Chap. 6
Ginzburg-Landau model and Landau theory

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From Ising model to the Ginzburg-Landau model

- Ising model
  - Uniaxial ferromagnet
  - One spin variable σ_c in each cubic cell
  - Energy of these spins---Cell Hamiltonian

$$\hat{H}[\sigma] = \frac{1}{2} J \sum_{c} \sum_{r}{}' (\sigma_c - \sigma_{c+r})^2$$

  - This is the simplest form to see energy is smaller if the spin agrees with its neighbors

From Ising model to the Ginzburg-Landau model

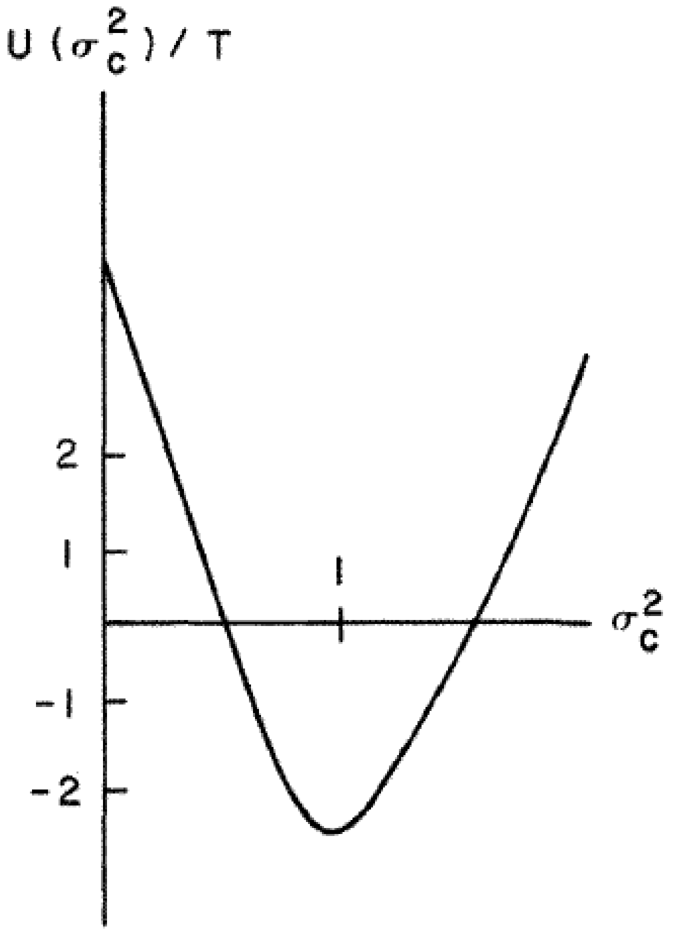
- Generalize the Hamiltonian slightly to

$$\hat{H}[\sigma] = \frac{1}{2} J \sum_{c} \sum_{r}{}' (\sigma_c - \sigma_{c+r})^2 + \sum_{c} U(\sigma_c{}^2)$$

  - Here σ_c is a continuous variable
  - $U(\sigma_c{}^2)$ is additional energy
  - $U(\sigma_c{}^2)$ is large except near $\sigma_c = \pm 1$

From Ising model to the Ginzburg-Landau model

- Fig.1 A sharp minimum of $U(\sigma_c^2)/T$ at 1 implies that the magnitude of σ_c is effectively restricted to nearly 1.



From Ising model to the Ginzburg-Landau model

- The XY model and Heisenberg model
  - If the spins are not restricted to point along one axis, we need to describe each spin by a vector
  - For Heisenberg model

$$\sigma_c = (\sigma_{1c}, \sigma_{2c}, \sigma_{3c})$$

  - For XY model

$$\sigma_c = (\sigma_{1c}, \sigma_{2c})$$

From Ising model to the Ginzburg-Landau model

- Cell Hamiltonian
  - Describe interactions between cell spins
  - Parameters sum up the relevant effects of the details within a scale smaller than a unit cell
- Block Hamiltonian
  - Describe interactions between block spins,each block consists of b^d unit cells. i.e. $b = 2, or\ 3$
  - Thus parameters in Block Hamiltonian sum up the relevant details within a scale of b lattice constants

