RIEMANN-HURWITZ FORMULA IN BASIC $\mathbf{Z}_S$-EXTENSIONS

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Abstract. We study the basic $\mathbf{Z}_S$-extension of imaginary abelian field and establish a formula on Hurwitz-type relations of $\lambda(p, S)$-invariants. Our result can be considered as a generalization of Y.Kida [5] .

1.Introduction.

Let p be a prime number and $\mathbf{F}$ be a CM-field. Let $\mathbf{F}_\infty$ be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{F}$. For every n, we have a unique subextension $\mathbf{F}_n$ of degree p^n over $\mathbf{F}$ in $\mathbf{F}_\infty$. We denote $\mathbf{F}^+$ the maximal real subfield of $\mathbf{F}$, and let h_n^- be the relative class number of $\mathbf{F}_n/\mathbf{F}_n^+$, then we have a well known result:

$$ord_p(h_n^-) = \mu^- p^n + \lambda^- n + \nu^-,$$

$\mu^- \geq 0, \lambda^- \geq 0$, and ν^- are integers, when n is sufficiently large.

Let $\mathbf{E}$ be a CM-field and a p-extension of $\mathbf{F}$, under the assumption $\mu_{\mathbf{F}}^- = 0$, Y.Kida([5]) proved a striking analogue of the classical Riemann-Hurwitz genus formula from the theory of compact Riemann surfaces, by describing the behavior of λ^- in p-extension. His result can be described as the following:

Theorem 0 (see [8,Theorem 4.1]). $\mu_{\mathbf{F}}^- = 0$ *if and only if* $\mu_{\mathbf{E}}^- = 0$, *and when this is the case*

$$\begin{aligned}\lambda_{\mathbf{E}}^- - \delta_{\mathbf{E}} =& [\mathbf{E}_\infty : \mathbf{F}_\infty](\lambda_{\mathbf{F}}^- - \delta_{\mathbf{F}}) \\ &+ \sum_{\omega'} (e(\omega'/\nu') - 1) - \sum_{\omega} (e(\omega/\nu) - 1),\end{aligned}$$

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where the summation is taken over all places ω' on $\mathbf{E}_\infty$ (resp. ω on $\mathbf{E}_\infty^+$) which do not lie above p and $\nu' = \omega'|_{\mathbf{F}_\infty}$ (resp. $\nu = \omega|_{\mathbf{F}_\infty^+}$), $e(\omega/\nu)$ (resp. $e(\omega'/\nu')$) is the ramification index of ω (resp. ω') over ν (resp. ν') and $\delta_{\mathbf{E}} = 1$ or 0 (resp. $\delta_{\mathbf{F}} = 1$ or 0) according $\mathbf{E}$ (resp. $\mathbf{F}$) contains ζ_p (or ζ_4 if $p = 2$) or not.

There are several ways to prove this result. K.Iwasawa([4]) showed us a proof by using Galois cohomology. W.Sinnott([8]) gave a proof by using p-adic L-function and J. Satoh([6]) obtained it by using the theory of Γ-transforms of rational functions. In this paper, we'll generalize the above result to basic $\mathbf{Z}_S$-extension when $\mathbf{E}$ and $\mathbf{F}$ are abelian.

Let $S = \{p_1, ..., p_s\}$ be a finite set of primes, $\mathbf{Z}_S = \prod_{l \in S} \mathbf{Z}_l$ and $\mathbf{Q}_S$ be the $\mathbf{Z}_S$-extension of $\mathbf{Q}$, $\mathbf{F}_S = \mathbf{F}\mathbf{Q}_S$ is called the basic $\mathbf{Z}_S$-extension of $\mathbf{F}$. Let $N = p_1^{n_1} \cdots p_s^{n_s}$ and $\mathbf{F}_N$ be the unique subextension of degree N of $\mathbf{F}_S$. Let h_N^- denote the relative class number of $\mathbf{F}_N/\mathbf{F}_N^+$. From a theorem of E. Friedman([2]), when $\mathbf{F}$ is an imaginary abelian number field, we have

$$ord_{p_i}(h_N^-) = \lambda^-(p_i, S)n_i + \nu^-(p_i, S),$$

where all n_i are sufficiently large and $p_i \in S$.

In this paper, using the relationship between $\lambda^-(p_i, S)$ and the $\lambda-$invariant of Dirichlet character of $\mathbf{F}$, we obtain the following main result.

