The Group Cohomology of the Universal Ordinary Distribution and Its Applications

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Preface

This thesis is a systematic study of the universal ordinary distribution, its group cohomology and application. We use tools from homological algebra, especially the spectral sequence method, to study the $\{\pm 1\}$-cohomology and the general group cohomology of the universal ordinary distribution. The former one is applied to study the index formula of the Stickelberger ideal, the latter one is used to study the cyclotomic Euler system.

We give an overall picture in Chapter 1. It consists of some history, an overview of research done in this thesis and an outlook to future study. In Chapter 2, we study Anderson's remarkable idea about constructing a Koszul-type torsion-free resolution of the universal ordinary distribution. We investigate further properties of this resolution and give necessary tools for the spectral sequence method. Our method in Chapters 3 and 4 is based on Anderson's resolution and the spectral sequence theory.

Chapter 3 is a detailed study of the $\{\pm 1\}$-cohomology of the universal ordinary distribution and the universal ordinary predistribution. By using the abstract index formula proposed by Anderson and proved here, we reprove of Sinnott's index formula for the Stickelberger ideal in a cyclotomic field.

In Chapter 4, we study the general group cohomology of the universal ordinary distribution of level r under the assumption r squarefree. We give a complete description of this group cohomology. In the 0-th and 1-st case, the cohomology groups have close connections with the cyclotomic Euler system. Though not completed yet, we explain briefly these connections in Chapter 5.

Contents

CHAPTER 1

Introduction

Some background The theory of distributions has a deep root in number theory, especially in the theory of cyclotomic fields. An extensive search could find the idea of distributions everywhere in the classical books by Lang [**20**] and Washington [**37**]. We see a couple of examples here.

(1). For the cyclotomic units $1-\zeta_r$, we know the relation

$$1-\zeta_r^a = \prod_{j=0}^{m/r-1}(1-\zeta_m^{a+rj}),\ \text{if } r \mid m, \tag{1.1}$$

which is fundamental in the study of cyclotomic units(see, for example Washington [**37**], Chapter 8).

(2). The first Bernoulli polynomial

$$B_1(X) = X - \frac{1}{2}$$

satisfies the following relation

$$B_1(\langle x\rangle) = \sum_{ry=x \mod \mathbb{Z}} B_1(\langle y\rangle),\ x\in\mathbb{Q} \tag{1.2}$$

where $\langle x\rangle$ means the fractional part of $x\in\mathbb{Q}$.

These phenomena prompt number theorists to introduce the definition of the universal ordinary distribution U_r. For positive integers k and r, the k-dimensional universal ordinary distribution of level r is the abelian group

$$U_r = U_r^k = \frac{\left\langle [a] : a\in \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k\right\rangle}{\left\langle [a]-\sum\limits_{pb=a}[b] :\ p\mid r \text{ prime}, a\in\frac{p}{r}\mathbb{Z}^k/\mathbb{Z}^k\right\rangle}.$$

Let $G_r = \mathrm{GL}_k(\mathbb{Z}/r\mathbb{Z})$. Viewing $a\in\mathbb{Q}^k/\mathbb{Z}^k$ as a k-dimensional column vector, U_r becomes a G_r-module under matrix multiplication.

One of the first appearances of the universal ordinary distribution is in Sinnott's paper [**35**]. To compute the index formulas of the circular units and the Stickelberger ideal in cyclotomic fields, Sinnott constructed a module U. He also computed the $\{\pm 1\}$-cohomology of this module U(though earlier papers by Schmidt [**30**] and Yamamoto [**40**] studied similar objects). These results are essential to the computation of the index formulas. Sinnott's method is very influential for later study in this subject.

Shortly after Sinnott's investigation, Kubert [**16**] gave the first systematic treatment of the universal ordinary distribution. He showed that Sinnott's module U is nothing but the 1-dimensional universal ordinary distribution U_r. He also showed that the universal ordinary distribution is a free abelian group. In [**17**], Kubert then studied the $\{\pm 1\}$-cohomology of U_r for any k and thereby generalized the 1-dimensional case to arbitrary dimension.

For the case $k = 2$, Kubert and Lang did an extensive study of U_r and its connections with modular units. Their results were included in the book Kubert-Lang [**19**].

Inspired by the success of Sinnott's index computation, many authors, for example, Galovich-Rosen [**10**] and Yin [**41**], obtained results in the function field case analogous to Sinnott's. The method, more or less, is the one used by Sinnott: construct a function field analogue of Sinnott's module U(i.e., the universal ordinary distribution in function fields) and then study the sign-cohomology of this module.

Sinnott's method is highly successful but in some way is rather complicated. The study of Sinnott's module U and its $\{\pm 1\}$-cohomology used a detailed analysis of the interactions of factors of r and also used a substantial amount of homological algebra. The idea behind his computation is illuminating, however, the actual index computation is an long intricate induction.

This situation was changed in Anderson [**1**]. In that paper, he gave another point of view on the index formula, and during the proof gave a basis for the universal ordinary distribution. Then in the course of the proof of a conjecture proposed by Yin [**41**], Anderson [**2**] constructed a torsion-free Koszul type complex. This complex is the starting point of Anderson's resolution, which he constructed in a secret work note [**3**] and now ultimately published in the appendix of Ouyang [**24**].

Briefly to say, Anderson's resolution $\mathbf{L}_r^\bullet$ for the universal ordinary distribution U_r is a graded free abelian group given by

$$L_r^p = \langle [a, g] : g \mid r, a \in \frac{g}{r}\mathbb{Z}^k/\mathbb{Z}^k, g \text{ squarefree and has } -p \text{ prime factors}\rangle$$

with the differential

$$d_r[a, g] = \sum_{p|g} (-1)^{|\{p'|g, p'<p\}|} \Big([a, \frac{g}{p}] - \sum_{pb=a} [b, \frac{g}{p}] \Big).$$

It is easy to see that $H^0(\mathbf{L}_r^\bullet) = U_r$, moreover, Anderson shows that the complex $\mathbf{L}_r^\bullet$ is acyclic in nonzero degree, i.e.,

$$H^n(\mathbf{L}_r^\bullet) = \begin{cases} 0, & \text{if } n \neq 0; \\ U_r, & \text{if } n = 0. \end{cases}$$

This complex possesses very nice properties, such as an explicit basis with good lattice structure etc. With these good properties, the study of the universal ordinary distribution becomes easier to handle.

An example here is the study of algebraic monomials in special values of the Γ-function. Previous studies by Deligne [**8**] and [**9**] applied his theorem of absolute Hodge cycles on abelian varieties, which certainly is quite advanced. Now by using Anderson's resolution, Das [**7**] studied the spectral sequences of a double complex which gives the $\{\pm 1\}$-cohomology of the universal ordinary distribution. By lifting the canonical basis of this $\{\pm 1\}$-cohomology group, Das obtained elementary proofs of some of Deligne's results about algebraic Γ-monomials, and used these cocycles to construct double coverings of cyclotomic fields.

Another side of the story is the theory of Euler systems. In [**26**], Rubin studied a certain family of cyclotomic units ξ_r indexed by certain squarefree integers r which he(after Kolyvagin [**15**]) called the Euler system. The study of the derivative classes generated by the Euler system gave an astonishingly simple proof of the Main Conjecture of Iwasawa theory. Those ξ_r's, in effect, form a 1-dimensional ordinary distribution of level r. Generalizing this observation, Rubin [**28**] introduced the concept of a universal Euler system which, in the cyclotomic case, is just the universal ordinary distribution. The derivative classes are just certain group cohomology classes with coefficients in the universal Euler system. Thus it is quite interesting to study $H^*(G_r, U_r)$.

What we do in this thesis This thesis is a systematic study of the universal ordinary distribution U_r and Anderson's resolution $\mathbf{L}_r^\bullet$. We study the $\{\pm 1\}$-cohomology of U_r and then use the results to give another proof of Sinnott's index formula about the Stickelberger ideal. We compute the G_r-cohomology of U_r for the case $k = 1$ and r squarefree. The results are then used to study the cyclotomic Euler system.

We start Chapter 2 by giving the definitions of the ordinary distribution and the universal ordinary distribution. The next two section are basically from Anderson's exposition in [**1, 2, 3, 24**]. We introduce a certain polynomial ring Λ. The free abelian group $\mathcal{A}_r = L_r^0$ becomes a Λ-module by a certain action. We then prove Theorem 2.2.3 which is due to Anderson(see Appendix of [**24**]). The corollary of Theorem 2.2.3 is a major result of Kubert [**16**](See also Washington [**37**], Chapter 12), but the proof here is much simpler. Then we construct Anderson's resolution $\mathbf{L}_r^\bullet$ and show that $\mathbf{L}_r^\bullet$ is acyclic in nonzero degree(Theorem 2.3.2, see also [**24**]). In §2.4, we study the order ideal structure of $\mathbf{L}_r^\bullet$ and U_r by using the explicit bases of $\mathbf{L}_r^\bullet$ and U_r. We also study various double complex structures for $\mathbf{L}_r^\bullet$ (resp. filtration structures for U_r). This study lays the foundation for the proofs of Theorem A and Theorem B in Chapter 4. In the last section of Chapter 2, we list basic properties of spectral sequences and group cohomology.

Chapter 3 is the result of a project proposed by Anderson to find a spectral-sequence-based proof of Sinnott's famous index formula(See [**35**], Theorem) about the Stickelberger ideal in cyclotomic fields. In that project, Anderson proposed an Abstract Index Formula (3.3), and defined a connecting map between Anderson's resolution of the universal ordinary distribution and the universal ordinary predistribution. We complete the project here by reproving Sinnott's formula using Anderson's resolution. We start with the definition of the regulator $\mathrm{reg}(A, B, \lambda)$ for two finite generated abelian groups A, B and an $\mathbb{R}$-linear isomorphism $\lambda : \mathbb{R}A \to \mathbb{R}B$. The regulator has a property(Proposition 3.1.5) similar to the Euler characteristic, namely, invariance under cohomology. We then show Theorem 3.1.6 by using this property. As suggested by Anderson, we study the spectral sequences of the double complexes whose total cohomologies are the $\{\pm 1\}$-cohomologies of the universal distribution and predistribution, respectively. We thus obtain the $\{\pm 1\}$-cohomologies of the universal distribution and predistribution in Theorem 3.4.1, which reproduces the results of Kubert [**17**]. By applying the Abstract Index Formula, we recover Sinnott's result in Theorem 3.5.1.

Chapter 4 is devoted to the study of $H^*(G_r, U_r)$, the general group cohomology of the universal ordinary distribution. Assuming that $k = 1$ and r is odd squarefree, we prove the following theorem:

THEOREM A . *We have*

$$H^n(G_r, U_r) = \bigoplus_{r'|r} H^{n+n_{r'}}_{r'}(G_r, \mathbb{Z})$$

where $n_{r'}$ *=number of prime factors of* r' *and*

$$H^n_{r'}(G_r, \mathbb{Z}) := \bigcap_{\ell|r'} \ker(H^n(G_r, \mathbb{Z}) \stackrel{res}{\to} H^n(G_{r/\ell}, \mathbb{Z})),$$

where $G_{r/\ell}$ *is viewed as a subgroup of* G_r*. In particular, in the case* $n = 0$*, we have*

$$H^0(G_r, U_r) = \mathbb{Z};$$

and in the case $n = 1$*, we have*

$$H^1(G_r, U_r) = \prod_{r'|r} \mathbb{Z}/m_{r'}\mathbb{Z}$$

where $m_{r'} = \gcd\{\ell - 1 : \ell \mid r'\}$.

The proof of Theorem A is a display of the power of spectral sequences. By using the resolution $\mathbf{L}_r^\bullet$ of U_r, we construct a double complex $\mathbf{K}_r^{\bullet,\bullet}$ whose total cohomology is $H^*(G_r, U_r)$. Studying the nontrivial spectral sequence of this double complex, we are able to find that it degenerates at E_2. Moreover, we find a quasi-isomorphism between $\mathbf{K}_r^{\bullet,\bullet}$ and a quotient complex $\mathbf{Q}_r^{\bullet,\bullet}$ of $\mathbf{K}_r^{\bullet,\bullet}$. With this quasi-isomorphism, we are able to prove Theorem A.

Our investigation doesn't stop here. For application to Euler systems, we study the group $H^0(G_r, U_r/MU_r)$, where M is a common factor of $\ell - 1$ for all prime factors ℓ of r. Assuming the familiarity with the derivative operator $D_{r'}$ here(see §4.2 for detail), we prove the following theorem:

THEOREM B . *The image of the family*

$$\Big\{D_{r'}\Big[\sum_{\ell|r'} \frac{1}{\ell}\Big] : \forall\ r'|r\Big\}$$

in U_r/MU_r is a $\mathbb{Z}/M\mathbb{Z}$-basis for $H^0(G_r, U_r/MU_r)$.

This result gives some rationale for Kolyvagin's ingenious construction of the derivative classes of the cyclotomic Euler system.

In Chapter 5, we investigate the connections of the universal ordinary distribution with the theory of Euler systems. This part is still not fully understood, but there is hope(for example, Theorem B) to believe that strong connections do exist.

A look to the future We finish the introduction with some look to the future on the study of the universal ordinary distribution. As said above, Anderson's resolution $\mathbf{L}_r^\bullet$ has very delicate structure. What we use in this thesis is only part of the features of $\mathbf{L}_r^\bullet$. From my point of view, there are still a few problems to think about:

(1). In Chapter 3, the Abstract Index Formula is a very powerful tool to study the index problem. It shouldn't only be applied to the Stickelberger ideal index. In the short run, one should replace $\theta = 1 + c$ by $1 - c$ and obtain results about the index of the circular units; in the long run, we might apply it to study more general index problems, for example, some generalization of Sinnott [**36**].

(2). In Chapter 4, we limit ourselves to the case $k = 1$ and r squarefree. However, it is of great interest to know if we can remove these restrictions. The study by Kubert and Lang [**19**] reveals a strong connection of modular units with the 2-dimensional universal ordinary distribution. With the connection of modular units to the elliptic Euler system, and the likeness between the cyclotomic Euler system and the elliptic Euler system(see Rubin [**25**], we couldn't help but speculate that some connection might exist between the 2-dimensional universal ordinary distribution and the elliptic Euler system.

Notation Throughout this thesis, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ will always mean the sets of positive integers, of integers, of rational numbers, of real numbers and of complex numbers. The prime numbers will be denoted by p, p_i, ℓ or ℓ_i.

For any finite set S, the cardinality of S will be denoted by $|S|$. The free abelian group generated by S will be denoted by $\langle S \rangle$.

For any complex, we denote the cochain complex(i.e., the differential has degree 1) with superscript $\bullet$ and chain complex(i.e., the differential has degree -1) with subscript $\bullet$. For any cochain complex $C^\bullet$, the complex $C^\bullet[n]$ is the complex with components $C^m[n] = C^{m+n}$. For any complex $C^\bullet$ of $\mathbb{Z}$-modules, we write $C^\bullet_M$ the module $C^\bullet \otimes \mathbb{Z}/M\mathbb{Z}$.

For any double complex $K^{\bullet,\bullet} = (K^{p,q}; d, \delta)$, we call the filtration

$$'F^p K^\bullet = \bigoplus_{p' \geq p} \bigoplus_q K^{p',q}$$

the *first* filtration or the filtration by p; we call the filtration

$$''F^q K^\bullet = \bigoplus_p \bigoplus_{q'' \geq q} K^{p,q''}$$

the *second* filtration or the filtration by q.

CHAPTER 2

Universal Ordinary Distribution and Anderson's Resolution

This chapter is devoted to the study of the universal ordinary distribution U_r and Anderson's resolution $\mathbf{L}_r^{\bullet}$ of U_r. First we introduce the definitions of the ordinary distribution and the universal ordinary distribution and give many examples from various parts of number theory. We then construct a resolution of free abelian groups(Anderson's resolution) for the universal ordinary distribution, which is of great importance to our later exploration in Chapters 3 and 4. Since the theory of double complexes and spectral sequences is a basic tool for our study, we give a brief introduction at the end of this chapter.

2.1. Ordinary distributions: Definitions and Examples

Let k be a fixed positive integer. Let A be any abelian group.

DEFINITION 2.1.1. A map $\phi : \mathbb{Q}^k/\mathbb{Z}^k \to A$ is called an *ordinary distribution* of *dimension* k if

$$\phi(a) = \sum_{nb=a} \phi(b),\ \forall\ a \in \mathbb{Q}^k/\mathbb{Z}^k, n \in \mathbb{N}.$$

For simplicity, we call it a *distribution.* Furthermore , if $\phi(a) = \phi(-a)$, then ϕ is called an *even* distribution; if $\phi(a) = \phi(-a)$, ϕ is called an *odd* distribution.

A map $\phi : (\mathbb{Q}^k/\mathbb{Z}^k)\backslash\{0\} \to A$ is called a *punctured* distribution if

$$\phi(a) = \sum_{nb=a} \phi(b),\ \forall\ a \in (\mathbb{Q}^k/\mathbb{Z}^k)\backslash\{0\}, n \in \mathbb{N}.$$

DEFINITION 2.1.2. Let r be a positive integer. A map $\phi : \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k \to A$ is called (ordinary) distribution of *level* r if

$$\phi(a) = \sum_{nb=a} \phi(b),\ \forall\ a \in \frac{n}{r}\mathbb{Z}^k/\mathbb{Z}^k, n \mid r.$$

Similarly we can define even(odd, punctured) distributions of level r.

Let $\mathcal{A}^k$ be the free abelian group equipped with a basis $[a]$ indexed by $\mathbb{Q}^k/\mathbb{Z}^k$. Fix a positive integer r, let $\mathcal{A}_r^k$ be the subgroup of $\mathcal{A}^k$ generated by the set $\{[a] : a \in \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k\}$. We write $a \in \mathbb{Q}^k/\mathbb{Z}^k$ as a column vector. For any r, let $G_r^k = \mathrm{GL}_k(\mathbb{Z}/r\mathbb{Z})$, then $\mathcal{A}_r$ becomes a G_r^k-module by the natural action $M([a]) = [Ma]$ for any $M \in G_r^k$. Moreover, note that $\mathcal{A}^k$ is the injective limit of $\mathcal{A}_r^k$, and is therefore naturally a $G^k = \mathrm{GL}_k(\hat{\mathbb{Z}})$-module, where $\hat{\mathbb{Z}}$ is the projective limit of $\mathbb{Z}/r\mathbb{Z}$.

Definition 2.1.3. For any positive integer k, let U^k be the quotient of $\mathcal{A}^k$ by the subgroup generated by all elements of the form

$$[a] - \sum_{nb=a} [b],\ a \in \mathbb{Q}^k/\mathbb{Z}^k.$$

The map

$$\upsilon : \mathbb{Q}^k/\mathbb{Z}^k \longrightarrow U^k,\ a \longmapsto [a]$$

is called the *universal ordinary distribution* of *dimension* k. By abuse of notation, we call U^k the universal ordinary distribution. U^k clearly inherits G^k-module structure.

Similarly, let U_r^k be the quotient of $\mathcal{A}_r$ by the subgroup generated by all elements of the form

$$[a] - \sum_{nb=a} [b], a \in \frac{n}{r}\mathbb{Z}^k/\mathbb{Z}^k,\ n \mid r.$$

The map

$$\upsilon : \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k \longrightarrow U_r^k, a \longmapsto [a]$$

is called the *universal ordinary distribution* of *level* r and dimension k. By abuse of notation, we also call U_r^k the universal distribution of level r, which is a G_r^k-module.

Note 2.1.4. From now on we drop the superscript k from our notation if the dimension k is clear from context.

By definition, for any distribution $\phi : \mathbb{Q}^k/\mathbb{Z}^k \to A$, there exists a unique homomorphism $\phi_* : U \to A$, such that $\phi = \phi_* \circ \upsilon$. In this sense, we say U(similarly U_r) is universal. Thus the properties of universal distribution should unveil properties of distributions. In the remaining part of this section, we give some examples of ordinary distributions, which come from various fields of number theory. These distributions play an important role in the theory of numbers and elliptic curves.

EXAMPLE 2.1.5. *Bernoulli distribution*: Let $B_1(X) = X - \frac{1}{2}$ be the *first Bernoulli polynomial*. For any $a \in \mathbb{Q}/\mathbb{Z}$, let

$$\mathbf{B}_1(a) = B_1(\langle a \rangle) = \langle a \rangle - \frac{1}{2},\ a \in \mathbb{Q}/\mathbb{Z}$$

where $\langle\ \rangle$ means the fractional part. Then $\mathbf{B}_1$ is an odd ordinary distribution of dimension 1. In Kubert [**17**], for any $k > 1$, Kubert also constructed k-dimensional distributions with the k-th Bernoulli polynomial. The Bernoulli distribution is an odd(resp. even) distribution if k is odd(resp. even).

EXAMPLE 2.1.6. For any $a \in (\mathbb{Q}/\mathbb{Z})\backslash\{0\}$, let

$$\phi(a) = -\frac{1}{2}\log|1 - e^{2\pi i a}|,$$

then ϕ is an even punctured distribution.

EXAMPLE 2.1.7. *Stickelberger distribution*: For $k = 1$, we identify $G_r = G_r^1 = \mathrm{Gal}(\mathbb{Q}(\mu_r)/\mathbb{Q})$. Let σ_t be the element of G_r sending a r-th root of unity to its t-th power. Let

$$St_r : \frac{1}{r}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}[G_r], a \mapsto \frac{1}{|G_r|}\sum_G \mathbf{B}_1(at)\sigma_t^{-1},$$

Then this distribution is an odd distribution of level r. Moreover, if we take the injective limit of St_r, then we obtain a distribution

$$\varinjlim St_r : \mathbb{Q}/\mathbb{Z} \longrightarrow \lim \mathbb{Q}[G_r].$$

The distributions from Examples 2.1.6 and 2.1.7 are critical to the study of cyclotomic units and the Stickelberger ideal.

EXAMPLE 2.1.8. *Kolyvagin distribution*: Let $\mathbf{e}$ be an injective homomorphism from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}^{ab\,\times}$. Let m be an odd integer and r be an integer whose prime factors are 1 modulo m. For any $a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$, let

$$\xi(a) = \Big(\mathbf{e}(a + \frac{1}{m}) - 1\Big)\Big(\mathbf{e}(a - \frac{1}{m}) - 1\Big).$$

Then ξ is an ordinary distribution of level r. This distribution appears in the construction of cyclotomic Euler system. We'll study it in more detail in Chapter 5.

EXAMPLE 2.1.9. *Sinnott's module*: Let G_r be given as in Example 2.1.7. For any $a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}$, let f_a be the order of a. Let $H_a = \{\sigma_t \in G_r : t \equiv f_a a \mod f_a, (t,r) = 1\}$ and let $s(H_a)$ denote the sum of the elements of H_a in $\mathbb{C}[G_r]$. For any prime p of r, let $\bar{\sigma}_p = \sum_\chi \bar{\chi}(p)e_\chi$, where χ is a primitive Dirichlet character of conductor dividing r and e_χ the idempotent related to χ. Now set

$$\mathcal{S}(a) = s(H_a)\sum_{p|f_a}(1-\bar{\sigma}_p).$$

$\mathcal{S}$ then gives a distribution of level r. Actually this distribution is isomorphic to the universal distribution U_r. We call it Sinnott's module since it first appeared in Sinnott's famous calculation [**35**] of the index formula in the cyclotomic fields.

EXAMPLE 2.1.10. *Siegel distribution*: Let $a = (a_1, a_2) \in \mathbb{Q}^2$, define

$$g_a = -q_\tau^{(1/2)B_2(a_1)} e^{2\pi i a_2(a_1-1)/2}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}),$$

where $z = a_1\tau + a_2$, $q_\tau = e^{2\pi i\tau}$ and $B_2(X) = X^2 - X + \frac{1}{6}$ is the 2nd Bernoulli polynomial. Now if $a \equiv a' \pmod{\mathbb{Z}^2}$, then $g_a \equiv g_{a'}$ modulo constants. If we let A be the group generated by the functions g_a modulo constants, then $g : a \mapsto g_a$ is an ordinary distribution. See Kubert Kubert1 for more details about this distribution.

2.2. The Λ-module $\mathcal{A}$

From now on, we concentrate on the study of the universal distribution. We fix the dimension k here. By definition, U and U_r are quotients of $\mathcal{A}$ and $\mathcal{A}_r$ respectively. Therefore it is necessary to study the abelian group $\mathcal{A}$ first. We equip $\mathcal{A}$ with a certain module structure besides the natural $\mathrm{GL}_k(\hat{\mathbb{Z}})$-module.

DEFINITION 2.2.1. A *supernatural number* is a formal product $\prod p^{n_p}$, where p runs over the set of prime numbers, and where n_p is an integer ≥ 0 or $+\infty$. In an obvious way, one defines the product and also the gcd and lcm of any family of supernatural numbers. We write the set of supernatural numbers as $\bar{\mathbb{N}}$ and consider $\mathbb{N}$ as a subset of $\bar{\mathbb{N}}$. We shall also call a supernatural number just a number.

Let Σ be the set of all primes of $\mathbb{N}$. We have the following table:

$$\begin{array}{ccc}
\{T : T \subseteq \Sigma\} & \Longleftrightarrow & \{g : g \in \bar{\mathbb{N}} \text{ squarefree}\} \\
T & \longleftrightarrow & \prod_{p\in T} p \\
\{p : p \mid g\} & \longleftrightarrow & g
\end{array}$$

By this one to one correspondence, we call the set associated to a squarefree supernatural number g the *support* of g and write it as T_g; conversely, we call the squarefree number associated to a given subset T the *number attached* to T and write it as g_T. We can easily see that the union(resp. intersection) of subsets of Σ corresponds to the lcm(resp. gcd) of squarefree supernatural numbers. For any number $r \in \mathbb{N}$, we say the *T-part* of r is the gcd of r and g_T^∞ and the *non-T part* $r/(r, g_T^\infty)$.