From Ising model to the Ginzburg-Landau model

- Construction of block spins
  - Label the blocks by the position vector x of the centers of the blocks
  - Defined as the net spin in a block

$$\sigma_x = b^{-d} \sum_c^x \sigma_c$$

$$\sigma(x) = L^{-d/2} \sum_{k<\Lambda} \sigma_k e^{ik\cdot x}$$

  - The sum in the second equation is in the first Brillouin zone

From Ising model to the Ginzburg-Landau model

- Construction of block spins
  - Block spin is simply the mean of the cell spins within the block labeled by x
  - That is to say the spatial resolution of the block Hamiltonian is b, whereas that of the cell Hamiltonian is 1

From Ising model to the Ginzburg-Landau model

- Ginzburg-Landau form
  - Starts by assuming a simple form for the block Hamiltonian

$$H[\sigma]/T = \int d^d x[a_0 + a_2\sigma^2 + a_4\sigma^4 + c(\nabla\sigma)^2 - h\cdot\sigma]$$

Where

$$\sigma^2 \equiv \sigma(x)\cdot\sigma(x) \equiv \sum_{i=1}^{n}(\sigma_i(x))^2$$

$$(\nabla\sigma)^2 \equiv \sum_{\alpha=1}^{d}\sum_{i=1}^{n}\left(\frac{\partial\sigma_i}{\partial x_\alpha}\right)^2$$

  - The coefficients $a_0, a_{2,} a_4, c$ are functions of T, and h is the applied magnetic field divided by T.

From Ising model to the Ginzburg-Landau model

- Ginzburg-Landau form
  - Fourier transformation
  - Relations between Fourier components σ_k and the spin configuration $\sigma(x)$

$$\sigma_k = L^{-2/d} \int d^d x e^{-ik\cdot x} \sigma(x)$$

$$\sigma(x) = L^{-2/d} \sum_k e^{ik\cdot x} \sigma_k$$

From Ising model to the Ginzburg-Landau model

- Ginzburg-Landau form
  - Fourier transformation

$$H[\sigma]/T = a_0 L^d + \sum_{k<\Lambda} \sigma_k \cdot \sigma_{-k} (a_2 + ck^2)$$

$$+L^{-d} \sum_{kk'k''<\Lambda} a_4 (\sigma_k \cdot \sigma_{k'})(\sigma_{k''} \cdot \sigma_{-k-k'-k''}) - L^{d/2} \sigma_0 \cdot h$$

  - Write in terms of σ_x

$$H[\sigma]/T = b^d \sum_x \left[a_0 + a_2 \sigma_x{}^2 + a_4 \sigma_x{}^4 + c \frac{b^{-2}}{2} \sum_y{}' (\sigma_x - \sigma_{x+y})^2 - h \cdot \sigma_x \right]$$

  - The sum over y is taken over the 2nd nearest neighbors of the block x

From Ising model to the Ginzburg-Landau model

- Meaning of various terms
  - Obviously $-h \cdot \sigma_x$ is the external field term
  - If we drop the $(\sigma_x - \sigma_{x+y})^2$ term and set h=0, we can see that

$$U(\sigma_x) = a_0 + a_2 \sigma_x{}^2 + a_4 \sigma_x{}^4$$

  - Each of the terms depends only on σ_x of one block, that means each block spin is statistically independent of other block spins.
  - Then we have a system of L^d / b^d non-interacting blocks

From Ising model to the Ginzburg-Landau model

- Meaning of various terms
  - We have no information about the coefficients except they must be smooth functions of T and other parameters
  - It will make no sense that if a_4 is negative because $U(\sigma_x)$ would the approach $-\infty$ as $\sigma_x \to \infty$ and the probability distribution $P \propto \exp(-H[\sigma]/T)$ would blow up

From Ising model to the Ginzburg-Landau model

- Meaning of various terms
  - When the gradient term $(\sigma_x - \sigma_{x+y})^2$ is included, then the block spins are no longer independent.
  - This term means interaction between neighboring blocks
  - For ferromagnets, this interaction will make a block spin parallel to its neighboring block spins.
  - The greater the difference among the block spins, the lager H/T becomes and hence the smaller the probability

Landau Theory

- Most probable value and Gaussian Approxi.
  - Consider a field φ and it Hamiltonian $H(\varphi)/kT$
  - The probability of the field is $p \propto e^{-H/kT}$
  - The most probable configuration φ_0 is given by

$$\frac{\partial H[\varphi]}{\partial \varphi} = 0$$

  - Near $\varphi_0, H/kT$ can be approximated by

$$H[\varphi]/kT = H[\varphi_0]/kT + \frac{1}{2\lambda^2}(\varphi - \varphi_0)^2$$

