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Newton polygons of L-functions of polynomials $x^d + ax^{d-1}$ with $p \equiv -1 \bmod d$

Yi Ouyang, Shenxing Zhang *

Wu Wen-Tsun Key Laboratory of Mathematics, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

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ABSTRACT

For prime $p \equiv -1 \bmod d$ and q a power of p, we obtain the slopes of the q-adic Newton polygons of L-functions of $x^d + ax^{d-1} \in \mathbb{F}_q[x]$ with respect to finite characters χ when p is larger than an explicit bound depending only on d and $\log_p q$. The main tools are Dwork's trace formula and Zhu's rigid transform theorem.

1. Main results

Let $q = p^h$ be a power of the rational prime number p. Let v be the normalized valuation on $\overline{\mathbb{Q}}_p$ with $v(p) = 1$. For a polynomial $f(x) \in \mathbb{F}_q[x]$, let $\hat{f} \in \mathbb{Z}_q[x]$ be its Teichmüller lifting. For a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of order p^{m_χ}, define the L-function

$$L^*(f, \chi, t) = \exp\left(\sum_{m=1}^{\infty} S_m^*(f, \chi)\frac{t^m}{m}\right), \tag{1}$$

* Corresponding author.
E-mail addresses: yiouyang@ustc.edu.cn (Y. Ouyang), zsxqq@mail.ustc.edu.cn (S. Zhang).

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where $S_m^*(f,\chi)$ is the exponential sum

$$S_m^*(f,\chi) = \sum_{x\in\mu_{q^m-1}} \chi(\mathrm{Tr}_{\mathbb{Q}_{q^m}/\mathbb{Q}_p} \hat{f}(x)) \tag{2}$$

and μ_n is the group of n-th roots of unity. Then $L^*(f,\chi,t)$ is a polynomial of degree $p^{m_\chi-1}d$ by Adolphson–Sperber [1] and Liu–Wei [4]. We denote $\mathrm{NP}_q(f,\chi,t)$ the q-adic Newton polygon of $L^*(f,\chi,t)$.

We fix a character $\Psi_1 : \mathbb{Z}_p \to \mathbb{C}_p^\times$ of order p, and denote $L^*(f,t) = L^*(f,\Psi_1,t)$ and $\mathrm{NP}_q(f,t) = \mathrm{NP}_q(f,\Psi_1,t)$. When $p \equiv 1 \bmod d$, it is well-known that $\mathrm{NP}_q(f,t)$ coincides the Hodge polygon with slopes $\{i/d : 0 \le i \le d-1\}$.

Let a be a nonzero element in $\mathbb{F}_q$. For $f(x) = x^d + ax^s$ $(s < d)$, Zhu obtained the slopes of $\mathrm{NP}_q(f,t)$ for p large enough under certain conditions in [9], but these conditions are not so easy to check. For $f(x) = x^d + ax$, Zhu and Ouyang–J. Yang obtained the slopes in [9, Theorem 1.1] and [5, Theorem 1.1], see also R. Yang [7, §1 Theorem] for earlier results.

In [2], Davis–Wan–Xiao gave a result on the behavior of the slopes of $\mathrm{NP}_q(f,\chi,t)$ when the order of χ is large enough. In this way for p sufficiently large, they can obtain the slopes of $\mathrm{NP}_q(f,\chi,t)$ based on the slopes of $\mathrm{NP}_q(f,\chi_0,t)$ with χ_0 a character of order p^2. In [5, Theorem 4.3], Ouyang–Yang showed that if the Newton polygon of $L^*(f,t)$ is sufficiently close to its Hodge polygon, the slopes of $\mathrm{NP}_q(f,\chi,t)$ for χ in general follow from the slopes of $\mathrm{NP}_q(f,t)$. As a consequence they obtained the slopes of $\mathrm{NP}_q(x^d + ax,\chi,t)$ when p is bigger than an explicit bound depending only on d and h.

Our main results are the following two theorems.

Theorem 1. *Let $f(x) = x^d + ax^{d-1}$ be a polynomial in $\mathbb{F}_q[x]$ with $a \ne 0$. Let $N(d) = \frac{d^2+3}{4}$ for $q = p$ and $\frac{d^2}{2}$ for general q. If $p \equiv -1 \bmod d$ and $p > N(d)$, the q-adic Newton polygon of $L^*(f,t)$ has slopes*

$$\{w_0, w_1, \dots, w_{d-1}\},$$

where

$$w_i = \begin{cases} \frac{(p+1)i}{d(p-1)}, & \text{if } i < \frac{d}{2}; \\ \frac{(p+1)i-d}{d(p-1)} = \frac{1}{2}, & \text{if } i = \frac{d}{2}; \\ \frac{(p+1)i-2d}{d(p-1)}, & \text{if } i > \frac{d}{2}. \end{cases}$$

Remark.

