im}R^{\mathrm{Cl}}\}}{\# N_{t;x}^{(1)}} = \left(1 - \frac{1}{2^{r+1}}\right) \cdot \frac{\eta_\infty(2)}{2^{(r+1)^2}\eta_{r+1}(2)^2}, \tag{2.29}$$

since

$$\frac{1}{2^r} \cdot \frac{\eta_\infty(2)}{2^{r^2}\eta_r(2)^2} + \left(1 - \frac{1}{2^{r+1}}\right) \cdot \frac{\eta_\infty(2)}{2^{(r+1)^2}\eta_{r+1}(2)^2} = \frac{\eta_\infty(2)}{2^{r(r+1)}\eta_r(2)\eta_{r+1}(2)}.$$

By Lemma 2.15 and [24, Remark 2.3, Equation (3.19)], in each equivalence class $[m] \subset N_{t,x}^{(1)}$, $\tau \in \mathrm{Im}R^C$ has probability $\frac{2^{t-1-r}}{2^{t-1}} = \frac{1}{2^r}$ if rank $R^C = t-1-r$. i.e.,

$$\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#\{m \in N_{t,x}^{(1)} \mid \text{rank } R^C = t-1-r, \tau \in \mathrm{Im}R^C\}}{\#\{m \in N_{t,x}^{(1)} \mid \text{rank } R^C = t-1-r\}} = \frac{1}{2^r}.$$

It is proved by Gerth in [5] that

$$\lim_{t\to\infty}\lim_{x\to\infty} \frac{\#\{m \in N_{t,x}^{(1)} \mid \text{rank } R^C = t-1-r\}}{\# N_{t;x}^{(1)}} = \frac{\eta_\infty(2)}{2^{r^2}\eta_r(2)^2}.$$

This implies the equation (2.28). The proof of the equation (2.29) is similar and we leave the detail to the reader. Thus

$$d_{\infty,r}^T = \frac{1}{2^r} \cdot \frac{\eta_\infty(2)}{2^{r^2}\eta_r(2)^2} + \left(1 - \frac{1}{2^{r+1}}\right) \cdot \frac{\eta_\infty(2)}{2^{(r+1)^2}\eta_{r+1}(2)^2} = \frac{\eta_\infty(2)}{2^{r(r+1)}\eta_r(2)\eta_{r+1}(2)}.$$

This completes the proof of Theorem 1.1. □

## 3 Study of $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$ for odd prime $l$

By the 2-rank formula (2.7), if $l$ is a prime, $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$ are trivial if $l \equiv \pm 3 \pmod 8$, and nontrivial cyclic 2-groups if $l \equiv \pm 1 \pmod 8$. In what follows, we assume $l \equiv \pm 1 \pmod 8$ is a prime. In this section, we shall study the structures of $\mathcal{T}_2(\pm l)$ and $\mathcal{T}_2(\pm 2l)$, or equivalently, the 2-power divisibility of their orders $t_2(\pm l)$ and $t_2(\pm 2l)$.

### 3.1 The imaginary case

**Theorem 3.1.** *Let $l \equiv \pm 1 \pmod 8$ be a prime. Then $\mathcal{T}_2(-l)$ and $\mathcal{T}_2(-2l)$ are non-trivial cyclic 2 groups, and*

(1) $t_2(-l) = 2$ *if* $l \equiv 7 \pmod 8$, $t_2(-l) = 4$ *if* $l \equiv 9 \pmod{16}$, *and* $t_2(-l) \geqslant 8$ *if* $l \equiv 1 \pmod{16}$.

(2) $t_2(-2l) = 2$ *if* $l \equiv 7 \pmod 8$ *or* $l \equiv 9 \pmod{16}$, *and* $t_2(-2l) \geqslant 4$ *if* $l \equiv 1 \pmod{16}$.

**Remark 3.2.** Based on numerical data, we find out that the conditions $t_2(-l) = 2^i$ for $i \geqslant 3$ and $t_2(-2l) = 2^i$ for $i \geqslant 2$ are not classified by congruence relations.

*Proof.* (1) Let $F = \mathbb{Q}(\sqrt{-l})$. We consider (i) $l \equiv 7 \pmod 8$ and (ii) $l \equiv 1 \pmod 8$ separately.
(i) In this case $2 \nmid h(-l)$ by genus theory. From the commutative diagram (2.2), we have

$$\mathcal{T}_2(-l) \cong \left(\left(\mathbb{Z}_2^\times \times \mathbb{Z}_2^\times\right)/\pm 1\right)[2^\infty] \cong \mathbb{Z}/2\mathbb{Z}.$$

(ii) In this case 2 is ramified and $F_\mathfrak{p} = \mathbb{Q}_2(\sqrt{-1})$.

Let $a = (\cdots, \underset{\mathfrak{p}}{1+\sqrt{-1}}, \cdots) \in \mathbb{A}_F^\times$ and $\hat{a}$ be its image in $\mathcal{A}_F$; here we recall that $\mathcal{A}_F$ is the group defined in (2.1) with $p = 2$. Then $a^4 = (\cdots, \underset{\mathfrak{p}}{-4}, \cdots) \in \overline{F^\times \prod_{v \notin S} U_v}$ and hence $\hat{a} \in \mathcal{T}_2(F)[4]$. Since $a^2 = (\cdots, \underset{\mathfrak{p}}{2\sqrt{-1}}, \cdots) \equiv (\cdots, \underset{\mathfrak{p}}{\sqrt{-1}}, \cdots) \bmod \left(\overline{F^\times \prod_{v \notin S} U_v}\right)$ and $\sqrt{-1}$ is nontrivial in $U_{1,\mathfrak{p}}/\{\pm 1\} \subset \mathcal{A}_F^{\text{pro}-2}$, we have $\hat{a}^2 \neq 0$ in $\mathcal{T}_2(F)$. Thus $\hat{a}$ is a generator of the cyclic group $\mathcal{T}_2(F)[4]$.

The 2-units $E_S$ of $F$ is generated by $-1$ and 2. Clearly, $2 \notin N(E_S)$. Write $l = 2g^2 - h^2$. Let $\alpha = h + \sqrt{-l}$. By Proposition 2.6, $L = F\left(\sqrt{-1}, \sqrt{l}, \sqrt{2}, \sqrt{\alpha}\right) = F\left(\sqrt{-1}, \sqrt{2}, \sqrt{\alpha}\right)$ is the maximal abelian extension of exponent 2 over $F$ unramified outside 2. The map

$$\mathcal{T}_2(F)[4] \to \mathrm{Gal}(F(\sqrt{-1})/F) \times \mathrm{Gal}(F(\sqrt{2})/F) \times \mathrm{Gal}(F(\sqrt{\alpha})/F)$$

has kernel $2\mathcal{T}_2(F)[8]$. Thus $t_2(-l) \geqslant 8$ if and only if the additive Artin symbols $[a, -1] = [a, 2] = [a, \alpha] = 0$. It is easy to see $[a, 2] = [a, -1] = 0$ since $l \equiv 1 \pmod 8$. We have

$$\begin{aligned}
[a, h + \sqrt{-l}] &= \left[\frac{1+\sqrt{-1}, h+\sqrt{-l}}{F_\mathfrak{p}}\right] = \left[\frac{-\sqrt{l}, h+\sqrt{-l}}{F_\mathfrak{p}}\right] + \left[\frac{-\sqrt{l}-\sqrt{-l}, h+\sqrt{-l}}{F_\mathfrak{p}}\right] \\
&= \left[\frac{-\sqrt{l}, 2g^2}{\mathbb{Q}_2}\right] + \left[\frac{-\sqrt{l}-\sqrt{-l}, h+\sqrt{-l}}{F_\mathfrak{p}}\right] \\
&= \left[\frac{\sqrt{l}, 2}{\mathbb{Q}_2}\right] + \left[\frac{-\sqrt{l}-\sqrt{-l}, h+\sqrt{-l}}{F_\mathfrak{p}}\right].
\end{aligned}$$

