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On the universal norm distribution

Yi Ouyang

Department of Mathematics, University of Toronto, 100 St. George St., Toronto, ON M5S 3G3, Canada
e-mail: youyang@math.toronto.edu

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Abstract. We introduce and study the universal norm distribution in this paper, which generalizes the concepts of universal ordinary distribution and the universal Euler system. We study the Anderson type resolution of the universal norm distribution and then use this resolution to study the group cohomology of the universal norm distribution.

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1. Introduction

Let r be a positive integer, the *universal ordinary distribution* of rank 1 and level r is well known to be the free abelian group

$$U_r = \frac{\langle [a] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \rangle}{\langle [a] - \sum_{pb=a} [b] : p \mid r, a \in \frac{p}{r}\mathbb{Z}/\mathbb{Z} \rangle}.$$

With a natural $G_r = \mathrm{Gal}(\mathbb{Q}(\mu_r)/\mathbb{Q})$ action on U_r, U_r becomes a G_r-module and plays a very important role in the study of cyclotomic fields, see for example Lang [4] or Washington [10] for more information. In particular, the sign cohomology of U_r gives key information about the indices of cyclotomic units and Stickelberger ideals as illustrated by Sinnott's original paper [9] and many following papers on this subject by different authors. The G_r-cohomology is found to be related to the cyclotomic Euler system, as shown by Anderson-Ouyang [1] about the Kolyvagin recursion in the universal ordinary distribution.

In the book [8], Rubin defined a generalization of the universal ordinary distribution, which he called the *universal Euler system*. It then was used to prove the Kolyvagin recursions satisfied by the Euler systems. However, there are other universal objects satisfying similar distribution relations. In the paper [6], we proposed a generalization of the universal ordinary distribution,

for which we called the *universal norm distribution*. We used it successfully to study Sinnott's index formula.

We further generalize the idea of the universal norm distribution in this paper, which treats the universal Euler systems as special cases. We study in detail the structure of the universal norm distribution in this paper. We also study in detail its group cohomology. In short, this paper generalizes the results of Ouyang [5] and the appendix of it by Anderson. The goal is to set up necessary tools to the study of the universal Kolyvagin recursion for the universal norm distribution(thus includes the universal Euler system case), which is a question raised in Anderson-Ouyang [1] and will be answered in a subsequent paper [7]. However, our study here is more than application to the universal Kolyvagin recursion. The pure homological setup here should offer us more freedom to the study of other arithmetic aspects of the universal norm distribution. Certainly we expect more studies in this direction.

The structure of this paper is as follows. We first introduce the definition of the universal norm distribution $\mathcal{U}_z$ in § 2 and give some examples in § 3. Basic properties of $\mathcal{U}_z$ are studied in § 4. A general phenomenon of every universal norm distribution $\mathcal{U}_z$ is Anderson's resolution $\mathcal{L}_z$ attached to it. We construct $\mathcal{L}_z$ in § 5 and prove it is indeed a resolution of $\mathcal{U}_z$ in Theorem 5.1, a generalization of the results by Anderson in the appendix of [5]. Because of the existence of Anderson's resolution $\mathcal{L}_z$, we can thus apply the double complex and spectral sequences method to study the group cohomology of the universal norm distribution $\mathcal{U}_z$. This is accomplished in § 7, in particular, in Theorem 7.5 and Theorem 7.8. For the universal ordinary distribution case, the two Theorems recover and generalize Theorem A in Ouyang [5].

The author got very first idea of this paper during his pleasant visit in IHES in Spring 2001. Part of the results here was reported in the number theory seminar in Penn State University in November 2001 and then in McMaster University in February 2002, and in the summer meeting of CMS at Laval University in June 2002. The author sincerely thanks the above organizations, Professors Robert Vaughn and Winnie Li at PSU, Professor Manfred Kolster at McMaster and Professors Kumar Murty and Pramath Sastry at Toronto for inviting me to give these talks. Last but not least, thanks always go to Professor Greg W. Anderson for his ideas and his influence.

2. Notations and Definitions

2.1 Basic Notations

Let X be a totally ordered set. Denote by x, x_i the elements in X.

Let Y be the set of all squarefree formal products of $x \in X$, i.e., the element $y \in Y$ has the form $x_1 \cdots x_n \cdots$ for $x_i \neq x_j \in X$. In particular, let $\mathbf{1} \in Y$

denote the element of which no $x \in X$ appears in the formal product. One can identify Y with the collection of all subsets of X, thus $\mathbf{1}$ is corresponding to the empty set. For every $y \in Y$, the *degree* $\deg y$ of y is define to be the number of elements $x \in X$ dividing y. Denote by y, y_i the elements in Y. If without further statement, we'll assume that y is finite, i.e., $\deg y < \infty$. Denote by Y_{fin} the set of all finite $y \in Y$.

Let Z be the set of all formal product of $x \in X$, i.e., the element $z \in Z$ has the form $x_1^{i_1} \cdots x_n^{i_n} \cdots$ with $i_j \in \mathbb{Z}_{\geq 0}$. For every $z = x_1^{i_1} \cdots x_n^{i_n} \cdots$, define the *degree* of z to be $\deg z = \sum_{j=1}^{n} i_j$. Denote by z, z', w the elements in Z and in particular by $\mathbf{z}$, $\mathbf{z}'$ the infinite elements(elements with infinite degree) in Z. The subset of all finite elements in Z will be denoted by Z_{fin}.

Apparently we have $X \subseteq Y \subseteq Z$. One can always keep in mind the example that X is the set of prime numbers, Y_{fin} is the set of all squarefree positive integers and Z_{fin} is the set of positive integers. We can thus imitate all the terminologies from traditional sense, for example, prime factors, factors, the greatest common divisors and etc.

For every $z \in Z$ and $x \in X$, the *valuation* of z at x is the highest power of x dividing z and is denoted by $v_x(z)$. For every $z \in Z$, there exists a unique $\bar{z} \in Y$($\bar{z}$ could be infinite) such that if $x \mid z$ then $x \mid \bar{z}$. We call $\bar{z}$ the *support* of z. For every $z \in Z$, if a factor $z' \mid z$ satisfies $\gcd(z', z/z') = 1$, z' is called a *stalk* of z and is denoted by $z' \mid_s z$. Note that the set of stalks of z has a one-to-one correspondence with the set of factors(and also stalks) of $\bar{z}$. Fix z, for each $y \mid \bar{z}$, let $z(y)$ be the stalk of z whose support is y. In particular, $z(x)$ is just $x^{v_x(z)}$.

For each pair $x \in X$ and $y \in Y$, we define the function $\omega : X \times Y \to \{1, 0, -1\}$ by

$$(x, y) \longmapsto \begin{cases} (-1)^{\#\{x' : x' < x\}}, & \text{if } x \mid y; \\ 0, & \text{if } x \nmid y. \end{cases}$$

Let G be a profinite group. Let A be a point set with discrete topology such that G acts continuously. Suppose there is a surjection $A \to Z_{\text{fin}}$ which induces a bijection between the orbits of A and elements $z \in Z_{\text{fin}}$. Let B_z be the corresponding orbit of z. Let H_z be the stabilizer of any $b \in B_z$. We assume $\{H_z : z \in Z_{\text{fin}}\}$ satisfies the following axioms:

- For every $z \in Z_{\text{fin}}$, the commutator $[G, G] \leq H_z$;
- For every $z' \mid z \in Z_{\text{fin}}$, $H_z \leq H_{z'}$;
- For z and z' in Z_{fin} and relatively prime, $H_{zz'} = H_z \cap H_{z'}$ and $G = H_z H_{z'}$.

By the first axiom, then H_z is a normal open subgroup of G and the quotient group $G_z = G/H_z$ is finite abelian. By the second axiom, for every $z' \mid z \in Z_{\text{fin}}$, $G_{z'}$ is a quotient group of G_z; by the last axiom, one see that for every $z' \mid_s z$, the quotient map $G_z \to G_{z'}$ is canonically split as $G_z = G_{z'} \times G_{z/z'}$, we thus

have the following canonical decomposition

$$G_z = \prod_{x|z} G_{z(x)}.$$

Let N_z be the sum of all elements $g \in G_z$ in the group ring $\mathbb{Z}[G_z]$. For z finite and $z' \mid z$, Let $g_{z'}$ denote the image of $g \in G_z$ in $G_{z'}$. Let $N^z_{z'}$ be the corresponding inflation map from $\mathbb{Z}[G_{z'}]$ to $\mathbb{Z}[G_z]$. For every infinite $\mathbf{z} \in Z$, let $G_{\mathbf{z}}$ be the inverse limit of G_z over all finite $z \mid_s \mathbf{z}$. Then $G_{\mathbf{z}}$ is actually the direct product of $G_{\mathbf{z}(x)}$ for every $x \mid \mathbf{z}$.

Write $B_z = \{[gz] : g \in G_z\}$, then

$$A = \bigcup_{z \in Z_{\text{fin}}} B_z = \{[gz] : g \in G_z, z \in Z_{\text{fin}}\},$$

and G_{x^n} acts trivially in B_z if $x \nmid z$. Thus A and $\{G_z : z \in Z_{\text{fin}}\}$ are uniquely determined by each other. Let $A_z = \bigcup_{z' \mid_s z, z' \in Z_{\text{fin}}} B_{z'}$ for every $z \in Z$.

For each pair $x \in X$ and $z \in Z$, the Frobenius element Fr_x is a given element in G whose restriction to G_{x^n} is the identity for every $n \in \mathbb{N}$.

