Finite-size scaling of energylike quantities in percolation

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(Received 22 January 2006; published 13 June 2006)

We study the bond-percolation model in two and three dimensions by Monte Carlo simulation, and investigate the finite-size scaling behavior of several quantities that account for fluctuations of the total numbers of clusters and occupied bonds, N_c and N_b, respectively. These quantities include $C_{2c}=(\langle N_c^2\rangle-\langle N_c\rangle^2)/L^d$ and $C_{cb}=(\langle N_cN_b\rangle-\langle N_c\rangle\langle N_b\rangle)/L^d$, where L is the linear system size and d is the spatial dimensionality. In statistical models with thermal fluctuations, C_{2c} and C_{cb} are specific heatlike quantities. Despite the absence of thermal fluctuations in percolation, we find that the leading finite-size scaling of C_{2c} and C_{cb} is described by the thermal critical exponent y_t-d. We also measure quantity $\kappa_b=2\langle N_bS_2\rangle/\langle S_2\rangle-\langle N_b\widetilde{S}_4\rangle/\langle\widetilde{S}_4\rangle-\langle N_b\rangle$ and an analogous quantity κ_c for N_c, where S_2 and $\widetilde{S}_4$ are quantities associated with the second and the fourth moments of cluster sizes, respectively. At criticality, we show that κ_b and κ_c diverge as L^{y_t} for $L\to\infty$. The analysis of the data of κ_b and κ_c yields $y_t=1.145(2)$ for the three-dimensional percolation, in good agreement with existing results.

DOI: 10.1103/PhysRevE.73.066116 PACS number(s): 05.50.+q, 64.60.Cn, 64.60.Fr, 75.10.Hk

I. INTRODUCTION

Percolation problems have been of great research interest to physicists and mathematicians [1], and a variety of applications has been reported [2]. Even after decades of investigation, it still continues as an active research field [3–12]. In this work, we restrict ourselves to properties of energylike quantities in the bond-percolation model.

The bond percolation on a regular lattice is defined as follows. For each edge one places a bond with probability $0\leqslant p\leqslant 1$; otherwise, the edge is empty. Two sites connected through a chain of occupied bonds are said to percolate, i.e., to be in the same cluster. Then various questions can be asked concerning the distribution of cluster sizes and the fractal dimension of clusters, etc.

It is fascinating that the bond percolation can be generalized to an infinite-range universality class of model: The random-cluster representation of the q-state Potts model [13,14]. The pertinent clusters are generally referred to as the Kasteleyn-Fortuin (KF) clusters [15,16]. The partition sum of the random-cluster model reads

$$Z(q,u)=\sum_b u^{N_b}q^{N_c}. \tag{1}$$

The sum is over all bond-variable configurations, and N_b and N_c are the total numbers of bonds and KF clusters, respectively. The weight for an occupied bond u relates to probability p as $u=p/(1-p)$. For the integer $q>1$, by introducing a spin variable $\sigma=0,1,\ldots,q-1$ for each lattice site and then summing up all the bond variables, one can exactly transform model (1) into the standard q-state Potts model; the coupling strength between a pair of nearest-neighbor equivalent Potts variables is $J=\ln(u+1)$. For $q\to 1$, the random-cluster model reduces to the bond-percolation model. In this case, the partition sum (1) is just constant, but its derivative with respect to physical parameters like q and u can still yield many intriguing phenomena; see, e.g., Refs. [17–19].

As a consequence of the mapping between the Potts model and the random-cluster model, much of the knowledge that has been gathered for the Potts model is directly applicable to percolation. For instance, the fractal dimension of percolation clusters can be exactly related to the magnetic renormalization exponent of the Potts model for $q\to 1$. In two dimensions, owing to the Coulomb gas theory [20] and the conformal field theory [21], it is known that the thermal and magnetic scaling dimensions of the percolation are $X_t=5/4$ and $X_h=5/48$, respectively.