  - Taylor expansion with $\lambda^{-2} = T^{-1}\left(\frac{\partial^2 H}{\partial \varphi^2}\right)_{\varphi=\varphi_0}$

Landau Theory

- Most probable value and Gaussian Approxi.
  - The probability is approximated as

$$e^{-H[\varphi_0]-(\varphi-\varphi_0)/2\lambda^2}$$

  - Partition sum $Z=\int d\varphi e^{-H[\varphi_0]-(\varphi-\varphi_0)/2\lambda^2}$

$$=e^{-H[\varphi_0]}+\frac{1}{\sqrt{2\pi\lambda}}$$

  - Free energy $f=-\ln Z=H[\varphi_0]-\frac{1}{2}\ln(2\pi\lambda^2)$
  - Generalize to more degrees of freedom is straightforward

Landau Theory

- Minimum of Ginzburg-Landau Hamiltonian
  - At the minimum point field $\varphi(x) = \varphi_0$ is const
  - That means the Fourier components

$$\sigma_k = 0 \quad \text{for} \;\; k \neq 0 \quad \text{and} \quad \sigma_0 = L^{-d/2}\varphi_0$$

  - The value can be found by setting

$$\frac{\partial H[\varphi_0]}{\partial \varphi_0} = 0$$

  - We get

$$2a_2\varphi_0 + 4a_4\varphi_0{}^3 - h = 0$$

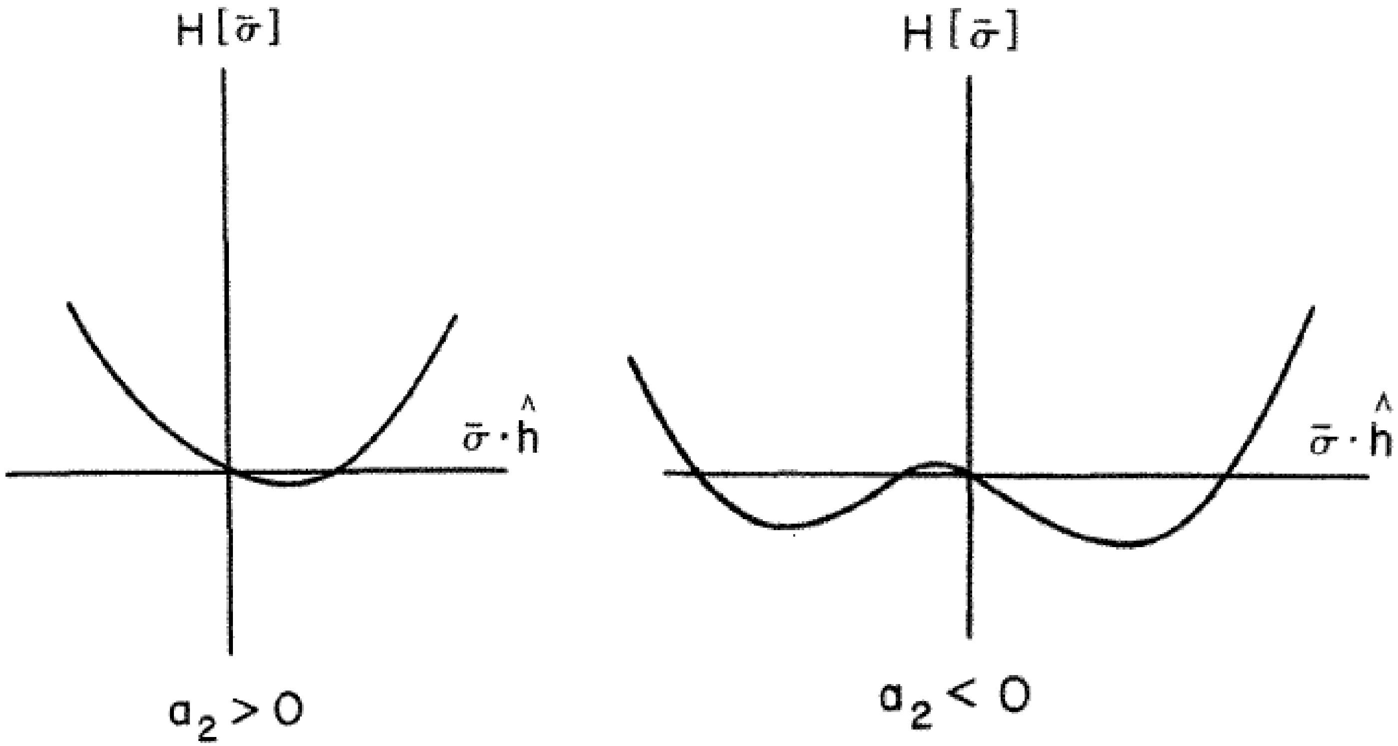
Landau Theory

- Minimum of Ginzburg-Landau Hamiltonian
  - When $a_2 > 0$, $\varphi_0 \approx h / 2a_2$
  - When $a_2 < 0$
    - $h = 0 \Rightarrow \varphi_0 = m_0 = \pm(-a_2 / 2a_4)^{1/2}$
    - $h > 0 \Rightarrow \varphi_0 = |m_0| + h / (8m_0{}^2 a_4)$
  - When $a_2 = 0$

$$\varphi_0 = \left(\frac{h}{4a_4} \right)^{1/3}$$

Landau Theory

- Minimum of Ginzburg-Landau Hamiltonian
  - Here φ_0 is denoted by $\overline{\sigma}$



Landau Theory

- Minimum of Ginzburg-Landau Hamiltonian
  - Critical point is at $a_2 = 0 \Rightarrow a_2 = a_2'(T - T_c) + \cdots$
  - When $h = 0, a_2 < 0$

$$H[\varphi_0]/T = L^d(a_0 + a_2\varphi_0^2 + a_4\varphi_0^4) = L^d(a_0 - \frac{a_2^2}{4a_4})$$

  - Energy density $e = H[\varphi_0]/L^d = a_0' - \frac{a_2'}{2a_4}(T - T_c)^2$
  - When $a_2 \geq 0, \varphi_0 = 0 \Rightarrow e = a_0'$