Theorem 1. *Fixed $p \in S$, let $\mathbf{E}$ and $\mathbf{F}$ be imaginary abelian number fields and $\mathbf{E}$ be a p-extension of $\mathbf{F}$, we have*

$$\begin{aligned}\lambda_{\mathbf{E}}^-(p, S) - \delta_{\mathbf{E}} =& [\mathbf{E}_S : \mathbf{F}_S](\lambda_{\mathbf{F}}^-(p, S) - \delta_{\mathbf{F}}) \\ &+ \sum_{\omega'}(e(\omega'/\nu') - 1) - \sum_{\omega}(e(\omega/\nu) - 1),\end{aligned}$$

where the summation is taken over all places ω' on $\mathbf{E}_S$ (resp. ω on $\mathbf{E}_S^+$) which do not lie above p and $\nu' = \omega'|_{\mathbf{F}_S}$ (resp. $\nu = \omega|_{\mathbf{F}_S^+}$), and $e(\omega/\nu)$ (resp. $e(\omega'/\nu')$) is the ramification index of ω (resp. ω') over ν (resp. ν') and $\delta_{\mathbf{E}} = 1$ or 0 (resp. $\delta_{\mathbf{F}} = 1$ or 0) according $\mathbf{E}$ (resp. $\mathbf{F}$) contains ζ_p (or ζ_4 if $p = 2$) or not.

2. Preliminaries.

Let $p \in S$ be a fixed prime number and put

$$q = \begin{cases} 4, & p = 2, \\ p, & p \neq 2. \end{cases}$$

Let ω_p be the Teichmüller character mod q. For every $m \in \mathbf{Z}$ with $(m, p) = 1$ and $m \neq \pm 1$, we have with $m_1 \in \mathbf{Z}_p, (m_1, p) = 1$ and n_m being a positive integer. We let $\mathbf{Q}^{(p)}$ denote the basic $\mathbf{Z}_p$-extension on $\mathbf{Q}$ and $T = S - \{p\}$.

Let $\mathcal{O}$ be a ring of integers of a finite extension over $\mathbf{Q}_p$ and let $f(X) = a_0 + a_1 X + \cdots \in \mathcal{O}[[X]]$ be a non zero power series. We define

$$\mu(f) = min\{ord_p a_i : i \geq 0\}, \quad \lambda(f) = min\{i \geq 0 : ord_p a_i = \mu(f)\}.$$

Clearly we have $\mu(fg) = \mu(f) + \mu(g), \lambda(fg) = \lambda(f) + \lambda(g)$, if f, g are non zero elements of $\mathcal{O}[[X]]$. So μ and λ can be defined in the quotient field of $\mathcal{O}[[X]]$ in a natural way.

Let $\mathbf{Z}_S^\times$ denote the unit group of $\mathbf{Z}_S$. So

$$\mathbf{Z}_S^\times = U_S \times V_S,$$

where V_S is the torsion part of $\mathbf{Z}_S^\times$ and $U_S = \prod_{l \in S}(1 + 2l\mathbf{Z}_l)$. Let $< >_S$ and ω_S denote the projections from $\mathbf{Z}_S^\times$ to U_S and V_S respectively. When $s = 1$, we have that ω_S is the Teichmüller character. Let θ be an odd primitive Dirichlet character with values in $\mathbf{C}_p$, where $\mathbf{C}_p$ is a fixed completion of algebraic closure of $\mathbf{Q}_p$. Any primitive Dirichlet character whose conductor is divisible only by the primes in S can be regarded as a character of $\mathbf{Z}_S^\times$. Such a character is called the second kind for S if it is trivial on V_S. For a character Ψ of the second kind for S, then we have the decomposition $\Psi = \Psi^{(p)}\Psi^{(T)}$, where $\Psi^{(p)}$ (resp. $\Psi^{(T)}$) is of the second kind for p (resp. T)(see [9]).

Let θ be an odd primitive Dirichlet character with values in $\mathbf{C}_p$. Fix u a generator of U_p. When $\theta\omega_p$ is not of the second kind for p, we define

$$\lambda(\theta) = \lambda(g_\theta(X-1)),$$

where

$$g_\theta(X-1) \in 2\mathcal{O}[[X-1]]$$

with

$$g_\theta(u^s - 1) = L_p(s, \theta\omega_p)$$

and $L_p(s, \theta\omega_p)$ is the p-adic L-function associated to $\theta\omega_p$. When $\theta\omega_p$ is of the second kind for p, we define $\lambda(\theta) = -1$. The following proposition is [6, Th.1].

