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Group cohomology of the universal ordinary distribution

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Abstract. For any odd squarefree integer r, we obtain a complete description of the $G_r = \mathrm{Gal}\big(\mathbb{Q}(\mu_r)/\mathbb{Q}\big)$ group cohomology of the universal ordinary distribution U_r in this paper. Moreover, for M a fixed integer dividing $\ell - 1$ for all prime factors ℓ of r, we study the cohomology group $H^*(G_r, U_r/MU_r)$. In particular, we explain the construction of the elements $\kappa_{r'}$ for $r'|r$ in Rubin [9], which come exactly from a certain $\mathbb{Z}/M\mathbb{Z}$-basis of the cohomology group $H^0(G_r, U_r/MU_r)$ through an evaluation map.

1. Introduction

Let $\{[a] : a \in \mathbb{Q}/\mathbb{Z}\}$ be a basis for a free abelian group $\mathbf{A}$. Then the (dimension 1) universal ordinary distribution U_r of level r for any positive integer r is given by

$$U_r = \frac{\left\langle [a] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \right\rangle}{\left\langle [a] - \sum_{\ell b = a} [b] : \ell | r \text{ prime}, a \in \frac{\ell}{r}\mathbb{Z}/\mathbb{Z} \right\rangle}.$$

For any $\sigma \in G_r := \mathrm{Gal}\big(\mathbb{Q}(\mu_r)/\mathbb{Q}\big)$ and $x \in \mathbb{Z}$, if $\sigma(\zeta) = \zeta^x$ for all $\zeta \in \mu_r$, set $\sigma([a]) = [xa]$. By this action, U_r becomes a G_r-module. The universal distribution is well known to be a free abelian group, moreover, for any integer $r'|r$, the natural map from $U_{r'}$ to U_r is a split monomorphism and thus $U_{r'}$ can be considered as a submodule of U_r.

The theory of the universal distribution plays an important role in the theory of cyclotomic fields. Detailed information can be found in the well-known textbooks by Lang [8] and Washington [14]. Most notably, Kubert [6] and [7], and Sinnott [13] studied the $\{\pm 1\}$-cohomology of U_r, from which Sinnott got his famous index formula about the cyclotomic units and the Stickelberger ideal.

Recently, Anderson [2] found a brand new way to compute the $\{\pm 1\}$-cohomology of the universal distribution U_r. He discovered a cochain complex which is a resolution of the universal distribution. To study a certain group cohomology of U_r, one studies instead a double complex related to this group cohomology. In this paper, we use Anderson's resolution to construct a double complex related to the G_r-cohomology of U_r and study the spectral sequence of the double complex. Supposing that r is some fixed odd squarefree integer, we prove the following theorem:

Theorem A (Abridged form). *The cohomology group*

$$H^n(G_r, U_r) = \bigoplus_{r'|r} H_{r'}^{n+n_{r'}}(G_r, \mathbb{Z})$$

where $n_{r'} =$ *number of prime factors of* r' *and*

$$H_{r'}^n(G_r, \mathbb{Z}) := \bigcap_{\ell|r'} \ker\bigl(H^n(G_r, \mathbb{Z}) \xrightarrow{\text{res}} H^n(G_{r/\ell}, \mathbb{Z})\bigr),$$

for $G_{r/\ell}$ *viewed as a subgroup of* G_r. *In particular, in the case* $n = 0$, *we have*

$$H^0(G_r, U_r) = \mathbb{Z};$$

in the case $n = 1$, *we have*

$$H^1(G_r, U_r) = \prod_{r'|r} \mathbb{Z}/m_{r'}\mathbb{Z}$$

where $m_{r'} = \gcd\{\ell - 1 : \ell | r'\}$.

We shall discuss the Unabridged form in §5. What's more, for any positive integer M which is a common factor of $\ell - 1$ over all prime factors ℓ of r, let σ_ℓ be a generator of the cyclic group G_ℓ and let

$$D_{r'} := \prod_{\ell|r'} \sum_{k=0}^{\ell-2} k\sigma_\ell^k,$$

then

Theorem B. *The image of the family*

$$\left\{ D_{r'} \left[\sum_{\ell|r'} \frac{1}{\ell} \right] : \forall r'|r \right\}$$

in U_r/MU_r *is a* $\mathbb{Z}/M\mathbb{Z}$*-basis for* $H^0(G_r, U_r/MU_r)$.

Theorem B has interesting applications in arithmetic. We follow the line given in Rubin [9]. Let $\mathbb{F} = \mathbb{Q}(\mu_m)^+$ be the maximal real subfield of $\mathbb{Q}(\mu_m)$, assume $\{\ell : \ell | r\}$ is a family of distinct odd primes which split completely in $\mathbb{F}/\mathbb{Q}$ and are $\equiv 1 \pmod{M}$ for a fixed integer M. Suppose that we have a G_r-homomorphism ξ from U_r to $\mathbb{F}(\mu_r)^\times$. Then ξ induces a map

$$H^n(\xi)\colon H^n(G_r, U_r/MU_r) \to H^n\big(G_r, \mathbb{F}(\mu_r)^\times/\mathbb{F}(\mu_r)^{\times M}\big)$$

for each $n \in \mathbb{Z}_{\geqq 0}$. In the case $n = 0$, since $H^0\big(G_r, \mathbb{F}(\mu_r)^\times/\mathbb{F}(\mu_r)^{\times M}\big) = \mathbb{F}^\times/\mathbb{F}^{\times M}$, we have the map

$$H^0(\xi)\colon H^0(G_r, U_r/MU_r) \to \mathbb{F}^\times/\mathbb{F}^{\times M}.$$

In particular, let $\mathbb{Q}^{\mathrm{ab}}$ be the abelian closure of $\mathbb{Q}$. Let $\mathbf{e}$ be an injective homomorphism from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}^{\mathrm{ab}\times}$. Put

$$\xi([a]) = \left(\mathbf{e}\left(a + \frac{1}{m}\right) - 1\right)\left(\mathbf{e}\left(a - \frac{1}{m}\right) - 1\right).$$

Then ξ is a G_r-homomorphism from U_r to $\mathbb{F}(\mu_r)^\times$. The image $H^0(\xi)\left(D_{r'}\left[\sum_{\ell | r'} \frac{1}{\ell}\right]\right)$ is just the Kolyvagin element $\kappa_{r'}$ as given in [9]. From this point of view, we can regard the Euler system as a system in the cohomology group $H^0(G_r, U_r/MU_r)$. This is the initial motivation for this paper.

This paper is organized in the following order. We give general notation in §2. In §3, we study Anderson's resolution in detail. In §4, a special G_r-projective resolution $\mathbf{P}_\bullet$ of $\mathbb{Z}$ is constructed and the group cohomology of $\mathbb{Z}$ and of $\mathbb{Z}/M\mathbb{Z}$ are given. With this projective resolution $\mathbf{P}_\bullet$, we construct a double complex in §5 whose total cohomology is the G_r-cohomology of U_r. The standard spectral sequence method is then used to compute the cohomology group $H^*(G_r, U_r)$. In §6, we study the lifting problem and prove Theorem B. At the end of this paper, we include an appendix by Prof. Anderson on his resolution.

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2. Notation

Fix a finite set S of cardinality $|S| = s$. Fix a family $\{\ell_i\colon i \in S\}$ of distinct odd prime numbers. Fix a positive integer M dividing $\ell_i - 1$ for all $i \in S$. Fix a total order ω of S. Put

- $r = r_S := \prod_{i \in S} \ell_i$,
- $G_S := \mathrm{Gal}\big(\mathbb{Q}(\mu_r)/\mathbb{Q}\big)$.

For each $i \in S$, put

- $G_i :=$ the inertia subgroup of G_S at ℓ_i,
- $\sigma_i :=$ a fixed generator of G_i,

- $N_i := \sum_{k=0}^{\ell_i - 2} \sigma_i^k$, $D_i := \sum_{k=0}^{\ell_i - 2} k\sigma_i^k$,
- $Fr_i :=$ the arithmetic Frobenius automorphism at ℓ_i in G_S/G_i.

For each subset $T \subseteqq S$, put

- $r_T := \prod_{i \in T} \ell_i$, $\mu_T := \mu_{r_T}$,
- $G_T := \prod_{i \in T} G_i \subset G_S$,
- $N_T := \prod_{i \in T} N_i$, $D_T := \prod_{i \in T} D_i$.

Put $R := \mathbb{Z}_{\geqq 0}[S]$. For any element $e = (e_i) \in R$, put

- $\deg e := \sum_i e_i$,
- $\operatorname{supp} e := \{i \in S : e_i \neq 0\}$,
- $\omega(e) := \big(\omega(e)_i\big) \in R$ where $\omega(e)_i = \sum_{j <_\omega i} e_j$.

For any $e, e' \in R$, put $\omega(e, e') := \sum_{j <_\omega i} e'_j e_i$.

For $a \in \mathbb{Q}/\mathbb{Z}$, the order of a (denoted by $\operatorname{ord} a$) means its order in $\mathbb{Q}/\mathbb{Z}$. For any set X, the cardinality of X is denoted by $|X|$, the free abelian group generated by X is denoted by $\langle X \rangle$, and the free $\mathbb{Z}/M\mathbb{Z}$-module generated by X is denoted by $\langle X \rangle_M$. The family of all subsets of X is denoted by 2^X. We call a subfamily $\mathscr{I}$ of 2^X an *order ideal* of X if for all $Y \in \mathscr{I}$, $2^Y \subseteqq \mathscr{I}$. For any pair of sets X and Y, the difference of X and Y is denoted by $X \backslash Y$.

For any complex $C^\bullet$, the complex $C^\bullet[n]$ is the complex with components $C^m[n] = C^{m+n}$. For any complex $C^\bullet$ of $\mathbb{Z}$-modules, $C^\bullet_M := C^\bullet \otimes \mathbb{Z}/M\mathbb{Z}$.

3. Universal ordinary distribution and its structure

3.1. Universal ordinary distribution and Anderson's resolution. Let $\{[a] : a \in \mathbb{Q}/\mathbb{Z}\}$ be a basis of a free abelian group $\mathbf{A}$. Recall that by Kubert [6], the (rank 1) universal ordinary distribution U is given by

$$U = \frac{\langle [a] : a \in \mathbb{Q}/\mathbb{Z} \rangle}{\left\langle [a] - \sum_{nb=a} [b] : n \in \mathbb{N} \right\rangle}.$$

For any positive number f, the universal ordinary distribution of level f is given by

$$U_f = \frac{\left\langle [a] : a \in \frac{1}{f}\mathbb{Z}/\mathbb{Z} \right\rangle}{\left\langle [a] - \sum_{pb=a} [b] : p | f, a \in \frac{p}{f}\mathbb{Z}/\mathbb{Z} \right\rangle}.$$

For any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})$, set $\sigma([a]) = [xa]$ if σ sends each f-th root of unity to its x-th power. By this action, U_f is a $G_f = \mathrm{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})$-module. Much has been studied about the structures of U and U_f, we list some basic properties here (for detailed proof, see Anderson [1], [2], Kubert [6], [7] and Washington [14]). First recall for any $a \in \frac{1}{f}\mathbb{Z}/\mathbb{Z}$, a can be written uniquely as

$$a \equiv \sum_{p|f} \sum_{v \in \mathbb{N}} \frac{a_{pv}}{p^v} \pmod{\mathbb{Z}}, \quad 0 \leqq a_{pv} \leqq p-1.$$

Then

Proposition 3.1. 1) *The universal ordinary distribution U_f is a free abelian group of rank $|G_f|$, the set $\{[a] : a \in \frac{1}{f}\mathbb{Z}/\mathbb{Z}, a_{p1} \neq p-1, \forall p|f\}$ is a $\mathbb{Z}$-basis for U_f.*

2) *For any factor g of f, the natural map from U_g to U_f is a split monomorphism. Moreover, by these natural maps, U is the direct limit of U_f for $f \in \mathbb{N}$ and thus U is free.*

In the sequel, for our convenience, the universal distribution U_r will be written as U_S and U_{r_T} as U_T. Now let

$$\mathbf{L}_S^{\bullet} = \left\langle [a, T] : a \in \frac{r_T}{r_S}\mathbb{Z}/\mathbb{Z}, T \subseteq S \right\rangle$$

be the free abelian group generated by the symbols $[a, T]$, and let

$$L_S^p = \left\langle [a, T] : |T| = -p, a \in \frac{r_T}{r_S}\mathbb{Z}/\mathbb{Z}, T \subseteq S \right\rangle,$$

then $\mathbf{L}_S^{\bullet}$ is a bounded graded module. Furthermore, for any $\sigma \in G_S$, set $\sigma[a, T] = [xa, T]$ if σ sends each r-th root of unity to its x-th power. By this action, $\mathbf{L}_S^{\bullet}$ becomes a G_S-module. Let

$$d[a, T] = \sum_{i \in T} \omega(i, T) \left([a, T \backslash \{i\}] - \sum_{\ell_i b = a} [b, T \backslash \{i\}] \right)$$

where

$$\omega(i, T) = \begin{cases} (-1)^{|\{j \in T : j <_\omega i\}|}, & \text{if } i \in T, \\ 0, & \text{if } i \notin T. \end{cases}$$

It is easy to check that $d^2 = 0$ and d is G_S-equivariant. Thus $(\mathbf{L}_S^{\bullet}, d)$ is a cochain complex. Note that the definition of d depends on ω. We'll write d_ω instead if we need to emphasize the order ω. The following proposition is given by Anderson:

Proposition 3.2. *The n-th cohomology of the complex $(\mathbf{L}_S^{\bullet}, d)$ is 0 for $n \neq 0$ and U_S for $n = 0$, furthermore, the map from L_S^0 to U_S is given by* $\mathfrak{u}$: $[a, \emptyset] \mapsto [a]$.