Note that

$$\left[\frac{\sqrt{l}, 2}{\mathbb{Q}_2}\right] = \begin{cases} 0 & \text{if } l \equiv 1 \pmod{16}, \\ 1 & \text{if } l \equiv 9 \pmod{16}. \end{cases}$$

For any $x, y \in F_\mathfrak{p}$, noting that $-1$ is a square in $F_\mathfrak{p}$, we have

$$0 = \left[\frac{\frac{x}{x+y}, \frac{y}{x+y}}{F_\mathfrak{p}}\right] = \left[\frac{xy, x+y}{F_\mathfrak{p}}\right] + \left[\frac{x+y, x+y}{F_\mathfrak{p}}\right] + \left[\frac{x, y}{F_\mathfrak{p}}\right].$$

Put $x = -\sqrt{l} - \sqrt{-l}, y = h + \sqrt{-l}$. Note that $x + y \in \mathbb{Q}_2$. It follows that $\left[\frac{x+y, x+y}{F_\mathfrak{p}}\right] = 0$. Thus,

$$\left[\frac{x, y}{F_\mathfrak{p}}\right] = \left[\frac{x+y, xy}{F_\mathfrak{p}}\right] = \left[\frac{x+y, 4g^2 l}{\mathbb{Q}_2}\right] = 0.$$

This proves (1).

(2) follows from the same argument used in the proof of (1). We omit the details here. □

### 3.2 The real case

We need the following formula of Coates (see [2, Appendix] or [7, Chapter III. 2.6.5]):

**Proposition 3.3.** *Let $K \neq \mathbb{Q}$ be a totally real number field. Assume that the Leopoldt Conjecture holds for $(p, K)$, i.e., $\delta_p(K) = 0$. Then*

$$\#\mathcal{T}_p(K) = (p\text{-adic unit}) \cdot \frac{p \cdot [K \cap \mathbb{Q}^{p,\mathrm{cyc}} : \mathbb{Q}] \cdot h(K) \cdot R_p(K)}{\sqrt{D_K} \cdot \prod_{\mathfrak{p}|p} N\mathfrak{p}}. \tag{3.1}$$

*Here $h(K)$ is the class number, $R_p(K)$ is the $p$-adic regulator and $D_K$ is the discriminant of $K$, $\mathbb{Q}^{p,\mathrm{cyc}}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and the product runs over all primes of $K$ lying above $p$ and $N$ is the norm map from $K$ to $\mathbb{Q}$.*

**Lemma 3.4.** *Assume $l \equiv \pm 1 \pmod 8$ is a prime. Let $\nu_2$ be the normalized 2-adic valuation and $\log_2$ be the 2-adic logarithmic map. For $m = l$ or $2l$, let $\varepsilon_m = a_m + b_m\sqrt{m}$ be the fundamental unit of $\mathbb{Q}(\sqrt{m})$. Then*

(1) $\nu_2(t_2(l)) = \nu_2(\log_2(\varepsilon_l)) - 1 = \nu_2(a_l) - 1$.

(2) $\nu_2(t_2(2l)) = \nu_2(h(2l)) + \nu_2(b_{2l}) - 1$.

*Proof.* (1) Let $F = \mathbb{Q}(\sqrt{l})$. Recall that the 2-adic regulator $R_2(F)$ is $\log_2(\varepsilon_l)$. By Coates' formula above, we have $\nu_2(t_2(l)) = \nu_2(\log_2(\varepsilon_l)) - 1$ as $2 \nmid h(l)$. It remains to show that $\nu_2(\log_2(\varepsilon_l)) = \nu_2(a_l)$. We shall use the basic property of logarithm that, for $x \in \overline{\mathbb{Q}}_2$, if $\nu_2(x-1) > 1$ then $\nu_2(\log_2(x)) = \nu_2(x)$.

If $l \equiv 1 \pmod 8$, then it is easy to see $a_l$ and $b_l$ are integers. It is also known that $N(\varepsilon_l) = a_l^2 - lb_l^2 = -1$. It follows that $4 \mid a_l$ and $b_l$ is odd. Thus $\nu_2(\varepsilon_l^2 - 1) = \nu_2(\varepsilon_l^2 + \varepsilon_l\bar{\varepsilon}_l) = 1 + \nu_2(a_l) \geqslant 3$. This implies that $\nu_2(\log_2(\varepsilon_l^2)) = \nu_2(\varepsilon_l^2 - 1) = 1 + \nu_2(a_l)$. Hence, $\nu_2(\log_2(\varepsilon_l)) = \nu_2(a_l)$.

If $l \equiv 7 \pmod 8$, we first prove that $a_l$ is even. In this case 2 is ramified in $F$, say $2\mathcal{O}_F = \mathfrak{p}^2$. Since $h(l)$ is odd, $\mathfrak{p}$ must be principal, say $\mathfrak{p} = (\pi)$ with $\pi \in \mathcal{O}_F$. Then $\pi^2/2$ is a unit, say $\varepsilon_l^k$. Note that $k$ must be odd. Otherwise, $\sqrt{2} \in F$, which is absurd. Then $(\pi\varepsilon_l^{-(k-1)/2})^2 = 2\varepsilon_l$ and hence $\pi\varepsilon_l^{-(k-1)/2} \in \mathcal{O}_F$. Write $\pi\varepsilon_l^{-(k-1)/2} = c + d\sqrt{l}$ with $c, d \in \mathbb{Z}$. Then $c$ and $d$ must be odd since $N(c + d\sqrt{l}) = 2$. Hence $a_l = \frac{c^2+d^2l}{2}$ is clearly even.

Thus, $b_l$ must be odd. Then $\nu_2(\varepsilon_l^4 - 1) = \nu_2(\varepsilon_l^4 - \varepsilon_l^2\bar{\varepsilon}_l^2) = 2 + \nu_2(a_lb_l) = 2 + \nu(a_l)$. Therefore, $\nu_2(\log_2(\varepsilon_l)) = \nu_2(a_l)$. This completes the proof of (1).

(2) Clearly $a_{2l}$ is odd and $b_{2l}$ is even. We have

$$\nu_2(\varepsilon_{2l}^4 - 1) = \nu_2(\varepsilon_{2l}^2 + \varepsilon_{2l}\bar{\varepsilon}_{2l}) + \nu_2(\varepsilon_{2l}^2 - \varepsilon_{2l}\bar{\varepsilon}_{2l}) = \nu_2(2a_{2l}) + \nu_2(2\sqrt{2l}b_{2l}) = \frac{5}{2} + \nu_2(b_{2l}).$$

Hence, $\nu_2(\log_2(\varepsilon_{2l})) = \frac{1}{2} + \nu_2(b_{2l})$. Then (2) follows from Coates' formula for $\mathcal{T}_2(\mathbb{Q}(\sqrt{2l}))$. $\square$

**Remark 3.5.** The proof of $a_l$ is even for $l \equiv 7 \pmod 8$ holds for $l \equiv 3 \pmod 4$. For a different proof of this fact, see [25].