$$C = \begin{cases} a_0' & T > T_c \\ a_0' - \frac{a_2'}{2a_4}(T - T_c)^2 & T < T_c \end{cases}$$

Landau Theory

- Minimum of Ginzburg-Landau Hamiltonian
  - Calculation of critical exponents

$$h = 0$$

$$\begin{cases} a_2 > 0, \varphi_0 = 0 \\ a_2 < 0, \varphi_0 = \left[\dfrac{a_2'(T_c - T)}{2a_4} \right]^{1/2} \end{cases} \Rightarrow \beta' = 1/2$$

$$a_2 = 0, \varphi_0 = \left(h / 4a_4 \right)^{1/3} \Rightarrow \delta = 1/3$$

$$\begin{cases} a_2 > 0, \varphi_0 = h / 2a_2 \Rightarrow \chi = \dfrac{1}{2a_2'(T - T_c)} \Rightarrow \gamma = 1 \\ a_2 < 0, \varphi_0 = |m_0| - h / 4a_2 \Rightarrow \chi = \dfrac{1}{4a_2'(T - T_c)} \Rightarrow \gamma' = 1 \end{cases}$$

  - Same as mean-field calculation

Chap. 7
Gaussian approximation for the Ginzburg-Landau model

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Gaussian approximation for T>Tc

- We set h=0 for simplicity, thus

$$\varphi_0 = 0 \qquad H[\varphi_0]/T = a_0 L^d$$

$$H[\varphi]/T \approx H[\varphi_0]/T + \sum_{k<\Lambda} (a_2 + ck^2)\sigma_k \cdot \sigma_{-k}$$

- Partition sum

$$Z = e^{-H[\varphi_0]/T} \int d\sigma_k e^{\sum_k (a_2+ck^2)|\sigma_k|^2}$$

$$= e^{-a_0 L^d} \prod_k \left(\frac{\pi}{a_2 + ck^2} \right)^{1/2}$$

- Free enegry

$$f = a_0 - \frac{1}{2} L^{-d} \sum_k \ln[\pi / (a_2 + ck^2)]$$

Gaussian approximation for T>Tc

$$G(k) = \langle |\sigma_k|^2 \rangle = \frac{1}{2}(a_2 + ck^2)$$

- Let $T = T_c$, thus $a_2 = 0$

$$G(k) \propto k^{-2} \Rightarrow \eta = 0$$

$$\chi \propto \lim_{k \to 0} G(k) \propto (T - T_c)^{-1} \propto \gamma = 1$$

Gaussian approximation for T>Tc

- Energy density and specific heat

$$e \propto \frac{\partial f}{\partial a_2} \propto \int d^d k (a_2 + ck^2)^{-1}$$

$$C \propto \frac{\partial e}{\partial a_2} \propto \int d^d k (a_2 + ck^2)^{-2}$$