Let $\Lambda = \mathbb{Z}[X_2, X_3, \cdots, X_p, \cdots] = \mathbb{Z}[X_p : p \in \Sigma]$ be the polynomial ring generated by indeterminates X_p for all prime numbers p. Moreover, let $\Lambda(T) = \mathbb{Z}[X_p : p \in T]$ for every subset T of Σ. For every positive integer $n = \prod p^{n_p}$, put

$$X_n = \prod {X_p}^{n_p}, \qquad Y_n = \prod (1 - X_p)^{n_p}.$$

We equip $\mathcal{A}$ with a Λ-module structure by the rule

$$X_p[a] = \sum_{pb=a} [b]$$

for every prime p and every $a \in \mathbb{Q}^k/\mathbb{Z}^k$. Let $\mathcal{A}(T)$ be the subgroup of $\mathcal{A}$ generated by symbols $[a]$ where $a \in \frac{1}{g_T^\infty}\mathbb{Z}^k/\mathbb{Z}^k$. It is easy to see that $\mathcal{A}(T)$ has a $\Lambda(T)$-module structure. One has

$$U = \mathcal{A}/\sum_p Y_p\mathcal{A} \text{ and } U_r = \mathcal{A}_r/\sum_{p|r} Y_p\mathcal{A}_r.$$

Recall that each $x \in \mathbb{Q}/\mathbb{Z}$ has a unique partial fraction expansion

$$x \equiv \sum_p \sum_v \frac{x_{pv}}{p^v} \pmod{\mathbb{Z}},$$

where the sum is extended over primes p and positive integers v, the coefficient x_{pv} are integers in the range $0 \le x_{pv} < p$ and $x_{pv} = 0$ for all but finite many pairs (p, v). Now for any $a = (a_1, ..., a_k)^t \in \mathbb{Q}^k/\mathbb{Z}^k$, we have a partial fraction

$$a \equiv \sum_p \sum_v \frac{a_{pv}}{p^v} \pmod{\mathbb{Z}^k},$$

where a_{pv} is a vector with all entries in the range $0, \cdots, p-1$. For each nonnegative integer n we define $\mathcal{R}_n$ to be the set of $a \in \mathbb{Q}^k/\mathbb{Z}^k$ such that there exist at most n prime numbers p such that $a_{p1} = (p-1, 0, \cdots, 0)$. In particular, $\mathcal{R}_0$ is the set of $a \in \mathbb{Q}^k/\mathbb{Z}^k$ such that $a_{p1} \neq (p-1, 0, \cdots, 0)$ for all prime numbers p.

LEMMA 2.2.2. *The number of elements in the set* $\mathcal{R}_0 \cap \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k$ *is equal to the number of primitive elements in* $\frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k$ *(i.e., elements whose order is* r*), which we denote by* $\varphi_k(r)$.

PROOF. Let $r = \prod_p p^{n_p}$, let $\mathbf{v}_1 = (1, 0, \cdots, 0)$, we define a map

$$f : \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k \longrightarrow \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k$$

where

$$a = \sum_p \sum_v \frac{a_{pv}}{p^v} \longmapsto \sum_p \sum_v \frac{a_{pv} + \mathbf{v}_1}{p^{n_p - v + 1}}.$$

This map is clearly one to one and sends $\mathcal{R}_0 \cap \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k$ to the set of primitive elements in $\frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k$. □

THEOREM 2.2.3. *(1). For each positive integer* r*, the collection*

$$\{X_n[a] : n \mid r,\ n \in \mathbb{N}, a \in \mathcal{R}_0 \cap \frac{n}{r}\mathbb{Z}^k/\mathbb{Z}^k\}$$

constitutes a basis for the free abelian group $\mathcal{A}_r$.

(2). For each positive integer r *and a fixed subset* T *of* Σ*, write* $r = r_1 r_2$ *where* r_1 *is the* T*-part of* r*. Then the collection*

$$\{X_n[a] : n \mid r_1,\ n \in \mathbb{N}, a \in \mathcal{R}_0 \cap \frac{n}{r_1}\mathbb{Z}^k/\mathbb{Z}^k + \frac{1}{r_2}\mathbb{Z}^k/\mathbb{Z}^k\}$$

constitutes a basis for the free abelian group $\mathcal{A}_r$.

(3). The collection

$$\{X_n[a] :\ n \in \mathbb{N}, n \mid g_T^{\infty}, a \in \mathcal{R}_0 \cap \mathcal{A}(T)\}$$

constitutes a basis for the free abelian group $\mathcal{A}(T)$*. In particular, the collection* $\{X_n[a] :\ n \in \mathbb{N}, a \in \mathcal{R}_0\}$ *constitutes a basis for the free abelian group* $\mathcal{A}$.

(4). As a $\Lambda(T)$*-module* $\mathcal{A}(T)$ *is free with a* $\Lambda(T)$*-basis* $\{[a] : a_1 \in \mathcal{R}_0 \cap \mathcal{A}(T)\}$*. In particular, as a* Λ*-module* $\mathcal{A}$ *is free with a* Λ*-basis* $\{[a] : a_1 \in \mathcal{R}_0\}$.

(5). In (1), (2) and (3), if we change X_n *to* Y_n*, the corresponding results still hold.*

PROOF. We first prove (1). By Lemma 2.2.2, the number of elements at the set in question is

$$\sum_{n \mid r} \varphi_k(n) = r^k,$$

hence it suffices to show that the given collection generates $\mathcal{A}_r$. For $n \geq 1$, suppose that $a \in \mathcal{R}_n$ and $a_{p1} = (p-1, 0, \cdots, 0)$, since

$$[a] = -\sum_{\substack{pb=pa \\ b \neq a}} [b] + X_p[pa],$$

where $b \in \mathcal{R}_{n-1}$ and $[pa] \in \mathcal{A}_{r/p}$. Now by double induction on n and r, (1) follows.

The proof of (2) is similar to (1). (3) and (4) follow directly from (1) and (2). For (5), note that the identity

$$X_n - (-1)^{\sum n_i} Y_n = \sum_{\substack{m|n \\ m \neq n}} c_{nm} X_m,$$

holds for any $n = \prod p_i^{n_i}$ and integer constants c_{nm}, therefore (5) follows immediately from (1), (2) and (3). □

Corollary 2.2.4. *The following hold:*

(1). For each positive integer r, the group U_r is free abelian and the family $\{[a]\}$ indexed by $a \in \frac{1}{r}\mathbb{Z}^k/\mathbb{Z}^k \cap \mathcal{R}_0$ gives rise to a basis for U_r.

(2). The group U is free abelian and the family $\{[a]\}$ indexed by $a \in \mathcal{R}_0$ gives rise to a basis for U.

(3). The natural map $U_r \to U$ is a split monomorphism.

Remark 2.2.5. Theorem 2.2.3 is due to Anderson [**24**], Corollary 2.2.4 is due to Kubert [**16**]. The proof given here is essentially Anderson's.

2.3. Anderson's resolution

2.3.1. Construction of the complexes $\mathbf{L}^\bullet$ and $\mathbf{L}^\bullet_{r,f}$. We now assign Σ a total order ω, which may or may not inherited from $\mathbb{N}$. Let g be a squarefree integer. Set

$$\omega(p,g) := \begin{cases} (-1)^{|\{q \in T_g : q <_\omega p\}|}, & \text{if } p \mid g; \\ 0, & \text{if } p \nmid g. \end{cases}$$

Let L be a free abelian group equipped with a basis $\{[a,g]\}$ indexed by pairs (a,g) with $a \in \mathbb{Q}^k/\mathbb{Z}^k$ and g a squarefree positive integer. We make L a graded abelian group by declaring the symbol $[a,g]$ to be of degree $-|T_g|$. Note that L

possesses a natural $\mathrm{GL}_k(\hat{\mathbb{Z}})$-module structure. For any prime p, set

$$d_p[a,g] := \omega(p,g)\Big(\left[a,\frac{g}{p}\right] - \sum_{pb=a}\left[b,\frac{g}{p}\right]\Big),$$

Now for any given squarefree supernatural number f, set

$$d_f = \sum_{p|f} d_p.$$

We have

$$d_f[a,g] = \sum_{p|(f,g)} \omega(p,g)\Big(\left[a,\frac{g}{p}\right] - \sum_{pb=a}\left[b,\frac{g}{p}\right]\Big),$$

In particular, we denote d_f by d if $T_f = \Sigma$.

LEMMA 2.3.1. *(1). For any prime p, $d_p^2 = 0$.*

(2). For distinct primes p and q, $d_p d_q + d_q d_p = 0$.

(3). For any squarefree supernatural number f, $d_f^2 = 0$.

PROOF. An easy calculation. □

By Lemma 2.3.1, we equip the group L with a grading and a differential d_f of degree 1 for any squarefree supernatural number f. We write L as $(\mathbf{L}^\bullet, d_f)$ to respect the graded structure of L and the differential d_f. Note that the map $[a,1] \mapsto [a]$ induces an isomorphism $H^0(\mathbf{L}^\bullet, d) \tilde{\to} U$.

Fix a positive integer r and a squarefree supernatural number f. Denote by $\mathbf{L}^\bullet_{r,f}$ the graded subgroup of L spanned by

$$\{[a,g] : g \mid (r,f), a \in \frac{g}{r}\mathbb{Z}^k/\mathbb{Z}^k\}.$$

It is clear that $\mathbf{L}^\bullet_{r,f}$ is d_f-stable, therefore $\mathbf{L}^\bullet_{r,f}$ is a cochain complex with differential d_f. Note that

$$\mathbf{L}^\bullet_{r,(r,f)} = \mathbf{L}^\bullet_{r,f}.$$

Thus without loss of generality, we can suppose that $f \mid r$. Now let $\bar{r}$ be the product of $p \mid r$, then for any f divisible by $\bar{r}$,

$$\mathbf{L}^\bullet_{r,f} = \mathbf{L}^\bullet_{r,\bar{r}}.$$

We write this complex as $\mathbf{L}^\bullet_r$. Note that the map $[a,1] \mapsto [a]$ induces an isomorphism $H^0(\mathbf{L}^\bullet_r) \tilde{\to} U_r$.

The remaining part of this section is devoted to prove the following theorem:

THEOREM 2.3.2 (Anderson [**24**], Theorem 2). *The following hold:*

(1). For each positive integer r and squarefree supernatural number f, the complex $\mathbf{L}^{\bullet}_{r,f}$ is acyclic in nonzero degree.

(2). The complex $(\mathbf{L}^{\bullet}, d_f)$ is acyclic in nonzero degree.

REMARK 2.3.3. From Theorem 2.3.2, $\mathbf{L}^{\bullet}_r$(resp. $\mathbf{L}^{\bullet}$) is a G_r-module(resp. G) resolution of the universal distribution U_r(resp. U). We call it *Anderson's resolution.* To be consistent, we write $H^0(\mathbf{L}^{\bullet}_{r,f})$ as $U_{r,f}$. By Theorem 2.2.3, then $U_{r,f}$ is a free abelian group generated by

$$\{[a] : a \in \frac{1}{r_1}\mathbb{Z}^k/\mathbb{Z}^k \cap \mathcal{R}_0 + \frac{1}{r_2}\mathbb{Z}^k/\mathbb{Z}^k\}$$

where r_1 is the T_f-part of r and r_2 the non T_f-part of r.

2.3.2. The noncommutative ring $\tilde{\Lambda}$. Let $\tilde{\Lambda}$ be the exterior algebra over Λ generated by a family of symbols $\{\Xi_p\}$ indexed by primes p. For squarefree positive integer $g = p_1 \cdots p_m$, $p_1 < \cdots < p_m$, put

$$\Xi_g := \Xi_{p_1} \wedge \cdots \wedge \Xi_{p_m} \in \tilde{\Lambda},$$

and declare Ξ_g to be of degree $-|T_g| = -m$, thereby defining a Λ-basis $\{\Xi_g\}$ for $\tilde{\Lambda}$ indexed by squarefree positive integers g and equipping $\tilde{\Lambda}$ with a Λ-linear grading. For a fixed subset T of Σ, let f be the number attached to T. Let $\tilde{\Lambda}(T)$ be the subalgebra of $\tilde{\Lambda}$ with a Λ-basis $\{\Xi_g\}$ such that $g \mid f$. Let d_p be the unique Λ-linear derivation of $\tilde{\Lambda}$ of degree 1 such that

$$d_p\, \Xi_p = Y_p$$

for a given p. One has

$$d_p\, \Xi_g = \omega(p, g) Y_p \Xi_{g/p}.$$

Set $d_f = \sum_{p|f} d_p$.

Now fix a positive integer r. Let r_1 be the T-part of r. Let $\tilde{\Lambda}_{r,f}$ be the graded subgroup of $\tilde{\Lambda}$ generated by all elements of the form $Y_h \Xi_g$ where gh divides r_1. It is clear that $\tilde{\Lambda}_{r,f}$ is d_f-stable. Furthermore, one has

LEMMA 2.3.4. *The complex $\tilde{\Lambda}_{r,f}$ is acyclic in nonzero degree.*

PROOF. For any factor r' of r_1, consider the subgroup $\tilde{\Lambda}_{r,f}(r')$ of $\tilde{\Lambda}_{r,f}$ generated by $\{Y_h\Xi_g : hg = r' \mid r\}$. $\tilde{\Lambda}_{r,f}(r')$ is d_f-stable and $\tilde{\Lambda}_{r,f}$ is a direct sum of $\tilde{\Lambda}_{r,f}(r')$. Now $\tilde{\Lambda}_{r,f}(r')$ is Koszul-type complex. Except the case $r' = 1$, $\tilde{\Lambda}_{r,f}(r')$ is acyclic. □

Now we equip $\mathbf{L}^\bullet$ with graded left $\tilde{\Lambda}$-module structure by the rules

$$\Xi_p[a,g] = \begin{cases} \omega(p,gp)[a,gp] & \text{if } p \nmid g \\ 0 & \text{if } p \mid g \end{cases}$$

and

$$X_p[a,g] = \sum_{pb=a} [b,g].$$

LEMMA 2.3.5. *One has*

$$d(\xi\eta) = (d\xi)\eta + (-1)^{\deg \xi}\xi(d\eta)$$

for all homogeneous $\xi \in \tilde{\Lambda}$ *and* $\eta \in \mathbf{L}$.

PROOF. By straightforward calculation. □

PROOF OF THEOREM 2.3.2. We have only to prove the first statement. Let $r = r_1 r_2$ where r_1 is the T_f-part of r. By Theorem 2.2.3 and a straightforward calculation that we omit, one has

$$\mathbf{L}_{r,f} = \bigoplus_{(a,g)} \tilde{\Lambda}_{g,f}[a,1]$$

where the direct sum is indexed by pairs (a,g) with $a \in \frac{1}{r_1}\mathbb{Z}^k/\mathbb{Z}^k \cap \mathbb{R}_0 + \frac{1}{r_2}\mathbb{Z}^k/\mathbb{Z}^k$ and g is the largest positive integer such that $a \in \frac{g}{r_1}\mathbb{Z}^k/\mathbb{Z}^k + \frac{1}{r_2}\mathbb{Z}^k/\mathbb{Z}^k$. Each of the subcomplexes $(\tilde{\Lambda}_{g,f}[a,1], d)$ is an isomorphic copy of $(\tilde{\Lambda}_{g,f}, d_T)$, and the latter we have already observed to be acyclic in nonzero degree by Lemma 2.3.4. □

2.4. Further study of Anderson's resolution

2.4.1. Order ideals and Anderson's resolution. Let r be a fixed positive integer. Let $\bar{r} := \prod_{p|r} p$. In the previous section, we studied the complex $\mathbf{L}^\bullet_{r,f}$ for any squarefree number f. As noted in that section, we assume that f divides$\bar{r}$. By Theorem 2.3.2, $\mathbf{L}^\bullet_{r,f}$ is acyclic in nonzero degree. In particular, the complex $\mathbf{L}^\bullet_r = \mathbf{L}^\bullet_{r,\bar{r}}$ is a resolution of the universal distribution of level r. In this section, we give more details concerning the complexes $\mathbf{L}^\bullet_{r,f}$ and $\mathbf{L}^\bullet_r$.

First let us consider the two complexes $\mathbf{L}^\bullet_{r,f}$ and $\mathbf{L}^\bullet_{r',f'}$. We see that

$$r' \mid r,\ f' \mid f \Rightarrow \mathbf{L}^\bullet_{r',f'} \subseteq \mathbf{L}^\bullet_{r,f}$$

Now for any two pairs (r_1, f_1) and (r_2, f_2), consider the sum and the intersection of the complexes $\mathbf{L}^\bullet_{r_1,f_1}$ and $\mathbf{L}^\bullet_{r_2,f_2}$. Similarly, consider the sum and the intersection of the groups U_{r_1,f_1} and U_{r_2,f_2}.

LEMMA 2.4.1. *For any two pairs (r_1, f_1) and (r_2, f_2), let r_0 be the gcd of r_1 and r_2, let f_0 be the gcd of f_1 and f_2, then*
(1). $\mathbf{L}^\bullet_{r_1,f_1} \cap \mathbf{L}^\bullet_{r_2,f_2} = \mathbf{L}^\bullet_{r_0,f_0}$.
(2). $U_{r_1,f_1} \cap U_{r_2,f_2} = U_{r_0,f_0}$.

PROOF. Consider the bases given in §2.2 and §2.3. □

Now for a fixed r, consider the set $\mathrm{Fac}_r = \{(h,f) : h \mid r, f \mid \bar{r}\}$. Suppose that $(h, f) \le (h', f')$ if $h \mid h'$ and $f \mid f'$. By this ordering, Fac_r becomes a distributive lattice. We recall a definition from combinatorics(see, for example Stanley [**29**]).

DEFINITION 2.4.2. Let $(\mathrm{Lat}, \le)$ be a lattice. An *order ideal* of Lat is a subset I of Lat such that if $x \in I$, then $y \in I$ for any $y \le x$. For any $x \in \mathrm{Lat}$, the *associated* order ideal I_x is defined to be the set $\{y \in \mathrm{Lat} : y \in x\}$.

REMARK 2.4.3. (1). For two order ideals $I_1,\ I_2$, then $I_1 \cup I_2$ and $I_1 \cap I_2$ are also order ideals.

(2). Each order ideal is uniquely determined by its set of maximal elements.

From now on, we concentrate on the lattice Fac_r. Suppose I is an order ideal of Fac_r. Put

$$\mathbf{L}^\bullet_r(I) = \sum_{(h,f)\in I} \mathbf{L}^\bullet_{h,f} \quad \text{and} \quad U_r(I) = \sum_{(h,f)\in I} U_{h,f}.$$

Note that for any $(h, f) \in \mathrm{Fac}_r$, $\mathbf{L}^\bullet_r(I_{h,f}) = \mathbf{L}^\bullet_{h,f}$ and $U_r(I_{h,f}) = U_{h,f}$. By Lemma 2.4.1, we have

PROPOSITION 2.4.4. *Let I_1 and I_2 be two order ideals of* Fac_r*, then*
(1). $\mathbf{L}^\bullet_r(I_1 \cap I_2) = \mathbf{L}^\bullet_r(I_1) \cap \mathbf{L}^\bullet(I_2)$, $U_r(I_1 \cap I_2) = U_r(I_1) \cap U_r(I_2)$.
(2). $\mathbf{L}^\bullet_r(I_1 \cup I_2) = \mathbf{L}^\bullet_r(I_1) + \mathbf{L}^\bullet_r(I_2)$, $U_r(I_1 \cup I_2) = U_r(I_1) + U_r(I_2)$.

The following theorem is a generalization of Theorem 2.3.2:

PROPOSITION 2.4.5. *The complex* $\mathbf{L}_r^\bullet(I)$ *is acyclic with the* 0*-cohomology* $U_r(I)$.

PROOF. We let $\tilde{\mathbf{L}}_r^\bullet(I)$ be the complex

$$0 \longrightarrow L_r^{-|T_r|}(I) \longrightarrow \cdots \longrightarrow L_r^0(I) \xrightarrow{\mathfrak{u}} U_r(I) \longrightarrow 0.$$

Hence it suffices to show that $\tilde{\mathbf{L}}_r^\bullet(I)$ is exact. Let x be a maximal element in the order ideal I. Let I' be the order ideal whose set of maximal elements is obtained from the set of maximal elements of I by excluding x, then

$$I = I' \cup I_x.$$

By Proposition 2.4.4, we have

$$\tilde{\mathbf{L}}_r^\bullet(I)/\tilde{\mathbf{L}}_r^\bullet(I_x) = \tilde{\mathbf{L}}_r^\bullet(I')/\tilde{\mathbf{L}}_r^\bullet(I' \cap I_x).$$

Now we prove the Proposition by induction on the cardinality of the set of maximal elements of I. If I has only one maximal element, this is just Theorem 2.3.2. In general, both I' and $I' \cap I_x$ have fewer maximal elements than I has. Thus the exactness of $\tilde{\mathbf{L}}_r^\bullet(I)$ follows from the exactness of the three complexes $\tilde{\mathbf{L}}_r^\bullet(I_x)$, $\tilde{\mathbf{L}}_r^\bullet(I')$ and $\tilde{\mathbf{L}}_r^\bullet(I' \cap I_x)$. □

Now tensoring $\mathbf{L}_r^\bullet$(resp. $\mathbf{L}_r^\bullet(I)$) by $\mathbb{Z}/M\mathbb{Z}$ for any positive integer M, since $\mathbf{L}_r^\bullet$ is composed of free abelian groups, immediately from Proposition 2.4.5, along with Theorems 2.2.3 and 2.3.2,

COROLLARY 2.4.6. *(1). One has*

$$H^n(\mathbf{L}_r^\bullet/M\mathbf{L}_r^\bullet) = \begin{cases} U_r/MU_r, & if\ n = 0; \\ 0, & if\ n \neq 0. \end{cases}$$

(2). Moreover, for any order ideal I *of* Fac_r*, one has*

$$H^n(\mathbf{L}_r^\bullet(I)/M\mathbf{L}_r^\bullet(I)) = \begin{cases} U_r(I)/MU_r(I), & if\ n = 0; \\ 0, & if\ n \neq 0. \end{cases}$$

2.4.2. A double complex structure when r is squarefree. In this subsection supposing that r is squarefree, i.e., $r = \bar{r}$, we construct a double complex with total complex $\mathbf{L}_r^\bullet$. Consider the sublattice $\mathrm{Div}_r = \{f : f \mid r\} \cong \{(f, f) : f \mid r\} \subseteq \mathrm{Fac}_r$. Let $\mathcal{I}$ be an order ideal of Div_r. Thus we can define

$$L_r^\bullet(\mathcal{I}) = \sum_{f \in \mathcal{I}} L_f^\bullet, \text{ and } U_r(\mathcal{I}) = \sum_{f \in \mathcal{I}} U_f.$$

If we change I by $\mathcal{I}$ in Propositions 2.4.4 and 2.4.5, the corresponding results still hold. In particular, we let $\mathcal{I}(n)$ be the order ideal generated by all f with $|T_f| \leq n$. Let $L_r^\bullet(n) = L_r^\bullet(\mathcal{I}(n))$ and $U_r(n) = U_r(\mathcal{I}(n))$.

For any $p \mid r$, for any $a \in \mathbb{Q}^k/\mathbb{Z}^k$ such that $p \nmid \mathrm{ord}(a)$, the Frobenius Fr_p is given by $a \mapsto pa$. For the symbol $[a, g]$ where $p \mid g$, let

$$d_{1p}[a, g] = -\omega(p, g) \sum_{v \in \mathbb{F}_p^k \setminus \{0\}} [Fr_p^{-1} a + \frac{v}{p}, \frac{g}{p}]$$

and

$$d_{2p}[a, g] = \omega(p, g)\Big([a, \frac{g}{p}] - [Fr_p^{-1}\, a, \frac{g}{p}]\Big);$$

if $p \nmid g$, let $d_{1p}[a, g] = d_{2p}[a, g] = 0$. Furthermore, we let $d_{1r} = \sum_{p \mid r} d_{1p}$ and $d_{2r} = \sum_{p \mid r} d_{2p}$, then easily we can check that $d_{1r}^2 = d_{2r}^2 = 0$ and $d_{1r} d_{2r} + d_{2\bar{r}} d_{1r} = 0$.

For any pair of factors $g \mid g' \mid r$, set

$$L_r(g', g) := \langle [a, g] : \mathrm{ord}\, a = r/g' \rangle,$$

For any $p \mid g$, the map

$$\varphi_p : L_r(g', g) \to L_r(g', g/p),\ \ [a, g] \mapsto [a, g/p]$$

defines a natural isomorphism between $L_r(g', g)$ and $L_r(g', g/p)$. Now for any $g \mid r$,

$$\mathbf{L}_g^\bullet = \bigoplus_{(g_1, g_2)} L_r(g_1, g_2), \text{ where } g_2 \mid g_1,\ \frac{r g_2}{g_1} \mid g.$$

Let $\Gamma(\mathcal{I}) := \{(g_1, g_2) : g_2 \mid g_1,\ \frac{r g_2}{g_1} \in \mathcal{I}\}$, then

$$\mathbf{L}_r^\bullet(\mathcal{I}) = \bigoplus_{(g_1, g_2) \in \Gamma(\mathcal{I})} L_r(g_1, g_2).$$

In general for any $p \mid r$, define

$$\varphi_p : L^p \to L^{p+1}, [a, g] \mapsto \chi_g(p)[a, g/p]$$

where $\chi_g(p) = 1$ if $p \mid g$ and 0 otherwise. Let $\varphi(L^p)$ be the subgroup of L^{p+1} generated by $\varphi_p(L^p)$ for all $p \in r$, inductively, let $\varphi^n(L^p)$ be the subgroup of L^{p+n} generated by $\varphi_p(\varphi^{n-1}(L^p))$ for all $p \in r$. By this setup, there is a filtration of L^p given by

$$\varphi^{s+p}(L^{-s}) \subseteq \varphi^{s+p-1}(L^{-s+1}) \subseteq \cdots \subseteq L^p.$$

This filtration enables us to define the double complex structure of $\mathbf{L}_r^\bullet$ compatible with the differentials d_{1r} and d_{2r}. For the element $[a,g] \in \mathbf{L}_r^\bullet$, we say $[a,g]$ is of bidegree (p_1,p_2) if $[a,g] \in \varphi^{p_2}(L^{p_1}) \backslash \varphi^{p_2+1}(L^{p_1-1})$, more explicitly, if

$$p_1 = |T_{\operatorname{ord} a}| - s,\ p_2 = s - |\operatorname{supp} a| - |T_g|.$$

Then we see that the elements of $L_r(g',g)$ are of bidegree $(-|T_{g'}|, |T_{g'}| - |T_g|)$. Let $L_r^{p_1,p_2}$ be the subgroup of $\mathbf{L}_r^\bullet$ generated by all symbols $[a,g]$ with bidegree (p_1,p_2), then

$$L_r^{p_1,p_2} = \bigoplus_{|T_g| = -p_1-p_2} \bigoplus_{\substack{|T_{g'}| = -p_1 \\ g|g'}} L_r(g',g).$$

Then we see that d_{1r} maps $L_r^{p_1,p_2}$ to $L_r^{p_1+1,p_2}$ and d_{2r} maps $L_r^{p_1,p_2}$ to $L_r^{p_1,p_2+1}$. Hence we construct a double complex $(\mathbf{L}_r^{\bullet,\bullet}; d_1, d_2)$ with the single total complex $\mathbf{L}_r^\bullet$. Note that the second filtration of $\mathbf{L}_r^\bullet$ is given by the map φ.