Proposition 1. *Let θ be an odd primitive Dirichlet character, τ be an even primitive Dirichlet character and $\mathcal{O}$ be the integer ring of the field generated over $\mathbf{Q}_p$ by the values of θ and τ. Suppose*

(1) τ has a p-power order and its conductor l is a prime number,

(2) for all $a \in \mathbf{Z}, \theta\tau(a) = \theta(a)\tau(a)$,

then

(i) If $\theta \neq \omega_p^{-1}$, we have

$$\lambda(\theta\tau) = \begin{cases} \lambda(\theta) + p^{n_l}/q, & if\ \theta(l) \equiv 1\ mod\ \wp \\ \lambda(\theta), & if\ \theta(l) \not\equiv 1\ mod\ \wp \end{cases}$$

where $\wp$ is a prime ideal of $\mathcal{O}$ above p.

(ii) If $\theta = \omega_p^{-1}$, we have

$$\lambda(\theta\tau) = \frac{p^{n_l}}{q} - 1.$$

Remark 1. This proposition can also be proved by using p-adic L-function (see [8, §2]).

Proposition 2. *Let θ be an odd primitive Dirichlet character of order prime to p, τ be an even primitive Dirichlet character of p-power order and $\theta\tau(a) = \theta(a)\tau(a)$. Suppose the conductor $f(\tau)$ of τ is prime to p. Write $f(\tau) = \prod_l l^{k_l}$, where $k_l \geq 1$ and l are primes. Then*

(i) $k_l = 1$, for all l.

(ii) if $\theta \neq \omega_p^{-1}$,

$$\lambda(\theta\tau) = \lambda(\theta) + \sum_{\substack{l \\ \theta(l)=1}} \frac{p^{n_l}}{q},$$

if $\theta = \omega_p^{-1}$,

$$\lambda(\theta\tau) = (\sum_l \frac{p^{n_l}}{q}) - 1.$$

Proof. (i) By Chinese Remainder Theorem, we have $\tau = \prod_l \tau_l$, where l^{k_l} is the conductor of τ_l and τ_l has p-power order.

If $k_l \neq 1$, consider the natural map:

$$i: \quad \mathbf{Z}/(l^{k_l}) \longrightarrow \mathbf{Z}/(l^{k_l - 1})$$

For any $x \in ker\ i$, x has order of l power. Thus $\tau_l(x)$ is an l power-th root of unity. Note τ_l has p-power order and $(p, l) = 1$, we have $\tau_l(x) = 1$. This is a contradiction because l^{k_l} is the conductor of τ_l.

(ii) When $\theta \neq \omega_p^{-1}$, it follows from Proposition 1 and (i) since $\theta\tau(l) \equiv 1\ mod\ \wp$ if and only if $\theta(l) = 1$. When $\theta = \omega_p^{-1}$, then $l \equiv 1 \bmod p$ since τ_l has p-power order. Therefore $\theta(l) \equiv 1 \bmod \wp$ and we are done by Proposition 1. $\square$

3. The Number of Splitting Primes.

Let $\mathbf{k}$ be a finite abelian extension of $\mathbf{Q}$. In this section, we compute the number of primes of $\mathbf{k}_S$ above a prime number l, which is closely related with the characters of Galois group. The character group of an abelian profinite group G means the set of continuous homomorphisms from G to the roots of unity in $\mathbf{C}_p^\times$ with the induced topology. We denote this character group as $G^\wedge$.

Now we take $\chi \in Gal(\mathbf{k}_S/\mathbf{Q})^\wedge$, then $ker\ \chi$ is a close subgroup with finite index of $Gal(\mathbf{k}_S/\mathbf{Q})$ (an open subgroup) and χ is essentially a usual Dirichlet character. Let $\mathbf{k}^\chi$ be the subfield of $\mathbf{k}_S$ fixed by $ker\ \chi$, then we define

$$\chi(l) = \begin{cases} 0 & \text{if } l \text{ is ramified in } \mathbf{k}^\chi, \\ \chi(Frob_l) & \text{if } l \text{ is unramified in } \mathbf{k}^\chi. \end{cases}$$

Keeping the above notations, we have the following lemma:

Lemma 1. *For any prime number l, we have*

(i) There are finitely many primes in $\mathbf{k}_S$ above l.

(ii) The number of primes above l in $\mathbf{k}_S$ is equal to

$$\#\{\chi \in Gal(\mathbf{k}_S/\mathbf{Q})^\wedge : \chi(l) = 1\}.$$

Proof. (i) First consider $S = \{p\}$. Let $\mathcal{Q}$ be a prime in $\mathbf{k}$ above l.

If $l = p$, it is trivial by [10, Lemma 13.3].