Remarks. 1. The above proposition (in a more general form suitable for a resolution for the distribution U_f for general f), though known by Anderson for quite a while, has no published proof by now. We put the proof in Appendix A, traces of the idea behind the proof can be found in Anderson [1], [2].

2. For the sake of this proposition, we call $(\mathbf{L}_S^\bullet, d)$ Anderson's resolution of the universal distribution U_S. This resolution has been used by Das [5] in his work about the algebraic Γ-monomials and double coverings of cyclotomic fields.

3.2. Double complex structure of $\mathbf{L}_S^\bullet$. A remarkable fact about Anderson's resolution $\mathbf{L}_S^\bullet$ is that it possesses an even more delicate double complex structure, which in turn gives a natural filtration for the universal distribution U_S. We start with a more careful look at $\mathbf{L}_S^\bullet$, which we'll denote by $\mathbf{L}^\bullet$ instead. For any $T \subseteqq S$, we always regard $\mathbf{L}_T^\bullet$ as a subcomplex of $\mathbf{L}^\bullet$. Moreover, for any order ideal $\mathscr{I}$ of S, put

$$\mathbf{L}^\bullet(\mathscr{I}) := \sum_{T \in \mathscr{I}} \mathbf{L}_T^\bullet \quad \text{and} \quad U_S(\mathscr{I}) := \sum_{T \in \mathscr{I}} U_T.$$

In particular, let $\mathscr{I}(n)$ be the order ideal consisting of all subsets T such that $|T| \leqq n$, and let $\mathbf{L}^\bullet(n) = \mathbf{L}^\bullet(\mathscr{I}(n))$ and $U_S(n) = U_S(\mathscr{I}(n))$. Note that $\mathbf{L}^\bullet(2^T) = \mathbf{L}_T^\bullet$. For any $a \in \frac{1}{r_S}\mathbb{Z}/\mathbb{Z}$, let

$$\operatorname{supp} a := \{i : \ell_i | \operatorname{ord} a\} \subseteqq S.$$

We see that

$$\mathbf{L}^\bullet = \langle [a, T] : \operatorname{supp} a \cap T = \emptyset \rangle.$$

Then $\mathbf{L}^\bullet(\mathscr{I})$ is the free abelian group generated by

$$\{[a, T] : T \cup \operatorname{supp} a \in \mathscr{I}, T \cap \operatorname{supp} a = \emptyset\},$$

and $U_S(\mathscr{I})$ is the free abelian group generated by

$$\{[a] : \operatorname{supp} a \in \mathscr{I}, a_{\ell_i 1} \neq \ell_i - 1 \text{ for all } i \in S\}.$$

Immediately we have

Proposition 3.3. *Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be two order ideals of S, then*:

1) $\mathbf{L}^\bullet(\mathscr{I}_1 \cap \mathscr{I}_2) = \mathbf{L}^\bullet(\mathscr{I}_1) \cap \mathbf{L}^\bullet(\mathscr{I}_2)$, $U_S(\mathscr{I}_1 \cap \mathscr{I}_2) = U_S(\mathscr{I}_1) \cap U_S(\mathscr{I}_2)$.

2) $\mathbf{L}^\bullet(\mathscr{I}_1 \cup \mathscr{I}_2) = \mathbf{L}^\bullet(\mathscr{I}_1) + \mathbf{L}^\bullet(\mathscr{I}_2)$, $U_S(\mathscr{I}_1 \cup \mathscr{I}_2) = U_S(\mathscr{I}_1) + U_S(\mathscr{I}_2)$.

Then

Proposition 3.4. *The complex $\mathbf{L}^\bullet(\mathscr{I})$ is acyclic with the* 0*-cohomology $U_S(\mathscr{I})$.*

Proof. We let $\tilde{\mathbf{L}}^\bullet(\mathscr{I})$ be the complex

$$0 \to L^{-s}(\mathscr{J}) \to \cdots \to L^0(\mathscr{J}) \xrightarrow{u} U_S(\mathscr{J}) \to 0.$$

Hence it suffices to show that $\tilde{\mathbf{L}}^\bullet(\mathscr{J})$ is exact. Let T be a maximal element in the order ideal $\mathscr{J}$. Let $\mathscr{J}'$ be the order ideal whose maximal element set is the set of maximal elements of $\mathscr{J}$ excluding T, then

$$\mathscr{J} = \mathscr{J}' \cup 2^T.$$

By Proposition 3.3, we have

$$\tilde{\mathbf{L}}^\bullet(\mathscr{J})/\tilde{\mathbf{L}}^\bullet(2^T) = \tilde{\mathbf{L}}^\bullet(\mathscr{J}')/\tilde{\mathbf{L}}^\bullet(\mathscr{J}' \cap 2^T).$$

Now we prove the proposition by induction on the cardinality of maximal elements of $\mathscr{J}$. If $\mathscr{J}$ has only one maximal element, this is just Proposition 3.2. In general, both $\mathscr{J}'$ and $\mathscr{J}' \cap 2^T$ have less maximal elements than $\mathscr{J}$ has. Thus the exactness of $\tilde{\mathbf{L}}^\bullet(\mathscr{J})$ follows from the exactness of the three complexes $\tilde{\mathbf{L}}^\bullet(2^T)$, $\tilde{\mathbf{L}}^\bullet(\mathscr{J}')$ and $\tilde{\mathbf{L}}^\bullet(\mathscr{J}' \cap 2^T)$. $\square$

Now we can construct a double complex whose total single complex is $(\mathbf{L}^\bullet, d)$. With abuse of notation, we'll write it as $\mathbf{L}^{\bullet,\bullet}$. For any pair of subsets T', T of S such that $T' \supseteq T$, set

$$L_{T',T} := \langle [a,T] : \operatorname{supp} a = S \backslash T' \rangle,$$

then $L_{T',T}$ is isomorphic to $\operatorname{Ind}_{G_{T'}}^{G_S} \mathbb{Z}$. Moreover, for any $i \in T$, the map

$$\varphi_i \colon L_{T',T} \to L_{T',T\backslash\{i\}}, \quad [a,T] \mapsto [a, T\backslash\{i\}]$$

defines a natural isomorphism between $L_{T',T}$ and $L_{T',T\backslash\{i\}}$. Now for any $T \subseteq S$,

$$\mathbf{L}_T^\bullet = \bigoplus_{T_1,T_2} L_{T_1,T_2}, \quad \text{where } T_2 \cup (S\backslash T_1) \subseteq T,\, T_2 \cap (S\backslash T_1) = \emptyset,$$

and if let $\Gamma(\mathscr{J}) := \{(T_1,T_2) : T_2 \cup (S\backslash T_1) \in \mathscr{J},\, T_2 \cap (S\backslash T_1) = \emptyset\}$, then

$$\mathbf{L}^\bullet(\mathscr{J}) = \bigoplus_{(T_1,T_2)\in\Gamma(\mathscr{J})} L_{T_1,T_2}.$$

In general for any $i \in S$, define

$$\varphi_i \colon L^p \to L^{p+1}, \quad [a,T] \mapsto \chi_T(i)[a, T\backslash\{i\}]$$

where χ_T is the characteristic function of T. Let $\varphi(L^p)$ be the subgroup of L^{p+1} generated by $\varphi_i(L^p)$ for all $i \in S$, inductively, let $\varphi^n(L^p)$ be the subgroup of L^{p+n} generated by $\varphi_i(\varphi^{n-1}(L^p))$ for all $i \in S$. By this setup, there is a filtration of L^p given by

$$\varphi^{s+p}(L^{-s}) \subseteq \varphi^{s+p-1}(L^{-s+1}) \subseteq \cdots \subseteq L^p.$$

This filtration enables us to define the double complex structure of $\mathbf{L}^\bullet$. For the element $[a,T] \in \mathbf{L}^\bullet$, we say $[a,T]$ is of bidegree (p_1,p_2) if $[a,T] \in \varphi^{p_2}(L^{p_1})\backslash\varphi^{p_2+1}(L^{p_1-1})$, more explicitly, if

$$p_1 = |\operatorname{supp} a| - s, \quad p_2 = s - |\operatorname{supp} a| - |T|.$$

Then we see that all elements of $L_{T',T}$ are of bidegree $(-|T'|, |T'| - |T|)$. Let L^{p_1,p_2} be the subgroup of $\mathbf{L}^\bullet$ generated by all symbols $[a, T]$ with bidegree (p_1, p_2), then

$$L^{p_1,p_2} = \bigoplus_{|T|=-p_1-p_2} \bigoplus_{\substack{|T'|=-p_1 \\ T' \supseteqq T}} L_{T',T}.$$

Set

$$d_1 \colon L^{p_1,p_2} \to L^{p_1+1,p_2}, \quad [a, T] \mapsto -\sum_{i\in T} \omega(i, T) N_i \left[Fr_i^{-1} a + \frac{1}{\ell_i}, T \backslash \{i\} \right],$$

$$d_2 \colon L^{p_1,p_2} \to L^{p_1,p_2+1}, \quad [a, T] \mapsto \sum_{i\in T} \omega(i, T)(1 - Fr_i^{-1})[a, T\backslash\{i\}],$$

It is easy to check that $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$. Hence we construct a double complex $(\mathbf{L}^{\bullet,\bullet}; d_1, d_2)$. Note that $d = d_1 + d_2$ and

$$L^p = \bigoplus_{p_1+p_2=p} L^{p_1,p_2},$$

thus $(\mathbf{L}^\bullet, d)$ is the single total complex of the double complex $(\mathbf{L}^{\bullet,\bullet}; d_1, d_2)$, with the second filtration given by φ. Thus the total cohomology of $(\mathbf{L}^{\bullet,\bullet}; d_1, d_2)$ is the cohomology of $(\mathbf{L}^\bullet, d)$.

Proposition 3.5. *The E_1 term of the spectral sequence arising from the double complex $(\mathbf{L}^{\bullet,\bullet}; d_1, d_2)$ by the first filtration (i.e., $H_{d_1}^{p_1}(\mathbf{L}^{\bullet,p_2})$) is*

$$E_1^{p_1,p_2} = \begin{cases} U_S(s-p_2)/U_S(s-p_2-1), & \text{if } p_1 = -p_2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the spectral sequence for the first filtration degenerates at E_1.

Proof. Note that

$$\mathbf{L}^\bullet(n) = \bigoplus_{p_2 \geqq s-n} L^{p_1,p_2}$$

then it is easy to see that $\mathbf{L}^{\bullet,p_2}[-p_2]$ is nothing but the quotient complex $\mathbf{L}^\bullet(s-p_2)/\mathbf{L}^\bullet(s-p_2-1)$. The short exact sequence

$$0 \to \mathbf{L}^\bullet(s-p_2-1) \to \mathbf{L}^\bullet(s-p_2) \to \mathbf{L}^{\bullet,p_2}[-p_2] \to 0$$

induces a long exact sequence

$$\cdots \to H^i\big(\mathbf{L}^\bullet(s-p_2)\big) \to H^i(\mathbf{L}^{\bullet,p_2}[-p_2]) \to H^{i+1}\big(\mathbf{L}^\bullet(s-p_2-1)\big) \to \cdots .$$

By Proposition 3.4, for $i \neq 0$ and -1, both $H^i\big(\mathbf{L}^\bullet(s-p_2)\big)$ and $H^{i+1}\big(\mathbf{L}^\bullet(s-p_2+1)\big)$ are 0, so is $H^i(\mathbf{L}^{\bullet,p_2}[-p_2])$. Therefore the above long exact sequence is just the exact sequence

$$0 \to H^{-1}(\mathbf{L}^{\bullet,p_2}[-p_2]) \to U_S(s-p_2-1) \to U_S(s-p_2) \to H^0(\mathbf{L}^{\bullet,p_2}[-p_2]) \to 0.$$

Since the map from $U_S(s-p_2-1)$ to $U_S(s-p_2)$ is injective, the proposition follows immediately. $\square$

Remark. It is an interesting problem to investigate the spectral sequence coming from the second filtration of $\mathbf{L}^{\bullet,\bullet}$.