- For $a_2, k \to 0, (a_2 + ck^2)^{-1}, (a_2 + ck^2)^{-2}$ are singular!!
- Let $k = k'/\xi, \xi^{-1} \equiv (a_2 / c)^{1/2} \to 0 \text{ if } a_2 \to 0$

$$C \propto \left[\int d^d k'(1 + k'^2)^{-2} \right] \xi^{4-d} \propto \xi^{4-d} \propto (T - T_c)^{-(2-d/2)}$$

$$\Rightarrow \alpha = 2 - d/2$$

Gaussian approximation for T<Tc

- Hamiltonian of Ginzburg-Landau form

$$H[\varphi]/T = \int d^d x[a_0 + a_2\varphi^2 + a_4\varphi^4 - h\varphi - c(\nabla\varphi)^2]$$

- We expand this Hamiltonian for T<Tc near φ_0
- Let $f(\varphi) = a_0 + a_2\varphi^2 + a_4\varphi^4 - h\varphi$ use the Taylor expansion, note that

$$\varphi_0 = m_0 - h/4a_2, m_0 = \left(\frac{-a_2}{2a_4}\right)^{1/2}$$

- Then we get

$$H[\varphi]/T \approx H[\varphi_0]/T$$

$$+\int d^d x[(-2a_2 + \frac{3}{2}\sqrt{\frac{2a_4}{-a_2}} \cdot h)(\varphi - \varphi_0)^2 - c(\nabla\varphi)^2]$$

Gaussian approximation for T<Tc

- Fourier transformation and let $b = -2a_2 + \frac{3}{2}\sqrt{\frac{2a_4}{-a_2}} \cdot h$

$$H[\varphi]/T \approx H[\varphi_0]/T + \int d^d k (b + ck^2) |\sigma_k|^2$$

- Thus

$$G(k) = \langle |\sigma_k|^2 \rangle \propto (b + ck^2)^{-1}$$

$$\text{when } h = 0, \quad G(k) \propto [2a_2(T_c - T) + ck^2]^{-1}$$

- And free energy

$$f = H[\varphi_0]/T - \frac{1}{2} L^{-d} \sum_k \ln[\pi/(b + ck^2)]$$

- Specific heat $C \propto \xi^{4-d}$

Correlation length and temperature dependence

- The singular temperature dependence of the quantities can be summarized in terms of correlation length $\xi = \sqrt{c / a_2}$

$$T > T_c \qquad G(k) = \frac{c\xi^2}{2(1 + k^2\xi^2)}$$

$$C = C_0 \xi^{4-d} + \cdots$$

$$T < T_c \qquad G(k) = \frac{c\xi^2}{4(1 + k^2\xi^2 / 2)}$$

$$C = C_0' \xi^{4-d} + \cdots$$

- Where C_0 and C_0' are both constants fixed by previous calculatiom

Correlation length and temperature dependence

- Correlation length measures the distance over which spin fluctuations are correlated
- Singular behavior of quantities for vanishing $|T-T_c|$ and k can be viewed as a result of $\xi \to \infty$.
- As far as the singular temperature dependence is concerned, ξ is the only relevant length.

Ginzburg Criterion

- The importance of fluctuation
  - Let's see the ratio

$$C_0\xi^{4-d} / \Delta C \sim [\zeta_T / |1-T/T_c|]^{2-d/2}$$

  - C_0 is the const fixed by calculation

$$\zeta_T = [(2\pi\xi_0)^{-d} / \Delta C]^{2/(4-d)}$$

$$\xi_0 \equiv (c / a_2' T_c)^{1/2}$$

  - When the temperature is close to Tc within a range $\zeta_T T_c$, the fluctuation are expected to be important
  - The smaller ζ_T is ,the smaller this range will be

Chap. 8
Renormalizaion group

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Motivation

- Study of symmetry transformations are proven to be extremely useful
- At the critical region, we want to ask under what S.T is a system invariant
- Microscopic details seem to make very little difference in critical phenomena suggests there’s some kind of symmetry properties.
- This desired transformations is know as renormalization group