For spatial dimensions $d>2$, exact results are scarce. Thus, approximation techniques, such as series expansions and Monte Carlo simulations, play an important role in the study of percolation and other statistical models. Analysis of the numerical data then heavily relies on the finite-size scaling formula of the free-energy density $f=\ln Z/L^d$, which reads near criticality

$$f(t,h,L)=L^{-d}f_s(tL^{y_t},hL^{y_h})+f_a(t,h). \tag{2}$$

Here, we have assumed that the total number of sites is L^d, with L being the linear system size. The symbols t and h, respectively, represent the thermal and the magnetic scaling fields, and their renormalization exponents are denoted by $y_t=d-X_t$ and $y_h=d-X_h$. The terms f_s and f_a represent the singular and analytical parts of the free energy density, respectively.

On this basis, scaling behavior of most physical quantities can be derived by differentiating Eq. (2) with respect to appropriate scaling fields. For instance, it can be shown that, for statistical models with thermal fluctuations, the bond-occupation density $n_b=N_b/L^d$ in Eq. (1) is an energylike quantity, and behaves as $n_b=n_{b0}+bL^{y_t-d}$ at criticality, where n_{b0} and b are constant. The specific heatlike quantity $C_{2b}=L^d(\langle n_b^2\rangle-\langle n_b\rangle^2)$, which accounts for the fluctuations of n_b, has the leading finite-size term proportional to L^{2y_t-d}. In the thermodynamic limit $L\to\infty$, whether C_{2b} diverges depends on the sign of the exponent $2y_t$-d. For the percolation

model, however, bonds at different sites are uncorrelated, and thus the terms with L^{y_t-d} and L^{2y_t-d} vanish. No information about y_t can be derived from the behavior of n_b and C_{2b}. Therefore, in order to determine the thermal exponent y_t, one normally has to consider how the scaling behavior of other quantities (particularly those magnetic ones) varies when probability p deviates from the critical value.

In this work, we reexamine the finite-size behavior of several quantities in percolation that are related to the second derivative of the free-energy density with respect to the thermal scaling field (they will be defined in Sec. II). We will show that despite the absence of thermal fluctuations in percolation the thermal exponent y_t still appears in the finite-size scaling of these quantities. We also define a quantity which can serve as a good candidate to determine y_t from simulations precisely at criticality.

II. SAMPLED QUANTITIES

The bond percolation involves the decomposition of the whole lattice into clusters. The size of a cluster c_i can be counted as the total number sites or occupied bonds in the cluster; here, we shall use the former. From the distribution of c_i, one can define various magnetic-like quantities. In particular, the second and the fourth moments of the cluster sizes read

$$S_2 = \frac{1}{L^{2d}} \sum_{i=1}^{N_c} c_i^2 \quad \text{and} \quad S_4 = \frac{1}{L^{4d}} \sum_{i=1}^{N_c} c_i^4, \tag{3}$$

where N_c is the total number of clusters, as mentioned in Eq. (1). For the Potts model, the quantity $L^d S_2$ is equivalent to the magnetic susceptibility, and thus has the scaling exponent $2y_h-d$ at criticality. For the Ising model ($q=2$), the quantity $\tilde{S}_4=3\langle S_2^2\rangle-2\langle S_4\rangle$ equals to the fourth moment of magnetization m^4. Further, on the basis of S_2 and S_4, various dimensionless ratios can be defined, such as

$$Q = \frac{\langle S_2\rangle^2}{3\langle S_2^2\rangle - 2\langle S_4\rangle} \equiv \frac{\langle S_2\rangle^2}{\langle \tilde{S}_4\rangle}. \tag{4}$$

Note that any linear combination of $\langle S_2^2\rangle$ and $\langle S_4\rangle$ can in principle be used in the denominator of Eq. (4). All these ratios are universal at criticality. It is known that the dimensionless ratio Q is a very good candidate to locate phase transitions in Monte Carlo studies, see, e.g., Ref. [12] and references therein.