Now tensoring $\mathbf{L}^\bullet$ with $\mathbb{Z}/M\mathbb{Z}$, since $\mathbf{L}^\bullet$ is a resolution of free abelian groups, by Proposition 3.1, Proposition 3.2 and Proposition 3.4, we have

Proposition 3.6. 1) *One has*

$$H^n(\mathbf{L}_M^\bullet) = \begin{cases} U_S/MU_S, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

2) *Moreover, for any order ideal $\mathscr{J}$ of S, one has*

$$H^n\big(\mathbf{L}_M^\bullet(\mathscr{J})\big) = \begin{cases} U_S(\mathscr{J})/MU_S(\mathscr{J}), & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

4. The cohomology groups $H^*(G_T, \mathbb{Z})$ and $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$

4.1. A projective resolution of $\mathbb{Z}$. We first have a convention here: Let X be a finite totally ordered set and $x \in X$. Suppose that to every $x \in X$ we have a module A_x associated to x. We call

$$A_X = A_{x_1} \otimes \cdots \otimes A_{x_n}$$

the standard tensor product of A_x over X if $X = \{x_1, \ldots, x_n\}$ and $x_1 < \cdots < x_n$. Similarly, we can define the standard tensor product of elements $a_x \in A_x$ and of complexes $A_x^\bullet$.

Let

$$(\mathbf{P}_{i\bullet}, \partial_i)\colon \cdots \xrightarrow{\partial_{i,j+1}} P_{i,j+1} \xrightarrow{\partial_{ij}} P_{ij} \cdots \xrightarrow{\partial_{i0}} P_{i0} \longrightarrow 0$$

with $P_{ij} = \mathbb{Z}[G_i]$ for any $j \geqq 0$, ∂_{ij} is the multiplication by $1 - \sigma_i$ if j is even and by N_i if j is odd. It is well known that $\mathbf{P}_{i\bullet}$ is a $\mathbb{Z}[G_i]$-projective resolution of the trivial module $\mathbb{Z}$. For any $T \subseteqq S$, let $\mathbf{P}_{T\bullet}$ be the standard tensor product of $\mathbf{P}_{i\bullet}$ over $i \in T$. It is well known by homological algebra that $\mathbf{P}_{T\bullet}$ is a $\mathbb{Z}[G_T]$-projective resolution of the trivial module $\mathbb{Z}$. Now for the collection $\{P_{i,e_i}\colon i \in T\}$, the standard product of P_{i,e_i} over T is a rank 1 free $\mathbb{Z}[G_T]$-module whose grade is $\sum_i e_i$. Now let $e \in R$ be the element whose i-th component is e_i if $i \in T$ and 0 if not, and write the standard product of P_{i,e_i} over T as $\mathbb{Z}[G_T][e]$, then

$$\mathbf{P}_{T\bullet} = \bigoplus_{\operatorname{supp} e \subseteqq T} \mathbb{Z}[G_T][e].$$

For any $x = (\cdots \otimes x_i \otimes \cdots) \in \mathbb{Z}[G_T][e]$, the differential is given by

$$\partial_T(x) = \sum_{i \in T} (-1)^{\omega(e)_i} \big(\cdots \otimes \partial_{i, e_i - 1}(x_i) \otimes \cdots\big).$$

In particular, for $T = S$ let

$$\mathbf{P}_\bullet = \mathbf{P}_{S\bullet} = \bigoplus_{e \in R} \mathbb{Z}[G_S][e].$$

For any $T' \subseteqq T$, we have a natural inclusion $\iota: \mathbb{Z}[G_{T'}][e] \hookrightarrow \mathbb{Z}[G_T][e]$ for any $e \in R$ such that $\operatorname{supp} e \subseteqq T'$. By this inclusion, $\mathbf{P}_{T'\bullet}$ becomes a subcomplex of $\mathbf{P}_{T\bullet}$.

Now we define a diagonal map $\Phi_T: \mathbf{P}_{T\bullet} \to \mathbf{P}_{T\bullet} \otimes \mathbf{P}_{T\bullet}$. First set

$$\Phi_{ie_i, ie_i'}: P_{i, e_i + e_i'} \to P_{ie_i} \otimes P_{ie_i'},$$

$$1 \mapsto \begin{cases} 1 \otimes 1, & \text{if } e_i \text{ even}, \\ 1 \otimes \sigma_i, & \text{if } e_i \text{ odd}, e_i' \text{ even}, \\ \displaystyle\sum_{0 \leqq m < n \leqq \ell_i - 2} \sigma_i^m \otimes \sigma_i^n, & \text{if } e_i \text{ odd}, e_i' \text{ odd}. \end{cases}$$

Then the map $\Phi_i: \mathbf{P}_{i\bullet} \to \mathbf{P}_{i\bullet} \otimes \mathbf{P}_{i\bullet}$ given by $\Phi_{ie_i, ie_i'}$ is the diagonal map for the cyclic group G_i (see Cartan-Eilenberg [4], p. 250–252). For any $e, e' \in R$ with support contained in T, consider the standard product $P_{e,e'}$ of $P_{ie_i} \otimes P_{ie_i'}$ over $i \in T$. The isomorphism

$$\alpha: P_{ie_i} \otimes P_{je_j'} \to P_{je_j'} P_{ie_i},$$

$$x \otimes y \mapsto (-1)^{e_i e_i'} y \otimes x$$

induces an isomorphism $\alpha: P_{e,e'} \to \mathbb{Z}[G_T][e] \otimes \mathbb{Z}[G_T][e']$ by

$$\big(\cdots (x_i \otimes y_i) \cdots\big) \mapsto (-1)^{\omega(e,e')} (\cdots x_i \cdots) \otimes (\cdots y_i \cdots).$$

On the other hand, the standard product of the diagonal maps $\Phi_{ie_i, ie_i'}$ over $i \in T$ defines a map $\beta: \mathbb{Z}[G_T][e + e'] \to P_{e,e'}$. We let $\Phi_{e,e'} = \alpha \circ \beta$ and let

$$\Phi_{T,p,q} = \sum_{\substack{e, e' : \deg e = p, \deg e' = q \\ \operatorname{supp} e + e' \subseteqq T}} \Phi_{e,e'}.$$

Then Φ_T defines the diagonal map from $\mathbf{P}_{T\bullet}$ to $\mathbf{P}_{T\bullet} \otimes \mathbf{P}_{T\bullet}$. This map enables us to compute the cup product structures.

4.2. The cohomology groups $H^*(G_T, \mathbb{Z})$ and $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$. Let

$$\mathbf{C}_i^\bullet = \operatorname{Hom}_{G_i}(\mathbf{P}_{i\bullet}, \mathbb{Z}),$$

then $\mathbf{C}_i^\bullet$ is the complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\ell_i - 1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\ell_i - 1} \cdots$$

with the initial term at degree 0. We denote by C_i^j the j-th term of $\mathbf{C}_i^\bullet$. By the theory of group cohomology,

$$H^*(G_i, \mathbb{Z}) = H^*(\mathbf{C}_i^\bullet).$$

Now for any $T \subseteqq S$, let $\mathbf{C}_T^\bullet$ be the standard tensor product of $\mathbf{C}_i^\bullet$ for $i \in T$. If write

$$\mathrm{Hom}_{G_T}(\mathbb{Z}[G_T][e], \mathbb{Z}) = \mathbb{Z}[e],$$

then

$$\mathbf{C}_T^\bullet = \mathrm{Hom}_{G_T}(\mathbf{P}_{T\bullet}, \mathbb{Z}) = \bigoplus_{\operatorname{supp} e \subseteqq T} \mathbb{Z}[e],$$

and

$$H^*(G_T, \mathbb{Z}) = H^*(\mathbf{C}_T^\bullet).$$

Moreover, for any $T' \subseteqq T$, the inclusion $\iota\colon \mathbf{P}_{T'\bullet} \hookrightarrow \mathbf{P}_{T\bullet}$ induces a map

$$\iota^*\colon \mathbf{C}_T^\bullet \to \mathbf{C}_{T'}^\bullet,$$

which is just the natural projection of

$$\bigoplus_{\operatorname{supp} e \subseteqq T} \mathbb{Z}[e] \to \bigoplus_{\operatorname{supp} e \subseteqq T'} \mathbb{Z}[e].$$

On the other hand, $G_{T'}$ can also be considered naturally as a quotient group of G_T, by this meaning, the inflation map is just the injection

$$\bigoplus_{\operatorname{supp} e \subseteqq T'} \mathbb{Z}[e] \hookrightarrow \bigoplus_{\operatorname{supp} e \subseteqq T} \mathbb{Z}[e].$$

Now for any $j \in \mathbb{Z}_{\geqq 0}$ even, let

$$\mathbf{C}_i^{\bullet j} = \begin{cases} \cdots 0 \longrightarrow C_i^0 \longrightarrow 0 \cdots, & \text{if } j = 0, \\ \cdots 0 \longrightarrow C_i^{j-1} \xrightarrow{\ell_i - 1} C_i^j \longrightarrow 0 \cdots, & \text{if } j > 0. \end{cases}$$

For any $e = (e_i) \in 2R$, i.e., e_i even for all $i \in S$, we let $\mathbf{C}_e^\bullet$ be the standard product $\mathbf{C}_i^{\bullet e_i}$ over $i \in S$. If $\operatorname{supp} e \subseteqq T$, then $\mathbf{C}_e^\bullet$ is a subcomplex of $\mathbf{C}_T^\bullet$ and

$$\mathbf{C}_T^\bullet = \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \subseteqq T}} \mathbf{C}_e^\bullet.$$

Figure 1 shows us what the decomposition looks like in the case $S = \{1, 2\}$. Denote by A_e the cohomology group $H^*(\mathbf{C}_e^\bullet)$ and A_e^n its n-th component. Then

$$H^*(G_T, \mathbb{Z}) = \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \subseteqq T}} A_e, \quad H^n(G_T, \mathbb{Z}) = \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \subseteqq T}} A_e^n.$$

$$\begin{array}{ccc}
\mathbf{C}^\bullet_{(0,4)}\Big| & \boxed{\mathbf{C}^\bullet_{(2,4)}} & \boxed{\mathbf{C}^\bullet_{(4,4)}} \\[2em]
\mathbf{C}^\bullet_{(0,2)}\Big| & \boxed{\mathbf{C}^\bullet_{(2,2)}} & \boxed{\mathbf{C}^\bullet_{(4,2)}} \\[2em]
\mathbf{C}^\bullet_{(0,0)}\cdot & \overline{\mathbf{C}^\bullet_{(2,0)}} & \overline{\mathbf{C}^\bullet_{(4,0)}}
\end{array}$$

Figure 1. The complex $\mathbf{C}^\bullet_S$ when $S = \{1, 2\}$.

We now study the abelian group A_e. First we need a lemma from linear algebra:

Lemma 4.1. *Let $v = (m_1, m_2, \dots, m_n)^t$ be an n-dimensional column vector with integer entries m_i, then the greatest common divisor of the m_i is 1 if and only if there exists an $n \times n$ matrix $A \in SL_n(\mathbb{Z})$ whose first column is v.*

Now suppose $\operatorname{supp} e = T = \{i_1, \dots, i_t\}$ and $|T| = t$. If $t = 0$, then $T = \emptyset$, it is easy to see that $A_e = A_e^0 = \mathbb{Z}$. Now if $T \neq \emptyset$, we claim that $\mathbf{C}_e^\bullet[\deg e - t]$ is isomorphic to the exterior algebra $\Lambda(x_1, \dots, x_t)$ with differential $d(x) = \sum(\ell_i - 1)x_i \wedge x$ and $\deg x_i = 1$. This claim is easy to check: First if $t = 1$, let $T = \{i\}$, then $\mathbf{C}_i^{\bullet e_i} = C^{e_i - 1} \oplus C^{e_i}$. This case is trivial. In general, if $\mathbf{C}_i^{\bullet e_i}[e_i - 1]$ is isomorphic to $\Lambda(x_i)$, the tensor product of $\mathbf{C}_i^{\bullet e_i}[e_i - 1]$ is nothing but $\mathbf{C}_e^\bullet[\deg e - t]$ and the tensor product of $\Lambda(x_i)$ is just $\Lambda(x_1, \dots, x_t)$, hence they are isomorphic to each other.

