Arithmetic Purity for Strong Approximation

Yongqi LIANG

Université Paris Diderot - Paris 7, France
joint work with
Yang CAO - Université Paris 11, Orsay France
Fei XU - Capital Normal University, Beijing China

17 May 2017

Emory University, Atlanta U.S.

Notation

- k : number field
- $\Omega_k = \Omega_k^f \sqcup \infty_k$ set of places
- k_v for $v \in \Omega_k$
- $\mathcal{O}_v \subset k_v$ for $v \in \Omega_k^f$
- $\mathbf{A}_k$ ring of adèles
- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles *without* S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ natural projection
- X : smooth variety over k (variety = separated scheme of finite type, geometrically integral)
- $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ the cohomological Brauer group

Weak approximation

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U

Weak approximation

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U

Weak approximation

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} U(k_v)$ with $M_v \subset U(k_v)$ open,

Weak approximation

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} U(k_v)$ with $M_v \subset U(k_v)$ open,
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} X(k_v)$, $M_v \subset X(k_v)$ open

Weak approximation

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} U(k_v)$ with $M_v \subset U(k_v)$ open,
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} X(k_v)$, $M_v \subset X(k_v)$ open
  $\exists x \in X(k) \cap [\prod_{S_0} M_v \times \prod_{v\notin S_0} X(k_v)]$

Weak approximation

- $X(k) \hookrightarrow \prod_{v\in\Omega} X(k_v)$ diagonally
- Weak approximation holds if $X(k)$ is dense w.r.t. product topology
- $\emptyset \neq U \subset X$ Zariski open
- weak approximation on $X \implies$ weak approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} U(k_v)$ with $M_v \subset U(k_v)$ open,
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} X(k_v)$, $M_v \subset X(k_v)$ open
  $\exists x \in X(k) \cap [\prod_{S_0} M_v \times \prod_{v\notin S_0} X(k_v)]$
  $\implies x \in U(k) \cap [\prod_{S_0} M_v \times \prod_{v\notin S_0} U(k_v)]$, don't need to care about $v \notin S_0$

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

Strong approximation

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

Strong approximation

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{U}(\mathcal{O}_v)$ with $M_v \subset U(k_v)$ open
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{U}(\mathcal{O}_v)$ with $M_v \subset U(k_v)$ open
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{X}(\mathcal{O}_v)$ with $M_v \subset X(k_v)$ open
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

Strong approximation

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{U}(\mathcal{O}_v)$ with $M_v \subset U(k_v)$ open
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{X}(\mathcal{O}_v)$ with $M_v \subset X(k_v)$ open
  $\exists x \in X(k) \cap [\prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{X}(\mathcal{O}_v)]$
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on X $\not\Rightarrow$ strong approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} \mathcal{U}(\mathcal{O}_v)$ with $M_v \subset U(k_v)$ open
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v\notin S_0} \mathcal{X}(\mathcal{O}_v)$ with $M_v \subset X(k_v)$ open
  $\exists x \in X(k) \cap [\prod_{S_0} M_v \times \prod_{v\notin S_0} \mathcal{X}(\mathcal{O}_v)]$
  $\implies x \in U(k)$, but $\not\Rightarrow x \in \mathcal{U}(\mathcal{O}_v)$ for $v \notin S_0$
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

- $X(k) \hookrightarrow X(\mathbf{A}_k^S)$ diagonally
- Strong approximation off S holds if $X(k)$ is dense w.r.t. *adélic topology*
- subtle difference between product topology and adélic topology:
  - strong approximation on $X \not\Rightarrow$ strong approximation on U
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{U}(\mathcal{O}_v)$ with $M_v \subset U(k_v)$ open
  $(u_v)_v \in \prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{X}(\mathcal{O}_v)$ with $M_v \subset X(k_v)$ open
  $\exists x \in X(k) \cap [\prod_{S_0} M_v \times \prod_{v \notin S_0} \mathcal{X}(\mathcal{O}_v)]$
  $\Longrightarrow x \in U(k)$, but $\not\Rightarrow x \in \mathcal{U}(\mathcal{O}_v)$ for $v \notin S_0$
- Example: $k = \mathbb{Q}$, $S \neq \emptyset$, $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m$
  X satisfies strong approximation off S
  U does not satisfy strong approximation off S

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)

1. étale fundamental groups
2. Brauer groups

- in such a case

Strong approximation

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
2. Brauer groups

- in such a case

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups

- in such a case

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$

Theorem (Minchev)

Let V be a variety defined over a number field k. If $V_{\bar{k}}$ is not simply connected $\pi_1^{\text{ét}}(V_{\bar{k}}) \neq 0$, then V can never satisfy strong approximation.