Let $\mathcal{O}$ be an integral domain and let Φ be its fractional field. Let $\mathcal{T}$ be a fixed $\mathcal{O}$-algebra which is torsion free and finitely generated as an $\mathcal{O}$-module. We suppose that $\mathcal{T}$ is a trivial G-module. For each $x \in X$, a polynomial

$$p(x; t) \in \mathcal{T}[t]$$

is chosen corresponding to x.

2.2 Definition of the universal norm distribution

Let $\mathcal{A}$ be the free $\mathcal{T}$-module generated by A, along with the G-action, $\mathcal{A}$ becomes a torsion free $\mathcal{T}[G]$-module. Let $\mathcal{B}_z$ be the $\mathcal{T}[G]$-submodule of A generated by B_z as $\mathcal{T}$-module for $z \in Z_{\text{fin}}$. Then $\mathcal{B}_z$ is nothing but a free rank 1 $\mathcal{T}[G_z]$-module with generator $[z]$. Let $\mathcal{A}_z$ be the $\mathcal{T}[G]$-submodule generated by A_z as $\mathcal{T}$-module for every $z \in Z$. Thus $\mathcal{A}_z$ has a natural $\mathcal{T}[G_{z'}]$-module structure for every $z \mid_s z'$.

Let $\lambda_{z(x)}$ be the $\mathcal{T}[G_z]$-homomorphism of $\mathcal{A}_z$ given by

$$\lambda_{z(x)} : [z'] \longmapsto \begin{cases} p(x; \mathrm{Fr}_x^{-1})[z'] - N_{z(x)}[z(x)z'], & \text{if } x \nmid z', \\ 0, & \text{if } x \mid z'. \end{cases}$$

Let $\mathcal{D}_z$ be the submodule of $\mathcal{A}_z$ generated by the images of $\lambda_{z(x)}(\mathcal{A}_{z/z(x)})$ for all $x \mid z$. Elements in $\mathcal{D}_z$ are called *distribution relations* in $\mathcal{A}_z$. The *universal norm distribution* $\mathcal{U}_z$ according to the above assumptions is defined to be the quotient $\mathcal{T}[G_z]$-module $\mathcal{A}_z/\mathcal{D}_z$, i.e., $\mathcal{A}_z$ modulo all distribution relations.

Note that for every $z \in Z$,

$$\mathcal{A}_z = \bigcup_{\substack{z' \text{ finite} \\ z' |_s z}} \mathcal{A}_{z'}.$$

For any $z' |_s z$, the apparent inclusion of $\mathcal{A}_{z'}$ to $\mathcal{A}_z$ induces an injection map from $\mathcal{U}_{z'}$ to $\mathcal{U}_z$. In Proposition 4.2(2), we'll see this injection actually is a splitting G_z-monomorphism.

3. Examples

We give a few examples about the universal norm distribution here.

3.1 The trivial case

The first case of the universal norm distribution is that $p(x;t) = 1$ for every $x \in X$. In this case, one easily see that $\mathcal{U}_z$ is generated by the images of B_z. Actually, $\mathcal{U}_z$ is nothing but isomorphic to the $\mathcal{T}$-module $\mathcal{B}_z = \mathcal{T}[G_z]$ (see the remark after Proposition 4.2). We call this type of universal norm distribution the *trivial universal norm distribution*.

3.2 The universal ordinary distribution

Recall that an *ordinary distribution* of level r for a positive integer r is a function f from $\frac{1}{r}\mathbb{Z}/\mathbb{Z}$ to an abelian group Ab satisfying

$$f(pa) = \sum_{i=0}^{p} f(a + \frac{i}{p}), \ \forall \text{ primes } p \mid r.$$

In the category of ordinary distributions, there exists a universal object, i.e., an abelian group U_r and a distribution relation $u : \frac{1}{r}\mathbb{Z}/\mathbb{Z} \to U_r$ such that for every f, there is a unique homomorphism $f^u : U_r \to Ab$, such that $f = f^u \circ u$. Usually one can write U_r as

$$\frac{\langle [a] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \rangle}{\langle [pa] - \sum_{i=0}^{p} [a + \frac{i}{p}] : \forall p \mid r \rangle}$$

and the map u sends a to $[a]$.

The universal ordinary distribution U_r is actually a universal norm distribution according to our language. Let X be the set of all prime numbers. Then Y_{fin} is the set of all squarefree positive integers and Z_{fin} is just the set of positive integers. Let $G = G_{\mathbb{Q}}$. Let $G_r = \text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$. The Frobenius element Fr_p is defined by the usual way. Let $\mathcal{O} = \mathbb{Z} = \mathcal{T}$ and thus $\Phi = \mathbb{Q}$. Let the polynomial $p(p;t) = 1 - t$ for all $p \in X$. Then

Proposition 3.1. *The corresponding universal norm distribution $\mathcal{U}_r$ is isomorphic to the universal ordinary distribution U_r by the identification of $\left[\frac{1}{r}\right] \in U_r$ and $[r] \in \mathcal{U}_r$.*

Proof. This fact is just Proposition 3.1(iv) of Ouyang [6]. We reproduce the proof here. Let $\tilde{\mathcal{A}}_r = \langle [a] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \rangle$. Let $\pi : \tilde{\mathcal{A}}_r \to \mathcal{U}_r$ be the G_r-homomorphism by

$$\left[\frac{1}{f}\right] \mapsto N_f^{f'}[\tilde{f}] = \sum_{\substack{\sigma \in G_{\tilde{f}} \\ \sigma|_{G_f}=1}} \sigma[\tilde{f}],$$

where for every $f \mid r$, $\tilde{f}$ is the unique stalk of r with the same prime factors of f. Now it is easy to verify that π is surjective and factors through U_r. By Proposition 4.1 which we'll prove later, $\mathcal{U}_r$ is a free abelian group with the same rank $\phi(r)$ as U_r, hence π induces an isomorphism from U_r to $\mathcal{U}_r$. □

3.3 The universal ordinary predistribution

Keep X, Y, Z, G, $\mathcal{O}$ and $\mathcal{T}$ the same as in § 3.2. Now let $p(p;t) = -t$ for $p \neq 2$ and let $p(2;t) = -t$, we call the resulting universal norm distribution the *universal ordinary predistribution*.

Proposition 3.2. *The universal ordinary predistribution is isomorphic to the integer ring of the cyclotomic number field $\mathbb{Q}(\mu_r)$ for each squarefree integer r.*

Proof. Define $\mathbf{e}_r : \mathcal{A}_r \to \mathcal{O}_{\mathbb{Q}(\mu_r)}$ by

$$[\sigma r'] \longmapsto \exp\left(\frac{2\pi i}{r'}\right)^{\sigma},$$

then immediately one has

(1) $\mathcal{D}_r \subseteq \ker \mathbf{e}_r$,
(2) e_r is surjective.

By Proposition 4.1 which we'll prove later, we know that $\mathcal{U}_r$ has $\mathbb{Z}$-rank $\phi(r)$, the same as $\mathcal{O}_{\mathbb{Q}(\nu_r)}$, thus $\mathbf{e}_r$ is an isomorphism. □

Remark 3.3. When r is not squarefree, then the above Proposition is actually not true. Indeed, the map $\mathbf{e}_r$ is not surjective in this case. This fact is pointed to the author by Prof. Anderson.

3.4 The universal Euler system

Let K be a fixed number field. Let p be a rational prime number. Let Φ be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be the ring of integer of Φ. Let T be a p-adic representation of G_K with coefficients in $\mathcal{O}$. Assume that T is unramified outside a finite set of primes of K.

Fix an ideal $\mathfrak{N}$ of K divisible by p and by all primes where T is ramified. Let X be the set of all primes x of K which is prime to $\mathfrak{N}$ and $K(x) \neq K(\mathbf{1})$, where $K(x)$ is the maximal p-extension inside the class field of K modulo x and $K(\mathbf{1})$ is the Hilbert p-class field of K. By class field theory, $K(x) \neq K(\mathbf{1})$ is a cyclic extension totally ramified at primes above x and unramified outside x.

Let Y and Z be defined following X. For every $y = x_1 \cdots x_n \in Y$, let $K(y)$ be the composite

$$K(y) = K(x_1) \cdots K(x_n).$$

Fix a $\mathbb{Z}_p^d$-extension K_∞/K which no finite prime splits completely. We write $K \subset_f F \subset K_\infty$ to indicate F/K a finite subextension of K_∞/K. For $K \subset_f F \subset K_\infty$, we let $F(y) = FK(y)$. Let $G_y = \mathrm{Gal}(F(y)/F(\mathbf{1})) \cong \mathrm{Gal}(K(y)/K(\mathbf{1}))$. Since $G_{x_1} \cap G_{x_2} = 1$ for every pair $x_1 \neq x_2$, we see that for any $y' \mid y$, $G_y = G_{y'} \times G_{y/y'}$. Let $G = G_{K(\mathbf{1})}$.

Let Fr_x denote a Frobenius of x in G_K, and let

$$p(x; t) = \det(1 - \mathrm{Fr}_x^{-1} t | T^*) \in \mathcal{O}[t].$$

Let $\mathcal{T} = \mathcal{O}[\mathrm{Gal}(F(\mathbf{1})/K)]$. With the above X, Y, $\mathcal{O}$, Φ and $p(x; t)$, the corresponding universal norm distribution $\mathcal{U}_y$(related to F) is called the *universal Euler system* of level (F, y). This definition is actually the same as the one introduced by Rubin in Chapter 4 of his book Euler systems [8]. Indeed, in Rubin's definition, the universal Euler system of level (F, y) is the quotient $\mathbf{Y}_{F,t}/\mathbf{Z}_{F,t}$ where $\mathbf{Y}_{F,t}$ is the free $\mathcal{O}[\mathrm{Gal}(F(y)/K)]$-module by generators $x_{F(y')}$ for $y' \mid y$, and $\mathbf{Z}_{F,t}$ are the relations

(1) $g x_{F(y')} = x_{F(y')}$ for $g \in \mathrm{Gal}(F(y)/F(y')) = G_{y/y'}$,
(2) $N_x x_{F(xy')} = p(x; \mathrm{Fr}_x^{-1}) x_{F(y')}$ for $xy' \mid y$.