Now let m_T be the greatest common divisor of $\ell_i - 1$ for $i \in T$, thus the greatest common divisor of $(\ell_i - 1)/m_T$ is 1, let A be the matrix given by Lemma 4.1 corresponding to the vector $(\dots, (\ell_i - 1)/m_T, \dots)$. Let $(y_1, \dots, y_t) = (x_1, \dots, x_t)A$. Then $\{y_1, \dots, y_t\}$ is a set of new generators for the above exterior algebra and we have $d(x) = m_T y_1 \wedge x$. We see easily that

$$H^*\big(\Lambda(x_1, \dots, x_t)\big) = (\mathbb{Z}/m_T\mathbb{Z})^{2^{t-1}}$$

and

$$H^j\big(\Lambda(x_1, \dots, x_t)\big) = (\mathbb{Z}/m_T\mathbb{Z})^{\binom{t-1}{j}}, \quad 0 \leqq j \leqq t - 1.$$

Combining the above analysis, we have

Proposition 4.2. *There exists a family of complexes*

$$\{\mathbf{C}_e^\bullet \subseteqq \mathbf{C}^\bullet = \operatorname{Hom}_{G_S}(\mathbf{P}_\bullet, \mathbb{Z}) : e \in 2R\}$$

such that:

1) *For each* $T \subseteqq S$, *we can identify* $\mathbf{C}_T^\bullet = \mathrm{Hom}_{G_T}(\mathbf{P}_{T\bullet}, \mathbb{Z})$ *with* $\bigoplus_{\substack{e \in 2R \\ \mathrm{supp}\, e \subseteqq T}} \mathbf{C}_e^\bullet$ *through the following splitting exact sequence*:

$$0 \to \bigoplus_{\substack{e \in 2R \\ \mathrm{supp}\, e \nsubseteq T}} \mathbf{C}_e^\bullet \to \mathbf{C}^\bullet \to \mathbf{C}_T^\bullet \to 0.$$

2) *The cohomology groups* $H^*(\mathbf{C}_e^\bullet) = A_e$ *and* $H^n(\mathbf{C}_e^\bullet) = A_e^n$ *are given by*:

(a) *If* $\mathrm{supp}\, e \neq \emptyset$, *let* m_e *be the greatest common divisor of* $\ell_i - 1$ *for* $i \in \mathrm{supp}\, e$, *then* A_e *is the abelian group* $(\mathbb{Z}/m_e\mathbb{Z})^{2^{|\mathrm{supp}\, e|-1}}$, *and*

$$A_e^n = \begin{cases} (\mathbb{Z}/m_e\mathbb{Z})^{\binom{|\mathrm{supp}\, e|-1}{j}}, & \text{if } n = \deg e - j \text{ and } 0 \leqq j \leqq |\mathrm{supp}\, e| - 1, \\ 0, & \text{otherwise}. \end{cases}$$

(b) *If* $\mathrm{supp}\, e = \emptyset$, *then* $A_e = A_e^0 = \mathbb{Z}$.

For the case $H^*(G, \mathbb{Z}/M\mathbb{Z})$, the situation is much easier. We have

Proposition 4.3. *There exists a family*

$$\{[e] \in H^*(G_S, \mathbb{Z}/M\mathbb{Z}) : e \in R\}$$

with the following properties:

1) *For each* $T \subseteqq S$ *and* $n \in \mathbb{Z}_{\geqq 0}$, *the restriction of the family*

$$\{[e] : e \in R, \mathrm{supp}\, e \subseteqq T, \deg e = n\}$$

to $H^n(G_T, \mathbb{Z}/M\mathbb{Z})$ *is a* $\mathbb{Z}/M\mathbb{Z}$*-basis of the latter.*

2) *For each* $T \subseteqq S$ *and* $e \in R$ *such that* $\mathrm{supp}\, e \nsubseteq T$, *the restriction of* $[e]$ *to* $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$ *vanishes.*

3) *One has the cup product structure in* $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$ *given by*

$$[e] \cup [e'] = (-1)^{\omega(e,e')} \prod_{\substack{i \in S \\ e_i e_i' \equiv 1(2)}} \left(\frac{\ell_i - 1}{2}\right) [e + e']$$

for all $e, e' \in R$.

Proof. The complex $\mathbf{C}_{M,i}^\bullet = \mathrm{Hom}_{G_i}(\mathbf{P}_{i\bullet}, \mathbb{Z}/M\mathbb{Z})$ by definition, is a complex with $\mathbf{C}_{M,i}^j = \mathbb{Z}/M\mathbb{Z}$ for $j \geqq 0$ and the differential 0. In general, $\mathbf{C}_{M,T}^\bullet = \mathbf{C}_T^\bullet \otimes \mathbb{Z}/M\mathbb{Z}$ is exactly the standard tensor product of $C_{M,i}^\bullet$ for all $i \in T$. Write

$$\mathbf{C}_{M,T}^\bullet = \mathrm{Hom}_{G_i}(\mathbf{P}_{T\bullet}, \mathbb{Z}/M\mathbb{Z}) = \sum_{\mathrm{supp}\, e \subseteqq T} \mathbb{Z}/M\mathbb{Z}[e].$$

Since now $\mathbf{C}_{M,T}^\bullet$ has differential 0, $H^*(\mathbf{C}_{M,T}^\bullet) = \mathbf{C}_{M,T}^\bullet$. The restriction map is easy to see. This finishes the proof of 1) and 2).

For the cup product, the diagonal map Φ_T given above naturally induces a map:

$$\mathbf{C}^{\bullet}_{M,T} \times \mathbf{C}^{\bullet}_{M,T} \to \mathbf{C}^{\bullet}_{M,T}$$

which defines the cup product structure. More specifically, the cup product map

$$\mathbb{Z}/M\mathbb{Z}[e] \times \mathbb{Z}/M\mathbb{Z}[e'] \to \mathbb{Z}/M\mathbb{Z}[e+e']$$

is induced from $\Phi_{e,e'}$. Now the claim follows quickly from the explicit expression of $\Phi_{e,e'}$. $\square$

5. Study of $H^*(G_S, U_S)$

5.1. The complex K. With the preparation in §3 and §4, set

$$\mathbf{K}^{\bullet,\bullet} := \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, \mathbf{L}^\bullet).$$

Let d and δ be the differentials of $\mathbf{K}^{\bullet,\bullet}$ induced by the differentials of d of $\mathbf{L}^\bullet$ and ∂ by of $\mathbf{P}_\bullet$ respectively. If we let

$$[a, T, e] := ([e] \mapsto [a, T]) \in \mathrm{Hom}_{G_S}(P_e, \langle [a, T] \rangle),$$

then

$$K^{p,q} = \left\langle [a, T, e] : a \in \frac{r_T}{r_S}\mathbb{Z}/\mathbb{Z}, |T| = -p, \deg e = q \right\rangle;$$

$$d[a, T, e] = \sum_{i \in T} \omega(i, T)\left([a, T\backslash\{i\}, e] - \sum_{\ell_i b = a} [b, T\backslash\{i\}, e]\right);$$

$$\delta[a, T, e] = (-1)^{|T|} \sum_{i \in S} (-1)^{\omega(e)_i} \cdot \begin{cases} (1-\sigma_i)[a, T, e+\varepsilon_i], & \text{if } e_i \text{ even,} \\ N_i[a, T, e+\varepsilon_i], & \text{if } e_i \text{ odd.} \end{cases}$$

For any $T \subseteqq S$, set

$$\mathbf{K}^{\bullet,\bullet}(T) = \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, \mathbf{L}^\bullet_T) = \langle [a, T', e] : [a, T'] \in \mathbf{L}^\bullet_T, e \in R \rangle$$

and

$$\mathbf{K}^{\bullet,\bullet}_T = \mathrm{Hom}_{G_T}(\mathbf{P}_{T\bullet}, \mathbf{L}^\bullet_T) = \langle [a, T', e] : [a, T'] \in \mathbf{L}^\bullet_T, e \in R, \operatorname{supp} e \subseteqq T \rangle.$$

Furthermore, for any order ideal $\mathscr{J}$, set

$$\mathbf{K}^{\bullet,\bullet}(\mathscr{J}) := \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, \mathbf{L}^\bullet(\mathscr{J})) = \sum_{T \in \mathscr{J}} \mathbf{K}^{\bullet,\bullet}(T),$$

and set

$$\mathbf{K}^{\bullet,\bullet}(n) := \mathrm{Hom}_{G_S}\big(\mathbf{P}_\bullet, \mathbf{L}^\bullet(n)\big).$$

Set

$$\mathbf{U}^\bullet := \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, U_S) = \frac{\left\langle [a,e] : a \in \frac{1}{r_S}\mathbb{Z}/\mathbb{Z}, e \in R \right\rangle}{\left\langle [a,e] - \sum_{\ell_i b = a} [b,e] : a \in \frac{\ell_i}{r_S}\mathbb{Z}/\mathbb{Z}, e \in R \right\rangle},$$

with differential δ induced by ∂. Correspondingly,

$$\mathbf{U}^\bullet(\mathscr{J}) := \frac{\left\langle [a,e] : a \in \frac{1}{r_T}\mathbb{Z}/\mathbb{Z} \text{ for some } T \in \mathscr{J}, e \in R \right\rangle}{\left\langle [a,e] - \sum_{\ell_i b = a} [b,e] : a \in \frac{\ell_i}{r_T}\mathbb{Z}/\mathbb{Z} \text{ for some } T \in \mathscr{J}, e \in R \right\rangle},$$

which is a subcomplex of $\mathbf{U}^\bullet$. We consider $\mathbf{U}^\bullet$ as the double complex $(\mathbf{U}^{\bullet,\bullet}; 0, \delta)$ concentrated on the vertical axis. From Proposition 3.2, we have a map

$$\mathfrak{u}\colon \mathbf{K}^{\bullet,\bullet} \to \mathbf{U}^{\bullet,\bullet}, \quad [a,T,e] \mapsto \begin{cases} [a,e], & \text{if } T = \emptyset, \\ 0, & \text{if } T \neq \emptyset. \end{cases}$$

Proposition 5.1. *The map* $\mathfrak{u}$ (*resp. its restriction*) *is a quasi-isomorphism between* $\mathbf{K}^{\bullet,\bullet}$ (*resp.* $\mathbf{K}^{\bullet,\bullet}(\mathscr{J})$) *and* $\mathbf{U}^{\bullet,\bullet}$ (*resp.* $\mathbf{U}^{\bullet,\bullet}(\mathscr{J})$). *Therefore*:

1) $H^*_{\mathrm{total}}(\mathbf{K}^{\bullet,\bullet}) = H^*(G_S, U_S)$, $H^*_{\mathrm{total}}\big(\mathbf{K}^{\bullet,\bullet}(\mathscr{J})\big) = H^*\big(G_S, U_S(\mathscr{J})\big)$.

2) $H^*_{\mathrm{total}}(\mathbf{K}^{\bullet,\bullet}_M) = H^*(G_S, U_S/MU_S)$, $H^*_{\mathrm{total}}\big(\mathbf{K}^{\bullet,\bullet}_M(\mathscr{J})\big) = H^*\big(G_S, U_S(\mathscr{J})/MU_S(\mathscr{J})\big)$.

Proof. Immediately from Proposition 3.2 (resp. Proposition 3.4 for $\mathscr{J}$), we see that $\ker \mathfrak{u}$ is d-acyclic, and hence by spectral sequence argument, it is $(d+\delta)$-acyclic. On the other hand, $\mathfrak{u}$ is surjective. Thus $\mathfrak{u}$ is a quasi-isomorphism. Now 1) follows directly from the quasi-isomorphism. For 2), just consider $\mathfrak{u} \otimes 1$, which is also a quasi-isomorphism. □

From Proposition 5.1, the G_S-cohomology of U_S is isomorphic to the total cohomology of the double complex $(\mathbf{K}^{\bullet,\bullet}; d, \delta)$. Therefore we can use the spectral sequence of the double complex $\mathbf{K}^{\bullet,\bullet}$ to study the G_S-cohomology of U_S. The spectral sequence of $\mathbf{K}^{\bullet,\bullet}$ from the second filtration has given us Proposition 5.1. Now we study the spectral sequence from the first filtration. Then

$$E_1^{p,q}(\mathbf{K}^{\bullet,\bullet}) = H^q_\delta(\mathbf{K}^{p,\bullet}) = H^q(G_S, L^p).$$

Now since

$$L^p = \bigoplus_{p_1+p_2=p} L^{p_1,p_2} = \bigoplus_{|T|=-p} \bigoplus_{T' \supseteq T} L_{T',T},$$

then

$$E_1^{p,q}(\mathbf{K}^{\bullet,\bullet}) = \bigoplus_{|T|=-p} \bigoplus_{T' \supseteqq T} H^q(G_S, L_{T',T}).$$

Recall

$$\Gamma(\mathscr{J}) = \{(T_1, T_2) : T_2 \cup (S \backslash T_1) \in \mathscr{J}, T_2 \cap (S \backslash T_1) = \emptyset\}$$

from §3. We have

$$E_1^{p,q}\big(\mathbf{K}^{\bullet,\bullet}(\mathscr{J})\big) = \bigoplus_{(T',T) \in \Gamma(\mathscr{J})} H^q(G_S, L_{T',T}).$$

5.2. A Lemma. Suppose that for any $T \subseteqq S$, there is an abelian group B_T associated to T, and set

$$A_T = \bigoplus_{T'' \subseteqq T} B_{T''}.$$

Then for any $T' \supseteqq T$, there is a natural projection from $A_{T'}$ to A_T. Now let $\mathscr{C}_{S,T}^{\bullet}$ be the cochain complex with components given by

$$\mathscr{C}_{S,T}^{n} = \bigoplus_{\substack{|T'|=s-n \\ T' \supseteqq (S \backslash T)}} A_{T'},$$

and differential d given by

$$\begin{aligned} d\colon A_{T'} &\to \bigoplus_{i \in T' \cap T} A_{T' \backslash \{i\}}, \\ x &\mapsto \sum_{i \in T' \cap T} \omega(i, T' \cap T) x|_{T' \backslash \{i\}}, \end{aligned}$$

where $x|_{T' \backslash \{i\}}$ is the projection of x in $A_{T' \backslash \{i\}}$. It is easy to verify that $\mathscr{C}_{S,T}^{\bullet}$ is indeed a chain complex. Furthermore, we have