If $l \neq p$, then $\mathcal{Q}$ is unramified in $\mathbf{k}_S/\mathbf{k}$. Write

$$l = \omega_p(l)(1 + p^{n_l} l_1).$$

Then the number of primes of $\mathbf{k}$ above $\mathcal{Q}$ is equal to

$$\begin{aligned} &\#(Gal(\mathbf{k}_S/\mathbf{k})/\overline{< Frob_{\mathcal{Q}} >}) \\ &\leq \#(Gal(\mathbf{Q}^{(p)}/\mathbf{Q})/\overline{< Frob_l >}) \cdot [\mathbf{k} : \mathbf{Q}] \\ &\leq p^{n_l}[\mathbf{k} : \mathbf{Q}] < \infty \end{aligned}$$

and we proved the case $s = 1$.

If $s > 1$, let $D(\mathcal{Q})$ be the decomposition group of $\mathcal{Q}$, then $D(\mathcal{Q})$ is a closed subgroup of $\mathbf{Z}_S$ and has the form $p_1^{t_1}\mathbf{Z}_{p_1} \times \cdots \times p_s^{t_s}\mathbf{Z}_{p_s}, 0 \leq t_i \leq \infty, i = 1, \cdots, s$, where $p_i^\infty \mathbf{Z}_{p_i} = 0$. It is sufficient to prove that $t_i < \infty, i = 1, \cdots, s$. If not, suppose $t_i = \infty$. Let $\mathbf{k}^{(p_i)} \subseteq \mathbf{L}$ be a basic $\mathbf{Z}_{p_i}$-extension of $\mathbf{k}$ and $D^{(p_i)}(\mathcal{Q})$ be the decomposition group of $\mathcal{Q}$ over $\mathbf{k}^{(p_i)}$. So we have

$$D^{(p_i)}(\mathcal{Q}) = D(\mathcal{Q})| \quad = 0$$

This is a contradiction to the case of $s=1$ and we proved (i).

(ii) Let $D(l)$ denote the decomposition group of a prime in $\mathbf{k}_S$ above l, then the number of primes in $\mathbf{k}_S$ above l is equal to

$$\begin{aligned}&\#(Gal(\mathbf{k}_S/\mathbf{Q})/D(l)) = \#((Gal(\mathbf{k}_S/\mathbf{Q})/D(l))^\wedge)\\ =&\#\{\chi \in Gal(\mathbf{k}_S/\mathbf{Q})^\wedge : \chi(l)=1\}\end{aligned}$$

This is the result as desired. □

Remark 2. Lemma 1 is not true for arbitrary $\mathbf{Z}_S$-extention, see [10, ex.13.2].

From Lemma 1, we immediately have the following lemma:

Lemma 2. *Suppose* $\mathbf{k}\cap\mathbf{Q}_S=\mathbf{Q}$*,* $p\in S$ *with* $p\nmid[\mathbf{k}:\mathbf{Q}]$*,* $T=S-\{p\}$ *and* l *is a prime number different from* p*. Then the number of prime ideals above* l *in* $\mathbf{kQ}_S$ *is*

$$\begin{aligned}&\#\{\chi\in Gal(\mathbf{kQ}_T/\mathbf{Q})^\wedge : \chi(l)=1\}\#\{\chi\in Gal(\mathbf{Q}^{(p)}/\mathbf{Q})^\wedge : \chi(l)=1\}\\ &=(p^{n_l}/q)\#\{\chi\in Gal(\mathbf{kQ}_T/\mathbf{Q})^\wedge : \chi(l)=1\}.\end{aligned}$$

Proof. By Lemma 1, it is sufficient to prove

$$\begin{aligned}&\#\{\chi\in Gal(\mathbf{kQ}_S/\mathbf{Q})^\wedge : \chi(l)=1\}\\ =&\#\{\chi\in Gal(\mathbf{kQ}_T/\mathbf{Q})^\wedge : \chi(l)=1\}\#\{\chi\in Gal(\mathbf{Q}^{(p)}/\mathbf{Q})^\wedge : \chi(l)=1\}.\end{aligned}$$

Since

$$Gal(\mathbf{kQ}_S/\mathbf{Q})\cong Gal(\mathbf{kQ}_T/\mathbf{Q})\times Gal(\mathbf{Q}^{(p)}/\mathbf{Q}),$$

we have

$$Gal(\mathbf{kQ}_S/\mathbf{Q})^\wedge\cong Gal(\mathbf{kQ}_T/\mathbf{Q})^\wedge\times Gal(\mathbf{Q}^{(p)}/\mathbf{Q})^\wedge,$$