(1) For general p, write $pi = dk_i + r_i$ with $1 \le i, r_i \le d-1$. If $r_i > s$ for any $1 \le i \le s$, then one can decide that the first $s+1$ slopes of $\mathrm{NP}_q(f,t)$ are $\{0, \frac{k_1+1}{p-1}, \dots, \frac{k_s+1}{p-1}\}$ by our method for sufficiently large p. For the rest of slopes, one needs to calculate the determinants of submatrices of "Vandermonde style" matrices.

(2) The slopes in our case coincide Zhu's result in [9].

Theorem 2. *Assume $f(x)$ and $N(d)$ as above. For any non-trivial finite character χ, if $p \equiv -1 \bmod d$ and $p > \max\{N(d), \frac{h(d^2-1)}{4d}+1\}$, the q-adic Newton polygon of $L^*(f,\chi,t)$ has slopes*

$$\{p^{1-m_\chi}(i+w_j) : 0 \le i \le p^{m_\chi-1}-1, 0 \le j \le d-1\}.$$

2. Preliminaries

2.1. Dwork's trace formula

We will recall Dwork's work for $f(x) = x^d + ax^{d-1}$. For general f, one can see [5, §2]. Let $\gamma \in \mathbb{Q}_p(\mu_p)$ be a root of the Artin–Hasse exponential series

$$E(t) = \exp\left(\sum_{m=0}^{\infty} p^{-m} t^{p^m}\right)$$

such that $v(\gamma) = \frac{1}{p-1}$. Fix a $\gamma^{1/d} \in \bar{\mathbb{Q}}_p$. Let

$$\theta(t) = E(\gamma t) = \sum_{m=0}^{\infty} \gamma_m t^m$$

be Dwork's splitting function. Then $v(\gamma_m) \ge m/(p-1)$, and $\gamma_m = \gamma^m/m!$ for $0 \le m \le p-1$. Let

$$F(x) = \theta(x^d)\theta(ax^{d-1}) = \sum_{i=0}^{\infty} F_i x^i,$$

then

$$F_i = \sum_{dm+(d-1)n=i} \gamma_m \gamma_n a^n.$$

One can see $m+n \ge i/d$ and $v(F_i) \ge \frac{i}{d(p-1)}$.

Set $A_1 = (F_{pi-j}\gamma^{(j-i)/d})_{i,j\ge 0}$. This is a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$ with

$$v(F_{pi-j}\gamma^{(j-i)/d}) \ge \frac{pi-j}{d(p-1)} + \frac{j-i}{d(p-1)} = \frac{i}{d}.$$

We extend the Frobenius φ to $\mathbb{Q}_q(\gamma^{1/d})$ with $\varphi(\gamma^{1/d}) = \gamma^{1/d}$.

Theorem 3 *(Dwork). Let $A_h = A_1\varphi(A_1)\cdots\varphi^{h-1}(A_1)$. Then*

$$L^*(f,t) = \frac{\det^{\varphi^{-1}}(I - tA_h)}{\det^{\varphi^{-1}}(I - tqA_h)}.$$

2.2. Zhu's rigid transformation theorem

Let $U_1 = (u_{ij})_{i,j\geq 0}$ be a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$. Then the Fredholm determinant $\det(I - tU_1)$ is well defined and p-adic entire (see [6]). Write

$$\det(I - tU_1) = c_0 + c_1 t + c_2 t^2 + \cdots .$$

For $0 \leq t_1 < t_2 < \cdots < t_s$, denote by $U_1(t_1, \ldots, t_s)$ the principal sub-matrix consisting of (t_i, t_j)-entries of U_1 for $1 \leq i, j \leq s$. In particular, denote $U_1[s] = U_1(0, 1, \ldots, s-1)$. Then we have $c_0 = 1$ and for $s \geq 1$,

$$c_s = (-1)^s \sum_{0\leq t_1<t_2<\cdots<t_s} \det U_1(t_1, t_2, \ldots, t_s).$$

Let $U_h = U_1\varphi(U_1)\cdots\varphi^{h-1}(U_1)$. Write

$$\det(I - tU_h) = C_0 + C_1 t + C_2 t^2 + \cdots .$$

Theorem 4. *(See [8, Theorem 5.3].) Suppose $(\beta_s)_{s\geq 0}$ is a strictly increasing sequence such that*

$$\beta_i \leq v(a_{ij}) \text{ and } \lim_{s\to+\infty} \beta_s = +\infty.$$

If

$$\sum_{s<i} \beta_s \leq v(\det U_1[i]) \leq \frac{\beta_i - \beta_{i-1}}{2} + \sum_{s<i} \beta_s$$

holds for every $1 \leq i \leq k$, then $v(C_i) = hv(\det U_1[i])$ for $1 \leq i \leq k$ and

$$\mathrm{NP}_q(\det(I - tA_h[k])) = \mathrm{NP}_p(\det(I - tA_1[k])).$$

3. Slopes of the Newton polygon of $L^*(f, \chi, t)$

From now on, we assume $p \equiv -1 \bmod d$ and write $p = dk - 1$.