Lemma 5.2. *For any $T \subseteqq S$,*

$$H^n(\mathscr{C}_{S,T}^{\bullet}, d) = \begin{cases} \bigoplus_{T' \supseteqq T} B_{T'}, & \textit{if } n = 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Let $\tilde{\mathscr{C}}_{S,T}^{\bullet}$ be the subcomplex of $\mathscr{C}_{S,T}^{\bullet}$ with the same components as $\mathscr{C}_{S,T}^{\bullet}$ except at degree 0, where

$$\tilde{\mathscr{C}}_{S,T}^{0} = \bigoplus_{T' \not\supseteq T} B_{T'}.$$

We only need to show that $\tilde{\mathscr{C}}_{S,T}^{\bullet}$ is exact. We show it by double induction to the cardinalities of S and T. If $T = \emptyset$, we get a trivial complex. If S consists of only one element, or if T consists only one element, it is also trivial to verify. In general, suppose $i_0 = \max\{i : i \in T\}$. Let $S_0 = S \backslash \{i_0\}$ and $T_0 = T \backslash \{i_0\}$. Then we have the following commutative diagram which is exact on the columns:

$$
\begin{tikzcd}
0 \arrow[r] & \tilde{\mathscr{C}}^0_{S,T_0} \arrow[r, "\bar{d}"] & \tilde{\mathscr{C}}^1_{S,T_0} \arrow[r, "\bar{d}"] & \cdots \tilde{\mathscr{C}}^{t-1}_{S,T_0} \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & \tilde{\mathscr{C}}^0_{S,T} \arrow[r, "d"] \arrow[u, "p"] & \tilde{\mathscr{C}}^1_{S,T} \arrow[r, "d"] \arrow[u, "p"] & \cdots \tilde{\mathscr{C}}^{t-1}_{S,T} \arrow[r, "d"] \arrow[u, "p"] & \tilde{\mathscr{C}}^t_{S,T} \arrow[r] \arrow[u, "p"] & 0 \\
0 \arrow[r] & \bigoplus_{S_0 \supseteqq T' \supseteqq T_0} B_{T'} \arrow[r, "d"] \arrow[u, "i"] & A_{S_0} \arrow[r, "d"] \arrow[u, "i"] & \cdots \tilde{\mathscr{C}}^{t-1}_{S_0,T_0} \arrow[r, "d"] \arrow[u, "i"] & \tilde{\mathscr{C}}^t_{S_0,T_0} \arrow[r] \arrow[u, "i"] & 0.
\end{tikzcd}
$$

Here p means projection and i means inclusion. The differential $\bar{d}$ is induced by the differential d of the second row. Notice that the third row is a variation of the chain complex $\tilde{\mathscr{C}}^{\bullet}_{S_0,T_0}$, the first row is the chain complex $\tilde{\mathscr{C}}^{\bullet}_{S,T_0}$. By induction, the first row and and the third row are exact, so is the middle one. $\square$

5.3. The study of E_2 terms. By §5.1, we know that

$$E_1^{p,q}(\mathbf{K}^{\bullet,\bullet}) = \bigoplus_{|T|=-p} \bigoplus_{T' \supseteqq T} H^q(G_S, L_{T',T}).$$

Now let's consider the induced differential $\bar{d}$ of d in the E_1 term. Since $d = d_1 + d_2$, we can also write $\bar{d} = \bar{d}_1 + \bar{d}_2$. We first look at $\bar{d}_2$, which is induced by the map

$$L_{T',T} \to \bigoplus_{i \in T} L_{T',T \backslash \{i\}},$$

$$[a, T] \mapsto \sum_{i \in T} \omega(i, T)(1 - Fr_i^{-1})[a, T \backslash \{i\}].$$

Since for any $i \in T$, $L_{T',T}$ and $L_{T',T\backslash\{i\}}$ are G_S-isomorphic by the map φ_i, and since for any $q \geqq 0$, $H^q(G, A)$ is a trivial G-module, we have

$$\bar{d}_2 = \sum_{i \in T} \omega(i, T)(1 - Fr_i^{-1})\bar{\varphi}_i = 0.$$

The map $\bar{d}_1$ is induced by the map

$$L_{T',T} \to \bigoplus_{i \in T} L_{T'\backslash\{i\},T\{i\}},$$

$$[a, T] \mapsto -\sum_{i \in T} \omega(i, T) N_i \left[Fr_i^{-1} a + \frac{1}{\ell_i}, T \backslash \{i\} \right].$$

For any $i \in T$, consider the map

$$\psi_i\colon L_{T',T} \to L_{T'\setminus\{i\},T\setminus\{i\}},$$
$$[a,T] \mapsto N_i\left[Fr_i^{-1}a + \frac{1}{\ell_i}, T\setminus\{i\}\right].$$

The map ψ_i is a G_S-homomorphism and therefore induces a map in G_S-cohomology:

$$H^q(\psi_i)\colon H^q(G_S, L_{T',T}) \to H^q(G_S, L_{T'\setminus\{i\},T\setminus\{i\}}).$$

We have the commutative diagram:

$$\begin{array}{ccc} L_{T',T} & \xrightarrow{\psi_i} & L_{T'\setminus\{i\},T\setminus\{i\}} \\ \Big\downarrow{\scriptstyle \theta_{T'}} & & \Big\downarrow{\scriptstyle \theta_{T'\setminus\{i\}}} \\ \mathbb{Z} & \xrightarrow{\text{res}} & \mathbb{Z} \end{array}$$

where the top row are G_S-modules, the left $\mathbb{Z}$ is a trivial $G_{T'}$-module, the right $\mathbb{Z}$ is a trivial $G_{T'\setminus\{i\}}$-module, and $\theta_{T'}$ is the homomorphism sending $\left[\frac{1}{r_{S\setminus T'}}, T\right]$ to 1 and $\left[\frac{x}{r_{S\setminus T'}}, T\right]$ to 0 if $x \neq 1$. Then the above diagram induces the following commutative diagram:

$$\begin{array}{ccc} H^q(G_S, L_{T',T}) & \xrightarrow{H^q(\psi_i)} & H^q(G_S, L_{T'\setminus\{i\},T\setminus\{i\}}) \\ \Big\downarrow{\scriptstyle \theta^*_{T'}} & & \Big\downarrow{\scriptstyle \theta^*_{T'\setminus\{i\}}} \\ H^q(G_{T'}, \mathbb{Z}) & \xrightarrow{\text{res}} & H^q(G_{T'\setminus\{i\}}, \mathbb{Z}) \end{array}$$

where $\theta^*_{T'}$ (and $\theta^*_{T'\setminus\{i\}}$) is the isomorphism given by Shapiro's lemma (see Serre [11], Chap. VII, §5, Exercise). We identify $H^q(G_S, L_{T',T})$ with $H^q(G_{T'}, \mathbb{Z})$, moreover, to keep track of T, we'll write $H^q(G_{T'}, \mathbb{Z})$ as $H^q(G_{T',T}, \mathbb{Z})$. Then we see that $H^q(\psi_i)$ is the restriction map from $H^q(G_{T',T}, \mathbb{Z})$ to $H^q(G_{T'\setminus\{i\},T\setminus\{i\}}, \mathbb{Z})$. The induced differential $\bar{d} = \bar{d}_1$ is exactly the map

$$H^q(G_{T',T}, \mathbb{Z}) \to \bigoplus_{i\in T} H^q(G_{T'\setminus\{i\},T\setminus\{i\}}, \mathbb{Z}),$$
$$x \mapsto -\sum_{i\in T} \omega(i,T)x_i,$$

where x_i is the restriction of x in $H^q(G_{T'\setminus\{i\},T\setminus\{i\}}, \mathbb{Z})$. Hence we have a cochain complex $\mathscr{C}(q; S, T)$

$$H^q(G_{S,T}, \mathbb{Z}) \xrightarrow{\bar{d}_1} \bigoplus_{i\in T} H^q(G_{S\setminus\{i\},T\setminus\{i\}}, \mathbb{Z}) \cdots \xrightarrow{\bar{d}_1} H^q(G_{S\setminus T,\emptyset}, \mathbb{Z}) \longrightarrow 0.$$

Note that the complex $E_1^{\bullet,q}(\mathbf{K}^{\bullet,\bullet})$ is just the direct sum of $\mathscr{C}(q; S, T)$ over all subsets T of S. Moreover, the complex $E_1^{\bullet,q}(\mathbf{K}^{\bullet,\bullet})(\mathscr{J})$ is the direct sum of $\tilde{\mathscr{C}}(q; S, T)$ over all subsets $T \in \mathscr{J}$.

Recall in Proposition 4.2, we obtained

$$H^q(G_T, \mathbb{Z}) = \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \subseteqq T}} A_e^q.$$

If let

$$A_T^q = H^q(G_T, \mathbb{Z}), \quad B_T^q = \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e = T}} A_e^q,$$

then we have $A_T^q = \bigoplus_{T'' \subseteqq T} B_{T''}^q$. The complex $\mathscr{C}(q; S, T)[-|T|]$ satisfies the conditions in Lemma 5.2, thus the n-th cohomology of the cochain complex $\mathscr{C}(q; S, T)$ is 0 if $n \neq -|T|$ and $\sum_{T' \supseteqq T} B_{T'}$ if $n = -|T|$. We have the following proposition:

Proposition 5.3. *One has*:

1) $E_2^{p,q}(\mathbf{K}^{\bullet,\bullet}) \cong \bigoplus_{|T|=-p} \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \supseteqq T}} A_e^q.$

2) $E_2^{p,q}\big(\mathbf{K}^{\bullet,\bullet}(\mathscr{I})\big) \cong \bigoplus_{\substack{|T|=-p \\ T \in \mathscr{I}}} \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \supseteqq T}} A_e^q.$

5.4. Proof of Theorem A. Finally we are in a position to prove Theorem A. Put

$$\mathbf{S}^{\bullet,\bullet} = \langle [a, T, e] \in \mathbf{K}^{\bullet,\bullet}, a \neq 0 \text{ if } \operatorname{supp} e \supseteqq T \rangle.$$

It is easy to verify that $\mathbf{S}^{\bullet,\bullet}$ is a subcomplex of $\mathbf{K}^{\bullet,\bullet}$ using the explicit formulas for d and δ given in §5.1. Set

$$\mathbf{Q}^{\bullet,\bullet} = \mathbf{K}^{\bullet,\bullet}/\mathbf{S}^{\bullet,\bullet} = \langle [0, T, e] : \operatorname{supp} e \supseteqq T \rangle.$$

Note that the differential of $\mathbf{Q}^{\bullet,\bullet}$ induced by d is 0. Moreover, set

$$\mathbf{S}^{\bullet,\bullet}(\mathscr{I}) := \mathbf{K}^{\bullet,\bullet}(\mathscr{I}) \cap \mathbf{S}^{\bullet,\bullet},$$

and

$$\mathbf{Q}^{\bullet,\bullet}(\mathscr{I}) := \mathbf{K}^{\bullet,\bullet}(\mathscr{I})/\mathbf{S}^{\bullet,\bullet}(\mathscr{I}) = \langle [0, T, e] : T \in \mathscr{I}, \operatorname{supp} e \supseteqq T \rangle.$$

Let f be the corresponding quotient map, then we have a commutative diagram:

$$\begin{array}{ccc}
\mathbf{K}_M^{\bullet,\bullet}(\mathscr{I}) & \xrightarrow{\text{inc}} & \mathbf{K}_M^{\bullet,\bullet} \\
\Big\downarrow f & & \Big\downarrow f \\
\mathbf{Q}_M^{\bullet,\bullet}(\mathscr{I}) & \xrightarrow{\text{inc}} & \mathbf{Q}_M^{\bullet,\bullet}.
\end{array}$$

We make the following claim:

Proposition 5.4. *The quotient map* $f\colon \mathbf{K}^{\bullet,\bullet} \to \mathbf{Q}^{\bullet,\bullet}$ *is a quasi-isomorphism. Moreover, the quotient map* $f\colon \mathbf{K}^{\bullet,\bullet}(\mathscr{J}) \to \mathbf{Q}^{\bullet,\bullet}(\mathscr{J})$ *is a quasi-isomorphism.*

Proof. Let

$$\mathscr{L}_T^\bullet := \langle [0,T,e] : e \in R \rangle = \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, L_{S,T}) \subseteqq \mathbf{K}^{\bullet,\bullet}$$

and let

$$\mathscr{L}_T'^\bullet := \langle [0,T,e] : \operatorname{supp} e \supseteqq T \rangle, \quad \mathscr{L}_T''^\bullet := \langle [0,T,e] : \operatorname{supp} e \subseteqq S\backslash\{i\},\ \text{for some } i \in T \rangle.$$