The following proposition collects results about the 2-class groups $\mathrm{Cl}_2(-l)$ and $\mathrm{Cl}_2(-2l)$ due to Gauss, Hasse [9], Brown [1] and others, most importantly due to Leonard-Williams [13], see [14, Theorem 4.2] for a proof about $\mathrm{Cl}_2(-2l)$.

**Proposition 3.6.** *Let $l$ be an odd prime. Then both $\mathrm{Cl}_2(-l)$ and $\mathrm{Cl}_2(-2l)$ are cyclic groups.*

(1) *$h_2(-l) = 1$ if $l \equiv 3 \pmod 4$, $h_2(-l) = 2$ if $l \equiv 5 \pmod 8$ and $h_2(-l) \geqslant 4$ if $l \equiv 1 \pmod 8$. Moreover, if $l \equiv 1 \pmod 8$, suppose $l = 2g^2 - h^2$, then $h_2(-l) = 4$ if and only if $g \equiv 3 \pmod 4$, $h_2(-l) = 8$ if and only if $\left(\frac{2h}{g}\right)\left(\frac{g}{l}\right)_4 = -1$.*

(2) *$h_2(-2l) = 2$ if $l \equiv \pm 3 \pmod 8$ and $h_2(-2l) \geqslant 4$ if $l \equiv \pm 1 \pmod 8$. Moreover,*

(i) *If $l \equiv 1 \pmod 8$, suppose $l = u^2 - 2v^2$ such that $u \equiv 1 \pmod 4$, then $h_2(-2l) = 4$ if and only if $u \equiv 5 \pmod 8$, $h_2(-2l) = 8$ if and only if $\left(\frac{u}{l}\right)_4 = -1$.*

(ii) *If $l \equiv 7 \pmod 8$, then $h_2(-2l) = 4$ if and only if $l \equiv 7 \pmod{16}$, and $h_2(-2l) = 8$ if and only if $l \equiv 15 \pmod{16}$ and $(-1)^{\frac{l+1}{16}}\left(\frac{2u}{v}\right) = -1$ where $(u, v) \in \mathbb{Z}_{>0}^2$ satisfying $l = u^2 - 2v^2$.*

We have the following theorem:

**Theorem 3.7.** *Assume $l \equiv 7 \pmod 8$ is a prime. Then $\mathcal{T}_2(l)$ and $\mathcal{T}_2(2l)$ are non-trivial 2-cyclic groups, $4 \mid t_2(l)$ and*

(1) $t_2(l)=4 \Leftrightarrow t_2(2l)=2 \Leftrightarrow h_2(-2l)=4 \Leftrightarrow l \equiv 7 \pmod{16}$;

(2) $t_2(l)=8 \Leftrightarrow t_2(2l)=4 \Leftrightarrow h_2(-2l)=8 \Leftrightarrow$ $l \equiv 15 \pmod{16}$ *and* $(-1)^{\frac{l+1}{16}}\left(\frac{2u}{v}\right)=-1$ *where* $(u,v)\in\mathbb{Z}^2_{>0}$ *is a solution of* $l=X^2-2Y^2$.

*Consequently, we always have* $t_2(l)\equiv 2t_2(2l)\equiv h_2(-2l) \pmod{16}$.

**Remark 3.8.** However, in general the three numbers $t_2(l)$, $2t_2(2l)$ and $h_2(-2l)$ are not equal if one (hence all) of them $\geqslant 16$. For example, let $l=223$, then $t_2(l)=16$, $2t_2(2l)=256$ and $h_2(-2l)=32$.

*Proof.* (1) We first study $t_2(l)$. As shown in the proof of Lemma 3.4, $\varepsilon_l=a_l+b_l\sqrt{l}=\frac{1}{2}(c+d\sqrt{l})^2$ where $c,d$ are odd integers and $N(c+d\sqrt{l})=c^2-d^2l=2$. In particular, $c^2\equiv 2 \pmod{d}$. It follows that every prime factor of $d$ is congruent to $\pm1 \pmod 8$. Hence $d^2\equiv 1 \pmod{16}$ and $\nu_2(a_l)=\nu_2(1+d^2l)$. For $l\equiv 7 \pmod 8$, $\nu_2(1+d^2l)\geqslant 3$, with equality if and only if $l\equiv 7 \pmod{16}$. By Lemma 3.4(1), $4\mid t_2(l)=2^{\nu_2(ld^2+1)-1}$, and $t_2(l)=4$ if and only if $l\equiv 7 \pmod{16}$.

Note that the Jacobi symbol $\left(\frac{2u}{v}\right)$ is independent on the choices of $u$ and $v$ (see [14, Lemma 4.1]). By the results of Leonard-Williams (Proposition 3.6(2)), we are left to show that if $l\equiv 15 \pmod{16}$, then

$$\nu_2(ld^2+1)=4 \Longleftrightarrow (-1)^{\frac{l+1}{16}}\left(\frac{2u}{v}\right)=-1.$$

Since $l=(u+\sqrt{2}v)(u-\sqrt{2}v)\mid ld^2=(c+\sqrt{2})(c-\sqrt{2})$, one of the prime elements $u\pm\sqrt{2}v$ must divides $c+\sqrt{2}$ in the Euclidean domain $\mathbb{Z}[\sqrt{2}]$.

(i) Suppose $\frac{c+\sqrt{2}}{u+\sqrt{2}v}\in\mathbb{Z}[\sqrt{2}]$. Note that $c+\sqrt{2}$ and $c-\sqrt{2}$ are coprime in $\mathbb{Z}[\sqrt{2}]$, the integers $\frac{c+\sqrt{2}}{u+\sqrt{2}v}$ and $\frac{c-\sqrt{2}}{u-\sqrt{2}v}$ are coprime, but their product is $d^2$ and $\mathbb{Z}[\sqrt{2}]$ has class number 1, hence there exist $s,t\in\mathbb{Z}$ and $\varepsilon\in\{1,1+\sqrt{2}\}$ such that

$$\frac{c+\sqrt{2}}{u+\sqrt{2}v}=\varepsilon(t-s\sqrt{2})^2.$$

Since the left hand side is totally positive, we must have $\varepsilon=1$. Comparing the coefficients of $\sqrt{2}$ gives

$$1=(t^2+2s^2)v-2tsu. \tag{3.2}$$

Note that $ts$ must be positive. We may assume that $t,s$ are both positive. Since $l=u^2-2v^2\equiv -1 \pmod{16}$, both $u$ and $v$ are odd. In fact, $v\equiv 1 \pmod 4$ by (3.2). Hence $\left(\frac{2u}{v}\right)=\left(\frac{-st}{v}\right)=\left(\frac{t}{v}\right)\left(\frac{s}{v}\right)$. Note that $d,t$ are odd. By quadratic reciprocity law, $\left(\frac{t}{v}\right)=\left(\frac{v}{t}\right)=\left(\frac{2}{t}\right)$. The last equality follows from (3.2). Write $s=2^rs_0$ with $2\nmid s_0$. If $s\equiv 2 \pmod 4$, then $v\equiv 5 \pmod 8$ and $\left(\frac{s}{v}\right)=\left(\frac{2\cdot s_0}{v}\right)=-\left(\frac{v}{s_0}\right)$. If $s\equiv 0 \pmod 4$, then $t^2\equiv 1 \pmod 8$ and $v\equiv 1 \pmod 8$. So $\left(\frac{s}{v}\right)=\left(\frac{s_0}{v}\right)=\left(\frac{v}{s_0}\right)=1$. If $s\equiv \pm1 \pmod 4$, then $\left(\frac{s}{v}\right)=\left(\frac{v}{s}\right)=1$. Hence

$$\left(\frac{s}{v}\right)=\begin{cases}-1, & \text{if } s\equiv 2 \pmod 4,\\ 1, & \text{otherwise.}\end{cases}$$

Therefore $\left(\frac{2u}{v}\right)=1$ if and only if $\pm d=t^2-2s^2\equiv \pm1 \pmod{16}$. This implies that $16\parallel ld^2+1$ if and only if $(-1)^{\frac{l+1}{16}}\left(\frac{2u}{v}\right)=-1$.

