Graphical Representations of Some Statistical Models

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- *Introduction to critical phenomena*
- *High-temperature Graph*
- *Low-temperature Graph*
- *Fortuin-Kasteleyn Representation*
- *Baxter-Kelland-Wu Representation and Stochastic Lowner Evolution*

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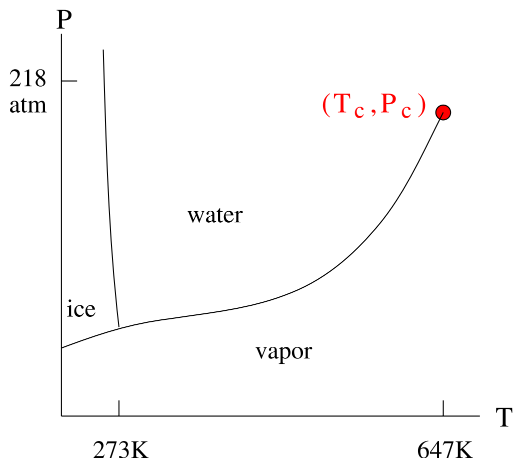
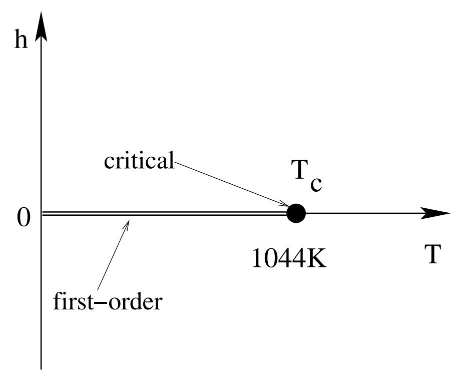
Introduction

- *Examples*





The liquid-gas critical point of H_2O: $T_c = 647K$, $p_c = 218$ atm.

The ferromagnetic point of Fe: $h_c = 0$, $T_c = 1044K$.

- *Statistical Models*

  - Ising model on Graph (V, E).

    1, Configuration Space $\vec{s} \in \{+1, -1\}^{|V|}$.

    2, *A priori* measure $\frac{1}{2}(\delta_{s,+1} + \delta_{s,-1})$.

    3, Hamiltonian

$$\mathcal{H}/k_{\mathrm{B}}T = -K \sum_{\langle i,j \rangle} s_i s_j - H \sum_k s_k$$

– Partition function $Z = \sum e^{-\mathcal{H}/k_b T}$

or free energy $F = -\ln Z$

– Physical quantities as derivatives of F.

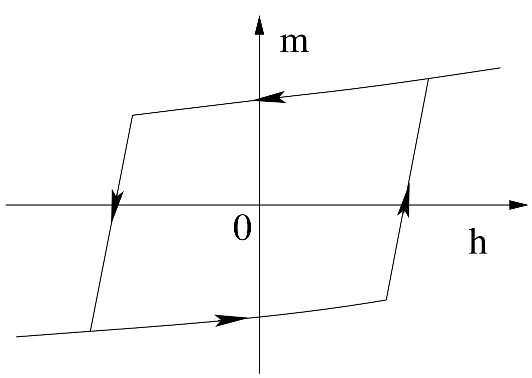
First derivative: ρ_{H_2O}, m

Second derivative: C, χ

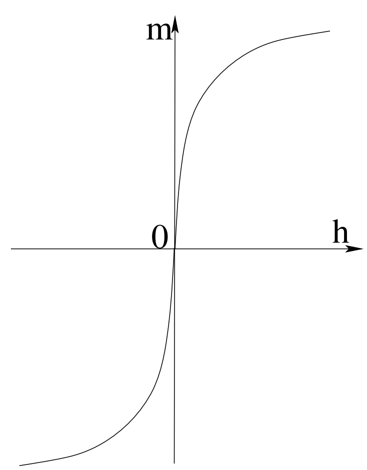
⋆ nth-order transition:

nth derivative of F is singular,

but $(n-1)$th derivative is analytic.