3.1. The case $\chi = \Psi_1$

Lemma 5. *Let $M(s) = (a_{ij})_{1\leq i,j\leq s}$ be an $s \times s$ matrix with entries*

$$a_{i,j} = \frac{a^{i+j}}{(ki - i - j)!(i + j)!}.$$

Then $v(\det M(s)) = 0$ for $1 \leq s \leq d - 1$.

Proof. Denote $x[0] = 1$ and $x[n] := x(x-1)\cdots(x-n+1)$ for $n \geq 1$. Then $x[n]$ is a polynomial of x of degree n and $\{(x+j)[t] : 0 \leq t \leq j-1\}$ is a basis of the space of polynomials of degree $\leq j-1$. Thus we can write

$$((k-1)x-1)[j-1] = c_0(j) + \sum_{t=1}^{j-1} c_t(j) \cdot (x+j)[t].$$

Let $x = -j$, we get

$$c_0(j) = ((k-1)(-j)-1)[j-1] = ((1-k)j-1)[j-1].$$

For any $1 \leq u \leq j-1$,

$$1 \leq (k-1)j + u < kj \leq k(d-1) \leq p.$$

Hence $p \nmid (1-k)j - u$ and $v(c_0(j)) = 0$.

Let $D = \mathrm{diag}\{a, a^2, \ldots, a^s\}$ and $M' = (a'_{ij})_{1 \leq i,j \leq s}$ with $a'_{ij} = a_{ij}a^{-i-j}$, then

$$M(s) = DM'D. \tag{3}$$

Let $a''_{ij} := (ki-i-1)!(i+s)!a'_{ij}$. Then

$$\begin{aligned} a''_{ij} &= (ki-i-1)[j-1] \cdot (i+s)[s-j] \\ &= \sum_{t=0}^{j-1} c_t(j) \cdot (i+j)[t] \cdot (i+s)[s-j], \\ &= \sum_{t=0}^{j-1} c_t(j) \cdot (i+s)[s-j+t] \\ &= \sum_{t=1}^{j} (i+s)[s-t] \cdot c_{j-t}(j). \end{aligned}$$

Define $c_{j-t}(j) := 0$ for $j < t$. Write $M'' = (a''_{ij})_{1 \leq i,j \leq s}$, $M_1 = ((i+s)[s-t])_{1 \leq i,t \leq s}$ and $M_2 = (c_{j-t}(j))_{1 \leq t,j \leq s}$. Then

$$M'' = M_1 M_2. \tag{4}$$

Write

$$x[n] = \sum_{t=0}^{n} c'_t(n) x^t,$$

then $c'_n(n) = 1$ and

$$(i+s)[s-j] = \sum_{t=0}^{s-j} c'_t(s-j)(i+s)^t.$$

Define $c'_t(n) := 0$ for $t > n$. Write $M_{11} = ((i+s)^{t-1})_{1\le i,t\le s}$ and $M_{12} = (c'_{t-1}(s-j))_{1\le t,j\le s}$. Then

$$M_1 = M_{11}M_{12}. \tag{5}$$

Notice that M_{11} is a Vandermonde matrix with determinant $\det M_{11} = \prod_{t=1}^{s} t^{s-t}$. One can also easily find

$$\det M_{12} = (-1)^{[s/2]} \quad \text{and} \quad \det M_2 = \prod_{i=1}^{s} c_0(i). \tag{6}$$

Now by (3), (4), (5) and (6),

$$\det M(s) = a^{s(s+1)}(-1)^{[s/2]} \prod_{i=1}^{s} \frac{i^{s-i}c_0(i)}{(ki-i-1)!(i+s)!}.$$

Hence $v(\det M(s)) = 0$. □

Denote $O(x)$ a number in $\overline{\mathbb{Q}}_p$ with valuation $\ge v(x)$ for $x \in \overline{\mathbb{Q}}_p$.

Lemma 6.