In the random-cluster model (1) for $q\neq 1$, the density of cluster number $n_c=N_c/L^d$ and the bond-occupation density $n_b=N_b/L^d$ are energylike quantities,[1] since the leading scaling behavior of n_c and n_b is governed by the thermal renormalization exponent y_t. In the bond percolation, however, the values of n_b and its fluctuations can be easily calculated and do not contain information for critical behavior. For instance, quantity C_{2b}, defined in Sec. I is equal to $zp(1-p)/4$, with z the coordination number of the lattice.

[1]In some of the literature, the bond density is defined as the number of occupied bonds per edge.

In the past decade, the cluster-number density n_c has received much research attention [4–6,10], in particular after the publication of Ref. [4]. It is known that, at criticality, the finite-size behavior of n_c reads $n_c=n_{c0}+b/L^d+\cdots$, where n_{c0} and b are constants and the "$\cdots$" represents higher-correction terms. It is further found that the correction-term amplitude b is universal. Using the conformal field theory and the Temperly-Lieb formula [22], exact values have been calculated for n_{c0} and b for several two-dimensional lattices for both the site- and bond-percolation model [4,5]. Recently, the exact value of n_c in the thermodynamic limit was calculated as a function of the bond probability p on infinitely long strips for various two-dimensional lattices [10].

In work by Kleban and Ziff [5], a set of cumulants C_n was studied for the critical bond percolation on the square lattice, as defined in the following. At criticality $u=u_c(q)$, the partition sum (1) for the random-cluster model can be rewritten as

$$Z_c(q) = \sum_b u_c(q)^{N_b} q^{N_c}. \tag{5}$$

The cumulants C_n are then obtained by differentiating the free energy $F=\ln Z$ with respect to parameter q, and read

$$C_n = \left(q\frac{d}{dq}\right)^n F_c(q). \tag{6}$$

In particular, C_1 and C_2 are, respectively,

$$C_1 = \langle N_c + gN_b\rangle \quad \text{and}$$
$$C_2 = L^{-d}[\langle (N_c+gN_b)^2\rangle - \langle N_c+gN_b\rangle^2], \tag{7}$$

where $g=qu_c'(q)/u_c(q)$ describes the slope of the critical line $u_c(q)$ as a function of q. For the random-cluster model on the square lattice, one has $u_c(q)=\sqrt{q}$ [13,14] and thus $g=1/2$.

At the critical point, the partition sum is supposed to factorize as $Z=Z_nZ_u$ [23], where Z_n is a nonuniversal factor, depending on lattice types, boundary conditions, etc., while Z_u encodes the universal information. It is then expected that, at criticality, the finite-size scaling behavior of C_n reads

$$C_n = c_{n0} + c_{n1}L^{-d} + \cdots, \tag{8}$$

and amplitudes c_{n1} are universal. This was already confirmed in Ref. [5] for percolation and in Ref. [24] for the general q-state Potts model. Several exact values of c_{n1} were calculated [10] for the percolation on the $L\times L$ torus geometry and on the infinitely long cylinder.

The cumulant C_2 can be divided into three parts as

$$C_2 = L^{-d}(\langle N_c^2\rangle - \langle N_c\rangle^2) + 2gL^{-d}(\langle N_cN_b\rangle - \langle N_c\rangle\langle N_b\rangle) + g^2L^{-d}(\langle N_b^2\rangle - \langle N_b\rangle^2) \equiv C_{2c} + 2gC_{cb} + g^2C_{2b}. \tag{9}$$

For $q\neq 1$, C_{2c}, C_{cb}, and C_{2b} are all specific heatlike quantities, and the finite-size scaling is governed by the exponent $2y_t-d$, while C_2 is described by Eq. (8). For percolation, C_{2b} does not have finite-size dependence, and is equal to $z^2p(1-p)/4$, as mentioned earlier. Since thermal fluctuations are absent for percolation, one would simply expect that the scaling behavior of C_{2c} and C_{cb}, as well as C_2, is all described by Eq. (8). Such an expectation is, however, incorrect. Our numerical data suggest that the leading finite-size