First-order



Second-order

- *Critical phenomena*

  Specific-heat : $C \propto |T - T_c|^{-\alpha}$

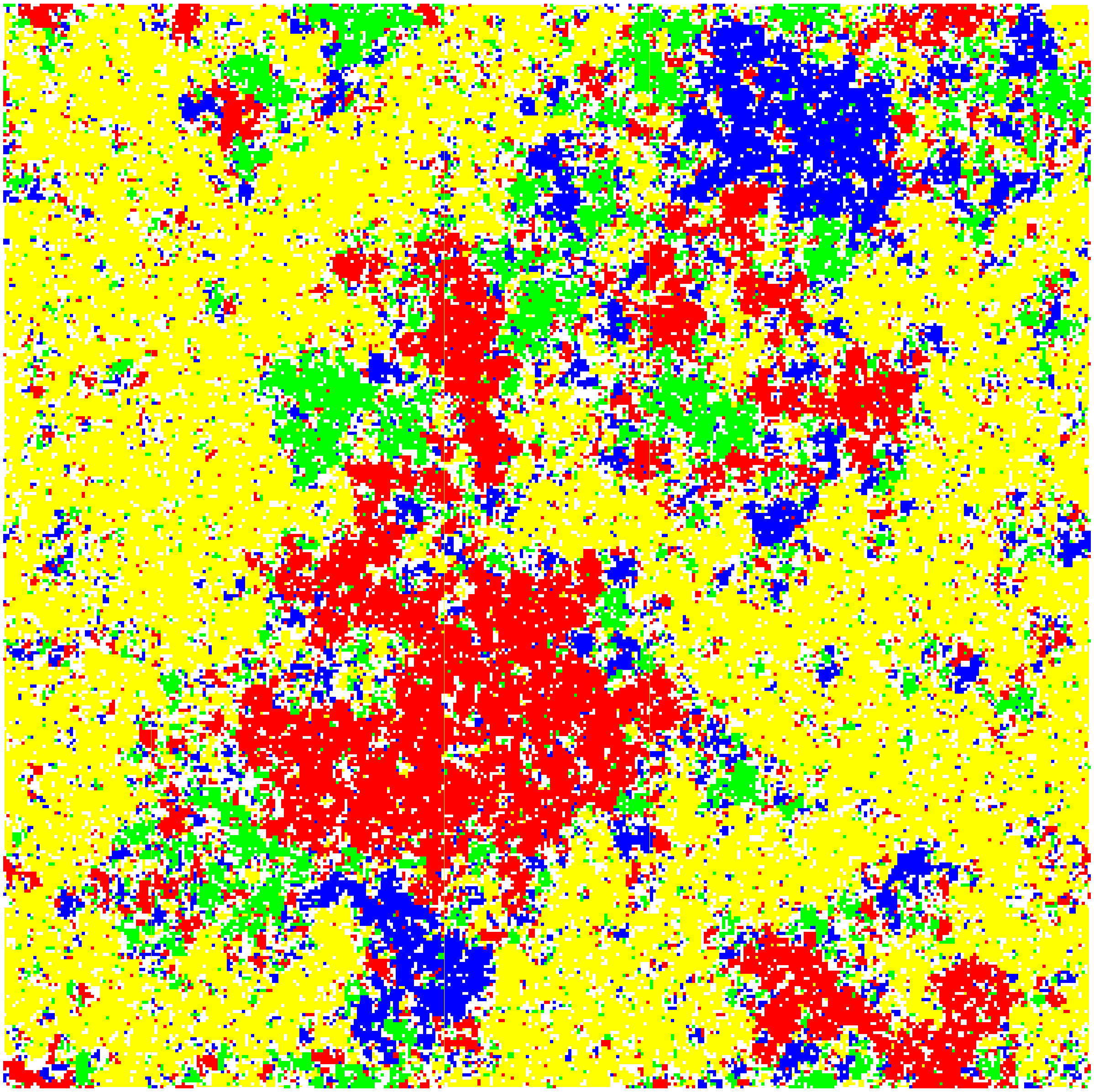
  Susceptibility : $\chi \propto |T - T_c|^{-\gamma}$

  Magnetization: $m \propto |T - T_c|^{\beta}$

  At T_c, correlation: $g(r) \propto r^{-2X}$

  $\star$ Universality: critical exponents, $\alpha, \gamma, \cdots$,

  take same values in different systems.