One see our definition clearly is isomorphic to Rubin's by identifying the symbols $[y']$ and $x_{F(y')}$.

3.5 Function field case: I

Let $K = F_q(T)$ and $R = F_q[T]$. For any $f(T) \in R$, let $K(f) = K(\lambda_f)$ be the cyclotomic function field of K related to f where λ_f is a division point

of f with respect to the Carlitz module. The Galois group G_f of $K(f)/K$ is known to be isomorphic to $(R/f)^\times$. Thus we can identify every $\sigma = \sigma_x \in G_f$ for some(a unique) $x \in (R/f)^\times$. The *ordinary distribution of level* f on the function field K is defined to be a map

$$\phi : \frac{1}{f}R/R \longrightarrow Ab = \text{abelian group}$$

satisfying

$$\phi(x) = \sum_{py=x} \phi(y), \forall p \mid f, x \in \frac{p}{f}R/R.$$

One can then talk about the *universal ordinary distribution* as the universal object to the category of ordinary distributions. As in the number theory counterpart, by abusing notation, we say the group

$$U_f = \frac{\langle [a] : a \in \frac{1}{f}R/R \rangle}{\langle [a] - \sum_{pb=a}[b] : p \mid f, a \in \frac{p}{f}R/R \rangle}$$

the universal ordinary distribution. U_f is naturally equipped with a G_f-action by sending $\sigma_x[a] = [xa]$. The distribution U_f, as shown to be a free abelian group of order $|G_f|$, plays a similar role to the universal ordinary distribution in the study of cyclotomic function field,

Now let $G = G_K = \mathrm{Gal}(K^{sep}/K)$. Let X be the set of all monic irreducible polynomials in K and then Z_{fin} is nothing but the set of all monic polynomials in R. Let A be the discrete set $\{[g \circ f] : f \in Z_{\text{fin}}, g \in G_f\}$. Then G acts on A by setting $g \circ [f] = [f]$ if $g \in G_{K(f)}$. Let $p(\wp, t) = 1 - t$ for every $\wp \in X$. For $\mathcal{O} = \mathcal{T} = \mathbb{Z}$, we then can define the universal norm distribution $\mathcal{U}_f$ as the G_f-module

$$\mathcal{U}_f = \frac{\langle [\sigma f'] : f' \mid_s f, \sigma \in G_{f'} \rangle}{\langle (1 - \mathrm{Fr}_p^{-1})[\sigma f'] - N_{f(p)}[\sigma f(p) f'] : f(p) f' \mid_s f, \sigma \in G_{f'} \rangle}.$$

Proposition 3.4. *The modules U_f and $\mathcal{U}_f$ are isomorphic as G_f-modules by identifying $[1/f'] \in U_f$ and $[f'] \in \mathcal{U}_f$.*

Proof. The proof is similar to Proposition 3.1. One can easily check that: (1). the map $[1/f'] \in U_f \mapsto [f'] \in \mathcal{U}_f$ is well defined; (2). this map is a G_f-morphism; (3). surjective; (4). both U_f and $\mathcal{U}_f$ have $\mathbb{Z}$-rank $|G_f|$(the latter follows from Proposition 4.1). □

3.6 Function field case: II

We now work on more generality. Let K be a fixed function field. Pick a place ∞ in K. Let R be the integer ring corresponding to the place ∞. Choose a sign

function sgn on K^*. Let ϕ be a sign-normalized Drinfeld module of rank 1. The field H^+ is defined to be the extension of K by adding all the coefficients of ϕ_a for $a \in R$.

For any ideal I of R, let $K(I)$ be the cyclotomic function field extension of K related to I(and related to the sign-normalized Drinfeld module ϕ). Let now X be the set of all prime ideals of R, then Z_{fin} can be considered as the set of all integral ideals of R. Let $G = \text{Gal}(K^{sep}/H^+)$. Let G_I be the Galois group of $K(I)/H^+$. We know that $G_I = (R/I)^\times$. For the discrete set $A = \{g \circ I : I \in Z_{\text{fin}}, g \in G_I\}$, G thus defines a natural continuous action on A satisfying the axioms of the universal norm distribution for $H_I = \text{Gal}(K^{sep}/K(I))$. For any $\wp \in X$, define the Frobenius element $\text{Fr}_\wp$ correspondingly. Let $\mathcal{O} = \mathbb{Z}[\text{Gal}(H^+/K)]$. We can now define the universal norm distribution by choosing a free finite $\mathcal{O}$-module $\mathcal{T}$ (with ring structure) and a set of polynomials $\{p(\wp, t)\}$ for every $\wp \in X$.

In particular, if let $K = F_q(T)$ and let $R = F_q[T]$. Let the sign normalized Drinfeld module be the usual Carlitz module. In this case H^+ is actually just K. let $p(\wp, t) = 1 - t$ for every $\wp \in X$. Then we are back to the special case in the previous section.

One notes that in our definition, $\mathcal{T}$ and $\{p(\wp, t)\}$ are not specified. This actually gives us an advantage for applications. The Euler system in the function field case, due to Feng-Xu [2] and Xu-Zhao [11], has been used to prove results about ideal class groups and Gras conjecture in the function field case. By choosing $\mathcal{T}$ and $\{p(\wp, t)\}$ (and sometimes even $\mathcal{O}$), we can formulate the universal Euler system in the fuction field case just as Rubin did for the number field case. However, more study is needed for applications.

4. Basic properties of the universal norm distribution $\mathcal{U}_Z$

Recall by our definition, for every $z \in Z$, $\mathcal{A}_z$ is a free $\mathcal{T}$-module generated by the set

$$A_z = \bigcup_{\substack{z' \text{ finite} \\ z'|_s z}} B_{z'} = \bigcup_{\substack{z' \text{ finite} \\ z'|_s z}} \{[gz'] : g \in G_{z'}\}.$$

If let B_n be the set of all elements

$$\{[gz] \in A : \text{the restriction} g_{z(x)} = 1 \text{ for exactly } n \text{ primes } x \mid z'\}$$

Then A_z is the disjoint union

$$A_z = \bigcup_{n \geq 0} \bigcup_{\substack{z' \text{ finite} \\ z'|_s z}} (B_n \cap B_{z'}) .$$

We have the following key proposition.

Proposition 4.1. *The free $\mathcal{T}$-module $\mathcal{A}_z$ for every $z \in Z$, possesses a $\mathcal{T}$-basis*

$$\{\lambda_{z''}[gz'] : z', z'z'' \mid_s z,\ z'z'' \in Z_{fin},\ [gz'] \in B_0\}$$

where $\lambda_{z''}$ is defined to be the product of $\lambda_{z(x)}$ for all $x \mid z''$.

Proof. Suppose that $[gz'] \in B_n \cap A_z$ for $n \geq 1$, then there exists a prime $x \mid z'$ such that $g_{z(x)} = 1$. One has

$$[gz'] = -\sum_{1 \neq g' \in G_{z(x)}} [gg'z'] - \lambda_{z(x)}[gz'/z(x)] + p(x; \mathrm{Fr}_x^{-1})[gz'/z(x)].$$

Thus

$$\langle B_n \rangle_{\mathcal{T}} \cap \mathcal{A}_z \subseteq \langle B_{n-1} \rangle_T \cap \mathcal{A}_z + \sum_{x|z} \lambda_{z(x)} A_{z/z(x)} + \sum_{x|z} \mathcal{A}_{z/z(x)}$$

where $\langle B_n \rangle_{\mathcal{T}}$ denotes the free $\mathcal{T}$-module generated by B_n. Thus by induction, the set given in the proposition generates $\mathcal{A}_z$. We just need to show the cardinality of this set agrees with the $\mathcal{T}$-rank of $\mathcal{A}_z$. For finite $z \in Z$, the $\mathcal{T}$-rank of $\mathcal{A}_z$ is

$$\sum_{z'|_s z} |G_{z'}| = \prod_{x|z} (|G_{z(x)}| + 1).$$

On the other hand, the cardinality of the set in the proposition is

$$\begin{aligned}
\sum_{z''|_s z} \sum_{z'|_s \frac{z}{z''}} |B_0 \cap B_{z'}| &= \sum_{z''|_s z} \sum_{z'|_s \frac{z}{z''}} \prod_{x|z'} (|G_{z(x)}| - 1) \\
&= \sum_{z''|_s z} \prod_{x|\frac{z}{z''}} |G_{z(x)}| \\
&= \prod_{x|z} (|G_{z(x)}| + 1).
\end{aligned}$$

This proved the case when z is finite. Taking the limit, then we have the proof for infinite $z \in Z$. □

Proposition 4.2.