Proposition 2.4.7. *The E_1 term of the spectral sequence arising from the double complex $(\mathbf{L}_r^{\bullet,\bullet}; d_1, d_2)$ by the first filtration(i.e., $H_{d_1}^{p_1}(\mathbf{L}_r^{\bullet,p_2})$) is*

$$E_1^{p_1,p_2} = \begin{cases} U_S(s-p_2)/U_S(s-p_2-1), & \text{if } p_1 = -p_2; \\ 0, & \text{otherwise.} \end{cases}$$

where $s = |T_r|$. Thus the spectral sequence for the first filtration degenerates at E_1.

Proof. Note that

$$\mathbf{L}_r^\bullet(n) = \bigoplus_{p_2 \geq s-n} L_r^{p_1,p_2}$$

then it is easy to see that $\mathbf{L}_r^{\bullet,p_2}[-p_2]$ is nothing but the quotient complex $\mathbf{L}_r^\bullet(s-p_2)/\mathbf{L}_r^\bullet(s-p_2-1)$. The short exact sequence

$$0 \longrightarrow \mathbf{L}_r^\bullet(s-p_2-1) \longrightarrow \mathbf{L}_r^\bullet(s-p_2) \longrightarrow \mathbf{L}_r^{\bullet,p_2}[-p_2] \longrightarrow 0$$

induces a long exact sequence

$$\cdots \to H^i(\mathbf{L}_r^\bullet(s-p_2)) \to H^i(\mathbf{L}_r^{\bullet,p_2}[-p_2]) \to H^{i+1}(\mathbf{L}_r^\bullet(s-p_2-1)) \to \cdots$$

By Proposition 2.4.5, for $i \neq 0$ and -1, both $H^i(\mathbf{L}_r^\bullet(s-p_2))$ and $H^{i+1}(\mathbf{L}_r^\bullet(s-p_2+1))$ are 0, so is $H^i(\mathbf{L}_r^{\bullet,p_2}[-p_2])$. Therefore the above long exact sequence is just the exact sequence

$$0 \to H^{-1}(\mathbf{L}_r^{\bullet,p_2}[-p_2]) \to U_r(s-p_2-1) \to U_r(s-p_2) \to H^0(\mathbf{L}_r^{\bullet,p_2}[-p_2]) \to 0,$$

Since the map from $U_r(s-p_2-1)$ to $U_r(s-p_2)$ is injective, the proposition follows immediately. □

Remark 2.4.8. It is an interesting problem to investigate the spectral sequence coming from the second filtration of $\mathbf{L}^{\bullet,\bullet}$.

2.4.3. Another double complex structure of $\mathbf{L}_r^\bullet$. . In the above subsection, we give a double complex structure for $\mathbf{L}_r^\bullet$ when r is squarefree. Actually in general, $\mathbf{L}_r^\bullet$ has another double complex structure. Write $d_{\bar r} = \sum_{p|r} d_p$. By Lemma 2.4.1, for any $p \mid r$, $d_p^2 = d_{\bar r/p}^2 = d_p d_{\bar r/p} + d_{\bar r/p} d_p = 0$. Hence

Proposition 2.4.9. *The complex $\mathbf{L}_r^\bullet$ is the total single complex of the double complex $(\mathbf{L}_r^{\bullet,\bullet}; d_p, d_{\bar r/p})$ given by*

$$L_r^{p,q} = \begin{cases} \langle [a,g] \in L_r^p : p \nmid g \rangle & \text{if } q = 0 \\ \langle [a,g] \in L_r^{p-1} : p \mid g \rangle & \text{if } q = -1 \\ 0 & \text{if otherwise.} \end{cases}$$

Moreover, we have $\mathbf{L}_r^{\bullet,0} = \mathbf{L}_{r,\bar r/p}^\bullet$ and $\mathbf{L}_r^{\bullet,1} \cong \mathbf{L}_{\bar r/p}^\bullet$.

Proof. Clear. □

Remark 2.4.10. This observation enables us to regard $\mathbf{L}_r^\bullet$ as a double complex. More generally, we can even consider $|T_{\bar r}|$-tuple complex structure in $\mathbf{L}_r^\bullet$. In the sequel, we won't need this double complex structure. We include it here for the hope that it could be used for future investigation on this topic.

2.5. Basic Theory of Spectral Sequences and Group Cohomology

Let R be a commutative ring. Any module in this section will be referred as a R-module. We outline basic theory of spectral sequences and group cohomology in this section. Our goal is to include necessary results for future study. For details, one should read the classical books such as Cartan-Eilenberg [**5**], Mac Lane [**22**] and Serre [**32**].

2.5.1. Basic theory of spectral sequences.

Definition 2.5.1. A *spectral sequence* $E = (E_r, d_r)$ is a sequence of bigraded modules $E_r, r \geq 1$ with a differential

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad r = 0, 1, \cdots$$

of bidegree $(r, 1-r)$ and with the isomorphism

$$H^*(E_r, d_r) \cong E_{r+1}, r = 0, 1, \cdots .$$

In practice, we always have $E_r = E_{r+1} = \cdots$ for $r \geq r_0$. We call this limit group E_∞ and say the spectral sequence (E_r) converges to E_∞.

Definition 2.5.2. Let E' be a second spectral sequence, a homomorphism $f : E \to E'$ is a family of homomorphism

$$f_r : E_r \longrightarrow E'_r, r = 0, 1, 2, \cdots ,$$

of bigraded modules, each of bidegree $(0, 0)$, such that $d_r f_r = f_r d_r$ and such that each f_{r+1} is the map induced by f_r on cohomology.

We now work with the most general sources of spectral sequences. Let $(K^\bullet, d)$ be a cochain complex(i.e., $\deg d = 1$). A *filtration* F of $K^\bullet$ is a family of subcomplexes $\{F^p K^\bullet$, subject to the conditions:

$$\cdots \subseteq F^p K^\bullet \subseteq F^{p+1} K^\bullet \subseteq \cdots , \qquad \bigcup F^p K^\bullet = K^\bullet,$$

and for convenient, set $F^\infty K^\bullet = 0$ and $F^{-\infty} K^\bullet = K^\bullet$. If there exists $p \in \mathbb{Z}$, such that $F^p K^\bullet = K^\bullet$, then the filtration F is called *bounded below*; if there exist $p \in \mathbb{Z}$, $F^p K^\bullet = 0$, then F is called *bounded above.* If F is both bounded above and below, then F is called *bounded.*

THEOREM 2.5.3. *Let $K^\bullet$ be a filtered complex as above. Then there exists a spectral sequence $\{E_r\}$ with*

$$
\begin{aligned}
E_0^{p,q} =& \frac{F^pK^{p+q}}{F^{p+1}K^{p+q}},\\
E_1^{p,q} =& H^{p+q}(Gr^pK^\bullet),\\
E_\infty^{p,q} =& Gr^p(H^{p+q}(K^\bullet)).
\end{aligned}
$$

REMARK 2.5.4. The last statement of the above proposition is usually written as $E_r \Rightarrow H^*(K^\bullet)$ and said as "the spectral sequence abuts to $H^*(K^\bullet)$".

Let $K_1^\bullet$ and $K_2^\bullet$ be two filtered complexes. Let $f : K_1^\bullet \to K_2^\bullet$ be a homomorphism compatible with the filtrations, i.e., $f(F^p(K_1^\bullet)) \subseteq F^p(K_2^\bullet)$. f clearly induces a homomorphism between the two spectral sequences. Moreover, we have a comparison theorem:

THEOREM 2.5.5. *Let $f : K_1^\bullet \to K_2^\bullet$ be given as above. Suppose that the two filtrations of $K_1^\bullet$ and $K_2^\bullet$ are bounded. Then if for certain index k, the induced map $f_k : E_r(K_1) \to E_r(K_2)$ is an isomorphism then the same holds for every finite index $r \geq k$ and $r = \infty$. $f^* : H(K_1) \to H(K_2)$ is also an isomorphism.*

REMARK 2.5.6. The hypothesis in the preceding theorem is much stronger than necessary. For more general results, see Cartan-Eilenberg [**5**], p318, Theorem 1.2.

A special but also the most common example of a filtered complex is one arising from a double complex. In a word, a *double complex* is a bigraded abelian group

$$K^{\bullet,\bullet} = \bigoplus_p \bigoplus_q K^{p,q}$$

equipped with anticommuting differentials d and δ of bidegree $(1,0)$ and $(0,1)$ respectively. We write it as $(K^{\bullet,\bullet}; d, \delta)$ hereafter. The *total single complex* $K_{total}^\bullet$ is then the complex with degree n component

$$K_{total}^n = \bigoplus_{p+q=n} K^{p,q}$$

and with differential $d + \delta$. The total complex comes with two natural filtrations:

$${}'F^pK = \bigoplus_{p' \geq p} K^{p',q},$$

and

$$''F^qK^{*,*} = \bigoplus_{q''\geq q} K^{p,q''}.$$

Corresponding to the above two filtrations, we have

$$'E_1^{p,q}(K) = H_\delta^q(K^{p,\bullet}),'' E_1^{p,q}(K) = H_d^p(K^{\bullet,q}).$$

$$'E_2^{p,q}(K) = H_d^p(H_\delta^q(K)),'' E_2^{p,q}(K) = H_\delta^q(H_d^p(K)).$$

REMARK 2.5.7. From now on in this thesis, we call $'F^pK$ the first filtration of K, or the filtration given by d(by p); we call $''F^pK$ the second filtration of K, or the filtration given by δ(by q);

2.5.2. Group cohomology. Let A be an abelian group. Let G be a group. Let A be an abelian group, equipped with a G-action. Then A becomes a $\mathbb{Z}[G]$-module. Consider the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$, let $P_\bullet$ be a projective resolution of $\mathbb{Z}$, then the group cohomology

$$H^q(G, A) := \mathrm{Ext}_G^q(\mathbb{Z}, A) = H^q(\mathrm{Hom}_G(P_\bullet, A)). \tag{2.1}$$

For each exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

then there exists a long exact sequence

$$\cdots \longrightarrow H^q(G, B) \longrightarrow H^q(G, C) \longrightarrow H^{q+1}(G, A) \longrightarrow H^{q+1}(G, B) \longrightarrow \cdots$$

From the definition, to compute the group cohomology, it is essential to choose a projective resolution of $\mathbb{Z}$ first. We first recall the “standard bar resolution” here.

Let $P_\bullet = \bigoplus_{i\geq 0} P_i$ where P_i is the free $\mathbb{Z}$-module

$$P_i = \bigoplus_{g_j\in G} \mathbb{Z} \cdot (g_0, \cdots, g_i)$$

with the G-operation by

$$g(g_0, \cdots, g_i) = (gg_0, \cdots, gg_i).$$

The homomorphism $\partial : P_i \to P_{i-1}$ is defined by

$$\partial(g_0, \cdots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, \cdots, \hat{g}_j, \cdots, g_i)$$

This complex is well known to be a projective resolution for the trivial module $\mathbb{Z}$. Now for any G-module A, form the complex $K^\bullet = \mathrm{Hom}_G(P_\bullet, A)$. An element of $K^i = \mathrm{Hom}_G(P_i, A)$ can then be identified with a function $f(g_0, \cdots, g_i)$ having values in A and satisfying the homogeneous condition

$$f(s \cdot g_0, \cdots, s \cdot g_i) = s \cdot f(g_0, \cdots, g_i).$$

Thus f is uniquely determined by its values at $(1, g_1, \cdots, g_1 \cdots g_i)$. Write

$$\hat{f}(g_1, \cdots g_i) = f(1, g_1, \cdots, g_1 \cdots g_i),$$

by the one to one correspondence of f and $\hat{f}$, we regard $\hat{f}$ as elements in K^i. Then the differential d induced by ∂ is given by

$$\begin{aligned} d\hat{f}(g_1, \cdots, g_{i+1}) =& g_1 \cdot \hat{f}(g_2, \cdots, g_{i+1}) \\ &+ \sum_{j=1}^{i} (-1)^j \hat{f}(g_1, \cdots, g_j g_{j+1}, \cdots, g_{i+1}) + (-1)^{i+1} \hat{f}(g_1, \cdots, g_i). \end{aligned}$$

We now compute the group cohomology of A in the case $q = 0$ and 1 :

(1). $q = 0$. In this case,

$$H^0(G, A) = \mathrm{Hom}_G(\mathbb{Z}, A) = A^G = \{a \in A : ga = a \text{ for all } g \in G\}.$$

(2). $q = 1$. A 1-cocycle is a map c of G into A satisfying the identity

$$c(gg') = gc(g') + c(g).$$

It is also called a crossed homomorphism. It is a coboundary if there exists $a \in A$ such that $c(g) = ga - a$ for all $g \in G$.

CHAPTER 3

$\{\pm 1\}$-cohomology of the Universal Distribution

Ever since Sinnott proved his famous result on the index of the Stickelberger ideal and the circular units of cyclotomic fields in [**35**], the sign(or $\{\pm 1\}$) cohomology of the universal ordinary distribution has been closely connected with the index formula. Actually the computation of the sign cohomology handled by Kubert [**17**] follows the idea employed in [**35**]. In this chapter, we use Anderson's resolution to give a brand new way to compute the sign cohomology of the universal ordinary distribution and predistribution. Not surprisingly, this new point of view gives a new proof of Sinnott's index formula for the Stickelberger ideal in cyclotomic fields. In this chapter, we assume that the dimension of the universal distribution is 1. However, our computation is also adaptable to the higher dimensional case. We also suppose that r is not $2 \mod 4$ and the number of prime factors of r is $s = |T_{\bar{r}}|$.

3.1. Regulators and an abstract index formula

3.1.1. Definition of regulator $\mathrm{reg}(A, B, \lambda)$**.** Let A and B be lattices in a finite dimensional vector space V over $\mathbb{R}$. Necessarily there exists some $\mathbb{R}$-linear automorphism ϕ of V such that $\phi(A) = B$. Put

$$(A : B)_V := |\det \phi|,$$

which is a positive real number independent of the choice of ϕ. We call it the *Sinnott symbol* of A to B. Context permitting, we drop the subscript and write simply $(A : B)$.

Note that

(1). For lattices $A,\ B \subseteq V$, if $B \subseteq A$, then $(A : B)_V = \#(A/B)$.

(2). Given lattices $A,\ B,\ C \subseteq V$, then $(A : B)(B : C) = (A : C)$.

(3). Let $f : V_1 \to V_2$ be an isomorphism of vector spaces. Let A and B be lattices in V_1, then $(A : B)_{V_1} = (f(A) : f(B))_{V_2}$.

For more results about the Sinnott symbol, see Sinnott [**35**] and [**36**].

Given a finitely generated abelian group A, we denote the tensor product $A\otimes\mathbb{R}$ by $\mathbb{R}A$. Now let two finitely generated abelian groups A and B, and an $\mathbb{R}$-linear isomorphism $\lambda : \mathbb{R}A \to \mathbb{R}B$ be given. Choose free abelian subgroups $A' \subseteq A$ and $B' \subseteq B$ of finite index. Then A' and B' are of the same rank and hence isomorphic. Choose any isomorphism $\phi : B' \to A'$, it can be naturally extended to an isomorphism $\mathbb{R}\phi : \mathbb{R}B' \to \mathbb{R}A'$, and make the evident identifications $\mathbb{R}A' = \mathbb{R}A$ and $\mathbb{R}B' = \mathbb{R}B$. Now put

$$\operatorname{reg}(A, B, \lambda) := \frac{|\det \mathbb{R}\phi \circ \lambda| \cdot \#B/B'}{\#A/A'}, \tag{3.1}$$

which is a positive real number independent of the choice of A', B' and ϕ. We call $\operatorname{reg}(A, B, \lambda)$ the *regulator* of λ with respect to A and B. We often write it $\operatorname{reg}\lambda$ in abbreviation.

Here we calculate a few examples of the regulator:

Example 3.1.1. If both A and B are finite, then $\operatorname{reg}(A, B, 0) = \#B/\#A$.

Example 3.1.2. Let $f : A \to B$ be any homomorphism of finitely generated abelian groups with finite kernel and cokernel, then $\operatorname{reg}(A, B, \mathbb{R}f) = \#\operatorname{coker} f/\#\ker f$.

Example 3.1.3. Let A, B and C be finitely generated abelian groups. Let $\lambda : \mathbb{R}A \to \mathbb{R}B$ and $\mu : \mathbb{R}B \to \mathbb{R}C$ be $\mathbb{R}$-linear isomorphisms. Then $\operatorname{reg}\mu \circ \lambda = \operatorname{reg}\mu \cdot \operatorname{reg}\lambda$.

Example 3.1.4. Let V be a finite dimensional $\mathbb{R}$-vector space. Let $A, B \subseteq V$ be lattices. Let $\alpha : \mathbb{R}A \to V$ and $\beta : \mathbb{R}B \to V$ be the natural isomorphisms induced by the inclusions $A \subseteq V$ and $B \subseteq V$ respectively. Then $\operatorname{reg}(A, B, \beta^{-1} \circ \alpha) = (B : A)_V$.

3.1.2. Regulators attached to maps of complexes. Consider bounded complexes of finitely generated abelian groups

$$(A, d_A) : \cdots \to A^i \to A^{i+1} \to \cdots$$

and

$$(B, d_B) : \cdots \to B^i \to B^{i+1} \to \cdots$$

and an isomorphism

$$\lambda : \mathbb{R}A \longrightarrow \mathbb{R}B$$

of bounded complexes of finite dimensional vector spaces. The map λ induces an isomorphism

$$H^i(\lambda) : H^i(\mathbb{R}A) \longrightarrow H^i(\mathbb{R}B)$$

in each degree i. Note that we also have $\mathbb{R}H^i(A) = H^i(\mathbb{R}A)$ and $\mathbb{R}H^i(B) = H^i(\mathbb{R}B)$.

PROPOSITION 3.1.5. *With the hypotheses above, then*

$$\prod_i (\text{reg } \lambda^i)^{(-1)^i} = \prod_i (\text{reg } H^i(\lambda))^{(-1)^i}. \tag{3.2}$$

PROOF. First we claim that there exist subcomplexes $A' \subseteq A$ and $B' \subseteq B$ satisfying the following conditions:

(1). A'^i and B'^i are free abelian groups of the same rank as A^i for all i.

(2). $H^i(A')$ and $H^i(B')$ are torsion free for all i.

(3). A' and B' are isomorphic complexes of abelian groups.

(4). The sequences

$$0 \to H^i(A') \to H^i(A) \to H^i(A/A') \to 0$$

and

$$0 \to H^i(B') \to H^i(B) \to H^i(B/B') \to 0$$

are exact for all i.

This claim can be proved by induction. First since A and B are bounded complexes of finitely generated abelian groups, without loss of generality we may suppose these complexes to be of the form

$$(A, d_A) : \cdots 0 \to A^{-n} \to \cdots \to A^{-1} \to A^0 \to 0 \cdots$$

and

$$(B, d_B) : \cdots 0 \to B^{-n} \to \cdots \to B^{-1} \to B^0 \to 0 \cdots .$$

Consider the subgroup $\text{im}(d_A : A^{-1} \to A^0)$ of A^0. Let r be the rank of im A^{-1} and let $\{e_1, \cdots, e_r\}$ be a maximal independent set in im A^{-1}. We can enlarge it into a maximal independent set $E_0 = \{e_1, \cdots, e_s\}$ of A^0. Set A'^0 be the subgroup generated by E_0. Then A^0/A'^0 is finite. Now consider the inverse image of

A'^0, it is a subgroup of A^{-1}. Moreover, it must have the same rank as A^{-1}. Since $\ker(d_A : A^{-1} \to A^0)$ is contained in the inverse image of A'^0, so is im $(d_A : A^{-2} \to A^{-1})$. Find $\{f_1, \cdots, f_s\} \subseteq A^{-1}$ such that $d_A(f_i) = e_i$. This set is an independent set in the inverse image of A'^0 and has only trivial intersection with $\ker(d_A : A^{-1} \to A^0)$. We select a maximal independent set in $\mathrm{im}(A^{-2} \to A^{-1})$, enlarge it to a maximal independent set in $\ker(A^{-1} \to A^0)$, together with $\{f_1, \cdots, f_s\} \subseteq A^{-1}$, we get a maximal independent set E_{-1} in the inverse image of A'^0. Denote the free subgroup generated by E_{-1} by A'^{-1}. Continuing this setup, we obtain a subcomplex A' of A such that A'^i is free, $(A/A')^i$ is finite and $H^i(A')$ is torsion free. Similarly for the complex B, we can construct a subcomplex B' of B such that B'^i is free, $(B/B')^i$ is finite and $H^i(B')$ is torsion free. Hence A' and B' satisfy conditions (1) and (2). But (3) and (4) easily follow from (1) and (2). Hence we proved our claim.

Now choose an isomorphism $\phi : B' \to A'$ of complexes. We have

$$\begin{aligned}\prod_i (\text{reg } \lambda^i)^{(-1)^i} &= \prod_i \left(\frac{|\det \mathbb{R}\phi^i \circ \lambda^i| \cdot \#(B/B')^i}{\#(A/A')^i} \right)^{(-1)^i} \\ &= \prod_i \left(\frac{|\det \mathbb{R}H^i(\phi) \circ H^i(\lambda)| \cdot \#H^i(B/B')}{\#H^i(A/A')} \right)^{(-1)^i} \\ &= \prod_i (\text{reg } H^i(\lambda))^{(-1)^i}.\end{aligned}$$

Here we use the following facts: (1). If A is a complex of finite abelian group, then

$$\prod_i (\#H^i(A))^{(-1)^i} = \prod_i (\#A^i)^{(-1)^i}.$$

(2). If V is a complex of $\mathbb{R}$-vector spaces, ϕ is an automorphism of V, then

$$\prod_i |\det \phi^i|^{(-1)^i} = \prod_i |\det H^i(\phi)|^{(-1)^i}.$$

□

3.1.3. The abstract index formula. Consider the following data $(V,\ L,\ G,\ \theta;\ d_1,\ d_2,\ \phi)$:

- A finite group G.
- A bounded graded finitely generated left $\mathbb{R}[G]$-module
$$V = \bigoplus_i V^i \text{ such that } V^i = 0 \text{ for } i > 0 \text{ and } i \ll 0,$$
equipped with two differentials d_1 and d_2 of degree 1.
- An $\mathbb{R}[G]$-linear isomorphism ϕ: (V, d_1) $\overset{\sim}{longrightarrow}$ (V, d_2) of cochain complexes.
- A lattice L^i in V^i for each i such that $L = \bigoplus_i L^i$ is G, d_1 and d_2-stable.
- $H^i_{d_1}(L) = H^i_{d_2}(L) = 0$ for all $i \neq 0$.
- $H^0_{d_1}(L)$ and $H^0_{d_2}(L)$ are free abelian groups.
- An arbitrary left ideal $\theta \subseteq \mathbb{Z}[G]$.

For any left $\mathbb{Z}[G]$-module M, let M^θ be the subgroup of M annihilated by θ. From these data, we have the following trivial consequences:

- $H^i_{d_1}(V^\theta) = H^i_{d_2}(V^\theta) = 0$ for all $i \neq 0$.
- $L^{i\theta}$ is a lattice in $V^{i\theta}$ for all i.
- $H^0_{d_2}(L)^\theta$ and $H^0_{d_2}(\phi L)^\theta$ are lattices in $H^0_{d_2}(V^\theta)$.

By Proposition 2.1, as suggested by Anderson [**3**], we have

Theorem 3.1.6 (Abstract Index Formula). *Data $(V, L, G, \theta; d_1, d_2)$ as above,*

$$(H^0_{d_2}(L)^\theta : H^0_{d_2}(\phi L)^\theta) = \prod_i |\det(\phi^i \mid V^{i\theta})|^{(-1)^i} \cdot I(L, d_1; \theta)^{-1} \cdot I(L, d_2; \theta), \tag{3.3}$$

where for any bounded complex of $\mathbb{Z}[G]$-modules A, we define

$$I(A; \theta) := \frac{\#\operatorname{coker}(H^0(A^\theta) \to H^0(A)^\theta)}{\#\operatorname{tor} H^0(A^\theta) \cdot \prod_{i\neq 0} \# H^i(A^\theta)^{(-1)^i}} \tag{3.4}$$

provided that the cardinalities of all the groups involved are finite.

Proof. Consider the complexes (L^θ, d_1) and (L^θ, d_2) with the restriction map $\phi : V^\theta \to V^\theta$. Note that:

(1). $\operatorname{reg}(L^{i\,\theta}, L^{i\,\theta}, \phi^i) = |\det(\phi^i \mid V^{i\theta})|$ for all i.

(2). Since $H^i_{d_1}(V^\theta) = H^i_{d_2}(V^\theta) = 0$ for all $i \neq 0$, $H^i_{d_1}(L^\theta)$ and $H^i_{d_2}(L^\theta)$ are both finite and $H^i(\phi) = 0$. Hence $\operatorname{reg}(H^i_{d_1}(L^\theta), H^i_{d_2}(L^\theta), H^i(\phi)) = \# H^i_{d_2}(L^\theta)/\# H^i_{d_1}(L^\theta)$ for all $i \neq 0$.