2. Brauer groups

- in such a case

- Why? $(X = \mathbb{A}^1,\ U = \mathbb{G}_m)$
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$

Theorem (Minchev)

Let V be a variety defined over a number field k. If $V_{\bar{k}}$ is not simply connected $\pi_1^{\text{ét}}(V_{\bar{k}}) \neq 0$, then V can never satisfy strong approximation.

2. Brauer groups

- in such a case

- Why? $(X = \mathbb{A}^1,\ U = \mathbb{G}_m)$
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite

- in such a case

- Why? $(X = \mathbb{A}^1,\ U = \mathbb{G}_m)$
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite
   - Brauer-Manin pairing: $V(\mathbf{A}_k) \times \mathrm{Br}(V) \to \mathbb{Q}/\mathbb{Z}$
   - $V(k) \subset \overline{V(k)} \subset V(\mathbf{A}_k)^{\mathrm{Br}} \subset V(\mathbf{A}_k)$

- in such a case

- Why? $(X = \mathbb{A}^1,\ U = \mathbb{G}_m)$
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite
   - Brauer-Manin pairing: $V(\mathbf{A}_k) \times \mathrm{Br}(V) \to \mathbb{Q}/\mathbb{Z}$
   - $V(k) \subset \overline{V(k)} \subset V(\mathbf{A}_k)^{\mathrm{Br}} \subset V(\mathbf{A}_k)$
   $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)$ while $U(\mathbf{A}_k)^{\mathrm{Br}} \subsetneq U(\mathbf{A}_k)$

- in such a case

- Why? $(X = \mathbb{A}^1,\ U = \mathbb{G}_m)$
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite
   - Brauer-Manin pairing: $V(\mathbf{A}_k) \times \mathrm{Br}(V) \to \mathbb{Q}/\mathbb{Z}$
   - $V(k) \subset \overline{V(k)} \subset V(\mathbf{A}_k)^{\mathrm{Br}} \subset V(\mathbf{A}_k)$
   $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)$ while $U(\mathbf{A}_k)^{\mathrm{Br}} \subsetneq U(\mathbf{A}_k)$

- in such a case

Strong approximation

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite

Question (Wittenberg 2014)

What happens if $Z = X \setminus U$ is of codimension ≥ 2 ?

- in such a case

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite

Question (Wittenberg 2014)

What happens if $Z = X \setminus U$ is of codimension ≥ 2 ?

- in such a case
  - Zariski-Nagata: $\pi_1^{\text{ét}}(X_{\bar{k}}) = \pi_1^{\text{ét}}(U_{\bar{k}})$

- Why? ($X = \mathbb{A}^1$, $U = \mathbb{G}_m$)
  Apart from the subtle adélic topology, two more reasons:

1. étale fundamental groups
   $\pi_1^{\text{ét}}(X_{\bar{k}}) = 0$ while $\pi_1^{\text{ét}}(U_{\bar{k}}) = \hat{\mathbb{Z}}$
2. Brauer groups
   $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$ while $\mathrm{Br}_1(U)/\mathrm{Br}(k) \simeq \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ is infinite

Question (Wittenberg 2014)

What happens if $Z = X \setminus U$ is of codimension ≥ 2 ?

- in such a case
  - Zariski-Nagata: $\pi_1^{\text{ét}}(X_{\bar{k}}) = \pi_1^{\text{ét}}(U_{\bar{k}})$
  - purity for étale cohomology: $\mathrm{Br}(X) = \mathrm{Br}(U)$

First example: the affine space

- $X = \mathbb{A}^n$ satisfies strong approximation off $S \neq \emptyset$
- What about the case when X is a semi-simple simply connected group?
- in these cases $\dfrac{\mathrm{Br}(X)}{\mathrm{Br}(k)} = \dfrac{\mathrm{Br}(X \setminus Z)}{\mathrm{Br}(k)} = 0$
- in general, should take into account the Brauer-Manin obstruction