(*1*) *The module $\mathcal{U}_z$ is a free $\mathcal{T}$-module with basis $B_0 \cap A_z$.*
(*2*) *For every $z' \mid_s z$, the natural injection of $\mathcal{U}_{z'}$ to $\mathcal{U}_z$ is a splitting G_z-monomorphism.*

Proof. Immediately from Proposition 4.1. □

Remark 4.3. From the above Proposition 4.2(1), one see that $\mathcal{U}_z$ is a free $\mathcal{T}$ -module of rank $|G_z|$. In particular, in the trivial universal norm distribution case, one see that the image of B_z in $\mathcal{U}_z$ actually is a basis of $\mathcal{U}_z$, thus $\mathcal{U}_z$ is isomorphic to $\mathcal{T}[G_z]$, which justifies the meaning of *trivial*.

Remark 4.4. From the above Proposition 4.2(2), we'll henceforth identify $\mathcal{U}_{z'}$ as a submodule of $\mathcal{U}_z$. In particular, for every $z \in Z$, we have

$$\mathcal{U}_z = \bigcup_{\substack{z' \text{ finite} \\ z' |_s z}} \mathcal{U}_{z'}.$$

This observation will be used to the study of the universal Kolyvagin recursion in Ouyang [7].

Proposition 4.5. *Let $w \mid z$ be a pair of elements in Z. Then the corestriction homomorphism* $\mathrm{Cor}_{w,z}$ *from $\mathcal{A}_w$ to $\mathcal{A}_z$ by*

$$[w'] \longmapsto N_{w'}^{z'}[z'](w' \mid_s w, z' \mid_s z, \bar{w}' = \bar{z}')$$

induces an embedding from $\mathcal{U}_w$ to $\mathcal{U}_z$. In particular, when $w \mid_s z$, this embedding is the natural injection as given in Proposition 4.2.

Proof. Write V_1(resp. W_1) the free $\mathcal{T}$-submodule of $\mathcal{A}_w$ (resp. $\mathcal{A}_z$) generated by $B_0 \cap A_w$(resp. $B_0 \cap A_z$). Write V_2(resp. W_2) the free $\mathcal{T}$-submodule of $\mathcal{A}_w$ (resp. $\mathcal{A}_z$) generated by other elements in the basis of $\mathcal{A}_w$ (resp. $\mathcal{A}_z$) given by Proposition 4.1. Then it is easy to check that $\mathrm{Cor}_{w,z}$ maps V_i to W_i injectively. Hence it induces a well defined embedding from $\mathcal{U}_w$ to $\mathcal{U}_z$. □

5. Anderson's resolution

5.1 Set up

Let $z \in Z$ be given. Let

$$\mathcal{L}_z = \bigoplus_{y \mid \bar{z}} \mathcal{A}_{z/z(y)}[y]$$

where y is finite and $[y]$ is a symbol depending only on y. If we write

$$[g'z'][y] = [g'z', y]$$

for elements in $\mathcal{A}_{z/z(y)}[y]$, then $\mathcal{L}_z$ is the free $\mathcal{T}$-module generated by the set

$$\{[a, y] : [a] \in A_{z/z(y)}, y \mid \bar{z}\}$$

We assign a grade in $\mathcal{L}_z$ by declaring

$$\deg[a, y] = -\deg y.$$

For any $g \in G_z$ and $[g'z'] \in A_{z/z(y)}$, declare the G_z-action as

$$g[g'z', y] := [g_{z'}g'z', y],$$

then $\mathcal{L}_z$ becomes a graded $\mathcal{T}[G_z]$-module. $\mathcal{L}_z$ is bounded above since all its non-negative components are 0. Moreover, $\mathcal{L}_z$ is bounded if and only if z is finite.

With abuse of notation, denote by $\lambda_{z(x)}$, $\lambda_{z'}$ the homomorphisms of $\mathcal{L}_z$ inheriting from the homomorphisms in $\mathcal{A}_z$ bearing the same names. Now let

$$d : \mathcal{L}_z \longrightarrow \mathcal{L}_z,\ [a, y] \longmapsto \sum_{x|y} \omega(x, y)\lambda_{z(x)}[a, y/x]$$

where ω is as defined in § 2.1. Clearly d commutes with G_z-actions. A straightforward calculation shows that $d^2 = 0$ and therefore d is a differential of degree 1. Define an $\mathcal{T}[G_z]$-homomorphism $\mathbf{u} : \mathcal{L}_z \to \mathcal{U}_z$ by

$$[a, y] \longmapsto \begin{cases} [a], & \text{if } y = \mathbf{1}; \\ 0, & \text{if } y \neq \mathbf{1}. \end{cases}$$

Regard $\mathcal{L}_z$ as a complex $\mathcal{L}_z^\bullet$ by the differential d, and regard $\mathcal{U}_z$ as a complex concentrated on 0-component. Then one can easily check that $\mathbf{u}$ is a homomorphism of complexes. Because of the following Theorem, we call the complex $(\mathcal{L}_z^\bullet, d)$(or simply $\mathcal{L}_z^\bullet$) *Anderson's resolution* of the universal norm system $\mathcal{U}_z$.

Theorem 5.1. *The homomorphism* $\mathbf{u}$ *is a quasi-isomorphism, i.e., the complex* $(\mathcal{L}_z^\bullet, d)$ *is acyclic for degree* $n \neq 0$ *and* $H^0(\mathcal{L}_z^\bullet, d) \cong \mathcal{U}_z$ *induced by* $\mathbf{u}$.

Proof. For any $a \in B_0 \cap B_{z/z(y)}$, consider the graded $\mathcal{T}$-submodule $C_a^\bullet$ of $\mathcal{L}_z^\bullet$ generated by

$$\{\lambda_w[a, y'],\ w \mid_s z, \bar{w}y' \mid y\}.$$

One can see that $C_a^\bullet$ is d-stable. Thus $C_a^\bullet$ is actually a subcomplex of $\mathcal{L}_z^\bullet$. By Proposition 4.1, $\mathcal{L}_z^\bullet$ is the direct sum of $C_a^\bullet$ for a over $B_0 \cap A_z$. We hence only have to study the complex $C_a^\bullet$. Now the theorem follows from Lemma 5.2. □

5.2 The Koszul complex $\tilde{C}_y^\bullet$

Let Λ be the polynomial ring

$$\Lambda = \mathcal{T}[Z] = \{\sum t_z z : t_z \in T, z \in Z\}.$$

Let $\tilde{C}_y^\bullet$ be the Koszul complex of Λ with the regular sequence $x_1 < \cdots < x_m$ where $y = x_1 \cdots x_m$. Thus $\tilde{C}_y^\bullet$ is the graded exterior algebra

$$\bigoplus_{y'|y} \Lambda e_{y'}$$

with

$$e_{y'} = e_{x_{i_1}} \wedge \cdots \wedge e_{x_{i_k}}, \text{ and } \deg e_{y'} = -\deg y' = -k$$

where

$$y' = x_{i_1} \cdots x_{i_k}, x_{i_1} < \cdots < x_{i_k}.$$

The corresponding differential is given by

$$d\, e_x = x.$$

5.3 Truncated Koszul subcomplex $C_y^\bullet$

Let $C_y^\bullet$ be the graded $\mathcal{T}$-submodule of $\tilde{C}_y^\bullet$ generated by all elements of the form $y''e_{y'}$ for all $y'y'' \mid y$. This submodule is stable under the differential, thus is a subcomplex of $\tilde{C}_y^\bullet$. Moreover, it is a direct summand of $\tilde{C}_y^\bullet$. By the general theory of Koszul complex, $C_y^\bullet$ is acyclic in nonzero degree and $H^0(C_y^\bullet)$ is a free $\mathcal{T}$-module generated by $e_{\mathbf{1}}$.

Lemma 5.2. *For any $a \in B_0 \cap B_{z/z(y)}$, the complex $C_a^\bullet$ is isomorphic to $C_y^\bullet$. Thus $C_a^\bullet$ is acyclic in nonzero degree and $H^0(C_a^\bullet)$ is a free $\mathcal{T}$-module generated by $[a, 1]$.*

Proof. Let $C_y^\bullet$ act on $C_a^\bullet$ by

$$x[a, y'] = \lambda_{z(x)}[a, y']$$

and

$$e_x[a, y'] = \begin{cases} (-1)^{|\{x' < x : x' \mid y'\}|}[a, xy'] & \text{if } x \nmid y'; \\ 0 & \text{if } x \mid w. \end{cases}$$

By straightforward calculation

$$d(\xi\eta) = (d\xi)\eta + (-1)^{\deg \xi}\xi(d\eta), \xi \in C_y^\bullet, \eta \in C_x^a.$$

Thus $C_a^\bullet = C_y^\bullet[a, 1]$. □

5.4 Compatibility

From Proposition 4.5, the injective corestriction homomorphism $\mathrm{Cor}_{w,z}$ from $\mathcal{A}_w$ to $\mathcal{A}_z$ induces a corestriction homomorphism Cor from $\mathcal{L}_w$ to $\mathcal{L}_z$ by

$$[a, y] \longmapsto \mathrm{Cor}_{w/w(y),z/z(y)}[a, y].$$

A straightforward calculation shows that Cor is compatible with the differential d. Now if let $\mathcal{Z}_z$ be the extended exact sequence of $\mathcal{U}_z$ to $\mathcal{L}_z$, i.e., $\mathcal{Z}_z$ is the sequence

$$\cdots \mathcal{L}_z^{-n} \to \cdots \to \mathcal{L}_z^0 \xrightarrow{\mathbf{u}} \mathcal{U}_z \to 0$$

then the corestriction map Cor is actually an injective chain homomorphism from $\mathcal{Z}_w$ to $\mathcal{Z}_z$ and is thus an embedding. When $w \mid_s z$, this embedding Cor is again a natural injection.