Through the map $L_{S,T} \to \mathbb{Z}$, $[0,T] \mapsto 1$, we have a commutative diagram

$$\begin{array}{ccccc}
\mathscr{L}_T^\bullet & = & \mathscr{L}_T'^\bullet & \oplus & \mathscr{L}_T''^\bullet \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{C}^\bullet & = & \bigoplus\limits_{\substack{e \in 2R \\ \operatorname{supp} e \supseteqq T}} \mathbf{C}_e^\bullet & \oplus & \bigoplus\limits_{\substack{i \in T, e \in 2R \\ \operatorname{supp} e \subseteqq S\backslash\{i\}}} \mathbf{C}_e^\bullet
\end{array}$$

where $\mathbf{C}^\bullet$ and $\mathbf{C}_e^\bullet$ are given in Proposition 4.2. By this diagram, we identify $\mathscr{L}_T^\bullet$ with $\mathbf{C}^\bullet$. By Proposition 4.2, we have

$$\ker\bigl(H^*(G_S,\mathbb{Z}) \to H^*(G_{S\backslash\{i\}},\mathbb{Z})\bigr) = H^*\Biggl(\bigoplus_{\substack{e \in 2R \\ i \in \operatorname{supp} e}} \mathbf{C}_e^\bullet\Biggr).$$

Then by the proof of Proposition 5.3,

$$\begin{aligned}
\ker(\bar{d}|_{H^q(\mathscr{L}_T^\bullet)}) &= \bigcap_{i \in T} \ker\bigl(H^*(G_S, L_{S,T}) \to H^*(G_S, L_{S\backslash\{i\}, T\backslash\{i\}})\bigr) \\
&= \bigcap_{i \in T} H^*\Biggl(\bigoplus_{\substack{e \in 2R \\ i \in \operatorname{supp} e}} \mathbf{C}_e^\bullet\Biggr) = H^*\Biggl(\bigcap_{i \in T} \bigoplus_{\substack{e \in 2R \\ i \in \operatorname{supp} e}} \mathbf{C}_e^\bullet\Biggr) \\
&= H^*\Biggl(\bigoplus_{\substack{e \in 2R \\ T \subseteqq \operatorname{supp} e}} \mathbf{C}_e^\bullet\Biggr) = H^*(\mathscr{L}_T'^\bullet)
\end{aligned}$$

where the second and the last identifications are made using the isomorphisms given in the commutative diagram above. Hence we have

$$E_2^{p,q}(\mathbf{K}^{\bullet,\bullet}) = \bigoplus_{|T|=-p} \ker(\bar{d}|_{H^q(\mathscr{L}_T^\bullet)}) = \bigoplus_{|T|=-p} H^q(\mathscr{L}_T'^\bullet).$$

On the other hand,

$$\mathbf{Q}^{\bullet,\bullet} = \bigoplus_{T \subseteqq S} \mathscr{L}_T'^\bullet.$$

Since $d = 0$ in $\mathbf{Q}^{\bullet,\bullet}$, the spectral sequence of $\mathbf{Q}^{\bullet,\bullet}$ by the first filtration (i.e., by d) degenerates at E_1. We have

$$E_1^{p,q}(\mathbf{Q}^{\bullet,\bullet}) = E_2^{p,q}(\mathbf{Q}^{\bullet,\bullet}) = \bigoplus_{|T|=-p} H^q(\mathscr{L}_T'^{\bullet}).$$

Since the projection map from $\mathscr{L}_T^{\bullet}$ to $\mathscr{L}_T'^{\bullet}$ in the commutative diagram above is nothing but the restriction of the quotient map f at $\mathscr{L}_T^{\bullet}$, by the above analysis, we get an isomorphism

$$f_2 \colon E_2^{p,q}(\mathbf{K}^{\bullet,\bullet}) \to E_2^{p,q}(\mathbf{Q}^{\bullet,\bullet}).$$

Thus the spectral sequence of $\mathbf{K}^{\bullet,\bullet}$ and $\mathbf{Q}^{\bullet,\bullet}$ are isomorphic at E_r for $r \geqq 2$. In our case, the first filtration is finite, thus strongly convergent, therefore f is a quasi-isomorphism (see Cartan-Eilenberg [4], page 322, Theorem 3.2).

The case $\mathscr{I}$ is similar. In this case,

$$E_2^{p,q}(\mathbf{K}^{\bullet,\bullet})(\mathscr{I}) = \bigoplus_{\substack{T \in \mathscr{I} \\ |T|=-p}} \ker(\bar{d}|_{H^q(\mathscr{L}_T^{\bullet})}) = \bigoplus_{|T|=-p} H^q(\mathscr{L}_T'^{\bullet}),$$

and

$$\mathbf{Q}^{\bullet,\bullet}(\mathscr{I}) = \bigoplus_{T \in \mathscr{I}} \mathscr{L}_T'^{\bullet}.$$

Now follow the same analysis as above. $\square$

For any subset T of S, set

$$H_T^*(G_S, \mathbb{Z}) := \bigcap_{i \in T} \ker\big(H^*(G_S, \mathbb{Z}) \xrightarrow{\text{res}} H^*(G_{S \setminus \{i\}}, \mathbb{Z})\big)$$

we see that

$$H^*(\mathscr{L}_T'^{\bullet}) \cong H_T^*(G_S, \mathbb{Z})$$

by the identification of $\mathscr{L}_T^{\bullet}$ and $\mathbf{C}^{\bullet}$. The following theorem is the main result in the paper:

Theorem A (Unabridged form). 1) *The cohomology group* $H^*(G_S, U_S)$ *is given by*

$$H^*(G_S, U_S) = \bigoplus_{T \subseteqq S} H_T^*(G_S, \mathbb{Z})[|T|] = \bigoplus_{T \subseteqq S} \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \supseteqq T}} A_e[|T|],$$

where $A_e[|T|]$ *represents the cohomology group* $H^*(\mathbf{C}_e^{\bullet}[|T|])$.

2) *The cohomology group* $H^*\big(G_S, U_S(\mathscr{I})\big)$ *is given by*

$$H^*\big(G_S, U_S(\mathscr{I})\big) = \bigoplus_{T \in \mathscr{I}} H_T^*(G_S, \mathbb{Z})[|T|] = \bigoplus_{T \in \mathscr{I}} \bigoplus_{\substack{e \in 2R \\ \operatorname{supp} e \supseteqq T}} A_e[|T|].$$

Proof. We only prove 1). The proof of 2) follows the same route. By Proposition 5.1 and Proposition 5.4, we know that

$$H^*(G_S, U_S) = H^*_{\mathrm{total}}(\mathbf{K}^\bullet) = H^*_{\mathrm{total}}(\mathbf{Q}^\bullet).$$

Now

$$H^n_{\mathrm{total}}(\mathbf{Q}^\bullet) = \bigoplus_{T \subseteqq S} H^{n+|T|}(\mathscr{L}'^\bullet_T).$$

Part 1) follows immediately. □

Remarks. 1) We can see that part 1) is actually a special case of part 2) when the order ideal $\mathscr{I}$ is 2^S.

2) By Theorem A, in the case $n = 0$, we have

$$H^0(G_S, U_S) = \mathbb{Z};$$

in the case $n = 1$, we have

$$H^1(G_S, U_S) = \prod_{T \subseteqq S} \mathbb{Z}/m_T\mathbb{Z}$$

where $m_T = \gcd(\ell_i - 1 : i \in T)$ as given in §4. It is likely the cohomology classes in $H^1(G_S, U_S)$ have a natural role to play in the Euler system method, but this role has not yet been worked out in detail.

In the case $\mathbb{Z}/M\mathbb{Z}$, we have

Theorem 5.5. *There exists a family*

$$\{c_{T,e} \in H^*(G_S, U_S/MU_S) : T \subseteqq S, e \in R, \operatorname{supp} e \supseteqq T\}$$

with the following properties:

1) *For each $n \in \mathbb{Z}_{\geqq 0}$, the subfamily*

$$\{c_{T,e} : T \subseteqq S, e \in R, \operatorname{supp} e \supseteqq T, \deg e = n + |T|\}$$

is a $\mathbb{Z}/M\mathbb{Z}$-basis for $H^n(G_S, U_S/MU_S)$.

2) *For any order ideal $\mathscr{I}$ of S, let $U_S(\mathscr{I}) = \sum_{T \in \mathscr{I}} U_T$. By the inclusion $U_S(\mathscr{I}) \hookrightarrow U_S$, $H^*\big(G_S, U_S(\mathscr{I})/MU_S(\mathscr{I})\big)$ can be considered as a submodule of $H^*(G_S, U_S/M_S)$. Furthermore, the subfamily*

$$\{c_{T,e} : T \in \mathscr{I}, e \in R, \operatorname{supp} e \supseteqq T\}$$

is a Z/MZ basis for $H^*\big(G_S, U_S(\mathscr{I})/MU_S(\mathscr{I})\big)$.

3) *One has cup product structure*

$$[e'] \cup c_{T,e} = (-1)^{\omega(e',e)} \prod_{\substack{i \in S \\ e_i e_i' \equiv 1(2)}} \left(\frac{\ell_i - 1}{2}\right) c_{T,e+e'}$$

for all $e, e' \in R$ *and* $T \subseteqq \operatorname{supp} e$.

Proof. 1) By Proposition 5.4, we have induced quasi-isomorphism:

$$f \otimes 1 : \mathbf{K}_M^{\bullet,\bullet} \to \mathbf{Q}_M^{\bullet,\bullet}.$$

Now since the induced differentials of d and δ in $\mathbf{Q}_M^{\bullet,\bullet}$ are 0. Consider the cocycle $[0, T, e]$ in $\mathbf{Q}_M^{\bullet,\bullet}$, there exists a cocycle $C_{T,e}$ (unique modulo boundary) which is the lifting of $[0, T, e]$ by the quotient map $f \otimes 1$. Hence $\mathfrak{u}(C_{T,e}) \otimes 1$ is a cocycle in the complex $\mathbf{U}_M^{\bullet}$. Let $c_{T,e}$ denote the cohomology element in $H^*(G_S, U_S/MU_S)$ represented by the cocycle $\mathfrak{u}(C_{T,e}) \otimes 1$. Then $\{c_{\mathrm{T},e} : \operatorname{supp} e \supseteqq T\}$ is a canonical $\mathbb{Z}/M\mathbb{Z}$-basis for the cohomology group $H^*(G_S, U_S/MU_S)$. This finishes the proof of 1).

2) Similar to 1), just consider the map $f \otimes 1 : \mathbf{K}_M^{\bullet,\bullet}(\mathscr{J}) \to \mathbf{Q}_M^{\bullet,\bullet}(\mathscr{J})$.

3) For the cup product, there is a natural homomorphism

$$\mathbb{Z}/M\mathbb{Z} \otimes U_S/MU_S \to U_S/MU_S,$$

therefore $H^*(G_S, U_S/MU_S)$ (and also $H^*\big(G_S, U_S(\mathscr{J})/MU_S(\mathscr{J})\big)$) has a natural $H^*(G_S, \mathbb{Z}/M\mathbb{Z})$-module structure. By the theory of spectral sequences (see, for example Brown [3], Chap. 7, §5), we have the cochain cup product

$$\mathbf{C}_M^{\bullet} \otimes \mathbf{K}_M^{\bullet,\bullet} \to \mathbf{K}_M^{\bullet,\bullet}.$$

By using the diagonal map Φ_S defined in §3, it is easy to check that:

$$\mathbf{C}_M^{\bullet} \otimes \mathbf{S}_M^{\bullet,\bullet} \subseteqq \mathbf{S}_M^{\bullet,\bullet},$$

hence we can pass the cup product structure to the quotient and have

$$\mathbf{C}_M^{\bullet} \otimes \mathbf{Q}_M^{\bullet,\bullet} \to \mathbf{Q}_M^{\bullet,\bullet}.$$

Now 3) follows immediately from the explicit expression of Φ_S. This concludes the proof. $\square$

6. Explicit basis of $H^0(G_S, U_S/MU_S)$

In §5, we obtained a canonical basis $\{c_{\mathrm{T},e} : \operatorname{supp} \supseteqq T\}$ for the cohomology group $H^*(G_S, U_S/MU_S)$. However, little is known yet for the explicit expression of the cocycle $c_{\mathrm{T},e}$ in the complex $\operatorname{Hom}_{G_S}(\mathbf{P}_\bullet, U_S/MU_S)$, which makes it necessary to study how to lift the cocycle $[0, T, e]$ in $\mathbf{Q}_M^{\bullet,\bullet}$ to the cocycle $C_{T,e}$ in $\mathbf{K}_M^{\bullet,\bullet}$. Unfortunately, we are unable to get a complete answer for this problem in this paper. We obtain a partial solution in the 0-cocycles case, however, which is enough for us to prove Theorem B.