(3). Now for $j = 0,\ 1$, consider the map $\alpha_j : H^0_{d_j}(L^\theta) \to H^0_{d_j}(L)^\theta$. We have $H^0(\phi) \circ \mathbb{R}\alpha_1 = \mathbb{R}\alpha_2 \circ H^0(\phi)$. Then

$$\begin{aligned}&\operatorname{reg}(H^0_{d_1}(L^\theta), H^0_{d_2}(L^\theta), H^0(\phi))\\ &\qquad = \operatorname{reg}(\alpha_1)\cdot \operatorname{reg}(\alpha_2)^{-1}\cdot \operatorname{reg}(H^0_{d_1}(L)^\theta, H^0_{d_2}(L)^\theta, H^0(\phi)),\end{aligned}$$

where

$$\operatorname{reg}(\alpha_j) = \frac{\#\operatorname{coker}(H^0_{d_j}(L)^\theta \to H^0_{d_j}(L^\theta))}{\#\operatorname{tor} H^0_{d_j}(L^\theta)}$$

and

$$\operatorname{reg}(H^0_{d_1}(L)^\theta, H^0_{d_2}(L)^\theta, H^0(\phi)) = (H^0_{d_2}(L)^\theta : H^0_{d_2}(\phi L)^\theta).$$

Now applying Formula (3.2) in Proposition 2.1 to the case $A = (L^\theta, d_1)$, $B = (L^\theta, d_2)$ and $\lambda = \phi$, we immediately get (3.3). $\square$

3.2. Spectral sequences revisited

Let G be a group and let $\mathbb{Z}[G]$ be the integral group ring of G. Let θ be a left ideal of $\mathbb{Z}[G]$. Let $M = \mathbb{Z}[G]/\theta$, let (P, ∂) :

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to 0$$

be a projective resolution of M. Assume that we have a complex of left G-modules

$$(A, d) : \cdots \to A^i \to A^{i+1} \to \cdots$$

satisfying

- $A^i = 0$ for $i > 0$ and $i \ll 0$.
- $H^i(A) = 0$ for $i \neq 0$.

Let $K^{p,q} = \operatorname{Hom}_G(P_q, A^p)$, therefore we have a double complex $K^{\bullet,\bullet} = (K^{p,q}; d, \delta)$ with the differentials d and δ induced by d and ∂ respectively. Let $K^\bullet$ be the total complex of $K^{\bullet,\bullet}$. From the theory of double complex as introduced in § 2.5, there exist two filtrations of the double complex $K^{\bullet,\bullet}$. For the first filtration, we have

$${}'E_2^{p,q} = H^p(\operatorname{Ext}^q_G(M, A));$$

for the second one,

$${}''E_2^{p,q} = \begin{cases} 0, & \text{if } p \neq 0;\\ \operatorname{Ext}^q_G(M, H^0(A)), & \text{if } p = 0.\end{cases}$$

Since the second case collapses at $p = 0$, we have

$$H^i(K^\bullet) = \mathrm{Ext}^i_G(M, H^0(A)).$$

From now on we will focus only on the first case. We omit the symbol $'$ from our notations. Then

$$E_2^{p,q} = H^p(\mathrm{Ext}^q_G(M, A)) \Rightarrow \mathrm{Ext}^{p+q}_G(M, H^0(A)).$$

Let $q = 0$, then

$$E_2^{p,0} = H^p(\mathrm{Ext}^0_G(M, A)) = H^p(A^\theta).$$

Lemma 3.2.1. $E_\infty^{0,0} = \mathrm{im}\ (H^0(A^\theta) \to H^0(A)^\theta)$.

Proof. Because $\mathrm{Fil}^1 K^\bullet$ is trivial, we have

$$E_\infty^{0,0} = \mathrm{Fil}^0 H^0(K^\bullet) = \mathrm{im}\ (H^0(\mathrm{Fil}^0 K^\bullet) \to H^0(K^\bullet)).$$

It is easy to see that $H^0(\mathrm{Fil}^0 K^\bullet) = A^{0\,\theta}$ and therefore

$$E_\infty^{0,0} = \mathrm{im}\ (A^{0\,\theta} \to H^0(A)^\theta).$$

Consider the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{0\,\theta} & \longrightarrow & K^{0,0} & \xrightarrow{\delta} & K^{0,1} \\ & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} \\ 0 & \longrightarrow & A^{-1\,\theta} & \longrightarrow & K^{-1,0} & \xrightarrow{\delta} & K^{-1,1} \end{array}$$

we see that $A^{-1\,\theta}$ is contained in the boundary of $K^0 = \bigoplus K^{p,-p}$. Furthermore, noting that $H^0(A^\theta) = \mathrm{coker}(A^{-1\,\theta} \to A^{0\,\theta})$, the lemma follows immediately. $\square$

Proposition 3.2.2. *Under the assumption above, if one has*

$$\#\,\mathrm{Ext}^1_G(M, H^0(A)) = \prod_q \# H^{1-q}(\mathrm{Ext}^q_G(M, A)), \tag{3.5}$$

then

$$I(A;\theta) = \prod_{\substack{p+q\le 0\\ q>0}} \# H^p(\mathrm{Ext}^q_G(M, A))^{(-1)^{p+q}} = \prod_{\substack{p+q\le 0\\ q>0}} (\# E_2^{p,q})^{(-1)^{p+q}}. \tag{3.6}$$

PROOF. First note that the given identity (3.5) is nothing but

$$\prod_q \# E_\infty^{1-q,q} = \prod_q \# E_2^{1-q,q}.$$

From the theory of spectral sequences, $H^\bullet(E_r) = E_{r+1}$, then

$$\# E_2^{p,q} \geq \# E_3^{p,q} \geq \cdots \geq \# E_\infty^{p,q}.$$

Hence by (3.5),

$$\# E_2^{1-q,q} = \# E_3^{1-q,q} = \cdots = \# E_\infty^{1-q,q},$$

which means that for $r \geq 2$,

$$\operatorname{im}(d_r : E_r^{1-q-r,q+r-1} \to E_r^{1-q,q}) = \operatorname{im}(d_r : E_r^{1-q,q} \to E_r^{1-q+r,q-r+1}) = 0.$$

Therefore we have a shorter complex:

$$\cdots \to E_r^{1-q-2r,q+2r-2} \to E_r^{1-q-r,q+r-1} \to 0.$$

Now we set to prove the following fact:

$$\prod_{\substack{p+q\leq 0 \\ (p,q)\neq(0,0)}} (\# E_r^{p,q})^{(-1)^{p+q}} \cdot \# \operatorname{tor} E_r^{0,0} \text{ is independent of } r. \tag{3.7}$$

Observe that in the set $\{E_r^{p,q} : p+q \leq 0,\ q \geq 0\}$, the only term not finite is $E_r^{0,0}$. If we substitute $E_r^{0,0}$ by its torsion, we get a collection of complexes composed of finite abelian groups and with differential d_r. The cohomology groups are $E_{r+1}^{p,q}$(or $\operatorname{tor} E_{r+1}^{0,0}$). By the invariance of Euler characteristic under cohomology, (3.7) is proved. Note that $E_\infty^{0,0}$ is free and

$$\prod_{\substack{p+q\leq 0 \\ (p,q)\neq(0,0)}} (\# E_\infty^{p,q})^{(-1)^{p+q}} = \# \operatorname{coker}(H^0(A^\theta) \to H^0(A)^\theta).$$

The formula (3.6) now follows immediately. □

3.3. The universal distribution and predistribution

From now on, we shall apply the abstract index formula to the study of Sinnott's index formula on the Stickelberger ideal. In this section, we are going to produce the data satisfying the hypothesis of Theorem 3.1.6. To achieve this goal, we first introduce the concept of the universal predistribution.

3.3.1. Universal predistribution. From the study in §§ 2.2 and 2.3, we have an abelian group $\mathcal{A}_r =< [a] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} >$ and operators X_n and Y_n on $\mathcal{A}_r$. The universal distribution of level r is given by

$$U_r = \mathcal{A}_r / \sum_{p|r} Y_p \mathcal{A}_r.$$

The Anderson's resolution of U_r is given by

$$\mathbf{L}_r^\bullet = \bigoplus L_r^p = \bigoplus_p < [a,g] : g \mid r, |T_g| = -p, a \in \frac{g}{r}\mathbb{Z}/\mathbb{Z} >$$

with differential

$$d_r[a,g] = \sum_{p|g} \omega(p,g) Y_p[a, g/p]$$

where $Y_p[a,g] = (1 - X_p)[a,g] = [a,g] - \sum_{pb=a}[b,g]$. This leads us to give the following definition:

Definition 3.3.1. The (dimension 1) *universal predistribution of level* r is the abelian group

$$\mathcal{O}_r = \mathcal{A}_r / \sum_{p|r} X_p \mathcal{A}_r.$$

The (dimension 1) *universal predistribution* is the abelian group

$$\mathcal{O} = \mathcal{A} / \sum_{p \text{ prime}} X_p \mathcal{A}_r.$$

Almost parallel to the theory of the universal distribution, we immediately have

Proposition 3.3.2. *(1). For each positive integer r, the group $\mathcal{O}_r$ is free abelian and the family $\{[a]\}$ indexed by $a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \cap \mathcal{R}_0$ gives rise to a basis for $\mathcal{O}_r$.*

(2). The group $\mathcal{O}$ is free abelian and the family $\{[a]\}$ indexed by $a \in \mathcal{R}_0$ gives rise to a basis for $\mathcal{O}$.

(3). The natural map $\mathcal{O}_r \to \mathcal{O}$ is a split monomorphism.

Proposition 3.3.3. *The complex $\mathbf{L}_r^\bullet$ with differential*

$$\hat{d}_r[a,g] = \sum_{p|g} \omega(p,g) X_p[a, g/p]$$

gives a free abelian resolution for the universal predistribution $\mathcal{O}_r$.

As known from Chapter 2, $\mathcal{A}_r$ has a $G_r = \mathrm{GL}_k(\mathbb{Z}/r\mathbb{Z})$-module structure. In the one-dimensional case, $G_r = \mathrm{GL}_1(\mathbb{Z}/r\mathbb{Z}) = (\mathbb{Z}/r\mathbb{Z})^\times$, we identify G_r with the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_r)/\mathbb{Q})$. Thus for $\sigma_t \in G_r$ such that $\sigma_t(\zeta_r) = \zeta_r^t$ for any r-th root of unity ζ_r, let $\sigma_t([a]) = [ta]$ and $\sigma_t([a,g]) = [ta,g]$, then U_r and $\mathcal{O}_r$ become G_r-modules and $(\mathbf{L}_r^\bullet, d_r)$ (resp. $(\mathbf{L}_r^\bullet, \hat{d}_r)$) becomes G_r-module resolution of U_r(resp. $\mathcal{O}_r$).

In Example 2.1.9, the Sinnott module is a model of universal distribution described as a submodule in $\mathbb{R}[G_r]$. Here we give an example of the universal predistribution.

Proposition 3.3.4. *The universal predistribution $\mathcal{O}_r$ is isomorphic to $\mathcal{O}_{\mathbb{Q}(\mu_r)}$, the integer ring of the cyclotomic field $\mathbb{Q}(\mu_r)$.*

Proof. Define $\mathbf{e}_r : \ \mathcal{A}_r \ \longrightarrow \ \mathcal{O}_{\mathbb{Q}(\mu_r)}$

$$\sum n_i[a_i] \longmapsto \sum n_i \exp(2\pi i a_i)$$

Then immediately we have

(a). $\mathbf{e}_r$ is surjective.

(b). $\ker \mathbf{e}_r \supseteq \langle \sum_{nb=a}[b], n|r, \ a \in \frac{n}{r}\mathbb{Z}/\mathbb{Z}\rangle$.

By (b), $\mathbf{e}_r$ induces a map from $\mathcal{O}_r$ to $\mathcal{O}_{\mathbb{Q}(\mu_r)}$. Since both $\mathcal{O}_r$ and $\mathcal{O}_{\mathbb{Q}(\mu_r)}$ are free abelian groups of the same rank $\varphi(r)$, by (a), the map induced by $\mathbf{e}_r$ is an isomorphism. $\square$

3.3.2. The data $(V_r, L_r, J, \theta; d_r, \hat{d}_r, \phi_r)$. In order to apply the abstract index formula, now we generate the data $(V_r, L_r, J; d_r, \hat{d}_r, \phi_r)$ satisfying the hypotheses of Theorem 3.1.6. We denote by L_r the abelian group structure of $\mathbf{L}_r^\bullet$. Let $L_{r,g}$ be the free abelian group generated by the symbol $[a,g]$. Let $V_{r,g}$, V_r^i and V_r be the $\mathbb{R}$-extensions of $L_{r,g}$, L_r^i and L_r respectively. Naturally the differentials d_r and $\hat{d}_r$ can be extended to V_r. Let $c = \sigma_{-1}$ be the complex conjugation in $\mathrm{Gal}(\mathbb{Q}(\mu_r)/\mathbb{Q})$. Let $J = \{1, c\}$ and let $\theta = (1+c)\mathbb{Z}[J]$. We only need to produce the connecting map ϕ_r.

Let $\phi_r : \mathbb{R} \otimes \mathbf{A}_r \to \mathbb{R} \otimes \mathbf{A}_r$ given by

$$[a] \longmapsto \sum_{n|r^\infty} \frac{[na]}{n}.$$

Then ϕ_r is an automorphism of $\mathbb{R}$-vector space $\mathbb{R}\mathbf{A}_r$, the inverse map is given by

$$\phi_r^{-1} : [a] \longmapsto \sum_{n|r^\infty} \frac{\mu(n)[na]}{n},$$

where $\mu(n)$ is the Möbius function. We have

LEMMA 3.3.5. *ϕ_r induces an isomorphism from $\mathbb{R}U_r$ to $\mathbb{R}\mathcal{O}_r$.*

PROOF. A straightforward calculation. □

Identifying V_r^0 and $\mathbb{R} \otimes \mathbf{A}_r$ under the map $[a, 1] \mapsto [a]$, we extend ϕ_r to V_r as follows:

$$\phi_r : V_r \longrightarrow V_r, \qquad [a, g] \longmapsto \sum_{\substack{n|r^\infty \\ (n,g)=1}} \frac{[na, g]}{n}.$$

Then ϕ_r is an automorphism of the vector space V_r with the inverse map ϕ_r^{-1} given by

$$[a, g] \longmapsto \sum_{\substack{n|r^\infty \\ (n,g)=1}} \frac{\mu(n)[na, g]}{n}.$$

The following proposition establishes the connection between (V_r, d_r) and $(V_r, \hat{d}_r)$.

PROPOSITION 3.3.6. *ϕ_r is an isomorphism from cochain complex (V_r, d_r) to cochain complex $(V_r, \hat{d}_r)$, i.e. ,*

$$\hat{d}_r \phi_r = \phi_r d_r.$$

PROOF. By direct calculation. □

The remaining part of this chapter is devoted to the study of the application of Theorem 3.1.6 to this data. First note that V_r^θ has a basis consisting of

$$\{[a, g] - [-a, g] : 0 < a < 1/2\}.$$

Denote by ϕ_r^θ the restriction of ϕ_r to V_r^θ. Then ϕ_r^θ is an automorphism of V_r^θ. We have

PROPOSITION 3.3.7.

$$\prod_i \det(\phi_r^\theta : V_r^{i\theta})^{(-1)^i} = \prod_{\chi \ odd} \prod_{p|r} (1 - \chi(p)p^{-1})^{-1}. \tag{3.8}$$

PROOF. First notice that $V_{r,g}$ is invariant under ϕ_r. Moreover, let $h = r/g$, for any $f \mid h$, define

$$V_{r,g}^f = \mathbb{R} \otimes \langle [x,g] : fx = 0 \rangle,$$

then clearly $V_{r,g}^f$ is invariant under ϕ_r. By definition, we have $V_{r,g}^h = V_{r,g}$. Put

$$V_{r,g}^{(f)} = V_{r,g}^f / \sum_{p|f} V_{r,g}^{f/p},$$

We can see that $V_{r,g}^{(f)}$ is a real vector space with a basis $\{[\frac{a}{f}, g] : (a,f) = 1\}$. Furthermore $V_{r,g}^{(f)}$ is a free $\mathbb{R}[G_f]$-module of rank 1. The induced map ϕ_r on $V_{r,g}^{(f)}$ is an automorphism. Let

$$T_{f,g} = T_{\bar{r}} - T_{\bar{f}} \cup T_g.$$

For each $p \in T_{f,g}$, define

$$\begin{aligned} \tau_p : V_{r,g}^{(f)} &\longrightarrow V_{r,g}^{(f)} \\ [x,g] &\longmapsto \sum_{n|p^\infty} \frac{[nx,g]}{n}. \end{aligned}$$

Note that $\tau_{p_i} \circ \tau_{p_j} = \tau_{p_j} \circ \tau_{p_i}$ and

$$\phi_r|_{V_{r,g}^{(f)}} = \tau_{p_1} \circ \cdots \circ \tau_{p_t}$$

where p_i passes through $T_{f,g}$. Moreover, the subspace $V_{m,g}^{(f)\theta}$ has a basis $\{[\frac{a}{f}, g] - [-\frac{a}{f}, g] : (a,f) = 1, 0 < a < f/2\}$. The restriction maps ϕ_r^θ and τ_p^θ have the relation:

$$\phi_r^\theta|_{V_{r,g}^{(f)\theta}} = \tau_{p_1}^\theta \circ \cdots \circ \tau_{p_t}^\theta.$$

Since the map τ_p is exactly the left multiplication map by the group ring element $\sum_i \frac{\sigma_p^i}{p^i}$ on $V_{r,g}^{(f)\theta}$, by [**36**] Lemma 1.2(b), we have

$$\det \tau_p^\theta := a_{p,f} = \prod_{\chi \text{ even} \in \hat{G}_f} \chi(\sum_i \frac{\sigma_p^i}{p^i}) = \begin{cases} (1 - p^{-c_{p,f}})^{-\varphi(f)/2c_{p,f}}, & if\ c_{p,f}\ odd; \\ (1 + p^{-c_{p,f}/2})^{-\varphi(f)/c_{p,f}}, & if\ c_{p,f}\ even. \end{cases}$$

where $c_{p,f}$ is the smallest number satisfying $p^{c_{p,f}} \equiv 1 \pmod{f}$. We have

$$\det(\phi_r^\theta :\ V_{r,g}^{(f)\theta}) = \prod_{p \in T_{f,g}} a_{p,f}.$$

Now by the Inclusion-Exclusion Principle, we have

$$\det(\phi_r^\theta : \sum_{p|f} V_{r,g}^{f/p\,\theta}) = \prod_{f'|f, f' \neq 1} \det(\phi_r : V_{r,g}^{f/f'\,\theta})^{-\mu(f')}.$$

Hence

$$\prod_{p\in T_{f,g}} a_{p,f} = \prod_{f'|f} \det(\phi_r : V_{r,g}^{f/f'\,\theta})^{\mu(f')}.$$

By the Möbius inverse formula,

$$\det(\phi_r : V_{r,g}^{\theta}) = \prod_{f|\frac{r}{g}} \prod_{p\in T_{f,g}} a_{p,f}$$

Therefore we have

$$\prod_i \det(\phi_r : V_r^{i\,\theta})^{(-1)^i} = \prod_{g|r} \Big(\prod_{f|\frac{r}{g}} \prod_{p\in T_{f,g}} a_{p,f}\Big)^{\mu(g)}.$$

Now let's look at the right hand side of the above identity. The exponent of $a_{p,f}$ is

$$\sum_{g|\frac{r}{f},\ (p,g)=1} \mu(g) = \sum_{g|\frac{r}{fp^\alpha}} \mu(g) = \begin{cases} 1, & \text{if } \frac{m}{fp^\alpha} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

here $p^\alpha \| r$. Write $r = r_p \cdot p^\alpha$, then

$$\prod_i \det(\phi_r : V_r^{i\,\theta})^{(-1)^i} = \prod_{p|r} a_{p,r_p} = \prod_{\chi\ odd} \prod_{p|r} (1-\chi(p)p^{-1})^{-1}.$$

$\square$

3.4. More spectral sequences

We apply the spectral sequence method in §3.2 to our data $(V_r, L_r, J, \theta; d_r, \hat{d}_r, \phi_r)$. Let $\mathbf{d} = d_r$ or $\hat{d}_r$. Let $M = \mathrm{coker}(\mathbb{Z}[J] \overset{1+c}{\to} \mathbb{Z}[J])$. Then M has a projective resolution

$$(P, \partial) : \cdots \xrightarrow{\partial_{q+1}} \mathbb{Z}[J]_{q+1} \xrightarrow{\partial_q} \mathbb{Z}[J]_q \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_0} \mathbb{Z}[J]_0 \longrightarrow 0$$

where $\mathbb{Z}[J]_q = \mathbb{Z}[J]$ and $\partial_q = 1 + (-1)^q \cdot c$. Now let

$$K^{p,q} = \begin{cases} \mathrm{Hom}_G(\mathbb{Z}[J]_q, L_r^p) := (L_r^p, q), & \text{if } q \geq 0 \\ 0, & \text{if } q < 0 \end{cases}$$

and let $K^{\bullet,\bullet} = (K^{p,q}; \mathbf{d}, \delta)$ where

$$\mathbf{d}(x,q) = (\mathbf{d}(x), q), \qquad \delta_q(x,q) = ((-1)^p(1+(-1)^q c)x, q+1).$$

From §3.2, the spectral sequence of the second filtration collapses at E_2, and

$$H^n(K^\bullet) = \mathrm{Ext}^n_J(M, H^0_{\mathbf{d}}(L_r)).$$

We introduce a complete resolution

$$(F,\partial): \cdots \xrightarrow{\partial_{q+1}} \mathbb{Z}[J]_{q+1} \xrightarrow{\partial_q} \mathbb{Z}[J]_q \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_0} \mathbb{Z}[J]_0 \xrightarrow{\partial_{-1}} \cdots$$

Let $F^{p,q} = \mathrm{Hom}_G(\mathbb{Z}[J]_q, L_r^p) = (L_r^p, q)$ for any q. Let $F^{\bullet,\bullet} = (F^{p,q}; \mathbf{d}, \delta)$. We can see that the double complex $K^{\bullet,\bullet}$ is a subcomplex of $F^{\bullet,\bullet}$. Moreover, for $n \geq 0$, by the degeneration of the second spectral sequence,

$$H^n(K^\bullet) = H^n(F^\bullet) = \mathrm{Ext}^n_J(M, H^0_d(L_r)) = \hat{H}^{n+1}(J, H^0_d(L_r)).$$

Now consider the first filtration. For $q > 0$, we always have

$${}'E_2^{p,q}(K) =' E_2^{p,q}(F).$$

Now we compute ${}'E_2^{p,q}(F)$(we drop $'$ in the sequel). First we show that the spectral sequence degenerates at E_2. For this purpose, let

$$SF^{p,q} = \begin{cases} (L_r^p, q), & \text{if } q \text{ even;} \\ (\beta(L_r^p), q), & \text{if } q \text{ odd.} \end{cases}$$

where

$$\beta([a,g]) = \begin{cases} [a,g], & \text{if } 2a \neq 0; \\ 2[a,g], & \text{if } 2a = 0. \end{cases}$$

It is easy to verify that $SF^{\bullet,\bullet} = (SF^{p,q}, d, \delta)$ is a subcomplex of $F^{\bullet,\bullet}$. Furthermore, the quotient complex $QF^{\bullet,\bullet} = F^{\bullet,\bullet}/SF^{\bullet,\bullet}$ has vertical differential 0, hence the spectral sequence of Q related to the first filtration degenerates at E_2(for the second filtration, degenerates at E_1). Now look the quotient map $f: F^{\bullet,\bullet} \to QF^{\bullet,\bullet}$. It induces maps

$$f_r : E_r^{p,q}(F^{\bullet,\bullet}) \longrightarrow E_r^{p,q}(QF^{\bullet,\bullet}).$$

We claim that f_1 is an isomorphism. Note that (recall $r \not\equiv 2 \mod 4$)

$$L_{r,g} = \begin{cases} \bigoplus_{2a\neq 0}(\mathbb{Z}[a,g] \bigoplus \mathbb{Z}[-a,g]) \bigoplus \mathbb{Z}[0,g] \bigoplus \mathbb{Z}[\frac{1}{2},g], & \text{if } r \text{ even;} \\ \bigoplus_{2a\neq 0}(\mathbb{Z}[a,g] \bigoplus \mathbb{Z}[-a,g]) \bigoplus \mathbb{Z}[0,g], & \text{if } r \text{ odd.} \end{cases}$$

Every subspace $\mathbb{Z}[a,g] \bigoplus \mathbb{Z}[-a,g]$ is a trivial $\mathbb{Z}[J]$-module, therefore has trivial cohomology. Since

$${}'E_1^{p,q}(F^{\bullet,\bullet}) = \bigoplus_{|T_g|=-p} \hat{H}^{q+1}(J, L_{r,g}),$$

f_1 is clearly an isomorphism. By this isomorphism, for all $r > 1$ the map f_r is an isomorphism. Since the first filtration is finite, hence bounded, by Theorem 2.5.5, the quotient map f is a quasi-isomorphism and the spectral sequence of the first filtration of $F^{\bullet,\bullet}$ degenerates at E_2. Now we compute $E_2^{p,q}(F^{\bullet,\bullet}) = E_2^{p,q}(QF^{\bullet,\bullet})$. Denote by x_g the cocycle represented by $[0, g]$ and by y_g the cocycle represented by $[1/2, g]$, then for $q > 0$,

$$E_1^{p,q} = \hat{H}^{q+1}(J, L_r^p) = \begin{cases} \bigoplus_g (\langle x_g \rangle \bigoplus \langle y_g \rangle), & \text{if } q \text{ odd, } r \text{ even;} \\ \bigoplus_g \langle x_g \rangle, & \text{if } q \text{ odd, } r \text{ odd;} \\ 0, & \text{if } q \text{ even.} \end{cases}$$

Here $\langle x \rangle$ represents the $\mathbb{Z}/2\mathbb{Z}$ vector space generated by x. Immediately we have $E_2^{p,q} = 0$ for q even. Now for q odd, if r is odd, the induced differential $\mathbf{d}^1$ is

$$x_g \xmapsto{d^1} 0, \qquad x_g \xmapsto{\hat{d}^1} \sum_{i=1}^{-p} x_{g/p_i};$$

if r is even, the induced differential $\mathbf{d}^1$ is

$$x_g \xmapsto{d^1} \delta_{2p_1} y_{g/2}, \qquad x_g \xmapsto{d^1} \delta_{2p_1} y_{g/2}.$$

and

$$x_g \xmapsto{\hat{d}^1} \sum_{i=1}^{-p} x_{g/p_i} + \delta_{2p_1} y_{g/2},\ y_g \xmapsto{\hat{d}^1} \sum_{i=1}^{-p} y_{g/p_i} + \delta_{2p_1} y_{g/2}.$$

Write $\mathfrak{X}_r^\bullet$ the chain complex $E^{\bullet,q}$ for q odd. This is well defined since this complex only depends on r. We calculate the cohomology groups for $\mathfrak{X}_r^\bullet$ for different r and d:

(1). r is odd and $\mathbf{d}^1 = d^1$. This is trivial:

$$(E_2^{p,q}, d) = E_1^{p,q} = (\mathbb{Z}/2\mathbb{Z})^{\binom{s}{-p}}.$$

(2). r is odd and $\mathbf{d}^1 = \hat{d}^1$. In this case, if $r = p^\alpha$, it is easy to see that $H^0(\mathfrak{X}_{p^\alpha}^\bullet, \hat{d}^1) = H^{-1}(\mathfrak{X}_{p^\alpha}^\bullet, \hat{d}^1) = 0$. Now if $r = r_1 r_2$ and $(r_1, r_2) = 1$, we can check

$$(\mathfrak{X}_r^\bullet, \hat{d}_r^1) = (\mathfrak{X}_{r_1}^\bullet, d_{2r_1}^1) \bigotimes (\mathfrak{X}_{r_2}^\bullet, d_{2r_2}^1).$$

By Künneth's formula, $H^p(\mathfrak{X}_r^\bullet, \hat{d}^1) = 0$. Therefore we have

$$(E_2^{p,q}, \hat{d}) = \cdots = (E_\infty^{p,q}, \hat{d}) = 0.$$

(3). r is even and $\mathbf{d}^1 = d^1$. Since $\mathcal{X}_r^\bullet$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space, by the formula above about d^1, we always have

$$\dim_{\mathbb{Z}/2\mathbb{Z}} \operatorname{im}(\mathcal{X}_r^p \to \mathcal{X}_r^{p+1}) = \binom{s-1}{-p-1},$$

By counting the $\mathbb{Z}/2\mathbb{Z}$-dimension, i.e.,

$$\dim E_2^{p,q} = \dim\ker(\mathcal{X}_r^p \to \mathcal{X}_r^{p+1}) - \dim\operatorname{im}(\mathcal{X}_r^{p-1} \to \mathcal{X}_r^p)$$

we have

$$(E_2^{p,q}, d) = (\mathbb{Z}/2\mathbb{Z})^{\binom{s}{-p}}.$$

(4). r is even and $\mathbf{d}^1 = \hat{d}^1$. In this case, if $r = 2^\alpha$,

$$H^0(\mathcal{X}_{2^\alpha}^\bullet, \hat{d}^1) = H^{-1}(\mathcal{X}_{2^\alpha}^\bullet, \hat{d}^1) = \mathbb{Z}/2\mathbb{Z}.$$

Now if $r = 2^\alpha r'$, $r' > 1$ odd, set

$$\mathcal{X'}_{r'}^p = \bigoplus_{g|r'} (\langle x_g\rangle \bigoplus \langle y_g\rangle)$$

and

$$d_2' : x_g \longmapsto \sum_{i=1}^{-p} x_{g/p_i},\ y_g \longmapsto \sum_{i=1}^{-p} y_{g/p_i}.$$

Then we have

$$(\mathcal{X}_r^\bullet, d_2^1) = (\mathcal{X}_{2^\alpha}^\bullet, d_2^1) \bigotimes (\mathcal{X'}_{r'}^\bullet, d_2').$$

Similar to the case (2), we can see $H^p(\mathcal{X'}_{r'}^\bullet, d_2') = 0$. By Künneth's formula again, $({}'E_2^{p,q}, \hat{d}) = H^p(\mathcal{X}_r^\bullet, \hat{d}^1) = 0$.