5.5 Connecting map for different norm distributions

Now fix X and $\mathcal{T}$, suppose that we have two sets of polynomials $\{p_1(x;t)\}$ and $\{p_2(x;t)\}$ in $\mathcal{O}[t]$, then we have two norm distributions $\mathcal{U}_{1,z}$ and $\mathcal{U}_{2,z}$, and two corresponding Anderson's resolutions $\mathcal{L}_{1,z}$ and $\mathcal{L}_{2,z}$. Then there exists a connecting homomorphism

$$\phi_{1,2} : \mathcal{L}_{1,z} \otimes_{\mathcal{O}} \Phi \longrightarrow \mathcal{L}_{2,z} \otimes_{\mathcal{O}} \Phi$$

by

$$[z', y] \longmapsto \sum_{w|_s z'} (-1)^{\deg \bar{w}} \prod_{x|w} \frac{p_2(x, \mathrm{Fr}_x^{-1}) - p_1(x, \mathrm{Fr}_x^{-1})}{|G_{z(x)}|} [z'/w, y].$$

By straightforward calculation, one can check that $\phi_{2,1}$ is the inverse of $\phi_{1,2}$, thus $\phi_{1,2}$ is actually an isomorphism, which induces isomorphisms between $\mathcal{U}_{1,z} \otimes_{\mathcal{O}} \Phi$ and $\mathcal{U}_{2,z} \otimes_{\mathcal{O}} \Phi$. In particular, if we let $p_1(x;t) \equiv 1$ for every $x \in X$, then $\mathcal{U}_{1,z} \otimes_{\mathcal{O}} \Phi$ is nothing but the module $\mathcal{T}[G_z]$, thus we have

Proposition 5.3. *The $\mathcal{T} \otimes_{\mathcal{O}} \Phi[G_z]$ module $\mathcal{U}_z \otimes_{\mathcal{O}} \Phi$ is free of rank 1 for every universal norm distribution.*

5.6 Double complex structure of $\mathcal{L}_z$

Set a bidegree in $\mathcal{L}_z$ by

$$\deg^{(2)}[z', y] = (\deg \bar{z}', -\deg \bar{z}' - \deg y).$$

We set

$$d_{1,x}[z', y] = -\omega(x, y) N_{z(x)}[z'z(x), y/x],$$
$$d_{2,x}[z', y] = \omega(x, y) p(x; \mathrm{Fr}_x)[z', y/x].$$

and let

$$d_x = d_{1,x} + d_{2,x}, \quad d_1 = \sum_x d_{1,x}, \quad d_2 = \sum_x d_{2,x}.$$

Lemma 5.4. (*1*) *For any* $x, x' \mid z$, $i = 1, 2$,

$$d_{i,x}^2 = d_{1,x}d_{2,x'} + d_{2,x'}d_{1,x} = 0.$$

(*2*) $d_1^2 = d_2^2 = d_1d_2 + d_1d_2 = 0$.
(*3*) $d_{i,x}$ *is* G_z*-stable.*

Proof. Straightforward. □

From the above lemma, we see that $\mathcal{L}_z$ is equipped with a multiple complex structure. In particular, $(\mathcal{L}_z^{\bullet,\bullet}; d_1, d_2)$ is a double complex corresponding to the above bigrading. We'll use this complex to study the group cohomology of $\mathcal{U}_z$ in § 7.

6. Preparation from homological algebra

6.1 Complex of type E

Let A be a free $\mathcal{O}$-module of finite rank. Let $\Lambda_A = \Lambda_A(x_1, \cdots, x_t)$ be the exterior algebra over A, with the differential d given by $d(x) = \sum_i m_i x \wedge x_i$ where $m_i \in \mathcal{O}$. For each $S \subseteq \{1, \cdots, t\}$, let m_S be the greatest common divisor of m_i for all $i \in S$. In particular, let m be the greatest common divisor of m_i for all $1 \leq i \leq t$.

Let $S = \{i_1, \cdots, i_s\}$ such that $i_1 \leq \cdots \leq i_s$. Let $\{e_S = x_{i_1} \wedge \cdots \wedge x_{i_s}\}$ be the standard basis of Λ_A. By linear algebra, in the Φ-vector space generated by $\{x_1, \cdots, x_t\}$, there exists another basis $\{y_1, \cdots, y_t\}$ such that $y_1 = \frac{1}{m}\sum_i m_i x_i$ and the transformation matrix is inside $SL(t, \mathbb{Z})$, thus $\{e'_S = y_{i_1} \wedge \cdots \wedge y_{i_s}\}$ is another basis for Λ_A. Hence one can easily show that $H^*(\Lambda_A)$ is a free graded A/mA-module generated by cocycles represented by e'_S for all S which contains 1, thus is a free A/mA-module of rank 2^{t-1}, with the i-th component a free A/mA-module of rank $\binom{t-1}{i-1}$(or 0 if $i = 0$).

6.2 The tensor projective resolution $P_{z\bullet}$

This setup is from Ouyang [5]. Fix an element $z \in Z$. Assuming that $G_{z(x)}$ is a cyclic group for every $x \mid z$. Let $\sigma_{z(x)}$ be a generator of $G_{z(x)}$. It is well known that the sequence

$$\cdots \mathbb{Z}[G_{z(x)}] \xrightarrow{N_{z(x)}} \mathbb{Z}[G_{z(x)}] \xrightarrow{1-\sigma_{z(x)}} \mathbb{Z}[G_{z(x)}] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is exact, where ϵ is the augmentation map. Let $P_{z(x)\bullet}$ be the resulting resolution for the trivial $\mathbb{Z}[G_{z(x)}]$-module $\mathbb{Z}$, we can thus write $P_{z(x)\bullet}$ as the graded module

$$\bigoplus_{n\geq 0} \mathbb{Z}[G_{z(x)}][x^n]$$

with the symbol $[x^n]$ is of degree n and the differential given by

$$\partial_{z(x)}[x^n] = \begin{cases} (1-\sigma_{z(x)})[x^{n-1}], & \text{if } n > 0 \text{ odd;} \\ N_{z(x)}[x^{n-1}], & \text{if } n > 0 \text{ even.} \end{cases}$$

Now let $P_{z\bullet}$ as the tensor product of $P_{z(x)\bullet}$ over all $x \mid z$. $P_{z\bullet}$ is the so called *tensor projective resolution* of the trivial $\mathbb{Z}[G_z]$-module $\mathbb{Z}$ with respect to the cyclic decomposition

$$G_z = \prod_{x|z} G_{z(x)} = \prod_{x|z} \langle \sigma_{z(x)} \rangle.$$

Let $[w]$ be an indeterminate for every $w \in Z$. Then the tensor resolution $P_{z\bullet}$ is the projective $\mathbb{Z}[G_z]$-resolution of the trivial module $\mathbb{Z}$ by

$$P_{z,n} = \bigoplus_{\substack{\bar{w}|z \\ \deg w = n}} \mathbb{Z}[G_z][w]$$

and the differential ∂_z is given by

$$\partial_z[w] = \sum_{x|w} (-1)^{\sum_{x'<x} v_{x'}w} \alpha_{z(x)}[w/x]$$

where $\alpha_{z(x)}$ is equal to $\sigma_{z(x)} - 1$ if $v_x w$ odd and $N_{z(x)}$ if $v_x w$ even. For any $z' \mid_s z$, one has a natural inclusion of $P_{z'\bullet}$ to $P_{z\bullet}$ by sending $[w]$ to $[w]$.

6.3 G_z-cohomology of trivial module A

Let A be a free $\mathcal{O}$-module with trivial G_z-structure. To compute its G_z-cohomology, it suffices to compute the cohomology

$$I^\bullet_{A,z} = \mathrm{Hom}_{\mathbb{Z}[G_z]}(P_{z\bullet}, A) = \bigoplus_{\substack{w \text{ finite} \\ \bar{w}|z}} A[w]$$

with the differential

$$\delta_z[w] = \sum_{x|z} (-1)^{\sum_{x'<x} v_{x'}w} a_{z(x)}[wx]$$

where $a_{z(x)}$ is equal to 0 if $v_x w$ even and to $|G_{z(x)}|$ if $v_x w$ odd. The inclusion of $P_{z'\bullet}$ to $P_{z\bullet}$ for $z' \mid_s z$ thus induces a projection from $I^\bullet_{A,z}$ to $I^\bullet_{A,z'}$. One see that $I^\bullet_{A,z'}$ is a direct summand of $I^\bullet_{A,z}$.

For any finite w with $\bar{w} \mid z$, let

$$I^\bullet_A[w^2] = \bigoplus_{w'|\bar{w}} A[w^2/w'],$$

then $I_A^\bullet[w^2]$ is a direct summand of $I_{A,z}^\bullet$ and

$$I_{A,z}^\bullet = \bigoplus_{\bar{w}|\bar{z}} I_A^\bullet[w^2].$$

If $w = 1$, the subcomplex $I_A^\bullet[w^2]$ is just a copy of A with the differential 0, thus the cohomology of it is A too. If $w \neq 1$, the subcomplex $I_A^\bullet[w^2]$ is of type E. Let m_w be the greatest common divisor of $|G_{z(x)}|$ for $x \mid w$, then $H^*(I_A^\bullet[w^2])$ is then a free graded A/m_wA-module of rank $2^{\deg \bar{w}-1}$. One see the $(2\deg w - \deg \bar{w} + i)$-th cohomology is just a free A/m_wA-module of rank $\binom{\deg \bar{w}-1}{i-1}$ for $1 \leq i \leq \deg w$ and 0 otherwise.