(i) *For* $i+j<d$, $F_{pi-j} = \gamma^{ki}(a_{ij}+O(\gamma))$.
(ii) *For* $i+j\ge d$, $v(F_{pi-j}) = ki-1$ *and*

$$F_{pi-(d-i)} = \frac{\gamma^{ki-1}(1+O(\gamma))}{(ki-1)!}.$$

Proof. Let

$$m = \begin{cases} ki-i-j, & \text{if } j<d-i; \\ ki-i-j+d-1, & \text{if } j\ge d-i, \end{cases}$$

$$n = \begin{cases} i+j, & \text{if } j<d-i; \\ i+j-d, & \text{if } j\ge d-i. \end{cases}$$

Then $pi-j = dm+(d-1)n$ and $0\le n\le d-1$. This lemma follows from

$$F_{pi-j} = \sum_{l\ge 0} \gamma_{m-(d-1)l}\gamma_{n+dl}a^{n+dl} = \gamma_m\gamma_n a^n(1+O(\gamma)) = \frac{\gamma^{m+n}a^n}{m!n!}(1+O(\gamma)). \qquad \square$$

Proposition 7. *For* $1 \le s \le d-1$*, the valuation of* $\det A_1[s+1]$ *is* $w_0 + w_1 + \cdots + w_s$*.*

Proof. Note that the first row of A_1 is $(1, 0, 0, \ldots)$. Let A be the matrix by deleting the first row and column of $A_1[s+1]$. Then $\det A_1[s+1] = \det A$.

Let $D_1 = \operatorname{diag}\{\gamma^{0/d}, \gamma^{1/d}, \ldots, \gamma^{s/d}\}$, $D_2 = \operatorname{diag}\{\gamma^{k-1}, \gamma^{2k-1}, \ldots, \gamma^{(d-1)k-1}\}$ and $B[s] = (\gamma^{1-ki} F_{pi-j})_{1 \le i,j \le s}$. Then $A = D_1^{-1} D_2 B[s] D_1$. It suffices to compute $v(\det B[s])$.

Note that for $s = d-1$,

$$B[d-1] = \begin{pmatrix} \gamma a_{11} + O(\gamma^2) & \cdots & \gamma a_{1,d-2} + O(\gamma^2) & \frac{1+O(\gamma)}{(k-1)!} \\ \vdots & \iddots & \frac{1+O(\gamma)}{(2k-1)!} & b_{2,d-1} \\ \gamma a_{d-2,1} + O(\gamma^2) & \iddots & \iddots & \vdots \\ \frac{1+O(\gamma)}{((d-1)k-1)!} & b_{d-1,2} & \cdots & b_{d-1,d-1} \end{pmatrix}$$

with $v(b_{ij}) = 0$. If $1 \le s \le \frac{d-1}{2}$, then

$$B[s] = \begin{pmatrix} \gamma a_{11} + O(\gamma^2) & \cdots & \gamma a_{s1} + O(\gamma^2) \\ \vdots & \ddots & \vdots \\ \gamma a_{s1} + O(\gamma^2) & \cdots & \gamma a_{ss} + O(\gamma^2) \end{pmatrix}$$

has determinant

$$\det B[s] = \gamma^s (\det M(s) + O(\gamma)).$$

The valuation of $\det B[s]$ is $s v(\gamma)$.

If $\frac{d}{2} \le s \le d-1$, then

$$B[s] = \begin{pmatrix} B[d-1-s] & P_1 \\ P_2 & Q \end{pmatrix}.$$

The valuation of any entry of $B[d-1-s]$, P_1, P_2 is $v(\gamma)$ and

$$Q \equiv \begin{pmatrix} 0 & & \frac{1}{(k-1)!} \\ & \iddots & \\ \frac{1}{((d-1)k-1)!} & & * \end{pmatrix} \bmod \gamma.$$

Thus Q is invertible over the ring of integers of $\mathbb{Q}_p(\gamma)$. The determinant

$$\det B[s] = \det Q \det(B[d-1-s] - P_1 Q^{-1} P_2) = \det Q \det B[d-1-s](1 + O(\gamma))$$

has valuation $(d-1-s)v(\gamma)$.