First example: the affine space

- $X = \mathbb{A}^n$ satisfies strong approximation off $S \neq \emptyset$

Theorem (D. Wei; Y. Cao & F. Xu)

Let Z be a Zariski closed subset of $\mathbb{A}^n$ such that $\operatorname{codim}(Z, X) \geq 2$. Then $\mathbb{A}^n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

- What about the case when X is a semi-simple simply connected group?
- in these cases $\dfrac{\mathrm{Br}(X)}{\mathrm{Br}(k)} = \dfrac{\mathrm{Br}(X \setminus Z)}{\mathrm{Br}(k)} = 0$
- in general, should take into account the Brauer-Manin obstruction

- $X = \mathbb{A}^n$ satisfies strong approximation off $S \neq \emptyset$

Theorem (D. Wei; Y. Cao & F. Xu)

Let Z be a Zariski closed subset of $\mathbb{A}^n$ such that $\operatorname{codim}(Z, X) \geq 2$. Then $\mathbb{A}^n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

- What about the case when X is a semi-simple simply connected group?
- in these cases $\dfrac{\mathrm{Br}(X)}{\mathrm{Br}(k)} = \dfrac{\mathrm{Br}(X \setminus Z)}{\mathrm{Br}(k)} = 0$
- in general, should take into account the Brauer-Manin obstruction

First example: the affine space

- $X = \mathbb{A}^n$ satisfies strong approximation off $S \neq \emptyset$

Theorem (D. Wei; Y. Cao & F. Xu)

Let Z be a Zariski closed subset of $\mathbb{A}^n$ such that $\operatorname{codim}(Z, X) \geq 2$. Then $\mathbb{A}^n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

- What about the case when X is a semi-simple simply connected group?
- in these cases $\dfrac{\operatorname{Br}(X)}{\operatorname{Br}(k)} = \dfrac{\operatorname{Br}(X \setminus Z)}{\operatorname{Br}(k)} = 0$
- in general, should take into account the Brauer-Manin obstruction

First example: the affine space

- $X = \mathbb{A}^n$ satisfies strong approximation off $S \neq \emptyset$

Theorem (D. Wei; Y. Cao & F. Xu)

Let Z be a Zariski closed subset of $\mathbb{A}^n$ such that $\operatorname{codim}(Z, X) \geq 2$. Then $\mathbb{A}^n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

- What about the case when X is a semi-simple simply connected group?
- in these cases $\dfrac{\mathrm{Br}(X)}{\mathrm{Br}(k)} = \dfrac{\mathrm{Br}(X \setminus Z)}{\mathrm{Br}(k)} = 0$
- in general, should take into account the Brauer-Manin obstruction

Brauer-Manin obstruction to strong approximation

- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles without S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ & $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ natural projections
- Consider $X(k) \subset \overline{X(k)} \subset pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$
- one may also consider larger codimension instead of 2, we will specify later.

- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles without S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ & $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ natural projections
- Consider $X(k) \subset \overline{X(k)} \subset pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$
- one may also consider larger codimension instead of 2, we will specify later.

- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles without S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ & $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ natural projections
- Consider $X(k) \subset \overline{X(k)} \subset pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$

Definition

(1)We say that X satisfies strong approximation with Brauer-Manin obstruction off S if $\overline{X(k)} = pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$.

- one may also consider larger codimension instead of 2, we will specify later.

- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles without S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ & $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ natural projections
- Consider $X(k) \subset \overline{X(k)} \subset pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$

Definition

(1)We say that X satisfies strong approximation with Brauer-Manin obstruction off S if $\overline{X(k)} = pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$.
(2) *Furthermore*, we say that X verifies arithmetic purity (for strong approximation with Brauer-Manin obstruction off S) if for all Zariski closed subset $Z \subset X$ such that $\mathrm{codim}(Z, X) \geq 2$, the complement $U = X \setminus Z$ still satisfies strong approximation with Brauer-Manin obstruction off S.

- one may also consider larger codimension instead of 2, we will specify later.