Denote $H^*(I_A^\bullet[w^2])$ by $H_{A,w}$. Then with the above analysis, one has

Proposition 6.1. *Fix a finite $z \in Z$ such that every $G_{z(x)}$ is cyclic for $x \mid z$. For a free $\mathcal{O}$-module A with trivial G_z-action, then we have*

(1) *For any $z' \mid_s z$, the cohomology group $H^*(G_{z'}, A)$ is a direct summand of $H^*(G_z, A)$.*
(2) *The cohomology group $H^*(G_z, A)$ is the direct sum of $H_{A,w}$ for every $\bar{w} \mid \bar{z}$ where: (a). For $w = 1$, $H_{A,w} = A$ is with grade 0; (b). For $w \neq 1$, $H_{A,w}$ is a free graded A/m_wA-module with the $(2\deg w - \deg \bar{w} + i)$-th component of rank $\binom{\deg \bar{w}-1}{i-1}$ for $1 \leq i \leq \deg \bar{w}$ and 0 for otherwise.*

Remark 6.2.

Now for a finite fixed $z \in Z$, suppose $M \in \mathcal{O}$a common divisor of $|G_{z(x)}|$ for every $x \mid z$. Then the case for G_z-cohomology of A/MA is much simpler. In this case,

$$H^*(G_z, A/MA) = H^*(I_{A,z}^\bullet/MI_{A,z}^\bullet),$$

and the differential in $I_{A,z}^\bullet/MI_{A,z}^\bullet$ is nothing but 0, thus $H^*(G_z, A/MA)$ as a graded module is isomorphic to $I_{A,z}^\bullet/MI_{A,z}^\bullet$. One has

Proposition 6.3. *There exists a family*

$$\{[w] \in H^*(G_z, A/MA) : w \text{ finite}, \bar{w} \mid z\}$$

with the following properties:

(1) *For any $z' \mid_s z$, the restriction of the family*

$$\{[w] : \bar{w} \mid \bar{z}', \deg w = n\}$$

to $H^n(G_{z'}, A/MA)$ is an A/MA-basis of the latter one.
(2) *The restriction of $[w]$ for $\bar{w} \nmid \bar{z}'$ to $H^*(G_{z'}, A/MA)$ is 0.*

7. G_z-cohomology of the universal norm distribution $\mathcal{U}_z$

In this section, we use tools developed in the previous sections to study the G_z-cohomology of the universal norm distribution $\mathcal{U}_z$ and of $\mathcal{U}_z/M\mathcal{U}_z$. We assume that $G_{z(x)}$ cyclic for every $z \in Z$ and M a common divisor of $|G_{z(x)}|$ for every $x \mid z$.

7.1 Setup of double complex $\mathbf{K}_z^{\bullet,\bullet}$

With preparations from the above two sections, we let

$$\mathbf{K}_z^{\bullet,\bullet} = \mathrm{Hom}_{G_z}(P_{z\bullet}, \mathcal{L}_z^{\bullet})$$

If we write $[a, y, w] = ([w] \mapsto [a, y])$, then $\mathbf{K}_z^{\bullet,\bullet}$ is the free graded $\mathcal{T}$-module with basis

$$\{[a, y, w] : y \mid \bar{z}, a \in A_{z/z(y)}, \bar{w} \mid \bar{z}\}$$

and with the double grading given by

$$\deg[a, y, w] = (-\deg y, \deg w).$$

The induced $\mathcal{T}[G_z]$-module structure is given by

$$g[a, y, w] = [gx, y, w]$$

for any $g \in G_z$. Use the same notations for the operators in $\mathbf{K}_z^{\bullet,\bullet}$ induced from $\mathcal{L}_z^{\bullet}$, i.e., $\lambda_{z(x)}$, λ_z and so on. Now the two differentials of $K_z^{\bullet,\bullet}$ are given by

$$d[a, y, w] = \sum_{x|y} \omega(x, y) \left(p(x; \mathrm{Fr}_x^{-1})[a, y/x, w] - N_{z(x)}[z(x)a, y/x, w]\right),$$

$$\delta[a, y, w] = (-1)^{\deg y} \sum_{x|z} (-1)^{\sum_{x'<x} v_{x'}(w)} a_{z(x)}[a, y, wx]$$

where $a_{z(x)}$ is equal to $1 - \sigma_{z(x)}$ if $v_x(w)$ even and $N_{z(x)}$ if $v_x(w)$ odd. Let $\mathbf{K}_z^{\bullet}$ be the single total complex of $\mathbf{K}_z^{\bullet,\bullet}$. and let $\mathbf{K}_z$ be the underlying module.

Let $\bar{\mathbf{K}}_z^{\bullet} = \mathrm{Hom}_{G_z}(P_{z\bullet}, \mathcal{U}_z)$. Then it is the quotient of free $\mathcal{T}$-module generated by

$$\{[a, w], a \in A_z, \bar{w} \mid \bar{z}\}$$

modulo relations generated by

$$\lambda_{z(x)}[a, w],\ a \in A_{z/z(x)},\ \bar{w} \mid \bar{z}, \forall\, x \mid z,$$

with the differential δ given by

$$\delta[a, w] = \sum_{x \in z} (-1)^{\sum_{x'<x} v_{x'} w} a_{z(x)}[wx].$$

We have the induced map

$$\mathbf{u} : \mathbf{K}_z^\bullet \longrightarrow \bar{\mathbf{K}}_z^\bullet,\ [a, y, w] \longmapsto \begin{cases} [a, w], & \text{if } y = \mathbf{1}; \\ 0, & \text{if } y \neq \mathbf{1}. \end{cases}$$

Proposition 7.1. *The homomorphism* $\mathbf{u}$ *is a quasi-isomorphism. Thus*

(1) $H^*(\mathbf{K}_z^\bullet, d + \delta) \cong H^*(G_z, \mathcal{U}_z)$.
(2) $H^*(\mathbf{K}_z^\bullet / M\mathbf{K}_z^\bullet, d + \delta) \cong H^*(G_z, \mathcal{U}_z / M\mathcal{U}_z)$.

Proof. By Theorem 5.1, $\ker \mathbf{u}$ is d-acyclic, by spectral sequence argument, it is hence $(d + \delta)$-acyclic. Thus $\mathbf{u}$ is a quasi-isomorphism. (1) follows immediately from the quasi-isomorphism. Since both $\mathbf{K}_z^\bullet$ and $\mathcal{U}_z$ are free $\mathcal{T}$-modules, the induced homomorphism $\bar{\mathbf{u}}$ from $\mathbf{K}_z^\bullet / M\mathbf{K}_z^\bullet$ to $\bar{\mathbf{K}}_z^\bullet / M\bar{\mathbf{K}}_z^\bullet$ is also a quasi-isomorphism and (2) follows immediately. □

7.2 Another double complex structure of $\mathbf{K}_z$

Keep $\mathbf{K}_z$ as the same bigraded module as in the previous section. Let's equip it with different differentials $(\tilde{d}, \tilde{\delta})$ as the following:

$$\begin{aligned} \tilde{d}[a, y, w] = &\sum_{x|y} \omega(x, y)(-1)^{\sum_{x'<x} v_{x'}(w)} \\ &\left(p(x; \mathrm{Fr}_x^-)[a, y/x, w] - N_{z(x)}[az(x), y/x, w]\right), \\ \tilde{\delta}[a, y, w] = &\sum_{x|z} (-1)^{\sum_{x'\leq x} v_{x'}(y)} (-1)^{\sum_{x'<x} v_{x'}(w)} a_{z(x)}[a, y, wx]. \end{aligned}$$

One can easily check that

$$\tilde{d}^2 = \tilde{\delta}^2 = \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d} = 0.$$

We define an involutive G_z-equivariant bigraded automorphism ϵ of $\mathbf{K}_z$ by the rule

$$\epsilon[a, y, w] = (-1)^{\sum_{x,x':x'<x} v_x(y) v_{x'}(w)}.$$

by a straightforward calculation, one finds that

$$\epsilon\, \tilde{d}\, \epsilon = d, \qquad \epsilon\, \tilde{\delta}\, \epsilon = \delta.$$

Thus ϵ induces an isomorphism between the cohomology of $(\mathbf{K}; \tilde{d}, \tilde{\delta})$ and the cohomology of $(\mathbf{K}; d, \delta)$, which is then isomorphic to the G_z-cohomology of $\mathcal{U}_z$.

In the sequel, we'll use the double complex $(\mathbf{K}; d, \delta)$ to study the cohomology of $\mathcal{U}_z$. However, the results obtained here is easy to adapt to the double complex $(\mathbf{K}; \tilde{d}, \tilde{\delta})$. The double complex $(\mathbf{K}; \tilde{d}, \tilde{\delta})$ will be used to the study of the universal Kolyvagin recursion in Ouyang [7].