Four-state Potts at criticality; the Hamiltonian reads

$$\mathcal{H}/k_{\mathrm{B}}T = -J\sum_{\langle i,j\rangle}\delta_{\sigma_i,\sigma_j}$$

High-temperature expansion of the zero-field Ising model

Parition function:

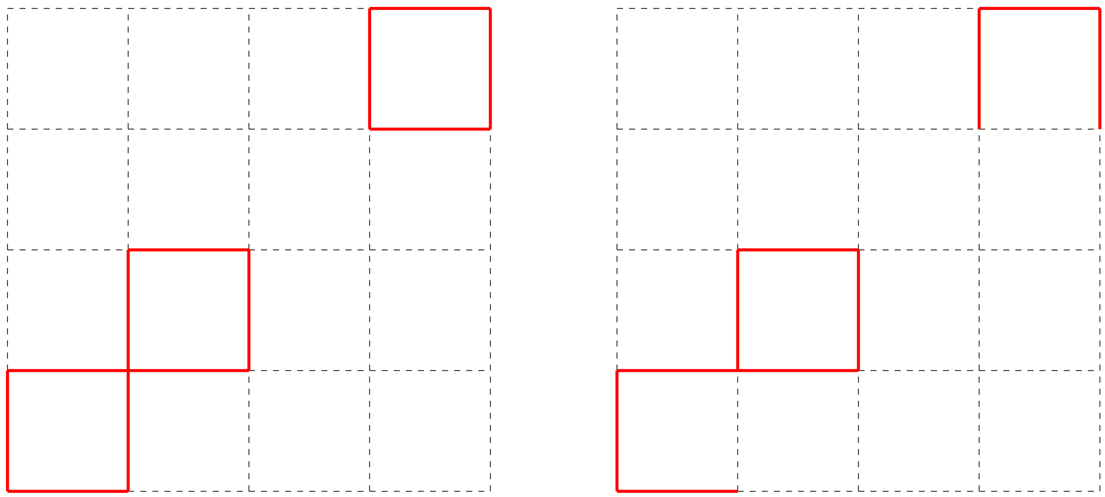
$$
\begin{aligned}
Z &= \sum_{\{\pm 1\}^{|V|}} e^{-\mathcal{H}} \\
&= \sum_{\{\pm 1\}^{|V|}} \prod_{\langle i,j\rangle} e^{K s_i s_j} \qquad (1)
\end{aligned}
$$

Identity

$$
\begin{aligned}
e^{K s_i s_j} &= \cosh K + s_i s_j \sinh K \\
&= \cosh K (1 + v s_i s_j) \;\; (v = \tanh K) \qquad (2)
\end{aligned}
$$

Insert (2) into (1), and expand the product $\prod$. Then, label each term in the expansion by a graph Λ:

- If factor 1 is taken, do nothing;
- If $v s_i s_j$, put a bond $\langle i, j\rangle$.



A closed graph　　　An non-closed graph

Parition function Z reads

$$Z = (\cosh K)^{|E|} \sum_{G \in \mathcal{G}_0} v^{b(G)} \sum_{\{\pm 1\}^{|V|}} \left(\prod_i s_i^{m_i(G)} \right)$$

– $m_i(G)$: number of bonds incident on vertex i.

$\star$: Any graph with at least one *odd* number m_i contributes to Z by *zero*.

Z becomes

$$Z = 2^{|V|} (\cosh K)^{|E|} \sum_{G \in \mathcal{G}_0, \partial G = 0} v^{b(G)}.$$

$\star$ High-T graph of $\langle s_i s_j \rangle$: Almost closed graphs with $\partial G = (i, j)$.