Therefore for any $\chi\in Gal(\mathbf{kQ}_S/\mathbf{Q})^\wedge$, we have $\chi=\chi_T\cdot\chi_p$, with $\chi_T\in Gal(\mathbf{kQ}_T/\mathbf{Q})^\wedge$, $\chi_p\in Gal(\mathbf{Q}^{(p)}/\mathbf{Q})^\wedge$ and $\chi(l)=\chi_T(l)\chi_p(l)$. Note $\chi_p(l)$ is a p-power root of unity and $\chi_T(l)$ is not, so we have

$$\chi(l)=1\Longleftrightarrow\chi_T(l)=1\ and\ \chi_p(l)=1$$

and Lemma 2 is proved. □

4. Proof of Theorem 1.

First let $\mathbf{k}$ be a finite abelian extension of $\mathbf{Q}$ and we use the following notations associated to $\mathbf{k}$:

$X_{\mathbf{k}}$ (resp. $X_{\mathbf{k}}^-$): the set of all (resp. odd) Dirichlet characters associated to $\mathbf{k}$.

$X_{\mathbf{k}}(l)$ (resp. $X^-_{\mathbf{k}}(l)$): all the elements of $X_{\mathbf{k}}$ (resp. $X^-_{\mathbf{k}}$) whose conductors are divisible by a prime number l.

$J_{\mathbf{k}}(l)$: all the elements of $X_{\mathbf{k}}$ whose conductors are prime to a prime number l.

We write $\chi_{\mathbf{k}}$ as an element of $X_{\mathbf{k}}$ and $\mathfrak{f}_{\mathbf{k}}$ as the conductor of $\mathbf{k}$. Let e, f and g denote the usual meaning as the ramification index, the residue class degree, the number of splitting primes respectively. For a prime number l, by [10. Th.3.7], we have

$$\#J_{\mathbf{k}}(l) = f_{\mathbf{k}}(l)g_{\mathbf{k}}(l) \quad \text{and} \quad \#(X_{\mathbf{k}}/J_{\mathbf{k}}(l)) = e_{\mathbf{k}}(l).$$

Now $\mathbf{E}$, $\mathbf{F}$ are the same as in section 1. Let $\mathbf{K}$ be the maximal p-extension of $\mathbf{Q}$ in $\mathbf{E}$ and $\mathbf{L}$ be the maximal extension of $\mathbf{Q}$ in $\mathbf{E}$ with $p \nmid [\mathbf{L} : \mathbf{Q}]$. ω (resp. ω') is a prime of $\mathbf{E}^+_S$ (resp. $\mathbf{E}_S$) which does not lie over the prime p, $\nu = \omega|_{\mathbf{F}^+_S}$ (resp. $\nu' = \omega'|_{\mathbf{F}_S}$) and $u = \omega|_{\mathbf{L}^+_S}$ (resp. $u' = \omega|_{\mathbf{L}_S}$).

Suppose $\omega|_{\mathbf{Q}} = l \neq p$. Since the residue field at u or u' has no finite p-extensions, it is clear that $f(\omega/u) = f(\omega'/u') = 1$. Furthermore

$$e_{\mathbf{K}}(l) = e(\omega'/u'), \quad e_{\mathbf{K}^+}(l) = e(\omega/u), \quad \text{and} \quad \#J_{\mathbf{K}^+} = g(\omega/u), \quad \#J_{\mathbf{K}} = g(\omega'/u')$$

We also note

1) It is easy to check that if Theorem 1 holds for two of $\mathbf{E}/\mathbf{F}, \mathbf{K}/\mathbf{F}$ and $\mathbf{E}/\mathbf{K}$, it holds for the third. This allows us to reduce ourselves to the case where $[\mathbf{F} : \mathbf{Q}]$ is not divisible by p for $p > 2$.

2) We can also assume $\mathbf{E} \cap \mathbf{F}_S = \mathbf{F}, \mathbf{F} \cap \mathbf{Q}_S = \mathbf{Q}$ and the conductor of $\mathbf{E}$ is not divisible by qp, since any number field between $\mathbf{E}$ and $\mathbf{E}_S$ has the same $\lambda(p, S)$-invariant as that of $\mathbf{E}$.