7.3 Multiple complex structure of $\mathbf{K}_z$

The underlying module $\mathbf{K}_z$ has abundant complex structures. For $x \mid z$, set

$$\begin{aligned}\deg_{1,x}([z', y, w]) &:= v_x(z'),\\ \deg_{2,x}([z', y, w]) &:= v_x(yz'),\\ \deg_{3,x}([z', y, w]) &:= v_x(w).\end{aligned}$$

We call $\deg_{i,x}([z', y, w])$ for $i = 1, 2, 3$ the (i, x)-degree of $[z', y, w]$. Make the degrees invariable with G_z action, then $\mathbf{K}_z$ is equipped with a multi-graded module structure. Let

$$\begin{aligned}d_{1,x}[a, y, w] &:= -\omega(x, y)N_{z(x)}[az(x), y/x, w]\\ d_{2,x}[a, y, w] &:= \omega(x, y)p(x; \mathrm{Fr}_x^{-1})[a, y/x, w]\\ d_{3,x}[a, y, w] &:= (-1)^{\deg y}(-1)^{\sum_{x'<x} v_{x'}w}a_{z(x)}[a, y, wx].\end{aligned}$$

The map $d_{i,x}$ is of (i, x)-degree $+1$. It is easy to check that for every $i, j = 1, 2, 3$ and $(i, x) \neq (j, x')$, one has

$$d^2 i, x = d_{i,x}d_{j,x'} + d_{j,x'}d_{i,x} = 0.$$

Thus $d_{i,x}$ are differentials of $\mathbf{K}_z$ observing the above multi-grading structure. One see that d is the sum of all $d_{i,x}$ for $i = 1, 2$ and $x \mid z$ and δ is the sum of $d_{3,x}$. The total degree of $\mathbf{K}_z$ is just the sum of all (i, x)-degrees. Thus we can use this multi-complex structure to study the total cohomology of $\mathbf{K}_z$ and hence the G_z-cohomology of $\mathcal{U}_z$.

Furthermore, note that any combination of $d_{i,x}$ is still a differential in $\mathbf{K}_z$. In particular, $d_i = \sum_{x|z} d_{i,x}$ for $i = 1, 2$ is the differential induced by the differential d_i in $\mathcal{L}_z$ when viewing $\mathcal{L}_z$ as a double complex. We have $d = d_1 + d_2$ and $\delta = \sum_{x|z} d_{3,x}$. Correspondingly, we can make $\mathbf{K}_z$ as a triple complex $\mathbf{K}_z^{\bullet,\bullet,\bullet}$ with differentials d_1, d_2 and δ. As a convention, we use $m, n, p = m + n$ and q to denote the corresponding degrees for the differentials d_1, d_2, d and δ. We shall use this triple complex structure of $\mathbf{K}_z$ to study the total cohomology of $\mathbf{K}_z^\bullet$.

7.4 Compatibility

For every $z' \mid_s z$, let $\mathbf{K}_{z'}$ be the submodule of $\mathbf{K}_z$ generated by

$$\{[a, y, w] : y \mid z', a \in B_{z'/z(y)}, \bar{w} \mid z'\}$$

and let $\mathbf{K}_z(z')$ be the submodule generated by

$$\{[a, y, w] : y \mid z', a \in B_{z'/z(y)}, \bar{w} \mid z\}$$

One can check that $\mathbf{K}_{z'}$ and $\mathbf{K}_z(z')$ are compatible with differentials. The $(d+\delta)$-cohomology of $\mathbf{K}_{z'}$ is just $H^*(G_{z'}, \mathcal{U}_{z'})$ and the $(d+\delta)$-cohomology of $\mathbf{K}_z(z')$ is $H^*(G_z, \mathcal{U}_{z'})$. Moreover, if using the embedding Cor defined in § 5.4 for Anderson's resolution, then for every $w \mid z$, one has a well defined embedding from $\mathbf{K}_w$ to $\mathbf{K}_z$.

7.5 The study of spectral sequences

We now discuss the G_z-cohomology of $\mathcal{U}_z$ and $\mathcal{U}_z/M\mathcal{U}_z$. We study the triple complex $(\mathbf{K}_z^{\bullet,\bullet,\bullet}; d_1, d_2, \delta)$, or rather, fix n, we study the double complex $(\mathbf{K}_z^{\bullet,n,\bullet}; d_1, \delta)$. Consider the spectral sequence

$$E_2^{m,q}(\mathbf{K}_z^{\bullet,n,\bullet}) = H_{d_1}^m H_\delta^q(\mathbf{K}_z^{\bullet,n,\bullet}).$$

Since $H_\delta^q(\mathbf{K}_z^{\bullet,n,\bullet})$ is just $H^q(G_z, \mathcal{L}_z^{\bullet,n})$, which is the direct sum of subcomplexes of the following form for all $y \mid \bar{z}$, $\deg y = -n$:

$$\begin{aligned} 0 \to H^q(G_z, [\mathcal{B}_{\mathbf{1}}, y]) &\xrightarrow{d_1^1} \cdots \xrightarrow{d_1^1} \\ \bigoplus_{\substack{y' \mid y \\ \deg y' = -p}} H^q(G_z, [\mathcal{B}_{z(y')}, y/y']) &\cdots \xrightarrow{d_1^1} H^q(G_z, [\mathcal{B}_{z(y)}, \mathbf{1}]) \to 0 \end{aligned} \tag{1}$$

where

$$[\mathcal{B}_{z'}, y'] := \langle [a, y'] : a \in B_{z'} \rangle_{\mathcal{T}} \cong \mathcal{B}_{z'}.$$

Note that for any $y' \mid y \mid \bar{z}$,

$$\mathcal{B}_{z(y)} = \mathcal{B}_{z(y')} \otimes_{\mathcal{T}} \mathcal{T}[G_{z(y/y')}].$$

One has a commutative diagram

$$\begin{array}{ccc} H^q(G_z, [\mathcal{B}_{z(y')}, y]) & \xrightarrow{-\omega(x,y)d_{1,x}^1} & H^q(G_z, [\mathcal{B}_{z(y'/x)}, y/x]) \\ \theta \downarrow & & \theta \downarrow \\ H^q(G_{z/z(y')}, [\mathcal{B}_{\mathbf{1}}, y]) & \xrightarrow{res} & H^q(G_{z/z(y'x)}, [\mathcal{B}_{\mathbf{1}}, y/x]) \end{array}$$

where θ is the isomorphism induced by Shapiro's Lemma. Note that $[\mathcal{B}_{\mathbf{1}}, y]$ is just one copy of $\mathcal{T}$ indexed by y, we write it as $\mathcal{T}[y]$. Through θ, the complex (1) is then quasi-isomorphic to

$$\begin{aligned} 0 \to H^q(G_z, \mathcal{T}[y]) &\cdots \to \\ \bigoplus_{\substack{y' \mid y \\ \deg y' = -p}} H^q(G_{z/z(y')}, \mathcal{T}[y/y']) &\cdots \to H^q(G_{z/z(y)}, \mathcal{T}[\mathbf{1}]) \to 0 \end{aligned} \tag{2}$$

with the differential

$$\tilde{d}(c) = -\sum_{x|y/y'} \omega(x, y/y') res_x c$$

for

$$c \in H^q(G_{z/z(y')}, \mathcal{T}[y/y']),\ res_x \text{ is the restriction of } c \text{ in } H^q(G_{z/z(xy')}, \mathcal{T}[y/y'x]).$$

If replace q in the complex (2) above by $*$, then we have a complex

$$0 \to H^*(G_z, \mathcal{T}[y]) \cdots \to \bigoplus_{\substack{y'|y \\ \deg y' = -p}} H^*(G_{z/z(y')}, \mathcal{T}[y/y']) \cdots \to H^*(G_{z/z(y)}, \mathcal{T}[\mathbf{1}]) \to 0 \qquad (3)$$

Lemma 7.2. *The complex* (3) *is acyclic except at the first cohomology while the first cohomology is the direct sum of free graded $\mathcal{T}/m_w\mathcal{T}$-modules $H_{\mathcal{T},w}$ for $y \mid w \mid z$, where $m_w = \gcd\{|G_{z(x)}| : x \mid w\}$ and the grading of $H_{T,w}$ is as stated in Proposition 6.1.*

Proof. Since $\mathcal{T}$ is a trivial G_z-module, we can apply the results of Proposition 6.1 here. The first cohomology is just

$$\bigcap_{x|y} \ker(H^*(G_z, \mathcal{T}) \to H^*(G_{z/z(x)}, \mathcal{T})),$$

which is nothing but the direct sum of $H_{\mathcal{T},w}$ for $y \mid w \mid z$ by Proposition 6.1. Apply Proposition 6.1 again, we see the complex (3) satisfies the conditions of Lemma 5.2 of Ouyang [5], Page 16. Following that lemma, we know other cohomology groups vanish for the complex (3). □

Write $H^q_{\mathcal{T},w}$ the q-th component of $H_{\mathcal{T},w}$, we thus have

Proposition 7.3. *For any fixed n, the $E_2^{m,q}$ term $H^m_{d_1} H^q_\delta(\mathbf{K}_z^{\bullet,n,\bullet})$ of the double complex $(\mathbf{K}_z^{\bullet,n,\bullet}; d_1, \delta)$, is then the direct sum of free $\mathcal{T}/m_w\mathcal{T}$-modules $H^q_{\mathcal{T},w}[y]$ where*

$$\deg y = -n, \qquad y \mid \bar{w} \mid z$$

and the $\mathcal{T}/m_w\mathcal{T}$-rank of $H^q_{\mathcal{T},w}[y]$ is $\binom{\deg \bar{w}-1}{i-1}$ if $q = 2\deg w - \deg \bar{w} + i$.