6.1. The triple complex structure of K. Recall from §3, $\mathbf{L}$ has a double complex structure, therefore we can make $\mathbf{K}$ a triple complex. Set

$$K^{p_1,p_2,q} := \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, L^{p_1,p_2}) = \langle [a,T,e] : [a,T] \in L^{p_1,p_2}, \deg e = q\rangle$$

with the differentials (d_1, d_2, δ) given by

$$d_1[a,T,e] = -\sum_{i\in T}\omega(i,T)N_i\left[Fr_i^{-1}a + \frac{1}{\ell_i}, T\backslash\{i\}, e\right],$$

$$d_2[a,T,e] = \sum_{i\in T}\omega(i,T)(1-Fr_i^{-1})[a,T\backslash\{i\},e],$$

and δ as given in the double complex $\mathbf{K}^{\bullet,\bullet}$. In this setup, we see that $\mathbf{K}(\mathscr{I})$ becomes a triple subcomplex of $\mathbf{K}$, moreover

$$\mathbf{K}(n) = \bigoplus_{p_2 \geqq s-n} K^{p_1,p_2,q}.$$

Correspondingly, we define triple complex structures on $\mathbf{K}_M$, $\mathbf{K}_M(\mathscr{I})$ and $\mathbf{K}_M(n)$. This triple complex structure enables us to construct different double complex structures in $\mathbf{K}$ and $\mathbf{K}_M$. By studying those double complexes, we can gather more information about $\mathbf{K}$. This method will be illustrated in the next subsection.

6.2. The double complex $(\mathbf{K}_M^{\bullet,p_2,\bullet}, \boldsymbol{d_1}, \boldsymbol{\delta})$. For fixed p_2, let

$$\mathbf{K}_M^{\bullet,p_2,\bullet} = \bigoplus_{p_1,q} K_M^{p_1,p_2,q},$$

with differentials d_1 and δ, then we get a double complex $(\mathbf{K}_M^{\bullet,p_2,\bullet}; d_1, \delta)$. Similarly, we can get the double complex $(\mathbf{K}_M^{\bullet,\bullet}; d_1+\delta, d_2)$ whose (p_1+q, p_2)-component is $\bigoplus K_M^{p_1,p_2,q}$. As before, for any $\mathscr{I}$, we have double complexes $\mathbf{K}_M^{\bullet,p_2,\bullet}(\mathscr{I})$ and $\bigoplus K_M^{p_1,p_2,q}(\mathscr{I})$ which are subcomplexes of $\mathbf{K}_M^{\bullet,p_2,\bullet}$ and $\bigoplus K_M^{p_1,p_2,q}$ respectively. First we have

Proposition 6.1. 1) *$H^*_{\mathrm{total}}(\mathbf{K}_M^{\bullet,p_2,\bullet}; d_1, \delta)$ is a free $\mathbb{Z}/M\mathbb{Z}$-module generated by cocycles $C'_{T,e}$ with leading term $[0,T,e]$ and the remainder with q-degree less than* $\deg e$ *over all pairs (T,e) satisfying $|T| = s-p_2$ and* $\mathrm{supp}\, e \supseteqq T$.

2) *Moreover, $H^*_{\mathrm{total}}\big(\mathbf{K}_M^{\bullet,p_2,\bullet}(\mathscr{I}); d_1, \delta\big)$ is a free $\mathbb{Z}/M\mathbb{Z}$-module generated by cocycles $C'_{T,e}$ with leading term $[0,T,e]$ and the remainder with q-degree less than* $\deg e$ *over all pairs (T,e) satisfying $T \in \mathscr{I}$, $|T| = s-p_2$ and* $\mathrm{supp}\, e \supseteqq T$.

Proof. We only prove 1). The proof of 2) is similar. First look at the spectral sequence of $\mathbf{K}_M^{\bullet,p_2,\bullet}$ with the second filtration (i.e., the filtration given by q), then

$$E_1^{p_1,q}(\mathbf{K}_M^{\bullet,p_2,\bullet}) = H^q(G_S, L_M^{p_1,p_2}).$$

Next for the differential d_1 induced on E_1, with the same analysis as in computing the E_2 terms of $(\mathbf{K}; d, \delta)$ (see §5, Proposition 5.3), we have

$$E_2^{p_1,q}(\mathbf{K}_M^{\bullet,p_2,\bullet}) = \begin{cases} \bigoplus\limits_{|T|=s-p_2} \bigoplus\limits_{\substack{e:\deg e=q \\ \operatorname{supp} e \supseteqq T}} \mathbb{Z}/M\mathbb{Z}, & \text{if } p_1 = -s, \\ 0, & \text{if } p_1 \neq -s. \end{cases}$$

Furthermore, let $(\mathbf{Q}_M^{\bullet,p_2,\bullet}; 0, 0)$ be the double complex generated by all symbols $[0, T, e]$ satisfying $|T| = s - p_2$ and $\operatorname{supp} e \supseteqq T$, which can be considered as a quotient complex of $\mathbf{K}_M^{\bullet,p_2,\bullet}$. As in the proof of Theorem A, the quotient map induces an isomorphism between cohomology groups. Let $C'_{T,e}$ be a cocycle in $\mathbf{K}_M^{\bullet,p_2,\bullet}$ with image $[0, T, e]$ in $\mathbf{Q}_M^{\bullet,p_2,\bullet}$, then the cocycle $C'_{T,e}$ is the sum of a leading term $[0, T, e]$ and a remainder contained in the direct sum of $K^{p'_1,p_2,q'}$ where $q' < \deg e$ and $p'_1 + q' = \deg e - s$. $\square$

Proposition 6.2. *The spectral sequence of the double complex* $(\mathbf{K}_M^{\bullet,\bullet}; d_1 + \delta, d_2)$ *with the first filtration, degenerates at* E_1. *The spectral sequence of the double complex* $\left(\mathbf{K}_M^{\bullet,\bullet}(\mathscr{I}); d_1 + \delta, d_2\right)$ *with the first filtration, degenerates at* E_1.

Proof. We only prove the first part. The E_1-terms of the spectral sequence are

$$E_1^{p_1+q,p_2}(\mathbf{K}_M^{\bullet,\bullet}) = H_{\text{total}}^{p_1+q}(\mathbf{K}_M^{\bullet,p_2,\bullet}; d_1, \delta).$$

Note that $|E_1^{p,q}| \geqq |E_2^{p,q}| \geqq \cdots \geqq |E_\infty^{p,q}|$ in general for any spectral sequence, then

$$\left| \bigoplus_{p_1+p_2+q=n} H_{\text{total}}^{p_1+q}(\mathbf{K}_M^{\bullet,p_2,\bullet}; d_1, \delta) \right| \geqq |H_{\text{total}}^n(\mathbf{K}_M^{\bullet,\bullet}, d + \delta)|.$$

By Theorem 5.5 and Proposition 6.1, the left hand side and the right hand side of the above inequality have the same number of elements, hence the inequality is actually an identity. Therefore, the spectral sequence of $\mathbf{K}_M^{\bullet,\bullet}$ with filtration given by $p_1 + q$ degenerates at E_1. $\square$

The advantage of studying the triple complex structure of the complex $\mathbf{K}_M$ is that we can obtain the $(-p_2)$-cocycles of $\mathbf{K}_M^{\bullet,p_2,\bullet}$ rather quickly. Recall that

$$(1 - \sigma_i) D_i = N_i \pmod{M}.$$

Now for the $(-p_2)$-cocycles $C'_{T,e}$, the pair (T, e) must satisfy $\deg e = |T|$ and therefore $e = e_T := \sum\limits_{i \in T} \varepsilon_i$. In this case, for any $i \in T$, we always have

$$\omega(i, T) = (-1)^{\omega(e_T)_i} = (-1)^{\omega(e_{T \setminus \{i\}})_i}.$$

First

$$\delta[0, T, e] = 0, \quad d_1[0, T, e_T] = -\sum_{i \in T} \omega(i, T) N_i \left[\frac{r_{T \setminus \{i\}}}{\ell_i}, T \setminus \{i\}, e_T\right],$$

then

$$\delta\left(\sum_{i\in T} D_i\left[\frac{r_{T\setminus\{i\}}}{\ell_i}, T\setminus\{i\}, e_T\setminus\{i\}\right]\right) = (-1)^{|T|} d_1[0, T, e_T].$$

Continue this procedure, we have

$$C'_{T,e_T} = \sum_{T'\subseteqq T} (-1)^{|T'|(2|T|-|T'|-1)/2} D_{T'}\left[\sum_{i\in T'} \frac{r_{T\setminus T'}}{\ell_i}, T\setminus T', e_{T\setminus T'}\right].$$

Apparently, we see that if $T \in \mathscr{J}$, then the cocycles C'_{T,e_T} are all contained in the subcomplex $\mathbf{K}_M^{\bullet,p_2,\bullet}(\mathscr{J})$. Combining the above results, we have

Proposition 6.3. 1) *The canonical basis* $\{C'_{T,e_T} : |T| = s - p_2\}$ *of* $H^{(-p_2)}(\mathbf{K}_M^{\bullet,p_2,\bullet})$ *is given by*

$$C'_{T,e_T} = \sum_{T'\subseteqq T} (-1)^{|T'|(2|T|-|T'|-1)/2} D_{T'}\left[\sum_{i\in T'} \frac{r_{T\setminus T'}}{\ell_i}, T\setminus T', e_{T\setminus T'}\right].$$

2) *If we restrict our attention in the subcomplex* $\mathbf{K}_M^{\bullet,p_2,\bullet}(\mathscr{J})$, *then* $H^{(-p_2)}\big(\mathbf{K}_M^{\bullet,p_2,\bullet}(\mathscr{J})\big)$ *has a canonical basis* $\{C'_{T,e_T} : |T| = s - p_2, T \in \mathscr{J}\}$.

6.3. Proof of Theorem B. First we claim that

$$D_T\left[\sum_{i\in T} \frac{1}{\ell_i}\right] \in H^0(G_S, U_S/MU_S) = (U_S/MU_S)^{G_S}.$$

We prove it by induction on $|T|$. For $T = \{j\}$, it is easy to see that $(1-\sigma_i)D_j\left[\frac{1}{\ell_j}\right] = 0$ for all $i \in S$. Now in general, for any $j \in T$,

$$(1-\sigma_j)D_T\left[\sum_{i\in T} \frac{1}{\ell_i}\right] = (Fr_j - 1)D_{T\setminus\{j\}}\left[\sum_{i\in T\setminus\{j\}} \frac{1}{\ell_i}\right]$$

which is 0 by induction, for $j \notin T$, it is obviously 0. Hence the claim holds.

Now we consider the double complex $(\mathbf{K}_M^{\bullet,\bullet}, d_1 + \delta, d_2)$. By Proposition 6.2, we know that $(\mathbf{K}_M^{\bullet,\bullet}, d_1 + \delta, d_2)$ degenerates at E_1 for the first filtration. By Proposition 6.3, $E_1^{-p_2,p_2}(\mathbf{K}_M^{\bullet,\bullet})$ is generated by $\{C'_{T,e_T} : |T| = s - p_2\}$. We plan to lift C'_{T,e_T} to a 0-cocycle in $\mathbf{K}_M^{\bullet,\bullet}$, which is guaranteed by the degeneration at E_1. Moreover, we can study the lifting C'_{T,e_T} in $\mathbf{K}_M^{\bullet,\bullet}(T)$. Therefore there exists a cocycle $\tilde{C}_{T,e_T}$ in $\mathbf{K}_M^{\bullet,\bullet}(T)$ with the leading term C'_{T,e_T} and the remainder contained in the direct sum of $K_M^{p'_1,p'_2,q'}(T)$ where $p'_1 + p'_2 + q' = 0$ and $p'_2 > p_2$. Hence the image $\mathrm{u}(\tilde{C}_{T,e_T})$ is exactly of the form

$$\pm D_T\left[\sum_{i\in T} \frac{1}{\ell_i}\right] + \mathrm{Re}(T),$$

where $\mathrm{Re}(T)$ is of the form

$$\mathrm{Re}(T) = \sum_{\substack{\mathrm{ord}(a)|r_T \\ \mathrm{ord}(a) \nmid r_T}} n_a[a].$$

Both $\mathrm{u}(\tilde{C}_{T,e_T})$ and $D_T\left[\sum_{i\in T}\frac{1}{\ell_i}\right]$ are 0-cocycles of U_S/MU_S, and hence is $\mathrm{Re}(T)$.

In order to prove Theorem B, it is sufficient to prove

$$(*) \qquad \mathrm{Re}(T) = \text{linear combination of } D_{T'}\left[\sum_{i\in T'}\frac{1}{\ell_i}\right] \text{ for } T' \subsetneqq T.$$

We show $(*)$ by induction on $|T|$. If $|T| = 1$, this is trivial. Now in general, without loss of generality, we may assume that $T = S$ and for any $T' \subsetneqq S$, $\mathrm{Re}(T')$ is a linear combination of $D_{T''}\left[\sum_{i\in T''}\frac{1}{\ell_i}\right]$ for $T'' \subsetneqq T'$. Then $\mathrm{u}(\tilde{C}_{T',e'_T})$ for any $T' \subsetneqq S$ is a linear combination of $D_{T''}\left[\sum_{i\in T''}\frac{1}{\ell_i}\right]$ with $T'' \subsetneqq T$. By Proposition 5.1, Proposition 6.2 and Theorem 5.5, $H^0\big(G_S, U_S(s-1)/MU_S(s-1)\big)$ is generated by $\{\mathrm{u}(\tilde{C}_{T',e'_T}) : T' \subsetneqq S\}$ and hence by $D_{T'}\left[\sum_{i\in T'}\frac{1}{\ell_i}\right]$. But obviously $\mathrm{Re}(S) \in U_S(s-1)/MU_S(s-1)$, so $(*)$ holds for $\mathrm{Re}(s)$. Theorem B is proved.