Motivation

- Study of symmetry transformations are proven to be extremely useful
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Variable transformation in probability theory

- Before we go to the Kadanoff transformation, first have a review
- Probability distribution $P(x,y)$, x and y are two random variables, define $z=1/2(x+y)$ then the probability distribution for z

$$P'(z)=\int dxdy\delta(z-\frac{1}{2}(x+y))P(x,y)$$

$$=\left\langle \delta(z-\frac{1}{2}(x+y)) \right\rangle_P$$

- $P'(z)$ does exactly the same job as $P(x,y)$,as far as he average values involving z are concerned

Variable transformation in probability theory

- Example: $P(x)$ and $P(y)$ are identical probability distribution for $x, y = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$
- Now we calculate the distribution for Z
- For a integer $z_0 \in [2, 20]$,we get

$$P(z = z_0) = \sum_{x,y} \delta(z_0, x + y) P(x) P(y)$$

Z		2	3	4	5	6	7	8	9	10
$P(z)\times 100$		1	2	3	4	5	6	7	8	9
Z	11	12	13	14	15	16	17	18	19	20
$P(z)\times 100$	10	9	8	7	6	5	4	3	2	1

Kadanoff transformation

- Transform cell Hamiltonian to block Hamiltonian

$$H[\sigma]/T = K_b \hat{H}[\sigma]/T$$

$$e^{-H[\sigma]/T} = \int e^{-\hat{H}[\sigma]/T} \prod_{i,x} \delta(\sigma_{ix} - b^{-d} \sum_c^x \sigma_{ic}) \prod_{j,c} d\sigma_{j,c}$$

- Where σ_x *and* σ_c are the block spin and cell spin, respectively and the index c runs over all cells and i,j from 1 to n (number of components)
- b indicates the block size is b times the cell size
- Still we can construct another block Hamiltonian

$$H''[\sigma]/T = K_s H[\sigma]/T$$

Kadanoff transformation

- We can see

$$K_s H[\sigma]/T = K_s K_b \hat{H}[\sigma]/T = K_{sb} \hat{H}[\sigma]/T$$

- In general

$$K_s K_{s'} = K_{ss'}$$

- Kadanoff transformations will play a major role in the construction of the renormalization group

Definition of renormalization group

- We take GL form for instance with h=0

$$P \propto e^{-H}$$

$$H = b^d \sum_x [u_2 \sigma_x{}^2 + u_4 \sigma_x{}^4 + \frac{1}{2} b^{-2} \sum_y{}' c(\sigma_x - \sigma_{x+y})^2]$$

- Use the triplet of parameters $\mu = (u_2, u_4, c)$ to label the probability distributions
- Parameter space: different values of parameters and every probability distribution is represented by a point in this parameter space

Definition of renormalization group

- We define a transformation $\mu' = R_s \mu$ by the following steps
  - 1,Apply the Kadanoff transformation

$$H''[\sigma] = K_s H[\sigma]$$

    - This step downgrades the spatial resolution of spin variations to sb, note that $H[\sigma]$ is the block Hamiltonian
  - 2,Relabel the block spins σ_x in $H''[\sigma]$ and multiply each of them by a constant λ_s

$$H'[\sigma] = (H''[\sigma])_{\sigma_x \to \lambda_s \sigma_x}$$

$$where \quad x' = x / s$$

    - Now the block size sb is shrink to b, back to original

Definition of renormalization group

- These two steps can be explicitly written as

$$e^{-H'[\sigma]/T} = \int e^{-H[\sigma'']/T} \prod_{x'} \delta(\lambda_s \sigma_x - s^{-d} \sum_y^x \sigma_y'') \prod_y d\sigma_y''$$

- Write H' in the GL form

$$H'[\sigma] = b^d \sum_{x'} [\frac{1}{2} c' b^{-2} \sum_{y'} (\sigma_{x'} - \sigma_{x'+y'})^2 + u_2' \sigma_{x'}^2 + u_4' \sigma_{x'}^4]$$

- New parameters

$$\mu' = (u_2', u_4', c')$$

- Thus define R_s

Definition of renormalization group

- The set of transformations $\{R_s, s \geq 1\}$ is called RG
- It's a semi-group, not a group since the inverse transformations are not defined.
- It has the property

$$R_s R_{s'} = R_{ss'}$$

only if $\lambda_s = s^a$ where a is independent of s.