7.6 The case $p(x;1) = 0$ for every $x \mid z$

In this subsection, we suppose that $p(x;1) = 0$ for every $x \mid z$. In this case, we can give a complete description of the G_z-cohomology of $\mathcal{U}_z$. Consider the $\mathcal{T}$-submodule $\mathbf{S}$ of $\mathbf{K}_z$ generated by

$$\{[a, y, w] : a \in B_{z/z(y)},\ y \mid z,\ \bar{w} \mid z, a \notin B_{\mathbf{1}} \text{ if } y \mid w\}.$$

Under the assumption $p(x;1) = 0$, one easily sees that $d_1\mathbf{S}, d_2\mathbf{S}, \delta\mathbf{S} \subseteq \mathbf{S}$, thus $\mathbf{S}$ is really a subcomplex of $\mathbf{K}_z$ with related double and triple complex structures. We let $\mathbf{Q}_z = \mathbf{K}_z/\mathbf{S}$, thus $\mathbf{Q}_z$ is a free $\mathcal{T}$-module generated by

$$\{[\mathbf{1}, y, w] : y \mid \bar{w} \mid z\}.$$

Note that the induced differential $d_1 = 0$ in $\mathbf{Q}_z$. We write the quotient map as ρ.

Proposition 7.4. *The quotient map ρ is a quasi-isomorphism.*

Proof. Consider the triple complex $(\mathbf{K}_z^{\bullet,\bullet,\bullet}; d_1, d_2, \delta)$ and the related triple complex $(\mathbf{Q}_z^{\bullet,\bullet,\bullet}; d_1, d_2, \delta)$. Fix d_2-degree n, we consider the double complex $(\mathbf{K}_z^{\bullet,n,\bullet}; d_1, \delta)$ and its quotient by ρ. Then ρ induces a map

$$\rho_2 : H_{d_1}^m(H_\delta^q(\mathbf{K}_z^{\bullet,n,\bullet})) \longrightarrow H_{d_1}^m(H_\delta^q(\mathbf{Q}_z^{\bullet,n,\bullet})).$$

We claim that ρ_2 is an isomorphism.

Assuming the claim, then $H_{total}^{m+q}(\mathbf{K}_z^{\bullet,n,\bullet}, d_1 + \delta)$ is isomorphic to $H_{total}^{m+q}(\mathbf{Q}_z^{\bullet,n,\bullet}, d_1 + \delta)$. Thus for the double complex $(\mathbf{K}_z^{\bullet,\bullet}; d_2, d_1 + \delta)$ and its quotient $(\mathbf{Q}_z^{\bullet,\bullet}; d_2, d_1 + \delta)$, the $E_2^{n,m+q}$-term $H_{d_2}^{m+q}(H_{d_1+\delta}^n(\mathbf{K}_z^{\bullet,\bullet}))$ is isomorphic to $H_{d_2}^{m+q}(H_{d_1+\delta}^n(\mathbf{Q}_z^{\bullet,\bullet}))$. ρ hence is a quasi-isomorphism. Noe that here we use the following fact about spectral sequences: a complex homomorphism is a quasi-isomorphism if in the corresponding weakly convergent spectral sequences, the E_r-terms are isomorphic for some positive integer r.

Now we show the isomorphism of ρ_2. Consider the complex $(L_y^\bullet, \delta)$ generated by $\{[\mathbf{1}, y, w] : \bar{w} \mid z\}$. This complex is exactly $\mathrm{Hom}(P_{z\bullet}, [\mathcal{B}_{\mathbf{1}}, y])$. Let $L_y'^\bullet$ and $L_y''^\bullet$ be the subcomplexes generated by $\{[\mathbf{1}, y, w] : y \mid w\}$ and by $\{[\mathbf{1}, y, w] : y \nmid w\}$ respectively. Thus $L_y^\bullet$ is the direct sum of $L_y'^\bullet$ and $L_y''^\bullet$. Correspondingly, $H^*(G_z, [\mathbf{1}, y])$ is the direct sum of $H^*(L_y'^\bullet, \delta)$ and $H^*(L_y''^\bullet, \delta)$. Now the kernel of d_1^1 at $H^q(G_z, [\mathbf{1}, y])$ in the complex (1), or equivalently, in the complex (2), is just $H^q(L_y'^\bullet, \delta)$. We see that $\mathbf{Q}_z^{\bullet,n,\bullet}$ is actually the direct sum of $L_y'^\bullet$(Note that $d_1 = 0$ in $\mathbf{Q}_z$). This proves the isomorphism of ρ_2. □

Theorem 7.5. *If for every $x \mid z$, $p(x;1) = 0$. Then $H^*(G_z, \mathcal{U}_z)$, the G_z-cohomology of the universal norm distribution $\mathcal{U}_z$is the direct sum of $H_{\mathcal{T},w}[y]$ where $H_{\mathcal{T},w}$ is as stated in Proposition 6.1 and*

$$y \mid \bar{w} \mid z.$$

Any element $c[y] \in H^*(G_z, \mathcal{U}_z)$ for $c \in H_{\mathcal{T},w}^q$ is of degree $q - \deg y$.

Remark 7.6. Let $\mathcal{U}_z = U_r$, the universal ordinary distribution of level r, if r is odd, then G_{p^i} is cyclic for every $p^i \| r$. We also see that $p(x;1) = 1-1 = 0$, hence the above theorem gives a complete description of $H^*(G_r, U_r)$ and generalizes Theorem A in Ouyang [5], where we need the condition r is squarefree.

7.7 The G_z-cohomology of $\mathcal{U}_z/M\mathcal{U}_z$

We suppose now that M is a common divisor of $|G_{z(x)}|$ and $p(x;1)$ for every $x \mid z$. Let $\mathbf{S}_z$ be the same as in § 7.6. Then $\mathbf{S}_z/M\mathbf{S}_z$ is a submodule of $\mathbf{K}_z/M\mathbf{K}_z$ generated by

$$\{[a, y, w] : a \in B_{z/z(y)},\ y \mid z,\ \bar{w} \mid z, a \notin B_{\mathbf{1}} \text{ if } y \mid w\}.$$

One easily sees that $\mathbf{S}_z/M\mathbf{S}_z$ is a subcomplex of $\mathbf{K}_z/M\mathbf{K}_z$ with respect to the multi-complex structure of $\mathbf{K}_z/M\mathbf{K}_z$. We let $\mathbf{Q}_z/M\mathbf{Q}_z$ be the quotient of $\mathbf{K}_z/M\mathbf{K}_z$ to $\mathbf{S}_z/M\mathbf{S}_z$, thus $\mathbf{Q}_z/M\mathbf{Q}_z$ is a free $\mathcal{T}/M\mathcal{T}$-module generated by

$$\{[\mathbf{1}, y, w] : y \mid \bar{w} \mid z\}.$$

Note that the induced differentials $d_1 = d_2 = d = \delta = 0$ in $\mathbf{Q}_z/M\mathbf{Q}_z$. Write the quotient map from $\mathbf{K}_z/M\mathbf{K}_z$ to $\mathbf{Q}_z/M\mathbf{Q}_z$ as ρ_M.

Proposition 7.7. *The homomorphism ρ_M is a quasi-isomorphism.*

Proof. Similar to the proof of Proposition 7.4 □

Theorem 7.8. *Let $M \in \mathcal{O}$ be a common divisor of $|G_{z(x)}|$ and $p(x;1)$ for all $x \mid z$. Then the cohomology group $H^*(G_z, \mathcal{U}_z/M\mathcal{U}_z)$ is a direct sum of rank one graded $\mathcal{T}/M\mathcal{T}$-modules $\langle c(y, w)\rangle$ where*

$$y \mid \bar{w} \mid z, \deg c(y, w) = \deg w - \deg y.$$

Proof. By the quasi-isomorphism of ρ_M in Proposition 7.7, the cohomology group $H^*(G_z, \mathcal{U}_z/M\mathcal{U}_z)$ is then just the total cohomology group of the complex $\mathbf{Q}_z/M\mathbf{Q}_z$. However, all induced differentials in $\mathbf{Q}_z/M\mathbf{Q}_z$ are 0, thus its cohomology is itself. Let $c(y, w)$ be the element in $H^*(G_z, \mathcal{U}_z/M\mathcal{U}_z)$ represented by the cocycle $[\mathbf{1}, y, w]$ in $\mathbf{Q}_z/M\mathbf{Q}_z$, we hence get the proof of the above theorem. □

Remark 7.9. With the automorphism ϵ in § 7.2, we easily see that

$$\rho_M : (\mathbf{K}_z^{\bullet,\bullet}/M\mathbf{K}_z^{\bullet,\bullet}; \tilde{d}, \tilde{\delta}) \to (\mathbf{Q}_z^{\bullet,\bullet}/M\mathbf{Q}_z^{\bullet,\bullet}; 0, 0)$$

is a quasi-isomorphism, thus Theorem 7.8 can be stated in the form of the double complex $(\mathbf{K}_z^{\bullet,\bullet}; \tilde{d}, \tilde{\delta})$.

We call the basis $\{c(y, w) : y \mid \bar{w} \mid z\}$ given in Theorem 7.8, the *canonical basis* for $H^*(G_z, \mathcal{U}_z/M\mathcal{U}_z)$. In particular, we write $c(y, y)$ as c_y. By the above theorem, we see that for every $z \in Z$,

$$H^0(G_z, \mathcal{U}_z/M\mathcal{U}_z) = \langle c_y : \ y \mid z\rangle_{\mathcal{T}/M\mathcal{T}}$$

is the union of all $H^0(G_{z'}, \mathcal{U}_{z'}/M\mathcal{U}_{z'})$ with $z' \mid_s z$ and z' finite. We'll use this fact in Ouyang [7] for the double complex $(\mathbf{K}_z^{\bullet,\bullet}; \tilde{d}, \tilde{\delta})$.

Remark 7.10. One can expect parallel result to Theorem B in Ouyang [5] holds here too. The answer is yes. However, we feel more appropriate to state it in Ouyang [7], as a natural consequence of the universal Kolyvagin recursion, just like the proof of the above Theorem B in Anderson and Ouyang [1].

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