Combining all the cases above, for $\mathbf{d} = d$, we have

$$(E_2^{p,q}, d)(F^{\bullet,\bullet}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\binom{s}{-p}}, & \text{if } q \text{ odd;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

For $\mathbf{d} = \hat{d}$, we have

$$(E_2^{p,q}, \hat{d})(F^{\bullet,\bullet}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } q \text{ odd},\ r = 2^\alpha,\ p = 0 \text{ or } -1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

By results of (3.9) and (3.10), we easily have

THEOREM 3.4.1. *The group G_r acts trivially on the cohomology groups $H^i(J, \mathcal{O}_r)$ and $H^i(J, U_r)$ for $i = 1$ or 2, moreover,*

$$H^1(J, \mathcal{O}_r) = H^2(J, \mathcal{O}_r) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } r = 2^\alpha; \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

and

$$H^1(J, U_r) = H^2(J, U_r) = (\mathbb{Z}/2\mathbb{Z})^{2^{s-1}}. \tag{3.12}$$

REMARK 3.4.2. 1. The second statement is first proved in Yamamoto [**40**]. The spectral sequence method employed here makes the calculation significantly simpler than those in [**40**] and in [**35**]. Moreover, this same spectral sequence method can also be applied to the universal distribution of higher dimension, thus we can recover the results in Kubert [**17**].

2. For any cyclic group $C \in G_r$ which has trivial intersection with G_{p^i} for all $p^i \| r$, we can also obtain a similar result without any extra difficulty.

PROPOSITION 3.4.3.

$$I(L_r, d; \theta) = \begin{cases} 2, & if\ s = 1; \\ 2^{2^{s-2}}, & \text{if } r > 1. \end{cases} \qquad I(L_r, \hat{d}; \theta) = \begin{cases} 2, & if\ r = 2^\alpha; \\ 1, & otherwise. \end{cases}$$

PROOF. Since $K^{\bullet,\bullet}$ and $F^{\bullet,\bullet}$ have same E_2-terms for $q > 0$, and since the (first) spectral sequence of $F^{\bullet,\bullet}$ degenerates at E_2, the condition in Proposition 3.2.2 is satisfied. For $\mathbf{d} = d$, by (3.9), then the exponent of 2 in $I(L_r, d; \theta)$ is equal to

$$\sum_{\substack{p+q \le 0 \\ q>0\ odd}} (-1)^{p+1} \binom{s}{-p} = \begin{cases} 1, & if\ s = 1; \\ 2^{s-2}, & if\ s > 1. \end{cases}$$

The case $\mathbf{d} = \hat{d}$ immediately follows from Proposition 3.2.2 and (3.10). $\square$

3.5. Sinnott's index formula

With all our efforts, now we can use the abstract index formula to to prove Sinnott's index formula on the Stickelberger ideal, i.e., we prove here the following theorem:

THEOREM 3.5.1 (See [**35**], Theorem). *Let $R = \mathbb{Z}[G_r]$ and let S be the Stickelberger ideal of $\mathbb{Q}(\mu_r)$. Then*

$$[R^- : S^-] = 2^a h^-,$$

where $a = 0$ if $s = 1$ and $a = 2^{s-2} - 1$ if $s > 1$.

NOTE 3.5.2. In this section, the subscript r is omitted from our notations(i.e., G is the Galois group G_r and so on). p is always a prime factor of r. The superscript "-" is in accordance with the superscript "θ" in the previous sections.

PROOF. We consider the following diagram:

$$\begin{tikzcd} \mathbb{R}U^- \arrow[r, "\varphi^-"] \arrow[dr, "\alpha^{(s)}"'] & \mathbb{R}\mathcal{O}^- \arrow[r, "\psi^{(x)}"] & \mathbb{R}\mathcal{O}^- \arrow[dl, "\beta"] \\ & \mathbb{R}[G]^- & \end{tikzcd}$$

where $x > 1$ and $\varphi^- = \varphi|_{\mathbb{R}U^-}$,

$$\psi^{(x)}([a] - [-a]) = \sum_{(n,r)=1} \frac{[na] - [-na]}{n^x},$$

$$\beta([a] - [-a]) = \frac{1}{2\pi i} \sum_{t \in (\mathbb{Z}/r\mathbb{Z})^\times} (\exp(2\pi i a t) - \exp(-2\pi i a t))\sigma_t^{-1}$$

and

$$\alpha^{(x)} = \beta \circ \psi^{(x)} \circ \varphi^-.$$

$\psi^{(x)}$ is well defined and all the above maps are isomorphisms of vector spaces. Then we have

$$\begin{aligned} &(R^- : \alpha^{(x)}(U^-)) \\ =&(R^- : \beta(\mathcal{O}^-)) \cdot (\beta(\mathcal{O}^-) : \beta\psi^{(x)}(\mathcal{O}^-)) \cdot (\beta\psi^{(x)}(\mathcal{O}^-) : \alpha^{(x)}(U^-)) \qquad (3.13) \\ =&(R^- : \beta(\mathcal{O}^-)) \cdot (\mathcal{O}^- : \psi^{(x)}(\mathcal{O}^-)) \cdot (\mathcal{O}^- : \varphi^-(U^-)) \end{aligned}$$

Here for the second equality, we use the property that if V_1 and V_2 are two vector spaces and f is an isomorphism from V_1 to V_2, then $(A : B)_{V_1} = (f(A) : f(B))_{V_2}$. Now for the three factors at the last line of (3.13), we have:

LEMMA 3.5.3.

$$(R^- : \beta(\mathcal{O}^-)) = \begin{cases} (2\pi)^{-\varphi(r)/2}\sqrt{d(K_r)/d(K_r^+)}, & \text{if } r \neq 2^\alpha; \\ \frac{1}{2}(2\pi)^{-\varphi(r)/2}\sqrt{d(K_r)/d(K_r^+)}, & \text{if } r = 2^\alpha. \end{cases} \tag{3.14}$$

PROOF OF LEMMA 3.5.3. We first consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^+ & \xrightarrow{i} & \mathcal{O} & \xrightarrow{1-c} & \mathrm{im}(1-c) & \longrightarrow & 0 \\ & & \| & & \uparrow i & & \uparrow i & & \\ 0 & \longrightarrow & \mathcal{O}^+ & \xrightarrow{i} & \mathcal{O}^+ \oplus \mathcal{O}^- & \xrightarrow{1-c} & 2\mathcal{O}^- & \longrightarrow & 0 \end{array}$$

where i is the natural inclusion map. By Theorem 3.4.1, if r is not a power of 2, $\mathcal{O}^- = \mathrm{im}(1-c)$; if r is a power of 2, then $\mathcal{O}^-/\,\mathrm{im}(1-c) = \mathbb{Z}/2\mathbb{Z}$. Therefore,

$$(\mathcal{O} : \mathcal{O}^+ \oplus \mathcal{O}^-) = (\mathrm{im}(1-c) : 2\mathcal{O}^-) = \begin{cases} 2^{\varphi(r)/2}, & \text{if } r \neq 2^\alpha; \\ 2^{\varphi(r)/2-1}, & \text{if } r = 2^\alpha. \end{cases}$$

Now let T be the map from $\mathbb{C}\mathcal{O}$ to $\mathbb{C}[G]$ such that $T([a]) = \sum_t \exp(2\pi ita)\sigma_t^-$, then we have $T|_{\mathbb{C}\mathcal{O}^-} = 2\pi i\beta|_{\mathbb{C}\mathcal{O}^-}$. Then on one hand,

$$(R^+ \oplus R^- : T(\mathcal{O}^+ \oplus \mathcal{O}^-)) = (R^+ : T(\mathcal{O}^+)) \cdot (R^- : T(\mathcal{O}^-))$$

on the other hand,

$$(R^+ \oplus R^- : T(\mathcal{O}^+ \oplus \mathcal{O}^-)) = (R^+ \oplus R^- : R) \cdot (R : T(\mathcal{O})) \cdot (\mathcal{O} : \mathcal{O}^+ \oplus \mathcal{O}^-).$$

But we know $(R^+ \oplus R^- : R) = 2^{-\varphi(r)/2}$, and by the definition of T, $(R : T(\mathcal{O})) = \sqrt{d(K)}$ and $(R^+ : T(\mathcal{O}^+)) = \sqrt{d(K^+)}$. Now the lemma follows from the above results and

$$(R^- : \beta(\mathcal{O}^-)) = (2\pi)^{-\varphi(r)/2}(R^- : 2\pi i\beta(\mathcal{O}^-)).$$

□

LEMMA 3.5.4. *Let $S = T_{\bar{r}} = \{p : p \mid r\}$, then*

$$(\mathcal{O}^- : \psi^{(x)}(\mathcal{O}^-)) = \prod_{\chi \ odd} L_S(x, \chi). \tag{3.15}$$

PROOF OF LEMMA 3.5.4. Note that if we let

$$\Theta_S(x) = \sum_{(n.m)=1} \frac{\sigma_n}{n^x},$$

then $\psi^{(x)}$ is just the left multiplication of $\Theta_S(x)$ on $\mathbb{R}\mathcal{O}^-$. By [**36**] Lemma 1.2(b), we have

$$(\mathcal{O}^- : \psi^{(x)}(\mathcal{O}^-)) = \prod_{\chi\ odd} \chi(\Theta_S(x)) = \prod_{\chi\ odd} L_S(x,\chi).$$

□

LEMMA 3.5.5.

$$(\mathcal{O}^- : \varphi^-(U^-)) = \begin{cases} 2^{-2^{s-2}} \prod_{p|r} \prod_{\chi\ odd} (1-\chi(p)^{-1})p^{-1}, & if\ s>1; \\ \frac{1}{2}, & if\ s=1,\ r\neq 2^\alpha; \\ 1, & if\ r=2^\alpha. \end{cases} \tag{3.16}$$

PROOF OF LEMMA 3.5.5. This follows from the abstract index formula (3.3), Proposition 3.3.7 and Proposition 3.4.3. □

Now let x approach 1, then

$$\begin{aligned} \lim_{x\to 1} \alpha^{(x)}([a]-[-a]) &= \lim_{x\to 1} \beta\psi^{(x)} H^0(\varphi)([a]-[-a]) \\ &= \frac{1}{2\pi i} \sum_t \sigma_t^{-1} \sum_{n\in\mathbb{N}} \frac{\exp(2n\pi i a t) - \exp(-2n\pi i a t)}{n} \\ &= \sum_t (\frac{1}{2} - \{at\})\sigma_t^{-1}. \end{aligned} \tag{3.17}$$

If we let $\alpha = \lim_{x\to 1} \alpha^{(x)}$, by (3.13), (3.14),(3.15) and (3.16), with the class number formula,

$$h^- = (2\pi)^{-\varphi(r)/2} \prod_{\chi\ odd} L(1,\chi) \sqrt{d(K_r)/d(K_r^+)}\, \omega\, Q,$$

and since $(U^- : (1-c)U) = 2^{2^{s-1}}$, then we have

$$\begin{aligned} (R^- : \alpha((1-c)U)) &= \lim_{x\to 1} (R^- : \alpha^{(x)}(U^-)) \cdot (U^- : (1-c)U) \\ &= \begin{cases} \frac{h^-}{\omega Q} \cdot 2^{2^{s-2}}, & if\ s>1; \\ \frac{h^-}{\omega Q}, & if\ s=1. \end{cases} \end{aligned} \tag{3.18}$$

But by (3.17), $\alpha((1-c)U)$ is nothing but $e^- S'$ in [**35**]. and by [**35**], Lemma 3.1, we have $(e^- S' : S^-) = \omega$. This is enough to finish the proof of the theorem.. □

CHAPTER 4

General Group Cohomology of the Universal Ordinary Distribution

This chapter is the core part of the thesis. We use Anderson's resolution to study the G_r-cohomology of U_r for any odd squarefree integer r. In §1, we offer a detailed study of the cohomology group $H^*(G_r, \mathbb{Z})$ and $H^*(G_r, \mathbb{Z}/M\mathbb{Z})$. In §2, we construct a double complex $K^{\bullet,\bullet}$ whose cohomology is exactly the group cohomology $H^*(G_r, U_r)$. We then study the spectral sequence of $K^{\bullet,\bullet}$ under the first filtration. This spectral sequence is shown to degenerate at E_2. We thus get a complete description of $H^*(G_r, U_r)$. In the last section, we give an explicit description of the 0-th G_r -cohomology group of U_r/MU_r where M is an integer dividing $\ell - 1$ for all primes ℓ dividing r. The results obtained will be used in the next chapter to provide a rationale for Kolyvagin's construction of "derivative classes".

4.1. The cohomology groups $H^*(G, \mathbb{Z})$

In this section, let G be a finite abelian group. By the structure theorem of finite abelian groups, then there exists a decomposition

$$G = \prod_{i=1}^{s} G_i \tag{4.1}$$

where $G_i =< \sigma_i >$ is a cyclic group of order m_i. We let $S = \{1, \cdots, s\}$. For any $T \subseteq S$, let $G_T = \prod_{i \in T} G_i \subseteq G$ and let $m_T = \gcd\{m_i : i \in T\}$. Let M be a given factor of m_S. Let $R = \mathbb{Z}_{\geq 0}[S]$. For any $e = (e_i) \in R$, set

- $\operatorname{supp} e = \{i \in S : e_i \neq 0\}$.
- $\deg e = \sum_{i \in S} e_i$.
- $\omega(e) = (\omega(e)_i) \in R$, where $\omega(e)_i = \sum_{j<i} e_j$.

For any $e, e' \in R$, set $\omega(e, e') = \sum_{j<i} e'_j e_i$.

We compute the cohomology group $H^*(G_T, \mathbb{Z})$ and $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$ in this section. Recall in § 2.5, to compute the group cohomology, it is necessary to find a projective resolution of the trivial G-module $\mathbb{Z}$. We explained in § 2.5 the standard bar resolution. In this section, we'll introduce another projective resolution of $\mathbb{Z}$, depending on the decomposition (4.1). This projective resolution is constructed by forming a tensor product. First we give the following definition:

Definition 4.1.1. Let $(X,\leq)$ be a finite totally ordered set and $x \in X$. Suppose that to every x of X we have a module A_x associated to x. We call

$$A_X = A_{x_1} \otimes \cdots \otimes A_{x_n}$$

the standard tensor product of A_x over X if $X = \{x_1, \cdots, x_n\}$ and $x_1 < \cdots < x_n$. Similarly, we can define the standard tensor product of elements $a_x \in A_x$ and of complexes $A_x^\bullet$.

4.1.1. A projective resolution of $\mathbb{Z}$.

Let

$$(\mathbf{P}_{i\bullet}, \partial_i): \quad \cdots \xrightarrow{\partial_{i,j+1}} P_{i,j+1} \xrightarrow{\partial_{ij}} P_{ij} \cdots \xrightarrow{\partial_{i0}} P_{i0} \longrightarrow 0$$

with $P_{ij} = \mathbb{Z}[G_i]$ for any $j \geq 0$, ∂_{ij} is the multiplication by $1-\sigma_i$ if j is even and by $\sum_{k=0}^{m_i-1} \sigma_i^k$ if j is odd. It is well known that $\mathbf{P}_{i\bullet}$ is a $\mathbb{Z}[G_i]$-projective resolution of trivial module $\mathbb{Z}$. For any $T \subseteq S$, let $\mathbf{P}_{T\bullet}$ be the standard tensor product of $\mathbf{P}_{i\bullet}$ over $i \in T$. It is well known by homological algebra that $\mathbf{P}_{T\bullet}$ is a $\mathbb{Z}[G_T]$-projective resolution of trivial module $\mathbb{Z}$. Now for the collection $\{P_{i,e_i} : i \in T\}$, the standard product of P_{i,e_i} over T is a rank 1 free $\mathbb{Z}[G_T]$-module whose grade is $\sum_i e_i$. Now let $e \in R$ be the element whose i-th component is e_i if $i \in T$ and 0 if not, and write the standard product of P_{i,e_i} over T as $\mathbb{Z}[G_T][e]$, then

$$\mathbf{P}_{T\bullet} = \bigoplus_{\operatorname{supp} e \subseteq T} \mathbb{Z}[G_T][e].$$

For any $x = (\cdots \otimes x_i \otimes \cdots) \in \mathbb{Z}[G_T][e]$, the differential is given by

$$\partial_T(x) = \sum_{i \in T} (-1)^{\omega(e)_i} (\cdots \otimes \partial_{i,e_i-1}(x_i) \otimes \cdots).$$

In particular, for $T = S$, let

$$\mathbf{P}_\bullet = \mathbf{P}_{S\bullet} = \bigoplus_{e \in R} \mathbb{Z}[G_S][e].$$

For any $T' \subseteq T$, we have a natural inclusion $\iota : \mathbb{Z}[G'_T][e] \hookrightarrow \mathbb{Z}[G_T][e]$ for any $e \in R$ such that $\operatorname{supp} e \subseteq T'$. By this inclusion, $\mathbf{P}_{T'\bullet}$ becomes a subcomplex of $\mathbf{P}_{T\bullet}$.

Now we define a diagonal map $\Phi_T : \mathbf{P}_{T\bullet} \to \mathbf{P}_{T\bullet} \otimes \mathbf{P}_{T\bullet}$. First set

$$\Phi_{ie_i,ie'_i} : P_{i,e_i+e'_i} \longrightarrow P_{ie_i} \otimes P_{ie'_i}$$
$$1 \longmapsto \begin{cases} 1 \otimes 1, & \text{if } e_i \text{ even;} \\ 1 \otimes \sigma_i, & \text{if } e_i \text{ odd, } e'_i \text{ even;} \\ \sum\limits_{0\le m<n\le m_i-1} \sigma_i^m \otimes \sigma_i^n, & \text{if } e_i \text{ odd, } e'_i \text{ odd;} \end{cases}$$

Then the map $\Phi_i : \mathbf{P}_{i\bullet} \to \mathbf{P}_{i\bullet} \otimes \mathbf{P}_{i\bullet}$ given by Φ_{ie_i,ie'_i} is the diagonal map for the cyclic group G_i (see Cartan-Eilenberg [**5**], P250-252). For any $e, e' \in R$ with support contained in T, consider the standard product $P_{e,e'}$ of $P_{ie_i} \otimes P_{ie'_i}$ over $i \in T$. The isomorphism

$$\alpha : P_{ie_i} \otimes P_{je'_j} \longrightarrow P_{je'_j} \otimes P_{ie_i}$$
$$x \otimes y \longmapsto (-1)^{e_i e'_i} y \otimes x$$

induces an isomorphism $\alpha : P_{e,e'} \to \mathbb{Z}[G_T][e] \otimes \mathbb{Z}[G_T][e']$ by

$$(\cdots(x_i \otimes y_i)\cdots) \longmapsto (-1)^{\omega(e,e')}(\cdots x_i \cdots) \otimes (\cdots y_i \cdots).$$

On the other hand, the standard product of the diagonal maps Φ_{ie_i,ie'_i} over $i \in T$ defines a map $\beta : \mathbb{Z}[G_T][e+e'] \to P_{e,e'}$. We let $\Phi_{e,e'} = \alpha \circ \beta$ and let

$$\Phi_{T,p,q} = \sum_{\substack{e,e':\deg\ e=p,\deg\ e'=q \\ \mathrm{supp}\, e+e' \subseteq T}} \Phi_{e,e'}.$$

Then Φ_T defines the diagonal map from $\mathbf{P}_{T\bullet}$ to $\mathbf{P}_{T\bullet} \otimes \mathbf{P}_{T\bullet}$. This map enables us to compute cup product structures.

4.1.2. The cohomology groups $H^*(G_T,\mathbb{Z})$ and $H^*(G_T,\mathbb{Z}/M\mathbb{Z})$. Let $\mathbf{C}_i^\bullet = \mathrm{Hom}_{G_i}(\mathbf{P}_{i\bullet},\mathbb{Z})$, then $\mathbf{C}_i^\bullet$ is the complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m_i} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m_i} \cdots$$

with the initial term at degree 0. We denote by C_i^j the j-th term of $\mathbf{C}_i^\bullet$. By the theory of group cohomology,

$$H^*(G_i,\mathbb{Z}) = H^*(\mathbf{C}_i^\bullet).$$

Now for any $T \subseteq S$, let $\mathbf{C}_T^\bullet$ be the standard tensor product of $\mathbf{C}_i^\bullet$ for $i \in T$. If write

$$\mathrm{Hom}_{G_T}(\mathbb{Z}[G_T][e], \mathbb{Z}) = \mathbb{Z}[e],$$

then

$$\mathbf{C}_T^\bullet = \mathrm{Hom}_{G_T}(\mathbf{P}_{T\bullet}, \mathbb{Z}) = \bigoplus_{\mathrm{supp}\, e \subseteq T} \mathbb{Z}[e].$$

and

$$H^*(G_T, \mathbb{Z}) = H^*(\mathbf{C}_T^\bullet).$$

Moreover, for any $T' \subseteq T$, the inclusion $\iota : \mathbf{P}_{T'\bullet} \hookrightarrow \mathbf{P}_{T\bullet}$ induces a map

$$\iota^* : \mathbf{C}_T^\bullet \longrightarrow \mathbf{C}_{T'}^\bullet,$$

which is just the natural projection of

$$\bigoplus_{\mathrm{supp}\, e \subseteq T} \mathbb{Z}[e] \longrightarrow \bigoplus_{\mathrm{supp}\, e \subseteq T'} \mathbb{Z}[e].$$

On the other hand, $G_{T'}$ can also be considered naturally as a quotient group of G_T, by this meaning, the inflation map is just the injection

$$\bigoplus_{\mathrm{supp}\, e \subseteq T'} \mathbb{Z}[e] \hookrightarrow \bigoplus_{\mathrm{supp}\, e \subseteq T} \mathbb{Z}[e].$$

Now for any $j \in \mathbb{Z}_{\geq 0}$ even, let

$$\mathbf{C}_i^{\bullet j} = \begin{cases} \cdots 0 \longrightarrow C_i^0 \longrightarrow 0 \cdots, & if\ j = 0; \\ \cdots 0 \longrightarrow C_i^{j-1} \xrightarrow{m_i} C_i^j \longrightarrow 0 \cdots, & if\ j > 0. \end{cases}$$

For any $e = (e_i) \in 2R$, i.e., e_i even for all $i \in S$, we let $\mathbf{C}_e^\bullet$ be the standard product $\mathbf{C}_i^{\bullet e_i}$ over $i \in S$. If $\mathrm{supp}\, e \subseteq T$, then $\mathbf{C}_e^\bullet$ is a subcomplex of $\mathbf{C}_T^\bullet$ and

$$\mathbf{C}_T^\bullet = \bigoplus_{\substack{e \in 2R \\ \mathrm{supp}\, e \subseteq T}} \mathbf{C}_e^\bullet.$$

Figure 1 shows us what the decomposition looks like in the case $S = \{1, 2\}$. Denote by A_e the cohomology group $H^*(\mathbf{C}_e^\bullet)$ and A_e^n its n-th component. Then

$$H^*(G_T, \mathbb{Z}) = \bigoplus_{\substack{e \in 2R \\ \mathrm{supp}\, e \subseteq T}} A_e, \ H^n(G_T, \mathbb{Z}) = \bigoplus_{\substack{e \in 2R \\ \mathrm{supp}\, e \subseteq T}} A_e^n.$$

We now study the abelian group A_e. First we need a lemma from linear algebra:

FIGURE 4.1. The complex $\mathbf{C}_S^\bullet$ when $S = \{1, 2\}$.