(ii) Suppose $\frac{c+\sqrt{2}}{u-\sqrt{2}v}\in\mathbb{Z}[\sqrt{2}]$. By similar argument, there exist two positive integers $t,s$ such that

$$1=2stu-(t^2+2s^2)v.$$

For this equation, $v\equiv 3 \pmod 4$ and $\left(\frac{2u}{v}\right)=\left(\frac{t}{v}\right)\left(\frac{s}{v}\right)$. One can repeat the argument above to obtain that $\left(\frac{t}{v}\right)=\left(\frac{2}{t}\right)$ and that

$$\left(\frac{s}{v}\right)=\begin{cases}-1, & \text{if } s\equiv 2 \pmod 4,\\ 1, & \text{otherwise.}\end{cases}$$

Again this implies that $16\parallel ld^2+1$ if and only if $(-1)^{\frac{l+1}{16}}\left(\frac{2u}{v}\right)=-1$.

(2) If $l\equiv 7 \pmod 8$, then $h(2l)$ is odd. By Lemma 3.4(2), $\nu_2(t_2(2l))=\nu_2(b_{2l})-1$. According to the last paragraph in [13, § 3], we have $h(-2l)\equiv b_{2l} \pmod{16}$. Then $t_2(2l)=\frac{h_2(-2l)}{2}=\frac{t_2(l)}{2}$ if $t_{2l}=2$ or 4. We just need to apply Proposition 3.6. □

**Proposition 3.9.** *Assume $l \equiv 1 \pmod 8$ is a prime.*

(1) *Write $l = 2g^2 - h^2$ with $g, h \in \mathbb{Z}_{>0}$. Then*

$$t_2(l) = 2 \Longleftrightarrow h_2(-l) = 4 \Longleftrightarrow g \equiv 3 \pmod 4; \tag{3.3}$$

$$t_2(l) = 4 \Longleftrightarrow \begin{cases} h_2(-l) = 8 & \text{if } l \equiv 1 \pmod{16} \\ h_2(-l) \geqslant 16 & \text{if } l \equiv 9 \pmod{16} \end{cases} \Longleftrightarrow (-1)^{\frac{l-1}{8}} \left(\frac{2h}{g}\right)\left(\frac{g}{l}\right)_4 = -1. \tag{3.4}$$

(2) *Write $l = u^2 - 2v^2$ with $u, v \in \mathbb{Z}_{>0}$ and $u \equiv 1 \pmod 4$. Then*

$$t_2(2l) = 2 \Longleftrightarrow \left(\frac{u}{l}\right) = -1, \tag{3.5}$$

$$t_2(2l) = 4 \Longleftrightarrow (-1)^{\frac{l-1}{8}} \left(\frac{u}{l}\right)_4 = -1. \tag{3.6}$$

*Proof.* (1) For $l \equiv 1 \pmod 8$, Williams [23] proved that

$$a_l \equiv \begin{cases} h(-l) + l - 1 \pmod{16}, & \text{if } h_2(-l) \geqslant 8; \\ 4(h(l) - 1) + l - 1 - h(-l) \pmod{16}, & \text{if } h_2(-l) = 4. \end{cases}$$

Hence we have

$$\begin{cases} 2t_2(l) \equiv h(-l) + l - 1 \pmod{16}, & \text{if } h_2(-l) \geqslant 8; \\ t_2(l) = 2, & \text{if } h_2(-l) = 4. \end{cases} \tag{3.7}$$

Applying Coates' formula (3.1), Lemma 3.4 and Proposition 3.6(1), we get the result.

(2) It follows from (3.1) that $t_2(2l)$ is equal to $\log_2(\varepsilon_{2l})h(2l)/(2\sqrt{2})$ up to a 2-adic unit. Denote by $R + S\sqrt{2l}$ the fundamental unit of norm 1 of $\mathbb{Q}(\sqrt{2l})$ and by $h^+(2l)$ the narrow class number of $\mathbb{Q}(\sqrt{2l})$. Then, $R + S\sqrt{2l} = \varepsilon_{2l}$ and $h^+(2l) = 2h(2l)$ if $N(\varepsilon_{2l}) = 1$; $R + S\sqrt{2l} = \varepsilon_{2l}^2$ and $h^+(2l) = h(2l)$ if $N(\varepsilon_{2l}) = -1$. Thus, by Lemma 3.4(2), we have

$$\nu_2(t_2(2l)) = \nu_2(h^+(2l)) + \nu_2(S) - 2.$$

The main theorem in [10] tells us that

$$\frac{S \cdot h^+(2l)}{2} \equiv 1 - l - h(-2l) \pmod{16}.$$

Then all results here directly follow the discussion in [13, § 2]. □

Now we can prove the density result about $\mathcal{T}_2(l)$ and $\mathcal{T}_2(2l)$:

*Proof of Theorem 1.3.* (1) We first show (1.4). In the case $e = 0$, then $l \equiv 1 \pmod 8$. Stevenhagen [20, Theorem 1] proved that $h_2(-l) \geqslant 8$ if and only if $l$ splits completely in $\mathbb{Q}(\zeta_8, \sqrt{1+i})$. Then by Chebotarev's density theorem,

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 1 \pmod 8,\ h_2(-l) = 4\}}{\#\{\ l \leqslant x : l \equiv 1 \pmod 8\}} = \lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 1 \pmod 8,\ h_2(-l) \geqslant 8\}}{\#\{\ l \leqslant x : l \equiv 1 \pmod 8\}} = \frac{1}{2}.$$

By (3.3) in Proposition 3.9, the case $i = 0$ follows.