3) By the above assumption, we have

$$[\mathbf{E}_S : \mathbf{F}_S] = [\mathbf{E} : \mathbf{F}], \ \mathbf{E} \cap \mathbf{Q}_S = \mathbf{Q}.$$

With the above notations, we have the following lemma:

Lemma 3.

$$\sum_{\omega'}(e(\omega'/\nu') - 1) - \sum_{\omega}(e(\omega/\nu) - 1)$$

$$= \begin{cases} \sum_l p^{n_l-1}\#X_{\mathbf{K}}(l)\#\{\chi_{\mathbf{F}}\Psi^{(T)} : \chi_{\mathbf{F}} \ odd, \chi_{\mathbf{F}}\Psi^{(T)}(l) = 1\}, & if \quad p > 2 \\ \sum_l 2^{n_l-2}\{\#X^-_{\mathbf{K}}(l) - [\mathbf{E} : \mathbf{F}]\#X_{\mathbf{F}\cap\mathbf{K}}(l)^-\}\#\{\chi_{\mathbf{L}}\Psi^{(T)} : \chi_{\mathbf{L}}\Psi^{(T)}(l) = 1\}, & if \quad p = 2 \end{cases}$$

where ω' (resp. ω) runs over all the primes in $\mathbf{E}_S$ (resp. $\mathbf{E}^+_S$) which do not lie over p, l runs over all the prime numbers different from p and $\Psi^{(T)}$ is taken over the characters of $Gal(\mathbf{Q}_S/\mathbf{Q})$.

Proof. Since

$$\begin{aligned}&\sum_{\omega'}(e(\omega'/u')-1)-\sum_{\omega}(e(\omega/u)-1)\\ =&\sum_{u'}g(\omega'/u')(e(\omega'/u')-1)-\sum_{u}g(\omega/u)(e(\omega/u)-1)\end{aligned} \qquad (*)$$

When $p>2$, then $\mathbf{F}=\mathbf{L}$, $\nu=u$, $\nu'=u'$ and $\mathbf{K}=\mathbf{K}^+$. By Lemma 2, we have

$$\begin{aligned}(*)&=\sum_{l\neq p}\#X_{\mathbf{K}}(l)\sum_{u'\cap\mathbf{Q}=l}1-\sum_{l\neq p}\#X_{\mathbf{K}}(l)\sum_{u\cap\mathbf{Q}=l}1\\ &=\sum_{l\neq p}\#X_{\mathbf{K}}(l)\#\{\chi_{\mathbf{F}}\Psi^{(T)}:\chi_{\mathbf{F}}\Psi^{(T)}(l)=1\}p^{n_l-1}\\ &-\sum_{l\neq p}\#X_{\mathbf{K}}(l)\#\{\chi_{\mathbf{F}^+}\Psi^{(T)}:\chi_{\mathbf{F}^+}\Psi^{(T)}(l)=1\}p^{n_l-1}\\ &=\sum_{l\neq p}\#X_{\mathbf{K}}(l)p^{n_l-1}\#\{\chi_{\mathbf{F}}\Psi^{(T)}:\chi_{\mathbf{F}}\ odd,\chi_{\mathbf{F}}\Psi^{(T)}(l)=1\}.\end{aligned}$$

For $p=2$, we have $\mathbf{F}\supset\mathbf{L}$ and $\mathbf{L}=\mathbf{L}^+$. So

$$\begin{aligned}(*)&=\sum_{l\neq p}\#X_{\mathbf{K}}(l)\sum_{u'|_{\mathbf{Q}}=l}1-\sum_{l\neq p}\#X_{\mathbf{K}^+}(l)\sum_{u|_{\mathbf{Q}}=l}1\\ &=\sum_{l\neq p}\#X^-_{\mathbf{K}}(l)\ 2^{n_l-2}\ \#\{\chi_{\mathbf{L}}\Psi^{(T)}:\chi_{\mathbf{L}}\Psi^{(T)}(l)=1\}\end{aligned} \qquad (1)$$

Let $\mathbf{E}=\mathbf{F}$, we have

$$\begin{aligned}&\sum_{\nu'}(e(\nu'/u')-1)-\sum_{\nu}(e(\nu/u)-1)\\ =&\sum_{l\neq p}2^{n_l-2}\#X^-_{\mathbf{K}\cap\mathbf{F}}(l)\#\{\chi_{\mathbf{L}}\Psi^{(T)}:\chi_{\mathbf{L}}\Psi^{(T)}(l)=1\}\end{aligned} \qquad (2)$$

Since

$$[\mathbf{E}_S:\mathbf{F}_S]=[\mathbf{E}:\mathbf{F}],\ f(\omega'/\nu')=1,$$

we have

$$e(\omega'/\nu')g(\omega'/\nu')=[\mathbf{E}:\mathbf{F}],$$

and

$$e(\omega'/u')=e(\omega'/\nu')e(\nu'/u'),$$

then

$$\begin{aligned}&[\mathbf{E}:\mathbf{F}]\sum_{\nu'}(e(\nu'/u')-1)\\ =&\sum_{\nu'}g(\omega'/\nu')(e(\omega'/u')-e(\omega'/\nu'))\\ =&\sum(e(\omega'/u')-e(\omega'/\nu')).\end{aligned}$$