LEMMA 4.1.2. *Let* $\mathbf{v} = (m_1, m_2, \cdots, m_n)^t$ *be an n-dimensional column vector with integer entries m_i, then the greatest common divisor of the m_i is 1 if and only if there exists an $n \times n$ matrix $B \in SL_n(\mathbb{Z})$ whose first column is v.*

PROOF. Well known. $\square$

Now suppose $\operatorname{supp} e = T = \{i_1, \cdots, i_t\}$ and $|T| = t$. If $t = 0$, then $T = \emptyset$, it is easy to see that $A_e = A_e^0 = \mathbb{Z}$. Now if $T \neq \emptyset$, we claim that $\mathbf{C}_e^\bullet[\deg e - t]$ is isomorphic to the exterior algebra $\Lambda(x_1, \cdots, x_t)$ with differential $d(x) = \sum m_i x_i \wedge x$ and $\deg x_i = 1$. This claim is easy to check: First if $t = 1$, let $T = \{i\}$, then $\mathbf{C}_i^{\bullet e_i} = C^{e_i - 1} \oplus C^{e_i}$. This case is trivial. In general, if $\mathbf{C}_i^{\bullet e_i}[e_i - 1]$ is isomorphic to $\Lambda(x_i)$, the tensor product of $\mathbf{C}_i^{\bullet e_i}[e_i - 1]$ is nothing but $\mathbf{C}_e^\bullet[\deg e - t]$ and the tensor product of $\Lambda(x_i)$ is just $\Lambda(x_1, \cdots, x_t)$, hence they are isomorphic to each other.

Now since the greatest common divisor of m_i/m_T is 1, let B be the matrix given by Lemma 4.1.2 corresponding to the vector $(\cdots, .m_i/m_T, \cdots)$. Let $(y_1, \cdots, y_t) = (x_1, \cdots, x_t)B$. Then $\{y_1, \cdots, y_t\}$ is a set of new generators for the above exterior algebra and we have $d(x) = m_T \, y_1 \wedge x$. We see easily that

$$H^*(\Lambda(x_1, \cdots, x_t)) = (\mathbb{Z}/m_T\mathbb{Z})^{2^{t-1}}$$

and

$$H^j(\Lambda(x_1, \cdots, x_t)) = (\mathbb{Z}/m_T\mathbb{Z})^{\binom{t-1}{j}}, 0 \leq j \leq t - 1.$$

Combining the above analysis, we have

PROPOSITION 4.1.3. *There exists a family of complexes*

$$\{\mathbf{C}_e^\bullet \subseteq \mathbf{C}^\bullet = \mathrm{Hom}_{G_S}(\mathbf{P}_\bullet, \mathbb{Z}) : e \in 2R\}$$

such that

(1). For each $T \subseteq S$, we can identify $\mathbf{C}_T^\bullet = \mathrm{Hom}_{G_T}(\mathbf{P}_{T\bullet}, \mathbb{Z})$ with $\bigoplus_{\substack{e\in 2R\\ \mathrm{supp}\, e\subseteq T}} \mathbf{C}_e^\bullet$ through the following splitting exact sequence

$$0 \longrightarrow \bigoplus_{\substack{e\in 2R\\ \mathrm{supp}\, e\not\subseteq T}} \mathbf{C}_e^\bullet \longrightarrow \mathbf{C}^\bullet \longrightarrow \mathbf{C}_T^\bullet \longrightarrow 0$$

(2). The cohomology groups $H^*(\mathbf{C}_e^\bullet) = A_e$ and $H^n(\mathbf{C}_e^\bullet) = A_e^n$ are given by:

(a). If $\mathrm{supp}\, e \neq \emptyset$, let m_e be the greatest common divisor of $\ell_i - 1$ for $i \in \mathrm{supp}\, e$, then A_e is the abelian group $(\mathbb{Z}/m_e\mathbb{Z})^{2^{|\mathrm{supp}\, e|-1}}$, and

$$A_e^n = \begin{cases} (\mathbb{Z}/m_e\mathbb{Z})^{\binom{|\mathrm{supp}\, e|-1}{j}}, & \text{if } n = \deg e - j \text{ and } 0 \leq j \leq |\mathrm{supp}\, e| - 1; \\ 0, & \text{if otherwise.} \end{cases}$$

(b). If $\mathrm{supp}\, e = \emptyset$, then $A_e = A_e^0 = \mathbb{Z}$.

For the case $H^*(G, \mathbb{Z}/M\mathbb{Z})$, the situation is much easier. We have

PROPOSITION 4.1.4. *There exists a family*

$$\{[e] \in H^*(G_S, \mathbb{Z}/M\mathbb{Z}) : e \in R\}$$

with the following properties:

(1). For each $T \subseteq S$ and $n \in \mathbb{Z}_{\geq 0}$, the restriction of the family

$$\{[e] : e \in R,\ \mathrm{supp}\, e \subseteq T,\ \deg e = n\}$$

to $H^n(G_T, \mathbb{Z}/M\mathbb{Z})$ is a $\mathbb{Z}/M\mathbb{Z}$-basis of the latter.

(2). For each $T \subseteq S$ and $e \in R$ such that $\mathrm{supp}\, e \not\subseteq T$, the restriction of $[e]$ to $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$ vanishes.

(3). One has the cup product structure in $H^*(G_T, \mathbb{Z}/M\mathbb{Z})$ given by

$$[e] \cup [e'] = (-1)^{\omega(e,e')} \prod_{\substack{i\in S\\ e_ie_i'\equiv 1(2)}} \binom{m_i}{2} [e + e']$$

for all $e, e' \in R$.

PROOF. The complex $\mathbf{C}^\bullet_{M,i} = \mathrm{Hom}_{G_i}(\mathbf{P}_{i\bullet}, \mathbb{Z}/M\mathbb{Z})$ by definition, is a complex with $\mathbf{C}^j_{M,i} = \mathbb{Z}/M\mathbb{Z}$ for $j \geq 0$ and the differential 0. In general, $\mathbf{C}^\bullet_{M,T} = \mathbf{C}^\bullet_T \otimes \mathbb{Z}/M\mathbb{Z}$ is exactly the standard tensor product of $C^\bullet_{M,i}$ for all $i \in T$. Write

$$\mathbf{C}^\bullet_{M,T} = \mathrm{Hom}_{G_i}(\mathbf{P}_{T\bullet}, \mathbb{Z}/M\mathbb{Z}) = \sum_{\mathrm{supp}\, e \subseteq T} \mathbb{Z}/M\mathbb{Z}[e].$$

Since now $\mathbf{C}^\bullet_{M,T}$ has differential 0, $H^*(\mathbf{C}^\bullet_{M,T}) = \mathbf{C}^\bullet_{M,T}$. The restriction map is easy to see. This finishes the proof of 1) and 2).

For the cup product, the diagonal map Φ_T given above naturally induces a map:

$$\mathbf{C}^\bullet_{M,T} \times \mathbf{C}^\bullet_{M,T} \longrightarrow \mathbf{C}^\bullet_{M,T}$$

which defines the cup product structure. More specifically, the cup product map

$$\mathbb{Z}/M\mathbb{Z}[e] \times \mathbb{Z}/M\mathbb{Z}[e'] \longrightarrow \mathbb{Z}/M\mathbb{Z}[e + e']$$

is induced from $\Phi_{e,e'}$. Now the claim follows quickly from the explicit expression of $\Phi_{e,e'}$. $\square$

4.2. Study of $H^*(G_r, U_r)$

From now on, we compute the cohomology group $H^*(G_r, U_r)$ for the dimension 1 level r universal distribution U_r when r is odd squarefree. We denote by ℓ or ℓ_i the prime factor of r. We want to use the results of §4.1. Similar to the decomposition (4.1), we have the decomposition:

$$G_r = \prod_{\ell \mid r} G_\ell. \tag{4.2}$$

This similarity enable us to observe the following correspondences:

(1). $i \in S \rightsquigarrow \ell_i \mid r$,
- $i < j \rightsquigarrow \ell_i <_\omega \ell_j$,
- $m_i \rightsquigarrow \ell_i - 1$,
- $G_i = \langle \sigma_i \rangle \rightsquigarrow G_{\ell_i} = \langle \sigma_{\ell_i} \rangle$.

(2). $T \subseteq S \rightsquigarrow g \mid r$,
- $m_T \rightsquigarrow m_g$,
- $G_T \rightsquigarrow G_g$,
- $P_{T\bullet} \rightsquigarrow P_{g\bullet}$.

(3). $R = \mathbb{Z}_{\geq 0}[S] \rightsquigarrow \{h : h \mid r^\infty\}$, $e \in R \rightsquigarrow h \mid r^\infty$,

- $\operatorname{supp} e \rightsquigarrow \bar{h} = \prod_{\ell | h} \ell$,
- $\deg e = \sum_i e_i \rightsquigarrow \deg h = \sum_\ell v_\ell(h)$,
- $\omega(e, e') = \sum_{j<i} e'_j e_i \rightsquigarrow \omega(h, h') = \sum_{\ell_j <_\omega \ell_i} v_{\ell_j}(h') v_{\ell_i}(h)$.

We define

$$N_\ell = \sum_{k=0}^{\ell-2} \sigma_\ell^k, \quad D_\ell = \sum_{k=0}^{\ell-2} i\sigma_\ell^k \in \mathbb{Z}[G_\ell].$$

Moreover, define

$$N_g = \prod_{\ell | g} N_\ell, \quad D_g = \prod_{\ell | g} D_\ell \in \mathbb{Z}[G_g].$$

4.2.1. The complex $\mathbf{K}_r$. Set

$$\mathbf{K}_r^{\bullet,\bullet} := \operatorname{Hom}_{G_r}(\mathbf{P}_{r\bullet}, \mathbf{L}_r^\bullet).$$

Let d and δ be the differentials of $\mathbf{K}_r^{\bullet,\bullet}$ induced by the differentials d of $\mathbf{L}_r^\bullet$ and ∂ of $\mathbf{P}_{\bullet r}$ respectively. If we let

$$[a, g, h] := ([h] \mapsto [a, T]) \in \operatorname{Hom}_{G_r}(P_h, \langle [a, g] \rangle),$$

then

$$K^{p,q} = \left\langle [a, g, h] : a \in \frac{g}{r}\mathbb{Z}/\mathbb{Z}, |T_g| = -p, \deg h = q \right\rangle;$$
$$d[a, g, h] = \sum_{\ell | g} \omega(\ell, g)\big([a, g/\ell, h] - \sum_{\ell b = a} [b, g/\ell, h]\big);$$
$$\delta[a, g, h] = (-1)^{|T_g|} \sum_{\ell | r} (-1)^{v_\ell(\omega(h))} \cdot \begin{cases} (1 - \sigma_\ell)[a, g, h\ell], & \textit{if } v_\ell(h) \textit{ even}; \\ N_\ell [a, g, h\ell], & \textit{if } v_\ell(h) \textit{ odd}. \end{cases}$$

For any $g \mid r$, set

$$\mathbf{K}_r^{\bullet,\bullet}(g) = \operatorname{Hom}_{G_r}(\mathbf{P}_{r\bullet}, \mathbf{L}_g^\bullet) = \left\langle [a, g', h] : [a, g'] \in \mathbf{L}_g^\bullet, h \mid r^\infty \right\rangle$$

and

$$\mathbf{K}_g^{\bullet,\bullet} = \operatorname{Hom}_{G_g}(\mathbf{P}_{g\bullet}, \mathbf{L}_g^\bullet) = \left\langle [a, g', h] : [a, g'] \in \mathbf{L}_g^\bullet, h \mid g^\infty \right\rangle.$$

Furthermore, for any order ideal $\mathcal{I}$, set

$$\mathbf{K}_r^{\bullet,\bullet}(\mathcal{I}) := \operatorname{Hom}_{G_r}(\mathbf{P}_{r\bullet}, \mathbf{L}_r^\bullet(\mathcal{I})) = \sum_{g \in \mathcal{I}} \mathbf{K}_r^{\bullet,\bullet}(g).$$

and set

$$\mathbf{K}_r^{\bullet,\bullet}(n) := \operatorname{Hom}_{G_r}(\mathbf{P}_{r\bullet}, \mathbf{L}_r^\bullet(n)).$$

Set

$$\mathbf{U}_r^\bullet := \mathrm{Hom}_{G_r}(\mathbf{P}_{r\bullet}, U_r) = \frac{\left\langle [a,h] : a \in \frac{1}{r}\mathbb{Z}/\mathbb{Z}, h \mid r^\infty \right\rangle}{\left\langle [a,h] - \sum_{\ell b = a}[b,h] : a \in \frac{\ell}{r}\mathbb{Z}/\mathbb{Z}, h \mid r^\infty \right\rangle},$$

with the differential δ induced by ∂. Correspondingly,

$$\mathbf{U}_r^\bullet(\mathfrak{I}) := \frac{\left\langle [a,h] : a \in \frac{1}{g}\mathbb{Z}/\mathbb{Z} \text{ for some } g \in \mathfrak{I}, h \mid r^\infty \right\rangle}{\left\langle [a,h] - \sum_{\ell b = a}[b,h] : a \in \frac{\ell}{g}\mathbb{Z}/\mathbb{Z} \text{ for some } g \in \mathfrak{I}, h \mid r^\infty \right\rangle},$$

which is a subcomplex of $\mathbf{U}_r^\bullet$. We consider $\mathbf{U}_r^\bullet$ as the double complex $(\mathbf{U}_r^{\bullet,\bullet}; 0, \delta)$ concentrated on the vertical axis. We have a map

$$\mathfrak{u} : \mathbf{K}_r^{\bullet,\bullet} \to \mathbf{U}_r^{\bullet,\bullet},\ [a,g,h] \mapsto \begin{cases} [a,h], & \text{if } g = 1; \\ 0, & \text{if } g \neq 1. \end{cases}$$

PROPOSITION 4.2.1. *The map $\mathfrak{u}$(resp. its restriction) is a quasi-isomorphism between $\mathbf{K}_r^{\bullet,\bullet}$ (resp. $\mathbf{K}_r^{\bullet,\bullet}(\mathfrak{I})$) and $\mathbf{U}_r^{\bullet,\bullet}$(resp. $\mathbf{U}_r^{\bullet,\bullet}(\mathfrak{I})$). Therefore*

(1). $H^*_{total}(\mathbf{K}_r^{\bullet,\bullet}) = H^*(G_r, U_r)$, $H^*_{total}(\mathbf{K}_r^{\bullet,\bullet}(\mathfrak{I})) = H^*(G_r, U_r(\mathfrak{I}))$.

(2). $H^*_{total}(\mathbf{K}_{r,M}^{\bullet,\bullet}) = H^*(G_r, U_r/MU_r)$,
$H^*_{total}(\mathbf{K}_{r,M}^{\bullet,\bullet}(\mathfrak{I})) = H^*(G_r, U_r(\mathfrak{I})/MU_r(\mathfrak{I}))$.

PROOF. Immediately from Theorem 2.3.2(resp. Proposition 2.4.5 for $\mathfrak{I}$), we see that $\ker \mathfrak{u}$ is d-acyclic, and hence, by spectral sequence argument, it is $(d+\delta)$-acyclic. On the other hand, $\mathfrak{u}$ is surjective. Thus $\mathfrak{u}$ is a quasi-isomorphism. Now (1) follows directly from the quasi-isomorphism. For (2), just consider $\mathfrak{u} \otimes 1$, which is also a quasi-isomorphism. □

From Proposition 4.2.1, the G_r-cohomology of U_r is isomorphic to the total cohomology of the double complex $(\mathbf{K}_r^{\bullet,\bullet}; d, \delta)$. Therefore we can use the spectral sequence of the double complex $\mathbf{K}_r^{\bullet,\bullet}$ to study the G_r-cohomology of U_r. The spectral sequence of $\mathbf{K}_r^{\bullet,\bullet}$ from the second filtration has given us Proposition 4.2.1. Now we study the spectral sequence from the first filtration. Then

$$E_1^{p,q}(\mathbf{K}_r^{\bullet,\bullet}) = H_\delta^q(\mathbf{K}_r^{p,\bullet}) = H^q(G_r, L^p).$$

Recall the double complex structure given in §2.4.2, we have

$$L^p = \bigoplus_{p_1+p_2=p} L^{p_1,p_2} = \bigoplus_{|T_g|=-p} \bigoplus_{g \mid g'} L_r(g', g),$$

then

$$E_1^{p,q}(\mathbf{K}_r^{\bullet,\bullet}) = \bigoplus_{|T_g|=-p} \bigoplus_{g|g'} H^q(G_r, L_r(g', g)).$$

For the double complex $\mathbf{K}_r^{\bullet,\bullet}(\mathcal{I})$, Recall

$$\Gamma(\mathcal{I}) = \{(g_1, g_2) : rg_2/g_1 \in \mathcal{I},\ g_2 \mid g_1\}$$

from §2.4.2. We have

$$E_1^{p,q}(\mathbf{K}_r^{\bullet,\bullet}(\mathcal{I})) = \bigoplus_{(g',g)\in\Gamma(\mathcal{I})} H^q(G_r, L_r(g', g)).$$

4.2.2. A Lemma. Let S be a totally ordered finite set in this subsection. Suppose that for any $T \subseteq S$, there is an abelian group B_T associated to T, and set

$$A_T = \bigoplus_{T''\subseteq T} B_{T''}.$$

Then for any $T' \supseteq T$, there is a natural projection from $A_{T'}$ to A_T. Now let $\mathcal{C}_{S,T}^{\bullet}$ be the cochain complex with components given by

$$\mathcal{C}_{S,T}^{n} = \bigoplus_{\substack{|T'|=s-n \\ T'\supseteq(S\setminus T)}} A_{T'},$$

and differential d given by

$$\begin{aligned} d:\ A_{T'} &\longrightarrow \bigoplus_{i\in T'\cap T} A_{T'\setminus\{i\}} \\ x &\longmapsto \sum_{i\in T'\cap T} \omega(i, T'\cap T) x|_{T'\setminus\{i\}}, \end{aligned}$$

where $x|_{T'\setminus\{i\}}$ is the projection of x in $A_{T'\setminus\{i\}}$. It is easy to verify that $\mathcal{C}_{S,T}^{\bullet}$ is indeed a chain complex. Furthermore, we have

LEMMA 4.2.2. *For any $T \subseteq S$,*

$$H^n(\mathcal{C}_{S,T}^{\bullet}, d) = \begin{cases} \bigoplus_{T'\supseteq T} B_{T'}, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Let $\tilde{\mathcal{C}}_{S,T}^{\bullet}$ be the subcomplex of $\mathcal{C}_{S,T}^{\bullet}$ with the same components as $\mathcal{C}_{S,T}^{\bullet}$ except at degree 0, where

$$\tilde{\mathcal{C}}_{S,T}^{0} = \bigoplus_{T'\not\supseteq T} B_{T'}.$$

We only need to show that $\tilde{\mathfrak{C}}^\bullet_{S,T}$ is exact. We show it by double induction to the cardinalities of S and T. If $T = \emptyset$, we get a trivial complex. If S consists of only one element, or if T consists only one element, it is also trivial to verify. In general, suppose $i_0 = \max\{i : i \in T\}$. Let $S_0 = S\backslash\{i_0\}$ and $T_0 = T\backslash\{i_0\}$. Then we have the following commutative diagram which is exact on the columns:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \tilde{\mathfrak{C}}^0_{S,T_0} & \xrightarrow{\bar{d}} & \tilde{\mathfrak{C}}^1_{S,T_0} & \xrightarrow{\bar{d}} & \cdots\tilde{\mathfrak{C}}^{t-1}_{S,T_0} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle p} & & \\
0 & \longrightarrow & \tilde{\mathfrak{C}}^0_{S,T} & \xrightarrow{d} & \tilde{\mathfrak{C}}^1_{S,T} & \xrightarrow{d} & \cdots\tilde{\mathfrak{C}}^{t-1}_{S,T} & \xrightarrow{d} & \tilde{\mathfrak{C}}^t_{S,T} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & \bigoplus_{S_0\supseteq T'\supseteq T_0} B_{T'} & \xrightarrow{d} & A_{S_0} & \xrightarrow{d} & \cdots\tilde{\mathfrak{C}}^{t-1}_{S_0,T_0} & \xrightarrow{d} & \tilde{\mathfrak{C}}^t_{S_0,T_0} & \longrightarrow & 0
\end{array}$$

Here p means projection and i means inclusion. The differential $\bar{d}$ is induced by the differential d of the second row. Notice that the third row is a variation of the chain complex $\tilde{\mathfrak{C}}^\bullet_{S_0,T_0}$, the first row is the chain complex $\tilde{\mathfrak{C}}^\bullet_{S,T_0}$. By induction, the first row and and the third row are exact, so is the middle one. □

We shall apply the above lemma to study the E_2 terms of $\mathbf{K}_r^{\bullet,\bullet}$. Again we'll use the one to one correspondence of ℓ, r, g to i, S, T.

4.2.3. The Study of E_2 terms. By §4.2.1, we know that

$$E_1^{p,q}(\mathbf{K}_r^{\bullet,\bullet}) = \bigoplus_{|T_g|=-p} \bigoplus_{g|g'} H^q(G_r, L_r(g', g)).$$

Now let's consider the induced differential $\bar{d}_r$ of d_r in the E_1 term. As we know in §2.4.2, $d_r = d_{1r} + d_{2r}$, we can write $\bar{d}_r = \bar{d}_{1r} + \bar{d}_{2r}$. We first look at $\bar{d}_{2r}$, which is induced by the map

$$\begin{aligned}
L_r(g', g) &\longrightarrow \bigoplus_{\ell|g} L_r(g', g/\ell), \\
[a, g] &\longmapsto \sum_{\ell|g} \omega(\ell, g)(1 - Fr_\ell^{-1})[a, g/\ell].
\end{aligned}$$

Since for any $\ell \mid g$, $L_r(g', g)$ and $L_r(g', g/\ell)$ are G_r-isomorphic by the map φ_ℓ given in §2.4.2, and since for any $q \geq 0$, $H^q(G, A)$ is a trivial G-module, we have

$$\bar{d}_{2r} = \sum_{\ell|g} \omega(\ell, g)(1 - Fr_\ell^{-1})\bar{\varphi}_\ell = 0.$$

The map $\bar{d}_{1r}$ is induced by the map

$$L_r(g', g) \longrightarrow \bigoplus_{\ell | g} L_r(g'/\ell, g/\ell),$$
$$[a, g] \longmapsto -\sum_{\ell | g} \omega(\ell, g) N_\ell [Fr_\ell^{-1} a + \frac{1}{\ell}, g/\ell].$$

For any $\ell \mid g$, consider the map

$$\psi_\ell : L_r(g', g) \longrightarrow L_r(g'/\ell, g/\ell),$$
$$[a, g] \longmapsto N_\ell [Fr_\ell^{-1} a + \frac{1}{\ell}, g/\ell].$$

The map ψ_ℓ is a G_r-homomorphism and therefore induces a map in G_r-cohomology:

$$H^q(\psi_\ell) : H^q(G_r, L_r(g', g)) \to H^q(G_r, L_r(g'/\ell, g/\ell)).$$

We have the commutative diagram:

$$\begin{array}{ccc} L_r(g', g) & \xrightarrow{\psi_\ell} & L_r(g'/\ell, g/\ell) \\ \downarrow{\scriptstyle \theta_{g'}} & & \downarrow{\scriptstyle \theta_{g'/\ell}} \\ \mathbb{Z} & \xrightarrow{res} & \mathbb{Z} \end{array}$$

where the top row are G_r-modules, the left $\mathbb{Z}$ is a trivial $G_{g'}$-module, the right $\mathbb{Z}$ is a trivial $G_{g'/\ell}$-module, and $\theta_{g'}$ is the homomorphism sending $[\frac{g'}{r}, g]$ to 1 and $[\frac{xg'}{r}, g]$ to 0 if $x \neq 1$. Then the above diagram induces the following commutative diagram:

$$\begin{array}{ccc} H^q(G_r, L_r(g', g)) & \xrightarrow{H^q(\psi_\ell)} & H^q(G_r, L_r(g'/\ell, g/\ell)) \\ \downarrow{\scriptstyle \theta^*_{g'}} & & \downarrow{\scriptstyle \theta^*_{g'/\ell}} \\ H^q(G_{g'}, \mathbb{Z}) & \xrightarrow{res} & H^q(G_{g'/\ell}, \mathbb{Z}) \end{array}$$

where $\theta^*_{g'}$(and $\theta^*_{g'/\ell}$) is the isomorphism given by Shapiro's lemma(See Serre [**32**], Chap. VII, §5, Exercise). We identify $H^q(G_r, L_r(g', g))$ with $H^q(G_{g'}, \mathbb{Z})$, moreover, to keep track of g, we'll write $H^q(G_{g'}, \mathbb{Z})$ as $H^q(G_{g',g}, \mathbb{Z})$. Then we see that $H^q(\psi_\ell)$ is the restriction map from $H^q(G_{g',g}, \mathbb{Z})$ to $H^q(G_{g'/\ell, g/\ell}, \mathbb{Z})$. The induced differential $\bar{d}_r = \bar{d}_{1r}$ is exactly the map

$$H^q(G_{g',g}, \mathbb{Z}) \longrightarrow \bigoplus_{\ell | g} H^q(G_{g'/\ell, g/\ell}, \mathbb{Z}),$$
$$x \longmapsto -\sum_{\ell | g} \omega(\ell, g) x_\ell.$$

where x_ℓ is the restriction of x in $H^q(G_{g'/\ell,g/\ell},\mathbb{Z})$. Hence we have a cochain complex $\mathcal{C}(q;r,g)$

$$H^q(G_{r,g},\mathbb{Z})\xrightarrow{\bar{d}_{1r}}\bigoplus_{\ell|g}H^q(G_{r/\ell,g/\ell},\mathbb{Z})\cdots\xrightarrow{\bar{d}_{1r}}H^q(G_{r/g,1},\mathbb{Z})\longrightarrow 0$$

Note that the complex $E_1^{\bullet,q}(\mathbf{K}_r^{\bullet,\bullet})$ is just the direct sum of $\mathcal{C}(q;r,g)$ over all factors g of r. Moreover, the complex $E_1^{\bullet,q}(\mathbf{K}_r^{\bullet,\bullet})(\mathcal{I})$ is the direct sum of $\mathcal{C}(q;r,g)$ over all $g\in\mathcal{I}$.