- $S \subset \Omega_k$ finite subset
  $\mathbf{A}_k^S$ adèles without S-components
  $pr^S : \mathbf{A}_k \to \mathbf{A}_k^S$ & $pr^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S)$ natural projections
- Consider $X(k) \subset \overline{X(k)} \subset pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$

Definition

(1)We say that X satisfies strong approximation with Brauer-Manin obstruction off S if $\overline{X(k)} = pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$.
(2) *Furthermore*, we say that X verifies arithmetic purity (for strong approximation with Brauer-Manin obstruction off S) if for all Zariski closed subset $Z \subset X$ such that $\mathrm{codim}(Z, X) \geq 2$, the complement $U = X \setminus Z$ still satisfies strong approximation with Brauer-Manin obstruction off S.

- one may also consider larger codimension instead of 2, we will specify later.

Arithmetic purity

- Analog: Arithmetic purity for weak approximation holds once X satisfies weak approximation with BM obstruction.
- No! even for rational varieties.
- Example (Y. Cao & F. Xu 2013):
- X fails arithmetic purity

- Analog: Arithmetic purity for weak approximation holds once X satisfies weak approximation with BM obstruction.

Question

Does arithmetic purity for strong approximation with BM obstruction holds in general?

- No! even for rational varieties.
- Example (Y. Cao & F. Xu 2013):
- X fails arithmetic purity

Arithmetic purity

- Analog: Arithmetic purity for weak approximation holds once X satisfies weak approximation with BM obstruction.

Question

Does arithmetic purity for strong approximation with BM obstruction holds in general?

- No! even for rational varieties.
- Example (Y. Cao & F. Xu 2013):
- X fails arithmetic purity

Arithmetic purity

- Analog: Arithmetic purity for weak approximation holds once X satisfies weak approximation with BM obstruction.

Question

Does arithmetic purity for strong approximation with BM obstruction holds in general?

- No! even for rational varieties.
- Example (Y. Cao & F. Xu 2013):
  $k = \mathbb{Q}$ or an imaginary quadratic field
  \- $X = \mathbb{G}_m \times \mathbb{A}^1$ satisfies str. approx. with BM obs. off ∞_k
  (Harari 2008, arithmetic duality theorems)
- X fails arithmetic purity

Arithmetic purity

- Analog: Arithmetic purity for weak approximation holds once X satisfies weak approximation with BM obstruction.

Question

Does arithmetic purity for strong approximation with BM obstruction holds in general?

- No! even for rational varieties.
- Example (Y. Cao & F. Xu 2013):
  $k = \mathbb{Q}$ or an imaginary quadratic field
  - $X = \mathbb{G}_m \times \mathbb{A}^1$ satisfies str. approx. with BM obs. off ∞_k (Harari 2008, arithmetic duality theorems)
  - $U = X \setminus \{one\ rational\ point\}$ does not satisfy str. approx. with BM obs. off ∞_k
- X fails arithmetic purity

Arithmetic purity

- Analog: Arithmetic purity for weak approximation holds once X satisfies weak approximation with BM obstruction.

Question

Does arithmetic purity for strong approximation with BM obstruction holds in general?

- No! even for rational varieties.
- Example (Y. Cao & F. Xu 2013):
  $k = \mathbb{Q}$ or an imaginary quadratic field
  - $X = \mathbb{G}_m \times \mathbb{A}^1$ satisfies str. approx. with BM obs. off ∞_k (Harari 2008, arithmetic duality theorems)
  - $U = X \setminus \{one\ rational\ point\}$ does not satisfy str. approx. with BM obs. off ∞_k
- X fails arithmetic purity

- One more example:
  $X = GL_n (n \geq 2)$ satisfies str. approx. with BM obs.
  (Demarche 2011)
- As a corollary of our main result (you will see it soon... be patient!):
- We should modify the question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$, does X verify arithmetic purity for str. approx. with BM obs.?

GL_n

- One more example:
  $X = GL_n(n \geq 2)$ satisfies str. approx. with BM obs. (Demarche 2011)
- As a corollary of our main result (you will see it soon... be patient!):
  X fails arithmetic purity if $k = \mathbb{Q}$ or an imaginary quadratic field
- We should modify the question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$, does X verify arithmetic purity for str. approx. with BM obs.?