The same is true for ω, u, ν. By (1) and (2), we obtain that

$$\sum_{\omega'}(e(\omega'/\nu')-1)-\sum_{\omega}(e(\omega/\nu)-1)$$
$$=\sum_{l\neq p}2^{n_l-2}\{\#X^-_{\mathbf{K}}(l)-[\mathbf{E}:\mathbf{F}]\#X^-_{\mathbf{K}\cap\mathbf{F}}(l)\}\#\{\chi_{\mathbf{L}}\Psi^{(T)}:\chi_{\mathbf{L}}\Psi^{(T)}(l)=1\}. \quad \square$$

Now we begin our proof of the main Theorem 1.

Proof. We know that for any imaginary abelian field $\mathbf{k}$, $\lambda(p,S)$ has the following relation (cf.[9]):

$$\lambda^-_{\mathbf{k}}(p,S)=\delta_{\mathbf{k}}+\sum_{\theta}\sum_{\Psi^{(T)}}\lambda(\theta\Psi^{(T)}),$$

where the outer sum is taken over all odd characters of $\mathbf{k}/\mathbf{Q}$ and the inner sum is taken over all $\Psi^{(T)}\in Gal(\mathbf{Q}_T/\mathbf{Q})^\wedge$ with $\lambda(\theta\Psi^{(T)})\neq 0$, and $\delta_{\mathbf{k}}=1$ if and only if ω_p is a character of $\mathbf{k}/\mathbf{Q}$. Therefore

$$\lambda^-_{\mathbf{E}}(p,S)-\delta_{\mathbf{E}}=\sum_{\chi_{\mathbf{E}}\ odd}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbf{E}}\Psi^{(T)})$$
$$=\sum_{\chi_{\mathbf{L}}}\sum_{\chi_{\mathbf{K}}}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbf{L}}\chi_{\mathbf{K}}\Psi^{(T)}) \qquad (**)$$

where $\chi_{\mathbf{K}}\chi_{\mathbf{L}}$ is odd.

When $p>2$, the conductor of $\chi\in Gal(\mathbf{K}/\mathbf{Q})^\wedge$ is not divisible by p since $\mathfrak{f}_{\mathbf{E}}$ is not divisible by p^2 and $[\mathbf{K}:\mathbf{Q}]$ is p-power. Note $\mathbf{L}=\mathbf{F}$ and $\mathbf{K}=\mathbf{K}^+$ in this case, by Proposition 1 and Proposition 2, we have

$$(**)=\sum_{\chi_{\mathbf{F}}\text{odd}}\sum_{\chi_{\mathbf{K}}}\sum_{\Psi^{(T)}}\Big(\lambda(\chi_{\mathbf{F}}\Psi^{(T)})+\sum_{\substack{l\mid f(\chi_{\mathbf{K}})\\ \chi_{\mathbf{F}}\Psi^{(T)}(l)=1}}p^{n_l-1}\Big)$$
$$=[\mathbf{E}:\mathbf{F}]\sum_{\chi_{\mathbf{F}}\text{odd}}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbf{F}}\Psi^{(T)})+\sum_{\chi_{\mathbf{F}}\text{odd}}\sum_{\Psi^{(T)}}\sum_{\substack{l\neq p\\ \chi_{\mathbf{F}}\Psi^{(T)}(l)=1}}\#X_{\mathbf{K}}(l)p^{n_l-1}$$
$$=[\mathbf{E}:\mathbf{F}](\lambda^-_{\mathbf{F}}(p,S)-\delta_{\mathbf{F}})+\sum_{l\neq p}p^{n_l-1}\#X_{\mathbf{K}}(l)\sum_{\chi_{\mathbf{F}}\text{odd},\chi_{\mathbf{F}}\Psi^{(T)}(l)=1}1$$
$$=[\mathbf{E}:\mathbf{F}](\lambda^-_{\mathbf{F}}(p,S)-\delta_{\mathbf{F}})+\sum_{l\neq p}p^{n_l-1}\#X_{\mathbf{K}}(l)\#\{\chi_{\mathbf{F}}\Psi^{(T)}:\chi_{\mathbf{F}}\ odd,\chi_{\mathbf{F}}\Psi^{(T)}(l)=1\}$$
$$=[\mathbf{E}:\mathbf{F}](\lambda^-_{\mathbf{F}}(p,S)-\delta_{\mathbf{F}})+\sum_{\omega'}(e(\omega'/\nu')-1)-\sum_{\omega}(e(\omega/\nu)-1).$$