Recall in Proposition 4.1.3, we obtained

$$H^q(G_g,\mathbb{Z})=\bigoplus_{h|g^{2\infty}}A_h^q.$$

If we let

$$A_g^q=H^q(G_g,\mathbb{Z}),\ B_g^q=\bigoplus_{\substack{h|g^{2\infty}\\ \bar{h}=g}}A_h^q.$$

then we have $A_g^q=\bigoplus_{g''|g}B_{g''}^q$. The complex $\mathcal{C}(q;r,g)[-|T_g|\,]$satisfies the conditions in Lemma 4.2.2, thus the n-th cohomology of the cochain complex $\mathcal{C}(q;r,g)$ is 0 if $n\neq-|T_g|$ and $\sum\limits_{g|g'}B_{g'}$ if $n=-|T_g|$. We have the following proposition:

Proposition 4.2.3. *One has*

1). $E_2^{p,q}(\mathbf{K}_r^{\bullet,\bullet})\cong\bigoplus\limits_{|T_g|=-p}\ \bigoplus\limits_{g|h|r^{2\infty}}A_h^q.$

2). $E_2^{p,q}(\mathbf{K}_r^{\bullet,\bullet}(\mathcal{I}))\cong\bigoplus\limits_{\substack{|T_g|=-p\\ g\in\mathcal{I}}}\ \bigoplus\limits_{g|h|r^{2\infty}}A_h^q.$

4.2.4. Proof of Theorem A. Finally we are in a position to prove the main theorem(Theorem A) in this paper. Put

$$\mathbf{S}_r^{\bullet,\bullet}=<[a,g,h]\in\mathbf{K}_r^{\bullet,\bullet},a\neq 0\text{ if }g\mid h>.$$

It is easy to verify that $\mathbf{S}_r^{\bullet,\bullet}$ is a subcomplex of $\mathbf{K}_r^{\bullet,\bullet}$ using the explicit formulas for d and δ given in §4.2.1. Set

$$\mathbf{Q}_r^{\bullet,\bullet}=\mathbf{K}_r^{\bullet,\bullet}/\mathbf{S}_r^{\bullet,\bullet}=<[0,g,h]:g\mid h>.$$

Note that the differential of $\mathbf{Q}_r^{\bullet,\bullet}$ induced by d_r is 0. Moreover, set

$$\mathbf{S}_r^{\bullet,\bullet}(\mathcal{I}):=\mathbf{K}_r^{\bullet,\bullet}(\mathcal{I})\cap\mathbf{S}_r^{\bullet,\bullet},$$

and

$$\mathbf{Q}^{\bullet,\bullet}(\mathcal{I})_r := \mathbf{K}_r^{\bullet,\bullet}(\mathcal{I})/\mathbf{S}_r^{\bullet,\bullet}(\mathcal{I}) =< [0,g,h]: \ g \in \mathcal{I}, \ g \mid h > .$$

Let f be the corresponding quotient map, then we have a commutative diagram:

$$\begin{array}{ccc} \mathbf{K}_r^{\bullet,\bullet}(\mathcal{I}) & \xrightarrow{inc} & \mathbf{K}_r^{\bullet,\bullet} \\ \big\downarrow f & & \big\downarrow f \\ \mathbf{Q}_r^{\bullet,\bullet}(\mathcal{I}) & \xrightarrow{inc} & \mathbf{Q}_r^{\bullet,\bullet} \end{array}$$

We make the following claim

PROPOSITION 4.2.4. *The quotient map $f : \mathbf{K}_r^{\bullet,\bullet} \to \mathbf{Q}_r^{\bullet,\bullet}$ is a quasi-isomorphism. Moreover, the quotient map $f : \mathbf{K}_r^{\bullet,\bullet}(\mathcal{I}) \to \mathbf{Q}_r^{\bullet,\bullet}(\mathcal{I})$ is a quasi-isomorphism.*

PROOF. Let

$$\mathcal{L}_g^\bullet :=< [0,g,h] : h \mid r^\infty >= \mathrm{Hom}_{G_r}(\mathbf{P}_{r\bullet}, L_r(r,g)) \subseteq \mathbf{K}_r^{\bullet,\bullet}$$

and let

$$\mathcal{L}_g^{'\bullet} :=< [0,g,h] : g \mid h >, \ \mathcal{L}_g^{''\bullet} :=< [0,g,h] : g \nmid h >$$

Through the map $L_r(r,g) \to \mathbb{Z}, \ [0,g] \mapsto 1$, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{L}_g^\bullet & = & \mathcal{L}_g^{'\bullet} & \bigoplus & \mathcal{L}_g^{''\bullet} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbf{C}^\bullet & = & \bigoplus_{g|h|r^{2\infty}} \mathbf{C}_h^\bullet & \bigoplus & \bigoplus_{g\nmid h|r^{2\infty}} \mathbf{C}_h^\bullet \end{array}$$

where $\mathbf{C}^\bullet$ and $\mathbf{C}_h^\bullet$ are those $\mathbf{C}^\bullet$ and $\mathbf{C}_e^\bullet$ given in Proposition 4.1.3. By this diagram, we identify $\mathcal{L}_g^\bullet$ with $\mathbf{C}^\bullet$. By Proposition 4.1.3, we have

$$\ker(H^*(G_r,\mathbb{Z}) \to H^*(G_{r/\ell},\mathbb{Z})) = H^*(\bigoplus_{\ell|h|r^{2\infty}} \mathbf{C}_h^\bullet).$$

Then by the proof of Proposition 4.2.3,

$$\begin{aligned} \ker(\bar{d}|_{H^q(\mathcal{L}_g^\bullet)}) =& \bigcap_{\ell|g} \ker(H^*(G_r, L_r(r,g)) \to H^*(G_r, L_r(r/\ell, g/\ell))) \\ =& \bigcap_{\ell|g} H^*(\bigoplus_{\ell|h|r^{2\infty}} \mathbf{C}_h^\bullet) = H^*(\bigcap_{\ell|g} \bigoplus_{\ell|h|r^{2\infty}} \mathbf{C}_h^\bullet) \\ =& H^*(\bigoplus_{g|h|r^{2\infty}} \mathbf{C}_h^\bullet) = H^*(\mathcal{L}_g^{'\bullet}) \end{aligned}$$

where the second and the last identifications are made using the isomorphisms given in the commutative diagram above. Hence we have

$$E_2^{p,q}(\mathbf{K}_r^{\bullet,\bullet}) = \bigoplus_{|T_g|=-p} \ker(\bar{d}_r|_{H^q(\mathcal{L}_g^\bullet)}) = \bigoplus_{|T_g|=-p} H^q(\mathcal{L}_g^{'\bullet}).$$

On the other hand,

$$\mathbf{Q}_r^{\bullet,\bullet} = \bigoplus_{g|r} \mathcal{L}_g^{'\bullet}.$$

Since $d_r = 0$ in $\mathbf{Q}_r^{\bullet,\bullet}$, the spectral sequence of $\mathbf{Q}_r^{\bullet,\bullet}$ by the first filtration(i.e., by d_r) degenerates at E_1. We have

$$E_1^{p,q}(\mathbf{Q}_r^{\bullet,\bullet}) = E_2^{p,q}(\mathbf{Q}_r^{\bullet,\bullet}) = \bigoplus_{|T_g|=-p} H^q(\mathcal{L}_g^{'\bullet}).$$

Since the projection map from $\mathcal{L}_g^\bullet$ to $\mathcal{L}_g^{'\bullet}$ in the commutative diagram above is nothing but the restriction of the quotient map f at $\mathcal{L}_g^\bullet$, by the above analysis, we get an isomorphism

$$f_2 : E_2^{p,q}(\mathbf{K}_r^{\bullet,\bullet}) \longrightarrow E_2^{p,q}(\mathbf{Q}_r^{\bullet,\bullet}).$$

Thus the spectral sequences of $\mathbf{K}_r^{\bullet,\bullet}$ and $\mathbf{Q}_r^{\bullet,\bullet}$ are isomorphic at E_r for $r \geq 2$. In our case, the first filtration is finite, by Theorem 2.5.5, therefore f is a quasi-isomorphism .

The case $\mathcal{I}$ is similar. In this case,

$$E_2^{p,q}(\mathbf{K}_r^{\bullet,\bullet})(\mathcal{I}) = \bigoplus_{\substack{g\in\mathcal{I}\\ |T_g|=-p}} \ker(\bar{d}_r|_{H^q(\mathcal{L}_g^\bullet)}) = \bigoplus_{|T_g|=-p} H^q(\mathcal{L}_g^{'\bullet}),$$

and

$$\mathbf{Q}_r^{\bullet,\bullet}(\mathcal{I}) = \bigoplus_{g\in\mathcal{I}} \mathcal{L}_g^{'\bullet}.$$

Now follow the same analysis as above. □

For any factor g of r, set

$$H_g^*(G_r, \mathbb{Z}) := \bigcap_{\ell|g} \ker(H^*(G_r, \mathbb{Z}) \to H^*(G_{r/\ell}, \mathbb{Z}))$$

we see that

$$H^*(\mathcal{L}_g^{'\bullet}) \cong H_g^*(G_r, \mathbb{Z})$$

by the identification of $\mathcal{L}_g^\bullet$ and $\mathbf{C}^\bullet$. The following theorem is the main result in the thesis:

THEOREM A . *(1). The cohomology group $H^*(G_r, U_r)$ is given by*

$$H^*(G_r, U_r) = \bigoplus_{g|r} H_g^*(G_r, \mathbb{Z})[\,|T_g|\,] = \bigoplus_{g|r} \bigoplus_{g|h|r^{2\infty}} A_h[\,|T_g|\,].$$

where $A_h[\,|T_g|\,]$ represents the cohomology group $H^*(\mathbf{C}_h^\bullet[\,|T_g|\,])$. More specifically, we have

$$H^n(G_r, U_r) = \bigoplus_{g|r} H_g^{n+|T_g|}(G_r, \mathbb{Z}).$$

(2). The cohomology group $H^*(G_r, U_r(\mathcal{I}))$ is given by

$$H^*(G_r, U_r(\mathcal{I})) = \bigoplus_{g\in\mathcal{I}} H_g^*(G_r, \mathbb{Z})[\,|T_g|\,] = \bigoplus_{g\in\mathcal{I}} \bigoplus_{g|h|r^{2\infty}} A_h[\,|T_g|\,].$$

More specifically, we have

$$H^n(G_r, U_r(\mathcal{I})) = \bigoplus_{g\in\mathcal{I}} H_g^{n+|T_g|}(G_r, \mathbb{Z}).$$

PROOF. We only prove (1). The proof of (2) follows the same route. By Proposition 4.2.1 and Proposition 4.2.4, we know that

$$H^*(G_r, U_r) = H^*_{total}(\mathbf{K}_r^\bullet) = H^*_{total}(\mathbf{Q}_r^\bullet).$$

Now

$$H^n_{total}(\mathbf{Q}^\bullet) = \bigoplus_{g|r} H^{n+|T_g|}(\mathcal{L}_g^{'\bullet}).$$

Part 1) follows immediately. □

REMARK 4.2.5. 1). We can see that Part (1) is actually a special case of Part (2) when the order ideal $\mathcal{I}$ is Div_r.

2). By Theorem A, in the case $n = 0$, we have

$$H^0(G_r, U_r) = \mathbb{Z};$$

in the case $n = 1$, we have

$$H^1(G_r, U_r) = \prod_{g|r} \mathbb{Z}/m_g\mathbb{Z}.$$

It is likely that the cohomology classes in $H^1(G_r, U_r)$ have a natural role to play in the cyclotomic Euler system method, but this role has not yet been worked out in detail.

In the case $\mathbb{Z}/M\mathbb{Z}$, we have

THEOREM 4.2.6. *There exists a family*

$$\{c_{g,h} \in H^*(G_r, U_r/MU_r) : g \mid r,\ \ h \mid r^\infty,\ \ g \mid h\}$$

with the following properties:

(1). For each $n \in \mathbb{Z}_{\geq 0}$*, the subfamily*

$$\{c_{g,h} : g \mid r,\ \ h \mid r^\infty,\ \ g \mid h, \deg h = n + |T_g|\}$$

is a $\mathbb{Z}/M\mathbb{Z}$*-basis for* $H^n(G_r, U_r/MU_r)$*.*

(2). For any order ideal $\mathfrak{I}$ *of* r*, let* $U_r(\mathfrak{I}) = \sum_{g\in\mathfrak{I}} U_g$*. By the inclusion* $U_r(\mathfrak{I}) \hookrightarrow U_r$*,* $H^*(G_r, U_r(\mathfrak{I})/MU_r(\mathfrak{I}))$ *can be considered as a submodule of* $H^*(G_r, U_r/M_r)$*. Furthermore, the subfamily*

$$\{c_{g,h} : g \in \mathfrak{I},\ \ h \mid r^\infty,\ \ g|h\}$$

is a Z/MZ *basis for* $H^*(G_r, U_r(\mathfrak{I})/MU_r(\mathfrak{I}))$*.*

(3). One has cup product structure

$$[h'] \cup c_{g,h} = (-1)^{\omega(h',h)} \prod_{v_{\ell_i}(hh')\equiv 1(2)} \left(\frac{\ell_i - 1}{2}\right) c_{g,hh'}$$

for all $h, h' \mid r^\infty$ *and* $g \mid h$*.*

PROOF. 1). By Proposition 4.2.4, we have induced quasi-isomorphism:

$$f \otimes 1 : \mathbf{K}^{\bullet,\bullet}_{r,M} \longrightarrow \mathbf{Q}^{\bullet,\bullet}_{r,M}$$

Now since the induced differentials of d_r and δ in $\mathbf{Q}^{\bullet,\bullet}_{r,M}$ are 0. Consider the cocycle $[0, g, h]$ in $\mathbf{Q}^{\bullet,\bullet}_{r,M}$, there exists a cocycle $C_{g,h}$(unique modulo boundary) which is the lifting of $[0, g, h]$ by the quotient map $f \otimes 1$. Hence $\mathfrak{u}(C_{g,h}) \otimes 1$ is a cocycle in the complex $\mathbf{U}^{\bullet}_{r,M}$. Let $c_{g,h}$ denote the cohomology element in $H^*(G_r, U_r/MU_r)$ represented by the cocycle $\mathfrak{u}(C_{g,h})\otimes 1$. Then $\{c_{g,h} : g \mid h\}$ is a canonical $\mathbb{Z}/M\mathbb{Z}$-basis for the cohomology group $H^*(G_r, U_r/MU_r)$. This finishes the proof of (1).

(2). Similar to (1), just consider the map $f \otimes 1 : \mathbf{K}^{\bullet,\bullet}_{r,M}(\mathfrak{I}) \to \mathbf{Q}^{\bullet,\bullet}_{r,M}(\mathfrak{I})$.

(3). For the cup product, there is natural homomorphism

$$\mathbb{Z}/M\mathbb{Z} \otimes U_r/MU_r \longrightarrow U_r/MU_r,$$

therefore $H^*(G_r, U_r/MU_r)$(and also $H^*(G_r, U_r(\mathfrak{I})/MU_r(\mathfrak{I}))$ has a natural $H^*(G_r, \mathbb{Z}/M\mathbb{Z})$-module structure. By the theory of spectral sequences(see, for example Brown [**4**], Chap. 7, §5), we have the cochain cup product

$$\mathbf{C}^{\bullet}_{r,M} \otimes \mathbf{K}^{\bullet,\bullet}_{r,M} \longrightarrow \mathbf{K}^{\bullet,\bullet}_{r,M}.$$

By using the diagonal map Φ_r defined in §4.1, it is easy to check that:

$$\mathbf{C}^{\bullet}_{r,M} \otimes \mathbf{S}^{\bullet,\bullet}_{r,M} \subseteq \mathbf{S}^{\bullet,\bullet}_{r,M},$$

hence we can pass the cup product structure to the quotient and have

$$\mathbf{C}^{\bullet}_{r,M} \otimes \mathbf{Q}^{\bullet,\bullet}_{r,M} \longrightarrow \mathbf{Q}^{\bullet,\bullet}_{r,M}.$$

Now (3) follows immediately from the explicit expression of Φ_r. This concludes the proof. □

4.3. Explicit basis of $H^0(G_r, U_r/MU_r)$

In §4.2, we obtained a canonical basis $\{c_{g,h} : h \mid r^\infty, g \mid h\}$ for the cohomology group $H^*(G_r, U_r/MU_r)$. However, little is known yet for the explicit expression of the cocycle $c_{g,h}$ in the complex $\mathrm{Hom}_{G_r}(\mathbf{P}_\bullet, U_r/MU_r)$, which makes it necessary to study how to lift the cocycle $[0, g, h]$ in $\mathbf{Q}^{\bullet,\bullet}_{r,M}$ to the cocycle $C_{g,h}$ in $\mathbf{K}^{\bullet,\bullet}_{r,M}$. Unfortunately, we are unable to get a complete answer for this problem in this paper. We obtain a partial solution in the 0-cocycles case, however, which is enough for us to prove Theorem B.

4.3.1. The triple complex structure of $\mathbf{K}_r$.

Recall from §2.4.3, $\mathbf{L}$ has a double complex structure, therefore we can make $\mathbf{K}_r$ a triple complex. Set

$$K^{p_1,p_2,q} := \mathrm{Hom}_{G_r}(\mathbf{P}_\bullet, L^{p_1,p_2}) = \langle [a, g, h] : [a, g] \in L^{p_1,p_2}, \deg h = q \rangle$$

with the differentials (d_{1r}, d_{2r}, δ) given by

$$d_{1r}[a, g, h] = -\sum_{\ell \mid g} \omega(\ell, g) N_\ell [Fr_\ell^{-1} a + \frac{1}{\ell}, g/\ell, h],$$

$$d_2[a, g, h] = \sum_{\ell \mid g} \omega(\ell, g)(1 - Fr_\ell^{-1})[a, g/\ell, h],$$

and δ as given in the double complex $\mathbf{K}_r^{\bullet,\bullet}$. In this setup, we see that $\mathbf{K}_r(\mathfrak{I})$ becomes a triple subcomplex of $\mathbf{K}_r$, moreover

$$\mathbf{K}_r(n) = \bigoplus_{p_2 \geq s-n} K^{p_1,p_2,q}.$$

Correspondingly, we have triple complex structures on $\mathbf{K}_{r,M}$, $\mathbf{K}_{r,M}(\mathfrak{I})$ and $\mathbf{K}_{r,M}(n)$. This triple complex structure enables us to construct different double complex structures in $\mathbf{K}_r$ and $\mathbf{K}_{r,M}$. By studying those double complexes, we can gather more information about $\mathbf{K}_r$. This method will be illustrated in the next subsection.

4.3.2. The double complex $(\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}, d_1, \delta)$. For fixed p_2, let

$$\mathbf{K}_{r,M}^{\bullet,p_2,\bullet} = \bigoplus_{p_1,q} K_{r,M}^{p_1,p_2,q},$$

with differentials d_1 and δ, then we get a double complex $(\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}; d_1, \delta)$. Similarly, we can get the double complex $(\mathbf{K}_{r,M}^{\bullet,\bullet}; d_1 + \delta, d_2)$ whose $(p_1 + q, p_2)$-component is $\bigoplus K_{r,M}^{p_1,p_2,q}$. As before, for any $\mathfrak{I}$, we have double complexes $\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}(\mathfrak{I})$ and $\bigoplus K_{r,M}^{p_1,p_2,q}(\mathfrak{I})$ which are subcomplexes of $\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}$ and $\bigoplus K_{r,M}^{p_1,p_2,q}$ respectively. First we have

PROPOSITION 4.3.1. *(1). $H^*_{total}(\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}; d_1, \delta)$ is a free $\mathbb{Z}/M\mathbb{Z}$-module generated by cocycles $C'_{g,h}$ with leading term $[0, g, h]$ and the remainder with q-degree less than* $\deg h$ *over all pairs (g, h) satisfying $|T_g| = s - p_2$ and $g \mid h$.*

(2). Moreover, $H^*_{total}(\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}(\mathfrak{I}); d_1, \delta)$ is a free $\mathbb{Z}/M\mathbb{Z}$-module generated by cocycles $C'_{g,h}$ with leading term $[0, g, h]$ and the remainder with q-degree less than $\deg e$ *over all pairs (g, h) satisfying $g \in \mathfrak{I}$, $|T_g| = s - p_2$ and $g \mid h$.*

PROOF. We only prove (1). The proof of (2) is similar. First look at the spectral sequence of $\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}$ with the second filtration(i.e., the filtration given by δ), then

$$E_1^{p_1,q}(\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}) = H^q(G_r, L^{p_1,p_2}).$$

Next for the differential d_{1r} induced on E_1, with the same analysis as in computing the E_2 terms of $(\mathbf{K}; d, \delta)$(see §4.2, Proposition 4.2.3), we have

$$E_2^{p_1,q}(\mathbf{K}_{r,M}^{\bullet,p_2,\bullet}) = \begin{cases} \displaystyle\bigoplus_{|T_g|=s-p_2} \bigoplus_{\substack{h:\deg h=q \\ g|h}} \mathbb{Z}/M\mathbb{Z}, & \text{if } p_1 = -s; \\ 0, & \text{if } p_1 \neq -s. \end{cases}$$

Furthermore, let $(\mathbf{Q}^{\bullet,p_2,\bullet}_{r,M}; 0, 0)$ be the double complex generated by all symbols $[0, g, h]$ satisfying $|T_g| = s - p_2$ and $g \mid h$, which can be considered as a quotient complex of $\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}$. As in the proof of Theorem A, the quotient map induces an isomorphism between cohomology groups. Let $C'_{g,h}$ be a cocycle in $\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}$ with image $[0, g, h]$ in the quotient $\mathbf{Q}^{\bullet,p_2,\bullet}_{r,M}$. Then $C'_{g,h}$ is the sum of a leading term $[0, g, h]$ and a remainder contained in the direct sum of $K^{p'_1,p_2,q'}$ where $q' < \deg h$ and $p'_1 + q' = \deg h - s$. □

PROPOSITION 4.3.2. *The spectral sequence of the double complex* $(\mathbf{K}^{\bullet;\bullet}_{r,M}; d_1 + \delta, d_2)$ *with the first filtration, degenerates at* E_1. *The spectral sequence of the double complex* $(\mathbf{K}^{\bullet;\bullet}_{r,M}(\mathfrak{I}); d_1 + \delta, d_2)$ *with the first filtration, degenerates at* E_1.

PROOF. We only prove the first part. The E_1-terms of the spectral sequence are

$$E_1^{p_1+q,p_2}(\mathbf{K}^{\bullet;\bullet}_{r,M}) = H^{p_1+q}_{total}(\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}; d_1, \delta).$$

Note that $|E_1^{p,q}| \geq |E_2^{p,q}| \geq \cdots \geq |E_\infty^{p,q}|$ in general for any spectral sequence, then

$$|\bigoplus_{p_1+p_2+q=n} H^{p_1+q}_{total}(\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}; d_1, \delta)| \geq |H^n_{total}(\mathbf{K}^{\bullet;\bullet}_{r,M}, d + \delta)|.$$

By Theorem A and Proposition 4.3.1, the left hand side and the right hand side of the above inequality have the same number of elements, hence the inequality is actually an identity. Therefore, the spectral sequence of $\mathbf{K}^{\bullet;\bullet}_{r,M}$ with filtration given by $p_1 + q$ degenerates at E_1. □

The advantage of studying the triple complex structure of the complex $\mathbf{K}_{r,M}$ is that we can obtain the $(-p_2)$-cocycles of $\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}$ rather quickly. Note that

$$(1 - \sigma_\ell)D_\ell = N_\ell \pmod{M}.$$

Now for the $(-p_2)$-cocycles $C'_{g,h}$, the pair (T, h) must satisfy $\deg h = |T_g|$ and therefore $g = h$. In this case, for any $\ell \mid g$, we always have

$$\omega(\ell, g) = (-1)^{v_\ell(\omega(g))}.$$

First

$$\delta[0, g, g] = 0,\ d_1[0, g, g] = -\sum_{\ell \mid g} \omega(\ell, g) N_\ell[\frac{g/\ell}{\ell}, g/\ell, g],$$

then

$$\delta(\sum_{\ell|g} D_\ell[\frac{g/\ell}{\ell}, g/\ell, g/\ell]) = (-1)^{|T_g|} d_1[0, g, g],$$

Continue this procedure, we have

$$C'_{g,g} = \sum_{g'|g} (-1)^{|T_{g'}|(2|T_g|-|T_{g'}|-1)/2} D_{g'}[\sum_{\ell|g'} \frac{g/g'}{\ell}, g/g', g/g'].$$

Apparently, we see that if $g \in \mathfrak{I}$, then the cocycles $C'_{g,g}$ are all contained in the subcomplex $\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}(\mathfrak{I})$. Combining the above results, we have

PROPOSITION 4.3.3. *1). The canonical basis $\{C'_{g,g} : |T_g| = s - p_2\}$ of the $\mathbb{Z}/M\mathbb{Z}$-module $H^{(-p_2)}(\mathbf{K}^{\bullet,p_2,\bullet}_{r,M})$ is given by*

$$C'_{g,g} = \sum_{g'|g} (-1)^{|T_{g'}|(2|T_g|-|T_{g'}|-1)/2} D_{g'}[\sum_{\ell|g'} \frac{g/g'}{\ell}, g/g', g/g'].$$

2). If we restrict our attention in the subcomplex $\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}(\mathfrak{I})$, then the $\mathbb{Z}/M\mathbb{Z}$-module $H^{(-p_2)}(\mathbf{K}^{\bullet,p_2,\bullet}_{r,M}(\mathfrak{I}))$ has a canonical basis $\{C'_{g,g} : |T_g| = s - p_2, g \in \mathfrak{I}\}$.