GL_n

- One more example:
  $X = GL_n (n \geq 2)$ satisfies str. approx. with BM obs.
  (Demarche 2011)
- As a corollary of our main result (you will see it soon... be patient!):
  X fails arithmetic purity if $k = \mathbb{Q}$ or an imaginary quadratic field
  (idea: remove a divisor of a certain fiber of $\mathrm{det} : \mathrm{GL_n} \to \mathbb{G}_\mathrm{m}$)
- We should modify the question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$, does X verify arithmetic purity for str. approx. with BM obs.?

- One more example:
  $X = GL_n (n \geq 2)$ satisfies str. approx. with BM obs.
  (Demarche 2011)
- As a corollary of our main result (you will see it soon... be patient!):
  X fails arithmetic purity if $k = \mathbb{Q}$ or an imaginary quadratic field
  (idea: remove a divisor of a certain fiber of $\mathrm{det} : \mathrm{GL_n} \to \mathbb{G}_\mathrm{m}$)
- We should modify the question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$, does X verify arithmetic purity for str. approx. with BM obs.?

- One more example:
  $X = GL_n (n \geq 2)$ satisfies str. approx. with BM obs. (Demarche 2011)
- As a corollary of our main result (you will see it soon... be patient!):
  X fails arithmetic purity if $k = \mathbb{Q}$ or an imaginary quadratic field
  (idea: remove a divisor of a certain fiber of $\mathrm{det} : \mathrm{GL_n} \to \mathbb{G}_\mathrm{m}$)
- We should modify the question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$, does X verify arithmetic purity for str. approx. with BM obs.?

No!

I am sorry...

Punctured elliptic curves

Harari-Voloch 2010: elliptic curve $E_{/\mathbb{Q}} : y^2 = x^3 + 3$, $E(\mathbb{Q}) = \mathbb{Z}$
$U = E \setminus \{O\}$ a hyperbolic curve
U does not satisfy str. approx. with BM obs. off $\infty_{\mathbb{Q}}$

- idea of proof:

Punctured elliptic curves

Harari-Voloch 2010: elliptic curve $E_{/\mathbb{Q}} : y^2 = x^3 + 3$, $E(\mathbb{Q}) = \mathbb{Z}$
$U = E \setminus \{O\}$ a hyperbolic curve
U does not satisfy str. approx. with BM obs. off $\infty_{\mathbb{Q}}$

- We generalise this to

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $\text{Ш}(E, k) < \infty$.

- idea of proof:

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $\text{Ш}(E, k) < \infty$.

- idea of proof:

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $Ш(E,k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $Ш(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $Ш(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. *If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.*
The converse is also true if $Ш(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S
  - Siegel: $\mathcal{U}(\mathcal{O}_{k,S})$ is finite

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k.
$U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $\text{Ш}(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S
  - Siegel: $\mathcal{U}(\mathcal{O}_{k,S})$ is finite
  - consider a sequence $pQ \in E(k) \subset E(\mathbf{A}_k)_\bullet$ where prime number p runs in a carefully chosen arithmetic progression

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k. $U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $\text{Ш}(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S
  - Siegel: $\mathcal{U}(\mathcal{O}_{k,S})$ is finite
  - consider a sequence $pQ \in E(k) \subset E(\mathbf{A}_k)_\bullet$ where prime number p runs in a carefully chosen arithmetic progression
  - a subsequence converges to $(x_v)_{v\in\Omega}$ in $E(\mathbf{A}_k)_\bullet$

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k. $U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k.
The converse is also true if $\text{Ш}(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S
  - Siegel: $\mathcal{U}(\mathcal{O}_{k,S})$ is finite
  - consider a sequence $pQ \in E(k) \subset E(\mathbf{A}_k)_\bullet$ where prime number p runs in a carefully chosen arithmetic progression
  - a subsequence converges to $(x_v)_{v \in \Omega}$ in $E(\mathbf{A}_k)_\bullet$
  - $(x_v)_v \perp \mathrm{Br}(E)$ and $\mathrm{Br}(E) = \mathrm{Br}(U)$

Punctured elliptic curves

Theorem

Let E be an elliptic curve defined over a number field k. $U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k. The converse is also true if $Ш(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S
  - Siegel: $\mathcal{U}(\mathcal{O}_{k,S})$ is finite
  - consider a sequence $pQ \in E(k) \subset E(\mathbf{A}_k)_\bullet$ where prime number p runs in a carefully chosen arithmetic progression
  - a subsequence converges to $(x_v)_{v\in\Omega}$ in $E(\mathbf{A}_k)_\bullet$
  - $(x_v)_v \perp \mathrm{Br}(E)$ and $\mathrm{Br}(E) = \mathrm{Br}(U)$
  - check that $x_v \in \mathcal{U}(\mathcal{O}_v)$ for $v \notin S$ and $(x_v)_v \in U(\mathbf{A}_k)_\bullet$