When $p=2$, $\mathbf{L}=\mathbf{L}^+$, $\mathbf{L}\subset\mathbf{F}$ and the conductor of each character of $\mathbf{K}$ is not divisible by 8. By [6. Th.1]

$$\sum\lambda(\chi_{\mathbf{K}})=\sum 2^{n_l-2}\#X^-_{\mathbf{K}}(l)-[\mathbf{K}^+:\mathbf{Q}].$$

Since $\mathbf{K}\cap\mathbf{F}$ is an imaginary abelian extension of $\mathbf{Q}$, we can choose a primitive odd character χ_0 of $Gal((\mathbf{F}\cap\mathbf{K})/\mathbf{Q})$ with order 2. Then, for any $\chi\in X_{\mathbf{K}}^-$, $\chi=\chi_0\tilde{\chi}$ with $\tilde{\chi}\in X_{\mathbf{K}^+}$. By Proposition 1 and Proposition 2, we have

$$\begin{aligned}&\sum_{\chi_{\mathbf{K}}\text{odd}}\sum_{\chi_{\mathbf{L}}\Psi^{(T)}\neq 1}\lambda(\chi_{\mathbf{K}}\chi_{\mathbf{L}}\Psi^{(T)})\\&=\sum_{l\neq p}2^{n_l-2}\#X_{\mathbf{K}}^-(l)\#\{\chi_{\mathbf{L}}\Psi^{(T)}\neq 1:\chi_{\mathbf{L}}\Psi^{(T)}(l)=1\}\\&+[\mathbf{K}^+:\mathbf{Q}]\sum_{\chi_{\mathbf{L}}\Psi^{(T)}\neq 1}\lambda(\chi_0\chi_{\mathbf{L}}\Psi^{(T)}).\end{aligned}$$

Therefore

$$\begin{aligned}(**)&=\sum_{\chi_{\mathbf{L}}}\sum_{\chi_{\mathbf{K}}\text{odd}}\sum_{\Psi^{(T)}}\lambda(\chi_{\mathbf{K}}\chi_{\mathbf{L}}\Psi^{(T)})\\&=\sum_{\chi_{\mathbf{K}}\text{odd}}\sum_{\chi_{\mathbf{L}}\Psi^{(T)}\neq 1}\lambda(\chi_{\mathbf{K}}\chi_{\mathbf{L}}\Psi^{(T)})+\sum_{\chi_{\mathbf{K}}\text{odd}}\lambda(\chi_{\mathbf{K}})\\&=[\mathbf{K}^+:\mathbf{Q}](\sum_{\chi_{\mathbf{L}}\Psi^{(T)}\neq 1}\lambda(\chi_0\chi_{\mathbf{L}}\Psi^{(T)})-1)\\&+\sum_{l\neq p}2^{n_l-2}\#X_{\mathbf{K}}^-(l)\#\{\chi_{\mathbf{L}}\Psi^{(T)}:\chi_{\mathbf{L}}\Psi^{(T)}(l)=1\}\end{aligned}\tag{3}$$

If we set $\mathbf{E}=\mathbf{F}$ in the above equality, then we obtain

$$\begin{aligned}&\lambda_{\mathbf{F}}^-(2,S)-\delta_{\mathbf{F}}\\&=[\mathbf{K}^+\cap\mathbf{F}:\mathbf{Q}](\sum_{\chi_{\mathbf{L}}\Psi^{(T)}\neq 1}\lambda(\chi_0\chi_{\mathbf{L}}\Psi^{(T)})-1)\\&+\sum_{l\neq p}2^{n_l-2}\#X_{\mathbf{K}\cap\mathbf{F}}^-(l)\#\{\chi_{\mathbf{L}}\Psi^{(T)}:\chi_{\mathbf{L}}\Psi^{(T)}(l)=1\}\end{aligned}\tag{4}$$

By $(3)-[\mathbf{E}^+:\mathbf{F}^+](4)$ we obtain the result as desired since $[\mathbf{E}^+:\mathbf{F}^+][\mathbf{K}^+\cap\mathbf{F}:\mathbf{Q}]=[\mathbf{K}^+:\mathbf{Q}]$ and $[\mathbf{E}:\mathbf{F}]=[\mathbf{E}^+:\mathbf{F}^+]=[\mathbf{E}_S:\mathbf{F}_S]$. □

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