MONSKY’S BOUND OF DE RHAM COHOMOLOGY

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ABSTRACT. This note is a detailed analysis of Monsky’s paper [2]. The aim is to give an explicit bound of the dimension of the De Rham cohomology $H^i_{dR}(k[x_1, \cdots, x_n|f])$ for $k$ an algebraic closed field of characteristic 0 and $f \in k[x_1, \cdots, x_n]$.

Let $k$ be an algebraic closed field of characteristic 0. Let $A$ be a finitely generated smooth algebra over $k$. One knows that $H^i_{dR}(A/k)$ is a finite dimensional $k$-vector space following the proof of Grothendieck. Monsky [2] gave another proof using local methods. This note is a detailed analysis of Monsky’s paper. The aim is to give an explicit bound of the dimension of the De Rham cohomology $H^i_{dR}(k[x_1, \cdots, x_n|f])$ for $f \in k[x_1, \cdots, x_n]$.

1. Finiteness for the homology of extended Koszul complex

Let $A$ be a vector space over $k$ and $D_i : A \to A$ ($1 \leq i \leq n$) be commuting $k$-linear maps. We recall that the Koszul complex $K^\bullet(A; D_1, \cdots, D_n)$ ($K^\bullet$ in short), is the graded group $A \otimes_k \Lambda^s(k^n)$, with the degree $-1$ differential given by

$$da(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{j=1}^s (-1)^{j-1} D_j(a)(e_{i_1} \wedge \cdots \hat{e}_{i_j} \cdots \wedge e_{i_s}).$$

The $i$-th homology group of $K^\bullet$ is denoted as $H_i(A; D_1, \cdots, D_n)$.

In this note, just as in [2], we are studying a special type of Koszul complex $K^\bullet(A; f; E_1, \cdots, E_n)$. Suppose $A$ is a $k$-algebra, let $E_1, \cdots, E_n$ be commuting $k$-linear derivations of $A$ and let $f \in A$. For every $i$, set

$$D_i : A \to A, \ g \mapsto E_i(g) + E_i(f)g$$

and set

$$K^\bullet(A; f; E_1, \cdots, E_n) := K^\bullet(A; D_1, \cdots, D_n).$$

Lemma 1.1. Let $R$ be a ring, $\epsilon \in R^*$ be a unit in $R$ and $m$ be a positive integer. Suppose for $1 \leq i \leq n$,

$$\phi_i : R[x_1, \cdots, x_n] \to R[x_1, \cdots, x_n]$$

are commuting $R$-linear maps such that

$$\phi_i(g) = \epsilon x_i^m \cdot g + \text{terms of lower degree}.$$

Then

1. $H_i(R[x_1, \cdots, x_n]; \phi_1, \cdots, \phi_n) = 0$ for $i > 0$.

2. $H_0(R[x_1, \cdots, x_n]; \phi_1, \cdots, \phi_n)$ is a free $R$-module of rank $m^n$ with a basis given by $\{x_i^j : 0 \leq i_j \leq m-1, 1 \leq j \leq n\}$, where $\overline{x}$ is the image of $x$ in $H_0$. 
We shall show that $K$.

For now on, we denote $A_n = k[x_1, \cdots, x_n]$ and

$A_{n+1}' = k[x_1, \cdots, x_n, x_{n+1}, x_{n+1}^{-1}] = A_n[x_{n+1}, x_{n+1}^{-1}]$.

Let $f \in A_n$ be a polynomial of $n$ variables. Suppose $\deg f = d > 0$. Set

$$f^* := f + x_{n+1} \sum_{i=1}^n x_i^{d+1} \in A_{n+1}.$$
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Proof. The first assertion is just Lemma 4 in §3 of [2], (1) also was proved there. (2) follows from Lemma 5 in §3 of [2]. The only thing we need to know is the dimension of $W_0$, for which Monsky gave a bound in Lemma 5 but there is a constant $c$ to be determined.

Recall that Monsky’s method is to use the standard Koszul exact sequence

$$\cdots \to W_{i+1} \to V_i \xrightarrow{\Delta_i} V_i \to W_i \to \cdots$$

where $V_i$ is given by Lemma 1.2 and $\Delta_i$ is induced by the operator $\phi_{n+1} : g \mapsto \frac{\partial g}{\partial x_{n+1}} + \frac{\partial f}{\partial x_n} : g$. Then

$$W_1 = \ker \Delta_0, \quad W_0 = \coker \Delta_0.$$

We only need to study $W_0$ here.

Let $O = k[x_{n+1}, x_n^{-1}]$. By Lemma 1.2, $V_0$ is a free $O$-module of rank $d^n$, with a basis given by $\{x_1^{n_1} \cdots x_n^{n_n} : 0 \leq i_j \leq d-1, 1 \leq j \leq n\}$ where $\bar{x}$ is the image of $x$ in $V_0$. For every $e_I = x_1^{n_1} \cdots x_n^{n_n}$, $u \in O$,

$$\Delta_0(u e_I) = \frac{\partial u}{\partial x_{n+1}} x_1^{n_1} \cdots x_n^{n_n} + \sum_{j=1}^{n} u \cdot x_1^{n_1} \cdots x_j^{n_j+1} \cdots x_n^{n_n}.$$  

It remains to know how to write $x_1^{n_1} \cdots x_n^{n_n}$ as a linear combination of the given basis. Note that for all $1 \leq j \leq n$, $i_j \geq 0$, and for any $l$ such that $1 \leq l \leq n$,

$$\phi_l(x_1^{n_1} \cdots x_n^{n_n}) = (d+1)x_{n+1} x_1^{n_1} \cdots x_l^{n_l+1} \cdots x_n^{n_n} + \frac{\partial f}{\partial x_l} x_1^{n_1} \cdots x_l^{n_l} x_n^{n_n} + i_l x_1^{n_1} \cdots x_l^{n_l-1} \cdots x_n^{n_n}.$$  