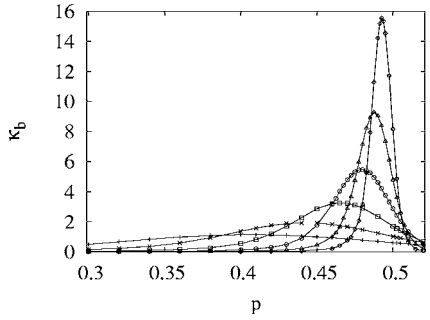

FIG. 1. Quantity κ_b for the bond percolation on the square lattice vs probability p. The data points +, ×, □, ○, △, and ◇ represent system sizes $L=8$, 16, 32, 64, 128, and 256, respectively.

behavior of C_{2c} and C_{cb} at criticality is governed by exponent y_t-d. This will be presented in detail later.

As a result, even for percolation, the numerical data C_{2c} and C_{cb} at criticality will allow one to extract information for y_t. Nevertheless, from the practical point of view, C_{2c} and C_{cb} are not good candidates for determining y_t, because they rapidly converge to constants.

In the study of the three-dimensional Ising model, a quantity $\kappa_p = L^d(2\langle em^2\rangle/\langle m^2\rangle - \langle em^4\rangle/\langle m^4\rangle - \langle e\rangle)$ was sampled in Refs. [25,26], where e and m are the energy and the magnetization density, respectively. It was shown that κ_p can be derived by differentiating the ratio $Q=\langle m^2\rangle^2/\langle m^4\rangle$ with respect to the thermal scaling field. The value of κ_p is proportional to the slope of Q at the critical point. It diverges as L^{y_t} in the thermodynamic limit. In comparison with the specific heat whose finite-size exponent $2y_t$-d is close to zero in many cases, it was found that κ_p is a rather good candidate for determining y_t.

Following a similar approach, we define for percolation

$$\kappa_b = 2\frac{\langle N_b S_2\rangle}{\langle S_2\rangle} - \frac{\langle N_b \tilde{S}_4\rangle}{\langle \tilde{S}_4\rangle} - \langle N_b\rangle, \tag{10}$$

on the basis of the dimensional ratio (4). Analogous quantities κ_c and κ_1 are defined for N_c and for $C_1 = N_c + gN_b$, respectively.

From the behavior of C_2, C_{2c}, and C_{cb}, we expect that there exist nontrivial correlations functions $\langle S_2 N_c\rangle$ and

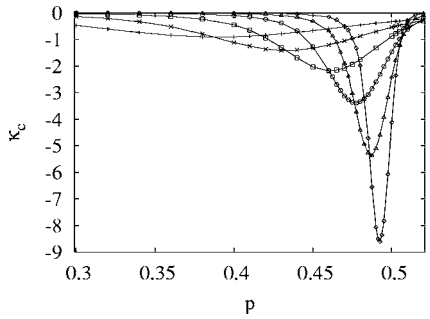

FIG. 2. Quantity κ_c for the bond percolation on the square lattice vs probability p. The data points +, ×, □, ○, △, and ◇ represent system sizes $L=8$, 16, 32, 64, 128, and 256, respectively.

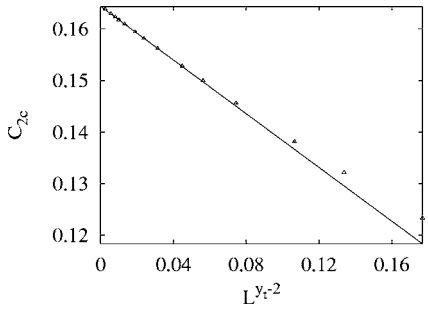

FIG. 3. Quantity C_{2c} for the square-lattice percolation at criticality $p_c=1/2$ vs $L^{y_t-2}=L^{-5/4}$. The statistical error bars are smaller than the size of data points.