Recently, Koymans ([11, Theorem 1.1]) proved that

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 1 \pmod 8 \text{ and } h_2(-l) = 8\}}{\#\{\ l \leqslant x : l \equiv 1 \pmod 8\}} = \frac{1}{4}.$$

As a corollary of [20, Theorem 1], we have that $l \equiv 9 \pmod{16}$ such that $h_2(-l) \geqslant 8$ if and only if the Frobenius of $l$ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{16}, \sqrt{1+i})/\mathbb{Q}$ acts trivially in $\mathbb{Q}(\zeta_8, \sqrt{1+i})$ and maps $\zeta_{16}$ to $-\zeta_{16}$. Hence

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 9 \pmod{16},\ h_2(-l) \geqslant 8\}}{\#\{\ l \leqslant x : l \equiv 9 \pmod{16}\}} = \lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 1 \pmod{16},\ h_2(-l) \geqslant 8\}}{\#\{\ l \leqslant x : l \equiv 1 \pmod{16}\}} = \frac{1}{2}.$$

If we can show

$$\lim_{x\to\infty}\frac{\#\{l\leqslant x: l\equiv 9 \pmod{16},\ h_2(-l)=8\}}{\#\{\ l\leqslant x: l\equiv 9 \pmod{16}\}}=\lim_{x\to\infty}\frac{\#\{l\leqslant x: l\equiv 9 \pmod{16},\ h_2(-l)\geqslant 16\}}{\#\{\ l\leqslant x: l\equiv 9 \pmod{16}\}}=\frac{1}{4}, \tag{3.8}$$

then

$$\lim_{x\to\infty}\frac{\#\{l\leqslant x: l\equiv 1 \pmod{16} \text{ and } h_2(-l)\geqslant 16\}}{\#\{\ l\leqslant x: l\equiv 1 \pmod{16}\}}=\lim_{x\to\infty}\frac{\#\{l\leqslant x: l\equiv 1 \pmod{16} \text{ and } h_2(-l)=8\}}{\#\{\ l\leqslant x: l\equiv 1 \pmod{16}\}}=\frac{1}{4}.$$

Hence the $i=1$ case follows from (3.4). It suffices to show (3.8).

Let

$$e_l=\begin{cases}1, & \text{if } h_2(-l)\geqslant 16,\\ -1, & \text{if } h_2(-l)=8,\\ 0, & \text{if } h_2(-l)=4.\end{cases}$$

By[11, Theorem 1.2], we have

$$\sum_{l\leqslant x,\ l\equiv 1 \pmod 8} e_l \ll x/\exp((\log x)^{0.1}). \tag{3.9}$$

Replacing the spin symbol $[w]$ in [11, Lemma 4.1, 4.2] by the twisted symbol $[w]':=[w]\cdot\lambda(w)$ for all totally positive elements $w$ of $\mathbb{Z}[\zeta_8]$, where $\lambda(w)=(-1)^{\frac{Nw-1}{8}}$ if $Nw\equiv 1 \pmod 8$ and 1 otherwise, one follows the argument there and obtains

$$\sum_{l\leqslant x,\ l\equiv 1 \pmod 8} (-1)^{\frac{l-1}{8}} e_l \ll x/\exp((\log x)^{0.1}). \tag{3.10}$$

Thus

$$\sum_{l\leqslant x, l\equiv 1 \pmod 8}\left(e_l-(-1)^{\frac{l-1}{8}}e_l\right)=2\sum_{l\leqslant x, l\equiv 9 \pmod{16}}\left(1_{16|h(-l)}-1_{8||h(-l)}\right)\ll x/\exp((\log x)^{0.1}).$$

Note that as $x\to+\infty$, $\log x=o(\exp((\log x)^{0.1})$, by Dirichlet's density theorem, then

$$\#\{l\leqslant x,\ l\equiv 9 \pmod{16},\ h_2(-l)=8\}\sim\#\{l\leqslant x,\ l\equiv 9 \pmod{16},\ h_2(-l)\geqslant 16\}\sim\frac{x}{32\log x}.$$

Hence we have (3.8).

In the case $e=1$, $l\equiv 7 \pmod 8$. By Theorem 3.7, the case $i=0$ follows from the fact that $t_2(l)=4$ if and only if $l\equiv 7 \pmod{16}$, and the case $i=1$ follows from the following result of Milovic [17, Theorem 1] on $h_2(-2l)$ that

$$\lim_{x\to\infty}\frac{\#\{\ l\leqslant x: l\equiv -1 \pmod 8,\ h_2(-2l)=8\}}{\#\{\ l\leqslant x: l\equiv -1 \pmod 8\}}=\frac{1}{4}.$$

(2) Case 7 (mod 8) for (1.5) follows from (1) and Theorem 3.7, and case 1 (mod 8) follows from Proposition 3.9(2) and [12, Theorem 1] with similar arguments for $t_2(l)$; we omit the details. □

**Remark 3.10.** We actually proved that for $i=1$ and 2,

$$\lim_{x\to\infty}\frac{\#\{l\leqslant x: l\equiv 1 \pmod{16},\ t_2(l)=2^i\}}{\#\{l\leqslant x: l\equiv 1 \pmod{16}\}}=\lim_{x\to\infty}\frac{\#\{l\leqslant x: l\equiv 9 \pmod{16},\ t_2(l)=2^i\}}{\#\{l\leqslant x: l\equiv 9 \pmod{16}\}}=\frac{1}{2^i}.$$

## 4 Distribution Conjectures for $\mathcal{T}_p$-groups of quadratic fields

### 4.1 Distribution conjecture of $\mathcal{T}_p$ in the full family

We first propose a distribution conjecture on the group structure of $\mathcal{T}_p(F)$ when $F$ varies in the family of all imaginary (resp. real) quadratic fields $\mathcal{F}_{\rm im}$ (resp. $\mathcal{F}_{\rm re}$).

For $5 \leqslant p \leqslant 47$, numerical data presented in [18, §5.2] suggested that of all real quadratic fields $\mathbb{Q}(\sqrt{m})$ such that $m \leqslant 10^9$ is squarefree, the proportion of fields with trivial $\mathcal{T}_p$-groups (so-called $p$-rational field) is close to $\eta_\infty(p)$. It was mentioned there that the authors also considered the distribution about the group structures of $\mathcal{T}_p$-groups, however, we did not find any further statement and subsequent studies in the literature.

Based on Theorem 1.1 and numerical data in the appendix, we propose the following conjecture:

**Conjecture 4.1.** Let $p$ be a prime. Let $\mathcal{F}_{\rm im}$ (resp. $\mathcal{F}_{\rm re}$) be the family of all imaginary (resp. real) quadratic fields. For each finite abelian $p$-group $G$, one has

$$\lim_{x\to\infty} \frac{\#\{F \in \mathcal{F}_{\rm im} \mid -D_F \leqslant x,\ 6\mathcal{T}_p(F) \cong G\}}{\#\{F \in \mathcal{F}_{\rm im} : -D_F \leqslant x\}} = \frac{\eta_\infty(p)/\eta_1(p)}{\#G \cdot \#\mathrm{Aut}(G)}; \tag{4.1}$$

$$\lim_{x\to\infty} \frac{\#\{F \in \mathcal{F}_{\rm re} \mid D_F \leqslant x,\ 6\mathcal{T}_p(F) \cong G\}}{\#\{F \in \mathcal{F}_{\rm re} : D_F \leqslant x\}} = \frac{\eta_\infty(p)}{\#\mathrm{Aut}(G)}. \tag{4.2}$$

Here $D_F$ is the discriminant of $F$, and we recall that $\eta_s(q) := \prod_{i=1}^{s}(1-q^{-i})$ for $s \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $q > 1$.

**Remark 4.2.** (1) For $p \geqslant 5$, we have $6\mathcal{T}_p(F) \cong \mathcal{T}_p(F)$ and hence the factor 6 can be removed from the statement of our conjecture. For $p = 2$ or 3, we have $6\mathcal{T}_p(F) = p\mathcal{T}_p(F)$. For $p = 5$ and 7, we have carried out numerical computation of $\mathcal{T}_p(F)$ with $|D_F| \leqslant 5 \times 10^7$; see Tables 1 - 4, which give strong evidence of Conjecture 4.1 in these cases.