Remark. One natural question to ask is if the bases of $H^0(G_S, U_S/MU_S)$ obtained in Theorem 5.5 and in Theorem B are the same. Unfortunately, they are not the same even in the case $|S| = 3$. Right now, we don't know too much about the explicit expression of the cocycles $c_{\mathrm{T},e}$. A deep understanding of those cocycles might tell us more about the arithmetic of the cyclotomic fields.

Appendix A. A resolution of the universal ordinary distribution

By *Greg W. Anderson* at Minneapolis

A.1. Basic definitions

A.1.1. The universal ordinary distribution. Let $\mathscr{A}$ be a free abelian group equipped with a basis $\{[x]\}$ indexed by $x \in \mathbb{Q} \cap [0,1)$. For all $x \in \mathbb{Q}$ put $[x] := [\langle x\rangle]$, where $\langle x\rangle$ is the unique rational number in the interval $[0,1)$ congruent to x modulo 1. The *universal ordinary distribution U* is defined to be the quotient of $\mathscr{A}$ by the subgroup generated by all elements of the form

$$[x] - \sum_{i=1}^{f}\left[\frac{x+i}{f}\right] \quad (f \in \mathbb{Z}_{>0}, x \in \mathbb{Q}).$$

A.1.2. The universal ordinary distribution of level *f*. Fix a positive integer f. Let $\mathscr{A}(f)$ be the subgroup of $\mathscr{A}$ generated by the set $\left\{[x] \mid x \in \frac{1}{f}\mathbb{Z}\right\}$. The *universal ordinary distribution* $U(f)$ of *level* f is defined to be the quotient of $\mathscr{A}(f)$ by the subgroup generated by all elements of the form

$$[x] - \sum_{i=1}^{g}\left[\frac{x+i}{g}\right] \quad \left(g \in \mathbb{Z}_{>0}, g \mid f, x \in \frac{g}{f}\mathbb{Z}\right).$$

The inclusions $\mathscr{A}(f) \subset \mathscr{A}$ induce a natural isomorphism $\varinjlim U(f) \xrightarrow{\sim} U$.

A.1.3. The ring Λ and its action on $\mathscr{A}$. Let Λ be the polynomial ring over $\mathbb{Z}$ generated by a family $\{X_p\}$ of independent variables indexed by primes p, and for each positive integer f, put

$$X_f := \prod X_{p_i}^{e_i} \in \Lambda, \quad Y_f := \prod (1 - X_{p_i})^{e_i} \in \Lambda$$

where $f = \prod_i p_i^{e_i}$ is the prime factorization of f. Each of the families $\{X_f\}$ and $\{Y_f\}$ is a basis for Λ as a free abelian group. We equip $\mathscr{A}$ with Λ-module structure by the rule

$$X_p[x] = \sum_{i=1}^{p}\left[\frac{x+i}{p}\right]$$

for all primes p and $x \in \mathbb{Q}$. One has

$$U = \mathscr{A}\Big/\left(\sum_p Y_p \mathscr{A}\right).$$

This last observation suggests that we can usefully resolve U by a procedure of Koszul type.

A.2. The structure of $\mathscr{A}$ as a Λ-module

A.2.1. Partial fraction expansions. Each $x \in \mathbb{Q}$ has a unique partial fraction expansion

$$x = x_0 + \sum_p \sum_i \frac{x_{pi}}{p^i}$$

where p ranges over primes, i ranges over positive integers, $x_0 \in \mathbb{Z}$, $x_{pi} \in \mathbb{Z} \cap [0, p)$, and all but finitely many of the coefficients x_{pi} vanish. For each nonnegative integer n, put

$$\mathscr{R}_n := \left\{ x \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{There exist at most } n \text{ primes} \\ p \text{ such that } x_{p1} = p - 1. \end{array} \right\} \cap [0, 1)$$

and let $\mathscr{A}_n$ be the subgroup of $\mathscr{A}$ generated by $\{[x] \mid x \in \mathscr{R}_n\}$.

Theorem 1. *The following hold*:

1. *For all positive integers f and n, one has*

$$\mathscr{A}_n \cap \mathscr{A}(f) \subseteqq \mathscr{A}_{n-1} \cap \mathscr{A}(f) + \sum_{p|f} X_p \mathscr{A}(f/p),$$

where the sum is extended over primes p dividing f.

2. *For each positive integer f, the family $\{X_g[x]\}$ indexed by the set*

$$\left\{(g,x) \in \mathbb{Z}_{>0} \times \mathscr{R}_0 \,\middle|\, g|f, x \in \frac{g}{f}\mathbb{Z}\right\}$$

is a basis for $\mathscr{A}(f)$.

3. *For each positive integer f, the family $\{Y_g[x]\}$ indexed by the set*

$$\left\{(g,x) \in \mathbb{Z}_{>0} \times \mathscr{R}_0 \,\middle|\, g|f, x \in \frac{g}{f}\mathbb{Z}\right\}$$

is a basis for $\mathscr{A}(f)$.

4. *The family $\{X_f[x]\}$ indexed by pairs $(f,x) \in \mathbb{Z}_{>0} \times \mathscr{R}_0$ is a basis for $\mathscr{A}$.*

5. *The family $\{Y_f[x]\}$ indexed by pairs $(f,x) \in \mathbb{Z}_{>0} \times \mathscr{R}_0$ is a basis for $\mathscr{A}$.*

6. *The free abelian group $\mathscr{A}$ is free as a Λ-module, and the family $\{[x]\}$ indexed by $x \in \mathscr{R}_0$ is a Λ-basis for $\mathscr{A}$.*

Proof. 1. For each $x \in \frac{1}{f}\mathbb{Z} \cap (\mathscr{R}_n \backslash \mathscr{R}_{n-1})$, there exists some prime p dividing f such that $x_{p1} = p - 1$, and one has

$$[x] = -\left(\sum_{i=1}^{p-1}\left[x + \frac{i}{p}\right]\right) + X_p[px],$$

whence the result.

2. The family $\{X_g[x]\}$ generates $\mathscr{A}(f)$ by what we have already proved. The family $\{X_g[x]\}$ is of cardinality

$$\sum_{g|f}\left|\mathscr{R}_0 \cap \frac{g}{f}\mathbb{Z}\right| = \sum_{g|f}|(\mathbb{Z}/(f/g)\mathbb{Z})^\times| = \sum_{g|f}|(\mathbb{Z}/g\mathbb{Z})^\times| = f.$$

Therefore the family $\{X_g[x]\}$ is a basis for $\mathscr{A}(f)$.

3.–6. These assertions follow trivially from what we have already proved. $\square$

Corollary 1. *The following hold*:

1. *For each $f \in \mathbb{Z}_{>0}$, the group $U(f)$ is free abelian and the family $\{[x]\}$ indexed by $x \in \frac{1}{f}\mathbb{Z} \cap \mathscr{R}_0$ gives rise to a basis for $U(f)$.*

2. *The group U is free abelian and the family $\{[x]\}$ indexed by $x \in \mathscr{R}_0$ gives rise to a basis for U.*

3. *The natural map $U(f) \to U$ is a split monomorphism.*

(*Thus the classical results of Kubert* [6] *are recovered.*)

Proof. Clear. $\square$

A.3. Construction of resolutions

A.3.1. The complex (L, d). Let L be a free abelian group equipped with a basis $\{[x, g]\}$ indexed by pairs (x, g) with $x \in \mathbb{Q} \cap [0, 1)$ and g a squarefree positive integer. For all $x \in \mathbb{Q}$ and squarefree integers g, put $[x, g] := [\langle x \rangle, g]$. For all $x \in \mathbb{Q}$ and increasing sequences $p_1 < \cdots < p_m$ of primes, we declare the symbol $[x, p_1 \cdots p_m]$ to be of degree $-m$ and we set

$$d[x, p_1 \cdots p_m] := \sum_{i=1}^{m} (-1)^{i-1} \left([x, p_1 \cdots p_{i-1} p_{i+1} \cdots p_m] - \sum_{j=1}^{p_i} \left[\frac{x+j}{p_i}, p_1 \cdots p_{i-1} p_{i+1} \cdots p_m \right] \right),$$

thereby equipping the group L with a grading and a differential d of degree 1. The map $[x, 1] \mapsto [x]$ induces an isomorphism $H^0(L, d) \xrightarrow{\sim} U$.

A.3.2. The subcomplexes $(L(f), d)$. Fix a positive integer f. We define $L(f)$ to be the graded subgroup spanned by the symbols of the form $[x, g]$ where g divides f and $x \in \frac{g}{f}\mathbb{Z}$. It is clear that $L(f)$ is d-stable. The map $[x, 1] \mapsto [x]$ induces an isomorphism $H^0(L(f), d) \xrightarrow{\sim} U(f)$.

A.3.3. The noncommutative ring $\tilde{\Lambda}$. Let $\tilde{\Lambda}$ be the exterior algebra over Λ generated by a family of symbols $\{\Xi_p\}$ indexed by primes p. For each increasing sequence $p_1 < \cdots < p_m$ of prime numbers, put

$$\Xi_{p_1 \cdots p_m} := \Xi_{p_1} \wedge \cdots \wedge \Xi_{p_m} \in \tilde{\Lambda},$$

and declare $\Xi_{p_1 \cdots p_m}$ to be of degree $-m$, thereby defining a Λ-basis $\{\Xi_h\}$ for $\tilde{\Lambda}$ indexed by squarefree positive integers h and equipping $\tilde{\Lambda}$ with a Λ-linear grading. Let d be the unique Λ-linear derivation of $\tilde{\Lambda}$ of degree 1 such that

$$d\Xi_p = Y_p$$

for all primes p. One then has

$$d\Xi_{p_1\cdots p_m} = \sum_{i=1}^{m}(-1)^{i-1} Y_{p_i}\Xi_{p_1\cdots p_{i-1}p_{i+1}\cdots p_m}$$

for all increasing sequences $p_1 < \cdots < p_m$ of prime numbers.

A.3.4. The subcomplexes $\big(\tilde{\Lambda}(f), d\big)$. Fix a positive integer f. The graded subgroup $\tilde{\Lambda}(f)$ generated by all elements of the form $Y_g\Xi_h$ where gh divides f is d-stable. It is not difficult to verify that the complex $\big(\tilde{\Lambda}(f), d\big)$ is acyclic in nonzero degree, and that $H^0\big(\tilde{\Lambda}(f), d\big)$ is a free abelian group of rank 1 generated by the symbol $\Xi_1 = 1$.

A.3.5. The action of $\tilde{\Lambda}$ on L. We equip L with graded left $\tilde{\Lambda}$-module structure by the rules

$$\Xi_p[x, p_1\cdots p_m] = \begin{cases} (-1)^{|\{i|p_i<p\}|}[x, pp_1\cdots p_m] & \text{if } p \notin \{p_1, \dots, p_m\}, \\ 0 & \text{if } p \in \{p_1, \dots, p_m\}, \end{cases}$$

and

$$X_p[x, p_1\cdots p_m] = \sum_{i=1}^{p}\left[\frac{x+i}{p}, p_1\cdots p_m\right]$$

for all primes p and increasing sequences $p_1 < \cdots < p_m$ of primes. By a straightforward calculation that we omit, one can verify that

$$d(\xi\eta) = (d\xi)\eta + (-1)^{\deg\xi}\xi(d\eta)$$

for all homogeneous $\xi \in \tilde{\Lambda}$ and $\eta \in L$.

Theorem 2. *The following hold*:

1. *For each positive integer f, the complex $\big(L(f), d\big)$ is acyclic in nonzero degree.*

2. *The complex (L, d) is acyclic in nonzero degree.*

Proof. We have only to prove the first statement. By Theorem 1 and a straightforward calculation that we omit, one has

$$L(f) = \bigoplus_{(x,g)} \tilde{\Lambda}(g)[x, 1]$$

where the direct sum is indexed by pairs (x, g) with $x \in \frac{1}{f}\mathbb{Z} \cap \mathscr{R}_0$ and g the largest positive integer such that $x \in \frac{g}{f}\mathbb{Z}$. Each of the subcomplexes $\big(\tilde{\Lambda}(g)[x, 1], d\big)$ is an isomorphic copy of $\big(\tilde{\Lambda}(g), d\big)$, and the latter we have already observed to be acyclic in nonzero degree. $\square$

A.3.6. Note on references. The construction of $(L(f), d)$ presented here is cobbled together from ideas presented in the author's papers [1] and [2], along with simplifications suggested by many conversations with Pinaki Das and Yi Ouyang on these topics.

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