$\langle S_2 N_b\rangle$. Thus, as in the Ising case, quantities κ_b and κ_c diverge as L^{y_t} for $L\to\infty$. On the other hand, according to the behavior of C_n, quantity κ_1 may converge to a constant, still with correction exponent −2.

III. TWO DIMENSIONS

In two dimensions, the nature of phase transition in percolation models is now well established. The thermal and magnetic exponents are $y_t=3/4$ and $y_h=91/48$, respectively, and the exponent for the least irrelevant scaling field is $y_i=-2$ [20,21]. Nevertheless, there still exist critical exponents whose exact values are unknown, such as the backbone exponent [27].

The critical points of site- and bond-percolation models on several regular lattices are also exactly known; for a review, see, e.g., Ref. [28]. These exact results can be obtained from dual symmetries and matching features, from the site-to-bond transformation, and from the star-triangular transformation.

In this work, we choose the bond-percolation model on the $L\times L$ square lattice with periodic boundary conditions in both directions. The self-duality of the square lattice yields the critical bond-occupation probability as $p_c=1/2$.

In comparison with other quantities such as n_c and S_2, quantities κ_b and κ_c were less well studied in the literature for percolation problems. Thus, as an illustration, we performed some simulations in the range $0.3\leqslant p\leqslant 0.52$ for system sizes $L=2^3$, 2^4, 2^5, 2^6, 2^7, and 2^8. The data of κ_b and κ_c

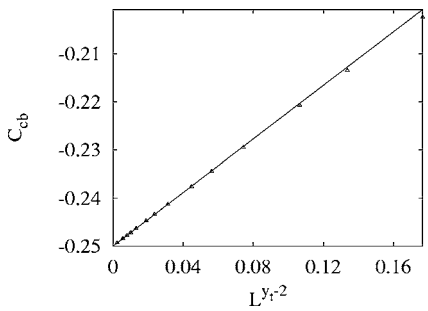

FIG. 4. Quantity C_{cb} for the square-lattice percolation at criticality $p_c=1/2$ vs $L^{y_t-2}=L^{-5/4}$. The statistical error bars are smaller than the size of data points.

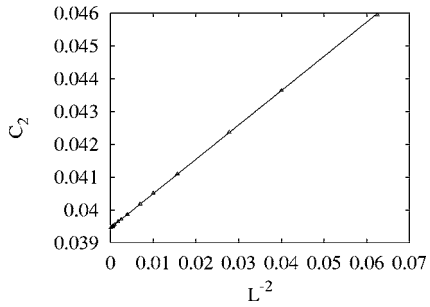

FIG. 5. Quantity C_2 for the square-lattice percolation at criticality $p_c=1/2$ vs L^{-2}. The statistical error bars are smaller than the size of data points.

are shown in Figs. 1 and 2, respectively. These figures suggest that the behavior of κ_b and κ_c resembles that of the specific heat in systems with thermal fluctuations.

Next, we performed extensive simulations at criticality $p_c=1/2$. The system took 14 values in the range $4\leqslant L\leqslant 128$. The number of samples for each system size is about 3×10^7. The data of C_{2c}, C_{cb}, and C_2 are shown in Figs. 3–5 respectively. The approximately straight lines in these figures indicate that the leading correction exponent for C_{2c} and C_{cb} is indeed y_t-d, while that of C_2 is -2. The data of κ_c and κ_b are shown in Fig. 6, and those of κ_1 are in Fig. 7. Again, these data confirm our expectation that, in the thermodynamic limit, κ_c and κ_b diverge as L^{y_t}, while κ_1 converges to a constant with correction exponent -2.