$\star$ High-T graph of the Potts model: Flow polynomial

Application:

- 1D Ising model: $Z = 2^L (\cosh K)^L$ (free)
  $Z = 2^L (\cosh K)^L [1+(\tanh K)^L]$ (periodic)

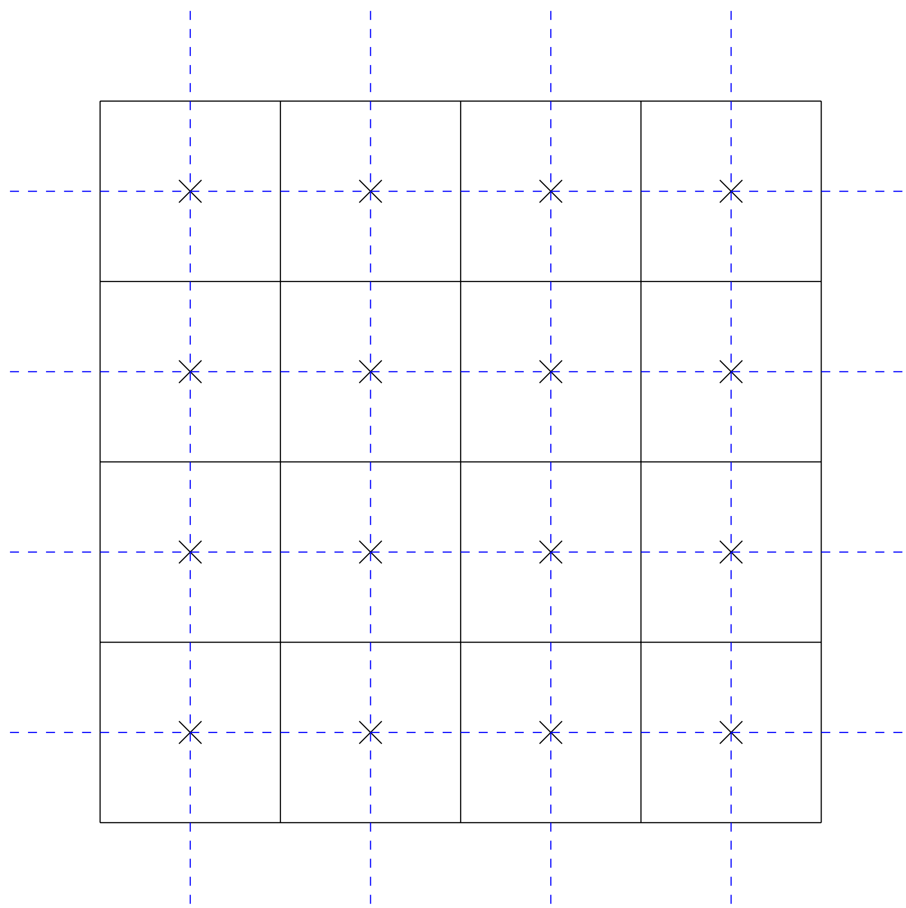
- Exact solution of the 2D Ising model

- Worm Algorithm (has considerable applications in quantum systems).

- O(n) loop model and its cluster simulations.