This shows that in $V_0$,

$$\frac{x_1^{n_1} \cdots x_l^{n_l+1} \cdots x_n^{n_n}}{(d+1)x_{n+1}} = -\frac{1}{(d+1)x_{n+1}} \left( \frac{\partial f}{\partial x_l} x_1^{n_1} \cdots x_l^{n_l} x_n^{n_n} + i_l x_1^{n_1} \cdots x_l^{n_l-1} \cdots x_n^{n_n} \right).$$

As $\deg f = d$, then $\deg \frac{\partial f}{\partial x_l} \leq d-1$. The total degree with respect to $x_1, \cdots x_n$ in the right hand side of the above identity is smaller than the total degree of the left hand side. Through this procedure, $x_1^{n_1} \cdots x_l^{n_l+1} \cdots x_n^{n_n}$ can be written as a linear combination of the given basis of $V_0$ in at most $n_1 + \cdots + n_d + (d-1) \leq n(d-1) + 1$ steps, therefore

$$\Delta_0(u e_I) = \frac{\partial u}{\partial x_{n+1}} e_I + u \sum_{J} c_{IJ} e_J$$

where $c_{IJ} \in O$ is a polynomial in $x_{n+1}$ whose degree is at most $(d-1)n + 1 = nd - n + 1$.

Now for $l, m \in \mathbb{N}$, let $U_{l,m}$ be the $k$-vector subspace of $V_0$ generated by $x_{n+1}^{e_I}$ where $-l \leq i \leq m$ and $e_I$ runs through the $O$-basis of $V_0$, then (1) show that $\Delta_0$ maps $U_{-j,j}$ to $U_{-j,(nd-n+1)+j}$. Now

$$\dim_k U_{-j,(nd-n+1)+j} - \dim U_{-j,j} + \dim W_1 \leq (nd - n + 1)d^n + d^n.$$  

The right hand side is a constant independent of $j$ and thus

$$\dim W_0 \leq (nd - n + 2)d^n.$$
Theorem 1.5. Let $d > 0$. We have
(1) $t_0(n, d) \leq (nd - n + 2)d^n$;
(2) $t_1(n, d) \leq d^n + t_0^2(n + 1, d + 2) \leq d^n + (nd + n + d + 3)(d + 2)^{n+1}$.
(3) For $i \geq 2$, $t_i(n, d) \leq h_{i-1}^*(n + 1, d + 2) \leq (d + 2i - 2)^{n+i-1} + t_0^2(n + i, d + 2i).

Proof. By the exact sequence of groups
$$0 \to A_{n+1} \to A_{n+1}' \to A_{n+1}'/A_{n+1} \to 0,$$
one has an exact sequence of Koszul complex and hence a long exact sequence of homology
$$\cdots \to W_i \to H_i(A_{n+1}'/A_{n+1}: f^*) \to H_{i-1}(A_{n+1}: f^*) \to W_{i-1} \to \cdots$$As $W_i = W_{i-1} = 0$ by Lemma 1.4 for $i > 2$, and $H_i(A_n; f) \cong H_i(A_{n+1}'/A_{n+1}: f^*)$ by Lemma 1.3, then we have
$$H_i(A_n; f) \cong H_{i-1}(A_{n+1}; f^*), \quad i > 2,$$and the exact sequence
$$0 \to H_2(A_n; f) \to H_1(A_{n+1}; f^*) \to W_1 \to$$$$H_1(A_n; f) \to H_0(A_{n+1}, f^*) \to W_0 \to H_0(A_n; f) \to 0.$$The theorem now follows from Lemma 1.4. □

2. The de Rham cohomology $H^i_{\text{DR}}((A_n)_f)$.

Let $f \in A_n$, the de Rham cohomology $H^i_{\text{DR}}((A_n)_f)$ is isomorphic with the homology group $H_{n-i}(K_n((A_n)_f; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})).$ We consider two cases:

(i) $f \in k^*$. In this case $(A_n)_f = A_n$. The Koszul complex $K_n(A_n; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ is the $n$-th tensor product $\bigotimes^n K_n(k[x_1]; \frac{\partial}{\partial x_1}).$ It is easy to see
$$H_1(K_n(k[x_1]; \frac{\partial}{\partial x_1})) = k, \quad H_i(K_n(k[x_1]; \frac{\partial}{\partial x_1})) = 0,$$hence
$$H_n(K_n(A_n; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) = k, \quad H_i(K_n(A_n; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) = k \ (i \neq n).$$

(ii) $\deg f = d > 0$. Then the exact sequence
$$0 \to A_n \to (A_n)_f \to (A_n)_f/A_n \to 0$$induces long exact sequence in homology. Note that
$$H_i(K_n((A_n)_f/A_n; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) \cong H_i(A_{n+1}; Tf; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial T})$$by Lemma 2.1 of [2], and by (i), we have
$$\dim_k H_i(K_n((A_n)_f; \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) \leq t_i(n + 1, d + 1) + 1.$$Thus if let
$$h_i(n, d) := \sup_{f \in A_n} \dim_k H^i_{\text{DR}}((A_n)_f),$$we have
Theorem 2.1. \( h_i(n, d) \leq t_{n-i}(n+1, d+1) + 1 \) where \( t_i(n, d) \) is bounded as given by Theorem 1.5.

Remark. One can see Katz [1] for more backgrounds and results about the sum of Betti numbers in arbitrary characteristic.

References