Finally, $A = D_1^{-1} D_2 B[s] D_1$ has valuation

$$\Big(\sum_{i=1}^{s} (ki-1) + \min\{s, d-1-s\}\Big) v(\gamma) = w_0 + w_1 + \cdots + w_s. \qquad \square$$

Proof of Theorem 1. For $1 \le s \le d-1$, we have

$$\begin{aligned} v(\det A_1[s+1]) &= \sum_{i \le s} w_i \\ &= \begin{cases} \frac{s(s+1)}{2d} + \frac{s(s+1)}{d(p-1)}, & \text{if } s \le (d-1)/2; \\ \frac{s(s+1)}{2d} + \frac{(d-s)(d-s-1)}{d(p-1)}, & \text{if } s \ge d/2; \end{cases} \\ &\le \frac{s(s+1)}{2d} + \frac{d^2-1}{4d(p-1)}. \end{aligned}$$

If $p > \frac{d^2+3}{4}$, then $\frac{d^2-1}{4d(p-1)} < 1/d$. For $0 \le t_0 < t_1 < \cdots < t_s$, assume $t_s \ne s$. Since

$$v(F_{pi-j}\gamma^{(j-i)/d}) \ge i/d,$$

we have

$$v(\det A_1[t_0, \dots, t_s]) \ge \frac{s^2+s+2}{2d} > v(\det A_1[s+1]).$$

Thus $v(c_{s+1}) = v(\det A_1[s+1]) = \sum_{i \le s} w_s$ and $\{w_0, w_1, \dots, w_{d-1}\}$ are slopes of $\mathrm{NP}_p(\det(I - tA_1))$.

If moreover $p > \frac{d^2}{2}$, then $p \ge \frac{d^2+1}{2}$ and $\frac{d^2-1}{4d(p-1)} \le \frac{1}{2d}$. Choose $\beta_i = i/d$ in Theorem 4, we have

$$v(C_{s+1}) = h(w_0 + w_1 + \cdots + w_s)$$

and

$$\mathrm{NP}_q(\det(I - tA_h[d])) = \mathrm{NP}_p(\det(I - tA_1[d])).$$

Thus $w_0, w_1, \dots, w_{d-1}$ are q-adic slopes of $\mathrm{NP}_q(\det^{\varphi^{-1}}(I - tA_h))$.

By Theorem 3,

$$\det{}^{\varphi^{-1}}(I - tA_h) = L^*(f,t) \det{}^{\varphi^{-1}}(I - tqA_h).$$

Since the valuation of any entry of A_h is ≥ 0, the q-adic slopes of $\det^{\varphi^{-1}}(I - tA_h)$ are ≥ 0 and the q-adic slopes of $\det^{\varphi^{-1}}(I - tqA_h)$ are ≥ 1. Thus any q-adic slope of $\det^{\varphi^{-1}}(I - tA_h)$ less than 1 must be a q-adic slope of $L^*(f,t)$. But $L^*(f,t)$ has degree d, hence $w_0, \dots, w_{d-1}$ are all slopes of $L^*(f,t)$. □

3.2. The case for general χ

Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial with degree d. Assume $p \nmid d$. Let $\mathrm{NP}(f,x)$ be the piecewise linear function whose graph is the q-adic Newton polygon of $\det(I - tA_h)$. Let $\mathrm{HP}(f,x)$ be the piecewise linear function whose graph is the polygon with vertices

$$(k, \frac{k(k-1)}{2d}), \quad k = 0, 1, 2, \ldots.$$

Then $\mathrm{NP}(f, x) \geq \mathrm{HP}(f, x)$ (cf. [3,5]). Set

$$\mathrm{gap}(f) = \max_{x \geq 0} \{\mathrm{NP}(f, x) - \mathrm{HP}(f, x)\}.$$

Theorem 8. *(See [5, Theorem 4.3].) Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{d-1} < 1$ denote the slopes of the q-adic Newton polygon of $L^*(f, t)$. If $\mathrm{gap}(f) < 1/h$, then the q-adic Newton polygon of $L^*(f, \chi, t)$ has slopes*

$$\{p^{1-m_\chi}(i + \alpha_j) : 0 \leq i \leq p^{m_\chi - 1} - 1, 0 \leq j \leq d - 1\}$$

for any non-trivial finite character χ.

Proof of Theorem 2. The slopes of $\mathrm{NP}(f, x)$ are

$$\{i + w_j : i \geq 0, 0 \leq j \leq d - 1\}.$$

Notice that

$$\sum_{i=0}^{d-1} w_i = \sum_{i=0}^{d-1} \frac{i}{d},$$

$\mathrm{NP}(f, x) - \mathrm{HP}(f, x)$ is a periodic function with period d. For $0 \leq k < d$,

$$\mathrm{NP}(f, x) - \mathrm{HP}(f, x) \leq \sum_{i \leq (d-1)/2} \frac{2i}{d(p-1)} \leq \frac{d^2 - 1}{4d(p-1)}.$$

If $p > \frac{h(d^2-1)}{4d} + 1$, then $\mathrm{gap}(f) < 1/h$ and this concludes the proof. □

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