Low-temperature expansion of the zero-field Ising model

Dual Lattice: $G \to G^*$

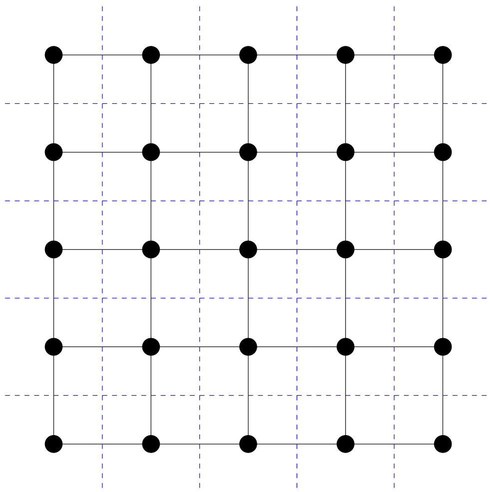


G: solid and black lines G^*: dashed and blue lines

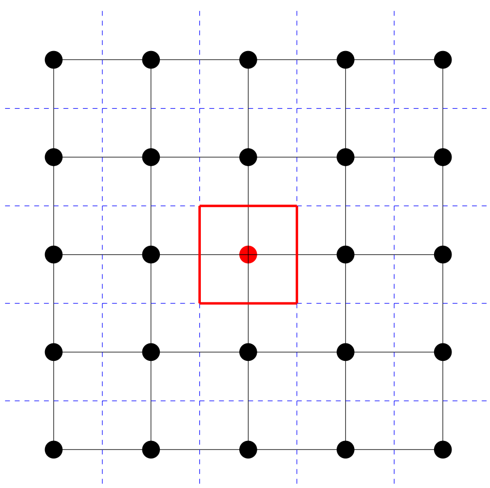
Partition function Z:

$$Z = \sum_{\{\pm 1\}^{|V|}} e^{-\mathcal{H}}$$

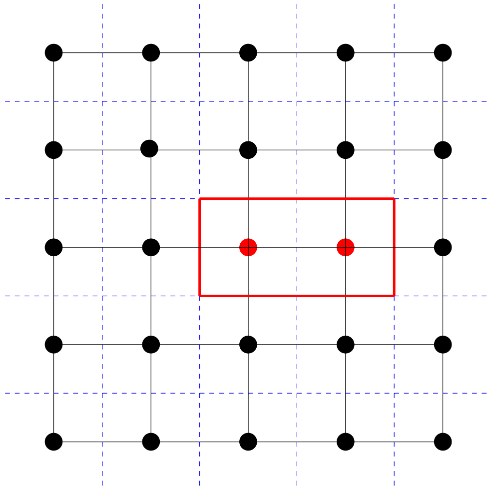
1) All spins DOWN: $e^{|E|K}$



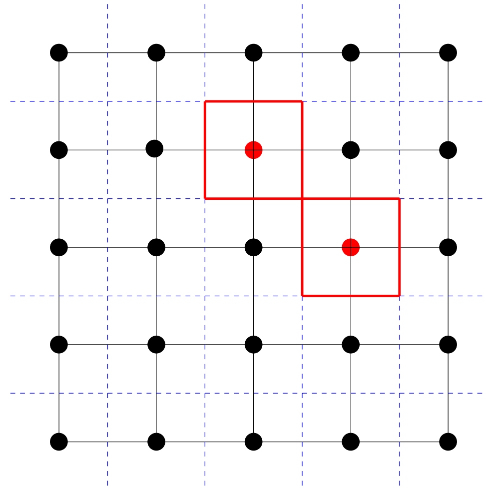
2) One spin UP: $e^{|E|K} * e^{-8K} = e^{|E|K} * u^4$



2) Two spins UP:



$e^{|E|K} * u^6$



$e^{|E|K} * u^8$

⋮

Z becomes

$$Z = e^{|E|K} \sum_{G^* \in \mathcal{G}_0^*, \partial G^* = 0} u^{b(G*)}.$$

Duality Relation:

$Z(\mathcal{G}; v) = \text{constant} \times Z(\mathcal{G}^*; u)$ if

$v = u$. Namely, $\tanh K = e^{-2K^*}$

Criticality on Square Lattice: $2K_c = \ln(1 + \sqrt{2})$.