4.3.3. Proof of Theorem B. In this subsection, we prove

THEOREM B . *The image of the family*

$$\left\{ D_{r'} \left[\sum_{\ell|r'} \frac{1}{\ell} \right] : \forall\ r'|r \right\}$$

in U_r/MU_r is a $\mathbb{Z}/M\mathbb{Z}$-basis for $H^0(G_r, U_r/MU_r)$.

PROOF. First we claim that

$$D_g[\sum_{\ell_i|g} \frac{1}{\ell}] \in H^0(G_r, U_r/MU_r) = (U_r/MU_r)^{G_r}.$$

We prove it by induction on g. For $g = \ell$, it is easy to see that $(1 - \sigma_\ell) D_\ell[\frac{1}{\ell}] = 0$ for all $\ell \mid r$. Now in general, for any $\ell_j \mid g$,

$$(1 - \sigma_{\ell_j}) D_g[\sum_{\ell|g} \frac{1}{\ell}] = (Fr_{\ell_j} - 1) D_{g/\ell_j}[\sum_{\ell|(g/\ell_j)} \frac{1}{\ell}]$$

which is 0 by induction, for $\ell_j \nmid g$, it is obviously 0. Hence the claim holds.

Now we consider the double complex $(\mathbf{K}^{\bullet,\bullet}_{r,M}, d_1 + \delta, d_2)$. By Proposition 4.3.2, we know that $(\mathbf{K}^{\bullet,\bullet}_{r,M}, d_1 + \delta, d_2)$ degenerates at E_1 for the first filtration. By Proposition 4.3.3, $E_1^{-p_2,p_2}(\mathbf{K}^{\bullet,\bullet}_{r,M})$ is generated by $\{C'_{g,g} : |T_g| = s - p_2\}$. We plan to lift

$C'_{g,g}$ to a 0-cocycle in $\mathbf{K}^{\bullet,\bullet}_{r,M}$, which is guaranteed by the degeneration at E_1. Moreover, we can study the lifting $C'_{g,g}$ in $\mathbf{K}^{\bullet,\bullet}_{r,M}(g)$. Therefore there exists a cocycle $\tilde{C}_{g,g}$ in $\mathbf{K}^{\bullet,\bullet}_{r,M}(g)$ with the leading term $C'_{g,g}$ and the remainder contained in the direct sum of $K^{p'_1,p'_2,q'}_{r,M}(g)$ where $p'_1+p'_2+q'=0$ and $p'_2>p_2$. Hence the image $\mathfrak{u}(\tilde{C}_{g,g})$ is exactly of the form

$$\pm D_g[\sum_{\ell|g}\frac{1}{\ell}]+Re(g),$$

where $Re(g)$ is of the form

$$Re(g)=\sum_{\substack{\mathrm{ord}(a)|g\\ \mathrm{ord}(a)\neq g}} n_a[a].$$

Both $\mathfrak{u}(\tilde{C}_{g,g})$ and $D_g[\sum_{\ell|g}\frac{1}{\ell}]$ are 0-cocycles of U_r/MU_r, and hence is $Re(g)$.

In order to prove Theorem B, it is sufficient to prove

$$(*): Re(g)=\text{linear combination of } D_{g'}[\sum_{\ell|g'}\frac{1}{\ell}] \text{ for } g'\mid g,\ g'\neq g.$$

We show $(*)$ by induction on g. If $g=\ell$, this is trivial. Now in general, without loss of generality, we may assume that $g=r$ and for any $g'\mid r$, $Re(g')$ is a linear combination of $D_{g''}[\sum_{\ell|g''}\frac{1}{\ell}]$ for $g''\mid g'$ but $g''\neq g$. Then $\mathfrak{u}(\tilde{C}_{g',g'})$ for any $g'\mid r, g'\neq r$ is a linear combination of $D_{g''}[\sum_{\ell|g''}\frac{1}{\ell_i}]$ with $g''\mid g'$. By Proposition 4.2.1, Proposition 4.3.2 and Theorem A, $H^0(G_r, U_r(s-1)/MU_r(s-1))$ is generated by $\{\mathfrak{u}(\tilde{C}_{g',g'}) : g'\mid r, g'\neq r\}$ and hence by $D_{g'}[\sum_{\ell|g'}\frac{1}{\ell}]$. But obviously $Re(r)\in U_r(s-1)/MU_r(s-1)$, so $(*)$ holds for $Re(r)$. Theorem B is proved. □

REMARK 4.3.4. One natural question to ask is if the bases of $H^0(G_r, U_r/MU_r)$ obtained in Theorem 4.2.6 and in Theorem B are the same. Unfortunately, they are not the same even in the case $|T_r|=3$. Right now, we don't know too much about the explicit expression of the cocycles $c_{g,h}$. A deep understanding of those cocycles might tell us more about the arithmetic of the cyclotomic fields.

CHAPTER 5

Connections with the Euler System

In this chapter, we give a brief introduction to the cyclotomic Euler system. We then discuss possible connections of the group cohomology of the universal distribution and the Euler system. Though the connections are still not fully understood, our investigation shows hope for future progress. We include the study we have done so far, and some problems for further investigation. In this chapter, for any $\mathbb{Z}[G]$-module A and any element $\alpha \in \mathbb{Z}[G]$, we denote by ${}_{\alpha}A$ the submodule $\{a \in A : \alpha a = 0\}$ and denote by A_α the quotient module $A/\alpha A$.

5.1. The cyclotomic Euler system

Fix a positive integer m, and let $\mathbb{F} = \mathbb{Q}(\mu_m)^+$. For any $r \in N$, $(r, m) = 1$, write $\mathbb{F}_r = \mathbb{F}(\mu_r)$ and $\mathcal{O}_r = \mathcal{O}_{\mathbb{F}(\mu_r)}$ (note that this notation is different than the one in Chapter 3). We identify G_r with the Galois group of $\mathbb{F}(\mu_r)/\mathbb{F}$. Let $\mathfrak{S}$ be the set of positive squarefree integers divisible only by primes in $\mathbb{Q}$ splitting completely in $\mathbb{F}/\mathbb{Q}$. Let $\mathfrak{s}$ be the supernatural number attached to $\mathfrak{S}$. We can define $\mu_{\mathfrak{s}}$ and $G_{\mathfrak{s}}$ correspondingly.

The cyclotomic Euler system, briefly to say, is a system of the elements

$$\{\xi_r \in \mathcal{O}_r \backslash \{0\} : r \in \mathbb{N}, r \mid \mathfrak{s}\}$$

satisfying the following two axioms:

(1). $N_\ell \xi_r = (Fr_\ell - 1)\xi_{r/\ell}$.

(2). $\xi_r \equiv \xi_{r/\ell}$ modulo every prime above ℓ.

Given a Euler system $\{\xi_r : r \mid \mathfrak{s}\}$, there exists a unique $G_{\mathfrak{s}}$-homomorphism ξ

$$\xi : U_{\mathfrak{s}} \longrightarrow \mathbb{F}_{\mathfrak{s}}^{\times}$$

satisfying

$$\xi\Big([\sum_{\ell \mid r} \frac{1}{\ell}]\Big) = \xi_r.$$

Remark 5.1.1. If we let ξ be the map given in Example 2.1.7 of §2.1, one can see that the associated Euler system is the one given by Rubin [**26**]. Thus $U_{\mathfrak{s}}$ plays a role here similar to that played by the *universal Euler system* defined in Rubin [**28**].

Now we make the simplifying assumption that $\xi_r \in \mathcal{O}_r^\times$ for all r(this assumption is not as bad as it looks, in application, we can always modify a given Euler system to satisfy this assumption). Now for any $r \mid \mathfrak{s}$, passing the map ξ to the G_r-cohomology, we have

$$H^*(\xi) : H^*(G_r, U_r) \longrightarrow H^*(G_r, \mathcal{O}_r^\times).$$

Fix an odd positive integer M. Let

$$\mathfrak{S}_M = \{r \in \mathfrak{S} : r \text{ is divisible only by primes} \equiv 1 \mod M\}$$

and let $\mathfrak{m}$ be the supernatural number attached to $\mathfrak{S}_M$. Hereafter we suppose that $r \mid \mathfrak{m}$. Then the map

$$U_r \xrightarrow{\xi} \mathcal{O}_r^\times \hookrightarrow \mathbb{F}_r^\times$$

induces a map

$$\kappa : H^0(G_r, U_r/MU_r) \longrightarrow H^0(G_r, \mathbb{F}_r^\times/\mathbb{F}_r^{\times M}).$$

From Theorem B, we know that $H^0(G_r, U_r/MU_r)$ has a $\mathbb{Z}/M\mathbb{Z}$-basis

$$\Big\{D_{r'}\Big[\sum_{\ell \mid r'} \frac{1}{\ell}\Big] : r' \mid r\Big\},$$

therefore the images $D_{r'}\xi_{r'}$ in $\mathbb{F}_r^\times/\mathbb{F}_r^{\times M}$ are invariant by G_r.

Furthermore, since $\mathbb{F}_r$ doesn't contain any M-th root of unity, we have an exact sequence

$$0 \longrightarrow \mathbb{F}_r^\times \xrightarrow{\times M} \mathbb{F}_r^\times \longrightarrow \mathbb{F}_r^\times/\mathbb{F}_r^{\times M} \longrightarrow 0$$

Passing to the G_r-cohomology, since

$$H^0(G_r, \mathbb{F}_r^\times) = \mathbb{F}^\times, \qquad H^1(G_r, \mathbb{F}_r^\times) = 0 (\text{by Theorem 90}),$$

we have an exact sequence

$$0 \longrightarrow \mathbb{F}^\times \xrightarrow{\times M} \mathbb{F}^\times \longrightarrow H^0(G_r, \mathbb{F}_r^\times/\mathbb{F}_r^{\times M}) \longrightarrow 0.$$

Thus $H^0(G_r, \mathbb{F}_r^\times/\mathbb{F}_r^{\times M}) = \mathbb{F}^\times/\mathbb{F}^{\times M}$. For any $r' \mid r$, we denote by $\kappa(r')$ the image of $D_{r'}\xi_{r'}$ in $\mathbb{F}^\times/\mathbb{F}^{\times M}$. Note that $\kappa(r')$ is independent the choice of r. The elements $\kappa(r)$ for $r \in \mathfrak{S}_M$ are called Kolyvagin's derivative classes.

Fix a prime λ of $\mathbb{F}$ above ℓ and a primitive root s modulo ℓ. Then s is also a primitive root modulo $\sigma\lambda$ for each $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$. Now for any $x \in \mathbb{F}^\times$ which is prime to ℓ, define $\varphi_\sigma(x) \in \mathbb{Z}/(\ell-1)\mathbb{Z}$ by

$$x \equiv s^{\varphi_\sigma(x)} \pmod{\sigma\lambda}.$$

For any $x \in \mathbb{F}^\times$, denote by $v_\lambda(x)$ the λ-valuation of x. The following proposition tells us about the ℓ-part prime factorization of $\kappa(r)$:

Proposition 5.1.2 (Proposition 2.4, Rubin [**26**]). *One has*
(1). If $\ell \nmid r$, then $v_{\sigma\lambda}(\kappa(r)) = 0$ for every $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$.
(2). If $\ell \mid r$, then $v_{\sigma\lambda}(\kappa(r)) = \varphi_{\sigma\lambda}(\kappa(r/\ell))$ for every $\sigma \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$.

In application, Proposition 5.1.2(especially the second relation) is the most important fact about the Kolyvagin classes $\kappa(r)$. Combining with Chebatarev's density theorem, Rubin [**27**] gave an elegant proof of the Main Conjecture over $\mathbb{Q}$ in Iwasawa theory. Moreover, in [**25**], Rubin proved the Main Conjecture over imaginary quadratic fields using second dimensional analogous Kolyvagin classes. As known from above, the $\kappa(r)$'s are the images of cocycles in $H^0(G_r, U_r/MU_r)$, thus if we can recover the above relations in the group cohomology of U_r, it might facilitate the study of the Euler systems. Our goal in this chapter is to find a way to write down the relations in Proposition 5.1.2 using cohomological language.

5.2. Investigation of connections

5.2.1. The cohomology group $H^*(G_\ell, U_r)$. We keep the assumptions of Chapter 4. For a given prime $\ell \mid r$, we discuss the cohomology group $H^*(G_\ell, U_r)$. In this case, since $G_\ell = \langle \sigma_\ell \rangle$ is cyclic, we choose the projective resolution $\mathbf{P}_{\ell\bullet}$ of trivial G_ℓ-module $\mathbb{Z}$ as

$$\cdots \longrightarrow \underset{2}{\mathbb{Z}[G_\ell]} \xrightarrow{N_\ell} \underset{1}{\mathbb{Z}[G_\ell]} \xrightarrow{1-\sigma_\ell} \underset{0}{\mathbb{Z}[G_\ell]} \longrightarrow 0.$$

We also have an acyclic complete complex $\hat{\mathbf{P}}_{\ell\bullet}$

$$\cdots \longrightarrow \underset{1}{\mathbb{Z}[G_\ell]} \xrightarrow{1-\sigma_\ell} \underset{0}{\mathbb{Z}[G_\ell]} \xrightarrow{N_\ell} \underset{-1}{\mathbb{Z}[G_\ell]} \longrightarrow \cdots$$

Thus the double complex $\mathbf{K}^{\bullet,\bullet} = \mathrm{Hom}_{G_\ell}(\mathbf{P}_{\ell\bullet}, L_r^\bullet)$ has total cohomology group $H^*(G_\ell, U_r)$, and the double complex $\hat{\mathbf{K}}^{\bullet,\bullet} = \mathrm{Hom}_{G_\ell}(\hat{\mathbf{P}}_{\ell\bullet}, L_r^\bullet)$ has total cohomology $\hat{H}^*(G_\ell, U_r)$. We consider the latter one. Follow the setup in Chapter 4, then

$$\hat{K}^{p,q} = \langle [a, g, q] : a \in \frac{g}{r}\mathbb{Z}/\mathbb{Z}, g \text{ squarefree} \mid r, |T_g| = -p\rangle$$

and the differentials induced are

$$d_r[a, g, q] = \sum_{\ell' | g} \omega(\ell', g)\Big([a, \frac{g}{\ell'}, q] - \sum_{\ell' b = a} [b, \frac{g}{\ell'}, q]\Big)$$

$$\delta_\ell[a, g, q] = (-1)^{q-1} \cdot \begin{cases} N_\ell[a, g, q], & \text{if } q \equiv 1 \mod 2; \\ (1 - \sigma_\ell)[a, g, q], & \text{if } q \equiv 0 \mod 2. \end{cases}$$

For the second filtration, then we have

$$E_1^{p,q}(\hat{\mathbf{K}}^{\bullet,\bullet}) = \hat{H}^q(G_\ell, L_r^p).$$

Now let's look the G_ℓ-module L_r^p, we have an isomorphism of G_ℓ-modules(recall the definition in §2.3)

$$L_r^p \cong L_{r/\ell}^p \bigoplus L_{r,r/\ell}^p$$

$$[a, g] \mapsto \begin{cases} ([a, g/\ell], 0), & \text{if } \ell \mid g; \\ (0, [a, g]), & \text{if } \ell \nmid g. \end{cases}$$

The module $L_{r/\ell}^p$ is a direct sum of trivial G_ℓ-module $\mathbb{Z}$, therefore

$$\hat{H}^q(G_\ell, L_{r/\ell}^p) = \begin{cases} 0, & \text{if } q \equiv 1 \mod 2; \\ L_{r/\ell}^p/(\ell - 1)L_{r/\ell}^p, & \text{if } q \equiv 0 \mod 2. \end{cases}$$

The module $L_{r,r/\ell}^p$, however, has the following structure

$$L_{r,r/\ell}^p \cong L_{r/\ell}^p \bigoplus \mathrm{Ind}_{\{1\}}^{G_\ell} L_{r/\ell}^p$$

$$[a, g] \mapsto \begin{cases} ([a, g], 0), & \text{if } \ell \nmid \mathrm{ord}\, a; \\ (0, [a, g]), & \text{if } \ell \mid \mathrm{ord}\, a. \end{cases}$$

The induced module $\mathrm{Ind}_{\{1\}}^{G_\ell} L_{r/\ell}^p$ has trivial Tate cohomology, thus $\hat{H}^q(G_\ell, L_{r,r/\ell}^p)$ is 0 if q is odd and is another copy of $L_{r/\ell}^p/(\ell - 1)L_{r/\ell}^p$ if q is even.

Look at the subcomplex $S^{\bullet,\bullet}$ of $\hat{\mathbf{K}}^{\bullet,\bullet}$ given by

$$S^{p,q} = \begin{cases} K^{p,q}, & \text{if } q \equiv 1 \mod 2; \\ (\ell-1)\langle [a,g,q] : \ell \nmid \operatorname{ord} a\rangle \bigoplus \langle [a,g,q] : \ell \mid \operatorname{ord} a\rangle, & \text{if } q \equiv 0 \mod 2. \end{cases}$$

The above consideration shows that $S^{\bullet,\bullet}$ is acyclic and $E_1^{\bullet,\bullet}(\hat{\mathbf{K}}^{\bullet,\bullet})$ is the quotient complex with the differentials $(\bar{d}_r, 0)$ induced by (d_r, δ_ℓ). Hence the quotient map from $\hat{\mathbf{K}}^{\bullet,\bullet}$ to $E_1^{p,q}(\hat{\mathbf{K}}^{\bullet,\bullet})$ is a quasi-isomorphism. Therefore this spectral sequence degenerates at E_2. Now we compute the E_2-terms.

We only need to consider the case q even. Then the complex $E_1^{\bullet,q}(\hat{\mathbf{K}}^{\bullet,\bullet})$ is isomorphic to a free graded $\mathbb{Z}/(\ell-1)\mathbb{Z}$-module $E^\bullet$ with a basis given by

$$\{[a,g] : \ell \nmid \operatorname{ord} a\},$$

and with the differential given by

$$\begin{aligned} d_{r1}[a,g] =& \sum_{\substack{\ell' \mid g \\ \ell' \neq \ell}} \omega(\ell',g)\Big([a,\frac{g}{\ell'}] - \sum_{\ell' b = a}[b,\frac{g}{\ell'}]\Big) + \omega(\ell,g)(1 - Fr_\ell^{-1})[a,\frac{g}{\ell}] \\ =& d_{r/\ell}[a,g] + d'_\ell[a,g] \end{aligned}$$

where $d'_\ell[a,g] = \omega(\ell,g)(1-Fr_\ell^{-1})[a,g/\ell']$. We check that

$$d_{r/\ell}^2 = d_\ell'^{\,2} = d_{r/\ell}d'_\ell + d'_\ell d_{r/\ell} = 0.$$

This gives us hints that the complex $E^\bullet$ might possess a double complex structure. For any symbol $[a,g]$ in $E_1^{p,q}(\hat{\mathbf{K}}^{\bullet,q})$, we declare $[a,g]$ is of bidegree (m,n) where

$$m = -|\{\ell' : \ell' \mid \gcd(g, r/\ell)\}|, \qquad n = -|\{\ell' : \ell' \mid \gcd(g,\ell\}|.$$

With this bigrading, the complex $E^\bullet$ indeed becomes a double complex $E^{\bullet,\bullet}$ with differentials $d_{r/\ell}$ and d'_ℓ. Actually one can see $E^{\bullet,\bullet}$ is nothing but the mapping cone defined by the map

$$1 - Fr_\ell^{-1} : \frac{L^p_{r/\ell}}{\ell-1} \longrightarrow \frac{L^p_{r/\ell}}{\ell-1}.$$

By studying the first filtration of $E^{\bullet,\bullet}$, the spectral sequence collapses at E_2 and

$$E_2^{m,n}(E^{\bullet,\bullet}) = \begin{cases} \operatorname{coker}(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)), & \text{if } m=0, n=0; \\ \ker(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)), & \text{if } m=0, n=-1; \\ 0, & \text{if otherwise.} \end{cases}$$

Thus for q even,

$$E_2^{p,q}(\hat{\mathbf{K}}^{\bullet,\bullet}) = \begin{cases} \operatorname{coker}(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)), & \text{if } p = 0; \\ \ker(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)), & \text{if } p = -1; \\ 0, & \text{if otherwise.} \end{cases}$$

Since this spectral sequences collapses at E_2, we have the following proposition

PROPOSITION 5.2.1.

$$\hat{H}^q(G_\ell, U_r) = \begin{cases} \operatorname{coker}(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)), & \textit{if } q \equiv 0 \mod 2; \\ \ker(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)), & \textit{if } q \equiv 1 \mod 2. \end{cases}$$

5.2.2. More on the prime factorization. Let P_r(resp. P) be the group generated by principal fractional ideals of $\mathbb{F}_r$ (resp. $\mathbb{F}$). Let I_r(resp. I) be the group generated by fractional ideals of $\mathbb{F}_r$ (resp. $\mathbb{F}$). Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_r^\times \longrightarrow \mathbb{F}_r^\times \longrightarrow P_r \longrightarrow 0.$$

Passing to the G_r-cohomology, we have

$$H^1(G_r, \mathcal{O}_r^\times) \cong \frac{P_r^{G_r}}{P}.$$

Thus we have

$$H^0(G_r, U_r/MU_r) \xrightarrow{Bock} {}_{\mathrm{M}}H^1(G_r, U_r) \xrightarrow{H^1(\xi)} {}_{\mathrm{M}}H^1(G_r, \mathcal{O}_r^\times) \cong {}_{\mathrm{M}}\Big(\frac{P_r^{G_r}}{P}\Big),$$

where "Bock" abbreviate Bockstein. Therefore we have a map

$$val: \; H^0(G_r, U_r/MU_r) \longrightarrow {}_{\mathrm{M}}\Big(\frac{P_r^{G_r}}{P}\Big) \longrightarrow {}_{\mathrm{M}}\Big(\frac{I_r^{G_r}}{I}\Big) \xrightarrow{\times M} \frac{I}{MI}.$$

We can show that the map val gives the same information about the prime factorization of $\kappa(r)$ as the valuation maps $v_{\sigma\lambda}$ give. Hence the map

$$H^1(\xi) : H^1(G_r, U_r) \longrightarrow \frac{P_r^{G_r}}{P}$$

contains all the information we want for prime factorizations. Since our goal is to recover Proposition 5.1.2, we also need to interpret the map φ_ℓ in the cohomological level. We speculate that we need a map from $H^1(G_r, U_r)$ to $H^1(G_{r/\ell}, U_{r/\ell})$.

5.2.3. Two maps ρ and ρ'. First we regard G_ℓ as a subgroup of G_r, and regard $G_{r/\ell}$ as the quotient group. Then Hochschild-Serre spectral sequences give the following exact sequence

$$0 \longrightarrow H^1(G_{r/\ell}, U_r^{G_\ell}) \longrightarrow H^1(G_r, U_r) \longrightarrow H^1(G_\ell, U_r)^{G_{r/\ell}} \longrightarrow H^2(G_{r/\ell}, U_r^{G_\ell})$$

By Proposition 5.2.1, we know that

$$H^1(G_\ell, U_r) = \ker(1 - Fr_\ell^{-1} : U_{r/\ell}/(\ell-1) \to U_{r/\ell}/(\ell-1)).$$

Thus we have a map

$$\rho : H^1(G_r, U_r) \to H^1(G_\ell, U_r)^{G_{r/\ell}} \hookrightarrow H^0(G_{r/\ell}, U_{r/\ell}/(\ell-1)) \to H^1(G_{r/\ell}, U_{r/\ell}).$$

Since

$$U_{r/\ell} \hookrightarrow U_r^{G_\ell} \hookrightarrow U_r,$$

we have

$$\iota : H^1(G_{r/\ell}, U_{r/\ell}) \longrightarrow H^1(G_{r/\ell}, U_r^{G_\ell}) \hookrightarrow H^1(G_r.U_r).$$

Let $\alpha_\ell = 1 - \ell Fr_\ell^{-1}$, we have a modulo ℓ map

$$U_r \longrightarrow \frac{U_{r/\ell}}{\alpha_\ell},$$
$$[a] \longmapsto [\ell a].$$

It is easy to check that the above map is a well-defined G_r-homomorphism. This modulo ℓ map thus induces a map

$$H^i(G_r, U_r) \longrightarrow H^i(G_r, U_{r/\ell}/\alpha_\ell)$$

for every $i \geq 0$.

Given two finite abelian groups G_1 and G_2 of order m_1 and m_2 respectively. Suppose that G_1 is cyclic with a generator τ. Let $G = G_1 \times G_2$. Let M be a G-module such that $H^0(G_1, M) = M$(i.e., M has trivial G_1-module structure). For any cross homomorphism $c : G \to M$, we have

$$c(\sigma\tau) = \sigma c(\tau) + c(\sigma) = \tau c(\sigma) + c(\tau).$$

Thus for any $\sigma \in G_2$, we have $c(\tau) = \sigma c(\tau)$, which is to say that $c(\tau) \in H^0(G_2, M)$. It is clear that $c(\tau)$ is independent the choice of c up to the coboundary, thus the map

$$c \longmapsto c(\tau)$$

is a well-defined map from $H^1(G, M)$ to $H^0(G_2, M)$.

Now applying the above discussion to the case $G_1 = G_\ell$, $G_2 = G_{r/\ell}$ and $M = U_{r/\ell}/\alpha_\ell$, we have a map

$$\rho' : H^1(G_r, U_r) \longrightarrow H^1(G_r, \frac{U_{r/\ell}}{\alpha_\ell}) \longrightarrow H^0(G_{r/\ell}, \frac{U_{r/\ell}}{\alpha_\ell}) \longrightarrow H^1(G_{r/\ell}, U_{r/\ell})$$

PROBLEM 5.2.2. What is the relationship between ρ and ρ'? How to describe images of elements of $H^1(G_r, U_r)$ in $H^1(G_{r/\ell}, U_{r/\ell})$?

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