(2) In the bad primes 2 and 3 case, when the bound is $5 \times 10^7$, the distributions of $2\mathcal{T}_2$ and $3\mathcal{T}_3$ are actually not quite good based on our computation, but this is expected just like the analogue phenomenon for the distributions of narrow 2-class groups and tame kernels of quadratic fields: the bound is not big enough. We gain confidence from recent breakthrough of Smith[21] on the distribution of narrow 2-class groups of quadratic fields, as well as the 4-rank density formula for $\mathcal{T}_2$ of imaginary quadratic fields we just proved here.

(3) If using the setting of local Cohen-Lenstra Heuristic, the weight function for $p$-class groups is $\omega_0$ for imaginary quadratic fields and $\omega_1$ for real ones where

$$\omega_i(G) = \frac{1}{(\#G)^i \cdot \#\mathrm{Aut}(G)}, \tag{4.3}$$

the weight functions for $\mathcal{T}_p$-groups are exactly the reverse order.

(4) For more general conjectures on distributions of $\mathcal{T}_p$-groups of quadratic fields, which are also in the spirit of the Cohen-Lenstra heuristics, see [15].

### 4.2 Distribution conjecture of $\mathcal{T}_2$ in sub-families

**Conjecture 4.3.** Assume all $l$ appeared below are primes. For each integer $i \geqslant 0$ and $e \in \{0,1\}$, then

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 1 \pmod{16}, t_2(-l) = 2^{i+3}\}}{\#\{l \leqslant x : l \equiv 1 \pmod{16}\}} = \frac{3}{4^{i+1}}, \tag{4.4}$$

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv 1 \pmod{16}, t_2(-2l) = 2^{i+2}\}}{\#\{l \leqslant x : l \equiv 1 \pmod{16}\}} = \frac{3}{4^{i+1}}, \tag{4.5}$$

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv (-1)^e \pmod{8},\ t_2(l) = 2^{i+1+e}\}}{\#\{l \leqslant x : l \equiv (-1)^e \pmod{8}\}} = \frac{1}{2^{i+1}}, \tag{4.6}$$

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv (-1)^e \pmod{8},\ t_2(2l) = 2^{i+1}\}}{\#\{l \leqslant x : l \equiv (-1)^e \pmod{8}\}} = \frac{1}{2^{i+1}}. \tag{4.7}$$

We shall present numerical evidence in Tables 5 - 10.

**Remark 4.4.** (1) Under the setting of extended local Cohen-Lenstra heuristic, one can interpret (4.4) more conceptually as follows. Let $\mathcal{M}_k = \{\mathbb{Z}/2^{i+k}\mathbb{Z} \mid i \geqslant 0\}$ for $k \geqslant 1$. For $G = \mathbb{Z}/2^{i+k}\mathbb{Z} \in \mathcal{M}_k$, then a direct computation gives

$$\frac{\omega_1(G)}{\sum_{H\in\mathcal{M}_k} \omega_1(H)} = \frac{3}{4^{i+1}}.$$

Thus (4.4) is equivalent to that the natural density of primes $l$ with $\mathcal{T}_2(-l) \cong G$ among all primes $\equiv 1 \pmod{16}$ is equal to the ratio of $\omega_1(G)$ to the total 1-weight of the space $\mathcal{M}_3$. For (4.5), the corresponding space is $\mathcal{M}_2$.

(2) One can also reformulate (4.6) and (4.7) by using the weight function $\omega_0$ and by noting the following identity:

$$\frac{\omega_0(G)}{\sum_{H\in\mathcal{M}_k}\omega_0(H)} = \frac{1}{2^{i+1}}, \quad \text{where} \quad G = \mathbb{Z}/2^{k+i}\mathbb{Z}.$$

In (4.6) (resp. (4.7)), the total space is $\mathcal{M}_{e+1}$ (resp. $\mathcal{M}_1$).

(3) By Lemma 3.4(1), (4.6) has the following equivalent form about the distribution of fundamental units: for each $i \geqslant 0$ and $e \in \{0,1\}$,

$$\lim_{x\to\infty} \frac{\#\{l \text{ prime}: \ l \leqslant x, \ l \equiv (-1)^e \pmod 8, \ \nu_2(a_l) = i+2+e\}}{\#\{l \text{ prime}: \ l \leqslant x, \ l \equiv (-1)^e \pmod 8\}} = \frac{1}{2^{i+1}}. \tag{4.8}$$

(4) Finally, for $l \equiv 1 \pmod 8$, (4.6) actually has a finer form: for $i \geqslant 0$ and $a \in \{1, 9\}$,

$$\lim_{x\to\infty} \frac{\#\{l \leqslant x : l \equiv a \pmod{16}, \ t_2(l) = 2^{i+1}\}}{\#\{l \leqslant x : l \equiv a \pmod{16}\}} = \frac{1}{2^{i+1}}. \tag{4.9}$$

The cases $i = 0$ and 1 were proved in Theorem 1.3. We actually speculate that this is the case for all sub-congruent classes $a \pmod{2^k}$ of $1 \pmod 8$.

In the case $a = 9$, let $\chi_l$ be the associated Dirichlet character of $\mathbb{Q}(\sqrt{l})$ and $L_2(s, \chi_l)$ be its 2-adic $L$-function, by the 2-adic class number formula (see [22, Theorem 5.24]) and Coates' order formula (3.1), (4.9) has the following equivalent form which was implicitly proposed by Shanks-Sime-Washington in [19, p. 1253]:

$$\lim_{x\to\infty} \frac{\#\{l \text{ prime} : l \leqslant x, l \equiv 9 \pmod{16} \text{ and } \nu_2(L_2(1,\chi_l)) = i+2\}}{\#\{l \text{ prime} : l \leqslant x \text{ and } l \equiv 9 \pmod{16}\}} = \frac{1}{2^{i+1}}. \tag{4.10}$$

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## References


1 Brown E A. The class number of $\mathbb{Q}(\sqrt{-p})$, for $p \equiv 1 \pmod 8$ a prime. P Am Math Soc, 1972, 31: 381-383

2 Coates J H. $p$-adic L-functions and Iwasawa's theory. In: Algebraic number fields: L-functions and Galois properties. Proc Sympos Univ Durham, London: Academic Press, 1977, 269-353

3 Colmez P. Résidu en $s = 1$ des fonctions zêta $p$-adiques. Invent Math, 1988, 91: 371-389

4 Cox D A. Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication. Second edition. Hoboken: John Wiley & Sons, Inc., 2013

5 Gerth F III. The 4-class ranks of quadratic fields. Invent Math, 1984, 77: 489-515.

6 Gras G. Groupe de Galois de la $p$-extension abélienne $p$-ramifiée maximale d'un corps de nombres. J Reine Angew Math, 1982, 333: 86-132

7 Gras G. Class field theory: From theory to practice. Translated from the French manuscript by Henri Cohen. Springer Monographs in Mathematics. Berlin: Springer-Verlag, 2003

8 Gras G. Practice of the incomplete $p$-ramification over a number field–History of abelian $p$-pamification. Communications in Advanced Mathematical Sciences, 2019, 2: 251-280