According to the least-squares criterion, we fitted the κ_c and κ_b data by

$$\kappa_j(L) = \kappa_{j0} + \kappa_{j1}L^{-d} + L^{y_t}(a_j + b_jL^{y_i}), \tag{11}$$

where we have used a single symbol κ_j to represent κ_c and κ_b. Terms with κ_{j0} account for the analytical contributions, and those with k_{j1} are from Eq. (8). The exponent y_i for the leading irrelevant scaling field was fixed at -2. The data for $L\geqslant 8$ are well described by Eq. (11). The fits of κ_c and κ_b yielded $y_t=0.753(4)$ and $a_c=-0.0618(5)$, and $y_t=0.752(4)$ and $a_b=0.1244(12)$, respectively. As expected, one has $a_b+2a_c=0$, so that the terms with L^{y_t} vanish for quantity κ_1 (κ_1 is defined in a similar form as Eq. (10) with N_b replaced by C_1). The estimate of y_t agrees well with the exact

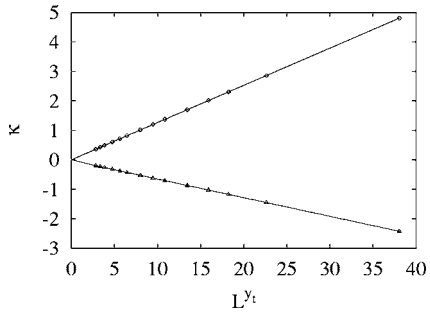

FIG. 6. κ_b and κ_c for the square-lattice percolation at criticality $p_c=1/2$ vs L^{y_t}. The thermal exponent is set at $y_t=3/4$. The data points $\triangle$ and $\diamond$ are for κ_c and κ_b, respectively. The statistical error bars are smaller than the size of data points.

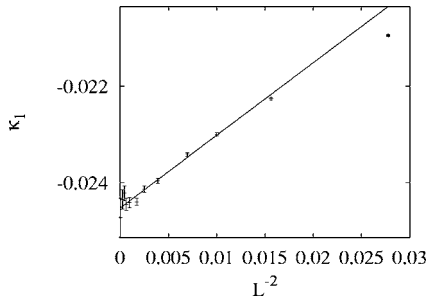

FIG. 7. κ_1 for the square-lattice percolation at criticality $p_c=1/2$ vs L^{-2}.

value $y_t=3/4$. The data of κ_1 for $L\geqslant 6$ were fitted by $\kappa_1(L)=\kappa_{10}+\kappa_{11}L^{-2}+\kappa_{11}L^{-3}$. We obtain $\kappa_{10}=-0.02456(8)$, $\kappa_{10}=0.170(8)$, and $\kappa_{20}=-1.4(3)$.

The data of C_{2c} and C_{cb} were fitted by

$$C_j(L) = c_{j0} + c_{j1}L^{-d} + L^{y_t-d}(a_j + b_jL^{y_i}), \tag{12}$$

where, as in Eq. (11), both C_{2c} and C_{cb} are represented by the symbol C_j. We determine $y_t=0.746(7)\approx 3/4$ from C_{2c} and $y_t=0.748(8)\approx 3/4$ from C_{cb}. The values of other parameters are $c_{2c0}=0.16445(2)$ and $c_{cb0}=-0.25000(4)$, $c_{2c1}=0.11(1)$ and $c_{cb1}=0.01(2)$, and $a_{2c}=-0.279(2)$ and $a_{cb}=-0.280(2)$. Again, one has $a_{2c}+a_{cb}=0$, so that $C_2=c_{20}+c_{21}L^{-2}+\cdots$. The analysis of the C_2 data yields $c_{20}=0.039\,44(4)$ and $c_{21}=0.1053(4)$. One has $c_{20}=c_{2c0}+2gc_{cb0}+g^2c_{2b}$, where $g=1/2$ and $c_{2b}=1/2$. The estimate of $c_{21}=0.1053(4)$ is in good agreement with the exact result $0.105\,437\ldots$ [5].

A. Theoretical explanation of the exponent y_t-d for C_{2c} and C_{cb}

Since thermal fluctuations are absent in percolation, the existence of the exponent y_t-d for C_{2c} and C_{cb} is somewhat unexpected; in particular, the thermal exponent y_t is known to be absent in the finite-size behavior of C_n for any n.