Fortuin-Kasteleyn Representation of the zero-field Ising model

Hamiltonian $\mathcal{H}$

$$\mathcal{H}/ = -K \sum_{\langle i,j \rangle} s_i s_j = -2K \sum_{\langle i,j \rangle} \delta_{s_i,s_j} + \text{constant}$$

Identity:

$$e^{2K\delta_{s_i,s_j}} = 1 + \mu\delta_{s_i,s_j} \quad (\mu = e^{2K} - 1)$$

Partition function Z:

$$\begin{aligned} Z &= \sum_{\{\pm 1\}^{|V|}} e^{-\mathcal{H}} \\ &= \sum_{\{\pm 1\}^{|V|}} \prod_{\langle i,j \rangle} (1 + \mu\delta_{s_i,s_j}) \end{aligned} \quad (3)$$

Expand the product $\prod$, label each term by the graph: if the factor 1 is taken, do nothing; if $\mu\delta_{s_i,s_j}$ is taken, put a bond on $\langle i, j \rangle$.

$\delta_{s_i,s_j} \Rightarrow$ Each component is of a unique color

Z becomes

$$Z = \sum_{G \in \mathcal{G}_0} \mu^{b(G)} 2^{k(G)}$$

$-b$: bond number, k: component number

Random-cluster model:

$$Z(q,\mu) = \sum_{G \in \mathcal{G}_0} \mu^{b(G)} q^{k(G)}$$

Some special cases:

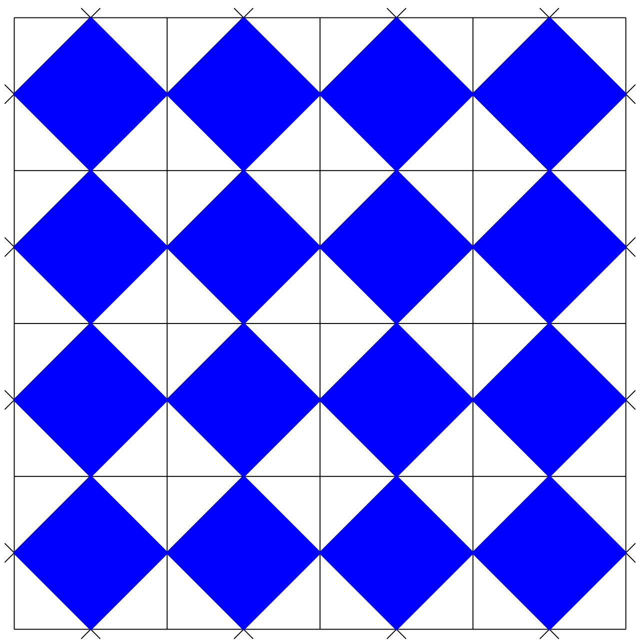
- $q \to 0$: spanning tree or spanning forests
- $\mu = -1$: Chromatic polynomial (such as The Four Color Prolem)

Monte Carlo Methods:

- Sweeny algorithm
- Swendsen-Wang-Chayes-Machta Algorithm

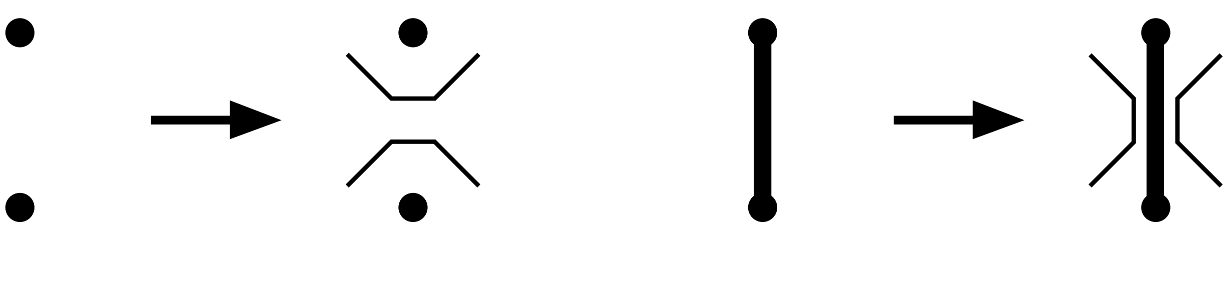
Baxter-Kelland-Wu (BKW) mapping of the random-cluster model