9 Hasse H. Über die Klassenzahl des Körpers $P(\sqrt{-2p})$ mit einer Primzahl $p \neq 2$. J Number Theory, 1969, 1: 231-234

10 Kaplan P, Williams K S. On the class numbers of $\mathbb{Q}(\sqrt{\pm p})$ modulo 16, for $p \equiv 1 \pmod 8$ a prime. Acta Arith, 1981, 40: 289-296

11 Koymans P. The 16-rank of $\mathbb{Q}(\sqrt{-p})$. Algebra Number Theory, 2020, 14: 37-65

12 Koymans P, Milovic D. On the 16-rank of class groups of $\mathbb{Q}(\sqrt{-2p})$ for primes $p \equiv 1 \pmod 4$. Int Math Res Notices, 2019, 23: 7406-7427

13 Leonard P A, Williams K S. On the divisibility of the class numbers of $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{-2p})$ by 16. Can Math Bull, 1982, 25: 200-206

14 Li J, Xu Y. On class numbers of pure quartic fields. Ramanujan J, 2021, 56: 235-248

15 Li J, Ouyang Y, Xu Y. Abelian $p$-ramification groups and new Cohen-Lenstra heuristics(in Chinese). Sci Sin Math, 2021, 51: 1635-1654
16 Li J, Yu C. The Chevalley-Gras formula over global fields. J Theor Nombr Bordx, 2020, 32: 525-544
17 Milovic D. On the 16-rank of class groups of $\mathbb{Q}(\sqrt{-8p})$ for $p \equiv -1 \pmod 4$. Geom Funct Anal, 2017, 27: 973C1016
18 Pitoun F, Varescon F. Computing the torsion of the $p$-ramified module of a number field. Math Comput, 2015, 84: 371-383
19 Shanks D C, Sime P J, Washington L C. Zeros of 2-adic L-functions and congruences for class numbers and fundamental units. Math Comput, 1999, 68: 1243-1255
20 Stevenhagen P. Divisibility by 2-powers of certain quadratic class numbers. J Number Theory, 1993, 43: 1-19
21 Smith A. $2^\infty$-Selmer groups, $2^\infty$-class groups, and Goldfeld's conjecture. ArXiv:1702.02325v2, 2017
22 Washington L C. Introduction to cyclotomic fields. Graduate Texts in Mathematics, vol. 83. Springer, 1997
23 Williams K S. On the class number of $\mathbb{Q}(\sqrt{-p})$ modulo 16, for $p \equiv 1 (mod\ 8)$ a prime. Acta Arith, 1981, 39: 381-398
24 Yue Q, Yu J. The densities of 4-ranks of tame kernels for quadratic fields. J Reine Angew Math, 2004, 567: 151-173
25 Zhang Z, Yue Q. Fundamental units of real quadratic fields of odd class number. J Number Theory, 2014, 137: 122-129

[theorem]Conjecture

## Appendix A Data for Conjecture 4.1

In Tables 1-4, we let the middle value be the ratio of field $F$ such that $\mathcal{T}_p(F) \cong G$ among all quadratic fields whose absolute discriminant $\leqslant B$, and $\mathbb{D}$ be the value predicted by Conjecture 4.1.

**Table 1** $\mathcal{T}_5$ of real quadratic fields

| $B$ \ $G$ | $\mathbb{Z}/5\mathbb{Z}$ | $\mathbb{Z}/25\mathbb{Z}$ | $(\mathbb{Z}/5\mathbb{Z})^2$ | $\mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}/25\mathbb{Z}$ | $(\mathbb{Z}/5\mathbb{Z})^3$ |
|---|---|---|---|---|---|
| $10^7$ | 0.1876 | 0.03694 | 1.375 e-3 | 3.277 e-4 | 0 |
| $2*10^7$ | 0.1880 | 0.03712 | 1.396 e-3 | 3.463 e-4 | 1.645 e-7 |
| $3*10^7$ | 0.1880 | 0.03727 | 1.416 e-3 | 3.439 e-4 | 2.193 e-7 |
| $4*10^7$ | 0.1880 | 0.03739 | 1.438 e-3 | 3.447 e-4 | 3.290 e-7 |
| $5*10^7$ | 0.1882 | 0.03740 | 1.453 e-3 | 3.430 e-4 | 2.632 e-7 |
| $\mathbb{D}$ | 0.1901 | 0.03802 | 1.584 e-3 | 3.802 e-4 | 5.110 e-7 |

**Table 2** $\mathcal{T}_7$ of real quadratic fields

| $B$ \ $G$ | $\mathbb{Z}/7\mathbb{Z}$ | $\mathbb{Z}/49\mathbb{Z}$ | $(\mathbb{Z}/7\mathbb{Z})^2$ | $(\mathbb{Z}/7\mathbb{Z})^3$ |
|---|---|---|---|---|
| $10^7$ | 0.1377 | 0.01950 | 3.622 e-4 | 0 |
| $2*10^7$ | 0.1382 | 0.01956 | 3.622 e-4 | 0 |
| $3*10^7$ | 0.1383 | 0.01963 | 3.713 e-4 | 0 |
| $4*10^7$ | 0.1385 | 0.01966 | 3.764 e-4 | 0 |
| $5*10^7$ | 0.1385 | 0.01968 | 3.833 e-4 | 5.483 e-8 |
| $\mathbb{D}$ | 0.1395 | 0.01992 | 4.151 e-4 | 2.477 e-8 |

**Table 3** $\mathcal{T}_5$ of imaginary quadratic fields

| $B$ \ $G$ | $\mathbb{Z}/5\mathbb{Z}$ | $\mathbb{Z}/25\mathbb{Z}$ | $(\mathbb{Z}/5\mathbb{Z})^2$ | $\mathbb{Z}/5\mathbb{Z}\times\mathbb{Z}/25\mathbb{Z}$ | $(\mathbb{Z}/5\mathbb{Z})^3$ |
|---|---|---|---|---|---|
| $10^7$ | 0.04558 | 1.767 e-3 | 6.185 e-5 | 6.580 e-7 | 0 |
| $2*10^7$ | 0.04584 | 1.789 e-3 | 6.004 e-5 | 1.645 e-6 | 0 |
| $3*10^7$ | 0.04604 | 1.801 e-3 | 6.152 e-5 | 2.084 e-6 | 0 |
| $4*10^7$ | 0.04613 | 1.809 e-3 | 6.424 e-5 | 2.385 e-6 | 0 |
| $5*10^7$ | 0.04618 | 1.915 e-3 | 6.659 e-5 | 2.237 e-6 | 0 |
| $\mathbb{D}$ | 0.04752 | 1.901 e-3 | 7.920 e-5 | 3.802 e-6 | 5.110 e-9 |

**Table 4** $\mathcal{T}_7$ of imaginary quadratic fields

| $B$ \ $G$ | $\mathbb{Z}/7\mathbb{Z}$ | $\mathbb{Z}/49\mathbb{Z}$ | $(\mathbb{Z}/7\mathbb{Z})^2$ | $(\mathbb{Z}/7\mathbb{Z})^3$ |
|---|---|---|---|---|
| $10^7$ | 0.02287 | 0.00043 | 3.619 e-6 | 0 |
| $2*10^7$ | 0.02297 | 0.00045 | 5.263 e-6 | 0 |
| $3*10^7$ | 0.02302 | 0.00045 | 5.593 e-6 | 0 |
| $4*10^7$ | 0.02307 | 0.00045 | 6.827 e-6 | 0 |
| $5*10^7$ | 0.02307 | 0.00045 | 7.435 e-6 | 0 |
| $\mathbb{D}$ | 0.02324 | 0.00047 | 9.883 e-6 | 8.425 e-11 |

## Appendix B Data for Conjecture 4.3

In the last six Tables, we let the middle value be the ratio of field $F$ such that $\mathcal{T}_p(F)\cong G$ among all quadratic fields whose absolute discriminant $\leqslant B$, and $\mathbb{D}$ be the value predicted by Conjecture 4.3.