To see where the exponent y_t can arise from, we rewrite the finite-scaling formula of the free energy density described by Eq. (2) as

$$f(u,q,L) = A(q)L^{-d}f_s\{[u-u_c(q)]L^{y_t}\}, \tag{13}$$

where we have neglected the analytical terms and those related to the magnetic and irrelevant scaling fields. The pa-

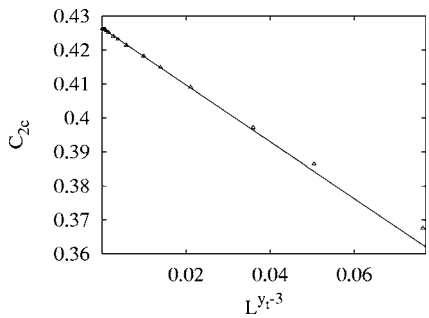

FIG. 8. Quantity C_{2c} for the critical bond percolation on the simple-cubic lattice vs L^{y_t-2}. The value of y_t was fixed at 1.145. The statistical error bars are smaller than the size of data points.

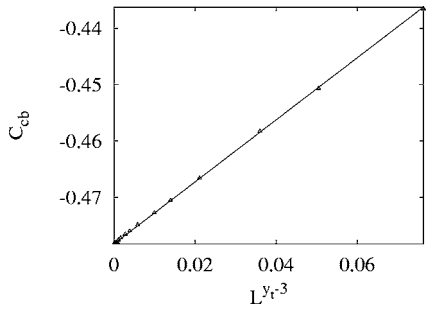

FIG. 9. Quantity C_{cb} for the critical bond percolation on the simple-cubic lattice vs L^{y_t-2}. The value of y_t was fixed at 1.145. The statistical error bars are smaller than the size of data points.

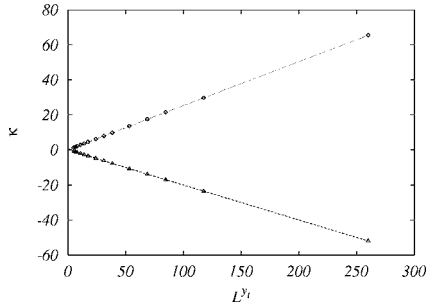

FIG. 10. κ_b and κ_c for the critical bond percolation on the simple-cubic lattice vs L^{y_t}. The thermal exponent is set at $y_t=1.145$. The data points $\triangle$ and $\diamond$ are for κ_c and κ_b, respectively. The statistical error bars are smaller than the size of data points.

rameter $A(q)$ is zero for $q=1$. At the critical point $u=u_c$, the first and the second derivative of Eq. (13) with respect to q can be expressed as

$$C_1(L) \equiv f'(L)_{q=1} = A'_{q=1}L^{-d}f_s(0) + A_{q=1}L^{y_t-d}f'_s(0) \quad (14)$$

and

$$C_2(L) \equiv f''(L)_{q=1} = A''_{q=1}L^{-d}f_s(0) + A'_{q=1}L^{y_t-d}f'_s(0) + A_{q=1}L^{2y_t-d}f''_s(0), \quad (15)$$

respectively. The scaling behavior of C_n indicates that $A^{(n)}(q=1)=0$. However, just as the division of C_2 into C_{2c}, C_{cb}, and C_{2b}, it is plausible that $A^{(n)}$ can correspondingly be divided into several nonzero terms. Such a scenario is in agreement with our numerical data.

IV. THREE DIMENSIONS

Exact information about critical phenomena is scarce in three-dimensional statistical models. Most of the results about critical exponents and locations of phase transitions are obtained from approximation techniques. For the bond- and site-percolation models on the simple-cubic lattice, the critical points have been located as $p_c=0.248\,812\,6(5)$ [29] and $p_c=0.311\,607\,7(4)$ [12,30], respectively. The thermal and magnetic exponents are determined [12,30,31] as $y_t=1.1450(7)$ and $y_h=2.5219(2)$. In addition to the critical exponents and amplitude ratios, it is known that the wrapping the crossing probabilities are also universal [6].