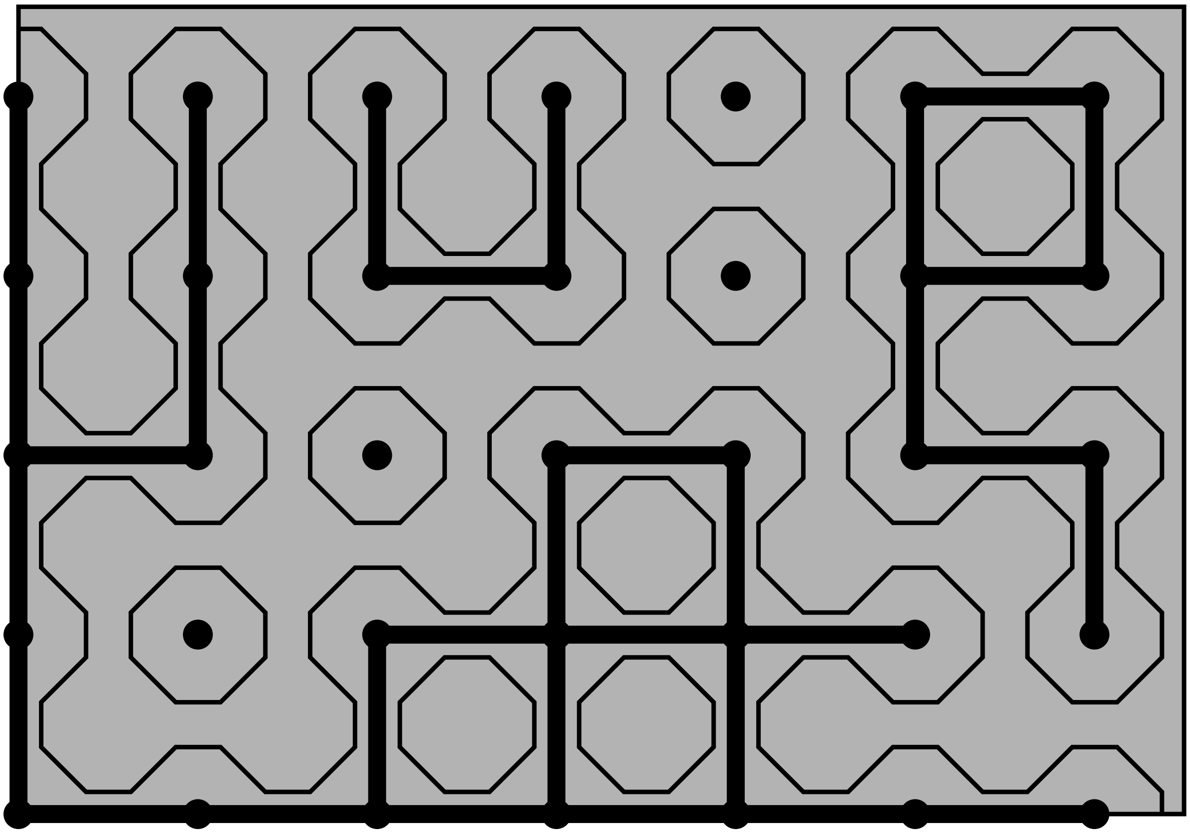
Medial Graph : $G \to \tilde{G}$



$\tilde{G}$–Water: blue area; Island: white

Operation:





Random-Cluster (RC) Model:

$$Z(q, \mu) = \sum_{G \in \mathcal{G}_0} \mu^{b(G)} (\sqrt{q})^{c(G)}$$

– $c(G)$: loop number

Application: Map the RC model onto the 8-vertex model.

Recent Development: Stochastic Lowner Evolution (SLE), by Oded Schramm (1999).

In the scaling limit, critical systems are not only *scale invariant*, but also conformally invariant.

In the scaling limit, BKW curves with appropriate boundary conditions are described by SEL_κ in Z^+:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}\,, g_0(z) = z\ ,$$

— $B_t, t \in [0, \infty)$: Brownian motion.