**Table 5** $\mathcal{T}_2$ of $\mathbb{Q}(\sqrt{-l}), l\equiv 1 \pmod{16}$ and $l$ is a prime

| $B$ \ $G$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/32\mathbb{Z}$ | $\mathbb{Z}/64\mathbb{Z}$ | $\mathbb{Z}/128\mathbb{Z}$ |
|---|---|---|---|---|---|
| $10^7$ | 0.7508 | 0.1867 | 0.04704 | 0.01172 | 2.905 e-3 |
| $2*10^7$ | 0.7501 | 0.1872 | 0.04708 | 0.01170 | 3.062 e-3 |
| $3*10^7$ | 0.7501 | 0.1878 | 0.04658 | 0.01169 | 2.977 e-3 |
| $4*10^7$ | 0.7498 | 0.1881 | 0.04666 | 0.01166 | 2.910 e-3 |
| $5*10^7$ | 0.7496 | 0.1880 | 0.04694 | 0.01160 | 2.934 e-3 |
| $\mathbb{D}$ | 0.75 | 0.1875 | 0.04688 | 0.01172 | 2.930 e-3 |

**Table 6** $\mathcal{T}_2$ of $\mathbb{Q}(\sqrt{-2l}), l \equiv 1 \pmod{16}$ and $l$ is a prime

| $B$ \ $G$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/32\mathbb{Z}$ | $\mathbb{Z}/64\mathbb{Z}$ |
|---|---|---|---|---|---|
| $10^7$ | 0.7508 | 0.1876 | 0.04611 | 0.01144 | 3.134 e-3 |
| $2*10^7$ | 0.7501 | 0.1886 | 0.04604 | 0.01142 | 3.075 e-3 |
| $3*10^7$ | 0.7501 | 0.1885 | 0.04611 | 0.01140 | 3.029 e-3 |
| $4*10^7$ | 0.7498 | 0.1885 | 0.04633 | 0.01153 | 3.032 e-3 |
| $5*10^7$ | 0.7496 | 0.1883 | 0.04655 | 0.01157 | 3.051 e-3 |
| $\mathbb{D}$ | 0.7500 | 0.1875 | 0.04688 | 0.01172 | 2.930 e-3 |

**Table 7** $\mathcal{T}_2$ of $\mathbb{Q}(\sqrt{l}), l \equiv 1 \pmod{8}$ and $l$ is a prime

| $B$ \ $G$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/32\mathbb{Z}$ | $\mathbb{Z}/64\mathbb{Z}$ |
|---|---|---|---|---|---|---|
| $10^7$ | 0.5002 | 0.2499 | 0.1245 | 0.06236 | 0.03169 | 0.01553 |
| $2*10^7$ | 0.5000 | 0.2499 | 0.1245 | 0.06255 | 0.03163 | 0.01567 |
| $3*10^7$ | 0.5005 | 0.2496 | 0.1246 | 0.06278 | 0.03115 | 0.01560 |
| $4*10^7$ | 0.5003 | 0.2496 | 0.1247 | 0.06278 | 0.03115 | 0.01564 |
| $5*10^7$ | 0.5001 | 0.2497 | 0.1247 | 0.06281 | 0.03116 | 0.01567 |
| $\mathbb{D}$ | 0.5000 | 0.2500 | 0.1250 | 0.06250 | 0.03125 | 0.01563 |

**Table 8** $\mathcal{T}_2$ of $\mathbb{Q}(\sqrt{l}), l \equiv 7 \pmod{8}$ and $l$ is a prime

| $B$ \ $G$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/32\mathbb{Z}$ | $\mathbb{Z}/64\mathbb{Z}$ | $\mathbb{Z}/128\mathbb{Z}$ |
|---|---|---|---|---|---|---|
| $10^7$ | 0.5000 | 0.2484 | 0.1260 | 0.06361 | 0.03103 | 0.01518 |
| $2*10^7$ | 0.5000 | 0.2494 | 0.1255 | 0.06265 | 0.03123 | 0.01534 |
| $3*10^7$ | 0.4998 | 0.2497 | 0.1252 | 0.06278 | 0.03109 | 0.01557 |
| $4*10^7$ | 0.4999 | 0.2497 | 0.1254 | 0.06246 | 0.03112 | 0.01570 |
| $5*10^7$ | 0.5001 | 0.2497 | 0.1254 | 0.06237 | 0.03116 | 0.01570 |
| $\mathbb{D}$ | 0.5000 | 0.2.500 | 0.1250 | 0.06250 | 0.03125 | 0.01563 |

**Table 9** $\mathcal{T}_2$ of $\mathbb{Q}(\sqrt{2l}), l \equiv 1 \pmod{8}$ and $l$ is a prime

| $B$ \ $G$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/32\mathbb{Z}$ | $\mathbb{Z}/64\mathbb{Z}$ |
|---|---|---|---|---|---|---|
| $10^7$ | 0.5006 | 0.2515 | 0.1237 | 0.06214 | 0.03100 | 0.01564 |
| $2*10^7$ | 0.5004 | 0.2511 | 0.1239 | 0.06219 | 0.03105 | 0.01576 |
| $3*10^7$ | 0.5001 | 0.2506 | 0.1245 | 0.06256 | 0.03093 | 0.01572 |
| $4*10^7$ | 0.5001 | 0.2505 | 0.1249 | 0.06233 | 0.03090 | 0.01564 |
| $5*10^7$ | 0.5000 | 0.2503 | 0.1252 | 0.06236 | 0.03083 | 0.01572 |
| $\mathbb{D}$ | 0.5000 | 0.2500 | 0.1250 | 0.06250 | 0.03125 | 0.01563 |

**Table 10** $\mathcal{T}_2$ of $\mathbb{Q}(\sqrt{2l}), l \equiv 7 \pmod 8$ and $l$ is a prime

| $B$ \ $G$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/32\mathbb{Z}$ | $\mathbb{Z}/64\mathbb{Z}$ |
|---|---|---|---|---|---|---|
| $10^7$ | 0.5000 | 0.2484 | 0.1253 | 0.06378 | 0.03129 | 0.01565 |
| $2*10^7$ | 0.5000 | 0.2.494 | 0.1253 | 0.06258 | 0.03137 | 0.01565 |
| $3*10^7$ | 0.4998 | 0.2497 | 0.1254 | 0.06258 | 0.03116 | 0.01575 |
| $4*10^7$ | 0.4999 | 0.2497 | 0.1252 | 0.06267 | 0.03129 | 0.01569 |
| $5*10^7$ | 0.5001 | 0.2497 | 0.1250 | 0.06268 | 0.03126 | 0.01573 |
| $\mathbb{D}$ | 0.5000 | 0.2500 | 0.1250 | 0.06250 | 0.03125 | 0.01563 |