This work chose the bond-percolation model on the $L\times L\times L$ simple-cubic lattice; periodic boundary conditions were imposed. Simulations were performed precisely at the estimated critical point $p_c=0.248\,812\,6$, and the precision of error margin is considered to be sufficient for the present purpose. The system sizes took 14 values in range $4\leqslant L\leqslant 128$. For each size, the number of samples is about 2×10^7.

We sampled various quantities, including n_c, C_{2c}, C_{cb}, κ_c, and κ_b etc. Not that we could not measure quantity C_n and κ_1, because no exact value of g is known. The data of C_{2c} and C_{cb} are shown in Figs. 8 and 9 respectively, and those of κ_c and κ_b are in Fig. 10. These approximately straight data lines confirm that the leading finite-scaling behavior of C_{2c} and C_{cb} is governed by exponent y_t-d, and that of κ_c and κ_b is by y_t.

The data of κ_c and κ_b were fitted by Eq. (11), and those of C_{2c} and C_{cb} were fitted by Eq. (12). The exponent for the leading irrelevant scaling field was fixed at $y_i=-1.2(2)$ [29]. The data for small system sizes $L\leqslant 5$ were excluded from the fits. The fit of C_{2c} yields $y_t=1.20(6)$, $c_{2c0}=0.4266(2)$, and $a_{2c}=-0.68(7)$; that of C_{cb} gives $y_t=1.21(8)$, $c_{cb0}=-0.4783(2)$, and $a_{cb}=0.40(7)$. From the data of κ_c and κ_b, we obtain $y_t=1.145(4)$ and $a_c=-0.200(3)$, and $y_t=1.145(2)$ and $a_b=0.253(2)$. The estimates of y_t are all in good agreement with the existing result $y_t=1.1450(7)$ [12].

Since the finite-size corrections of cumulants C_2 defined in Eq. (6) are governed by the exponent $-d$, the terms with L^{y_t-d} in C_{2c} and C_{cb} must compensate with each other; the same argument is also expected for κ_c and κ_b. This allows us to estimate the parameter $g=q[\ln u_c(q)]'$. We then have $g=-a_{2c}/2a_{cb}=0.85(17)$ and $g=-a_c/a_b=0.79(2)$; they agree with each other.

V. DISCUSSION

We investigate energylike quantities, such as C_2, C_{2c}, and C_{cb}, in the two- and three-dimensional bond-percolation model. Even though no thermal fluctuation is present in percolation (e.g., the dynamical exponent z is always zero for any Monte Carlo method), the finite-size scaling behavior of C_{2c} and C_{cb} is governed by the exponent y_t-d. The terms with L^{y_t-d} in C_{2c} and C_{cb} compensate with each other, such that the behavior of C_2 is described by Eq. (12).

For the random-cluster model for $q\neq 1$, although the leading finite-size scaling behavior of C_{2c}, C_{cb}, and C_{2b} is governed by $2y_t$-d, our results for percolation indicate that terms with L^{y_t-d} should also exist. We further show that, at criticality, quantities κ_c and κ_b diverge as L^{y_t} for $L\to\infty$. They can serve as good candidates to determine the thermal exponent y_t in Monte Carlo studies of percolation problems.

ACKNOWLEDGMENTS

One of us (Y.D.) is indebted to Jouke R. Heringa, Henk W.J. Blöte, and Bob Ziff for valuable discussions. This research was supported in part by U.S. National Science Foundation Grant No. PHY-0424082.

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[31] In Ref. [12], a typo exists in the abstract where the critical exponents were written as $y_t=1.1437(6)$ and $y_h=2.5219(2)$. The correct answers should be $y_t=1.1450(7)$ and $y_h=2.5226(1)$, as in Table III. Further, the value 1.030(6) at the fourth column in Table III should be at the last column on the same line.