Theorem

Let E be an elliptic curve defined over a number field k. $U = E \setminus \{O\}$. If $E(k)$ has positive rank, then U does not satisfy str. approx. with BM obs. off ∞_k. The converse is also true if $Ш(E, k) < \infty$.

- idea of proof: choose $\mathcal{U}$ a good $\mathcal{O}_k$-model of U
  - take $Q \in E(k)$ a point of infinite order
  - Q is a S-integral point of U for a certain finite set S
  - Siegel: $\mathcal{U}(\mathcal{O}_{k,S})$ is finite
  - consider a sequence $pQ \in E(k) \subset E(\mathbf{A}_k)_\bullet$ where prime number p runs in a carefully chosen arithmetic progression
  - a subsequence converges to $(x_v)_{v\in\Omega}$ in $E(\mathbf{A}_k)_\bullet$
  - $(x_v)_v \perp \mathrm{Br}(E)$ and $\mathrm{Br}(E) = \mathrm{Br}(U)$
  - check that $x_v \in \mathcal{U}(\mathcal{O}_v)$ for $v \notin S$ and $(x_v)_v \in U(\mathbf{A}_k)_\bullet$
  - by p-adic logarithm, check that $(x_v)_v$ can not be approximated by global (S-integral) points of U

Punctured Abelian varieties

the above 1-dimensional result + fibration argument $\Longrightarrow$ arithmetic purity results

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
- We should modify our question to avoid Abelian varieties.

the above 1-dimensional result + fibration argument $\implies$ arithmetic purity results

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
- We should modify our question to avoid Abelian varieties.

Punctured Abelian varieties

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
- We should modify our question to avoid Abelian varieties.

Punctured Abelian varieties

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
- We should modify our question to avoid Abelian varieties.

Punctured Abelian varieties

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
- We should modify our question to avoid Abelian varieties.

Punctured Abelian varieties

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
  $(E \times A) \setminus \{O\}$ str. approx. BM
- We should modify our question to avoid Abelian varieties.

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
  $(E \times A) \setminus \{O\}$ str. approx. BM $+$ $A(k)$ discrete in $A(\mathbf{A}_k)$
- We should modify our question to avoid Abelian varieties.

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
  $(E \times A) \setminus \{O\}$ str. approx. BM $+$ $A(k)$ discrete in $A(\mathbf{A}_k)$
  $\Longrightarrow$ the punctured fiber $E \setminus \{O\}$ str. approx. BM
- We should modify our question to avoid Abelian varieties.

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
  $(E \times A) \setminus \{O\}$ str. approx. BM $+$ $A(k)$ discrete in $A(\mathbf{A}_k)$
  $\Longrightarrow$ the punctured fiber $E \setminus \{O\}$ str. approx. BM
  contradicting the previous theorem
- We should modify our question to avoid Abelian varieties.

Theorem

Let E be an elliptic curve of rank ≥ 1 over a number field k. Let A be an Abelian variety of rank 0. Than $(E \times A) \setminus \{O\}$ does not satisfy str. approx. with BM obs. off ∞_k.

- $A \times E$ fails arithmetic purity (even for arbitrarily large codimension).
- Remark. Similar statement holds for tori and semi-abelian varieties.
- proof by contradiction:
  $(E \times A) \setminus \{O\}$ str. approx. BM $+$ $A(k)$ discrete in $A(\mathbf{A}_k)$
  $\Longrightarrow$ the punctured fiber $E \setminus \{O\}$ str. approx. BM
  contradicting the previous theorem
- We should modify our question to avoid Abelian varieties.

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$ and $\mathrm{Pic}(X_{\bar{k}})$ is finitely generated, does X verify arithmetic purity for str. approx. with BM obs. (off S)?

Question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$ and $\mathrm{Pic}(X_{\bar{k}})$ is finitely generated, does X verify arithmetic purity for str. approx. with BM obs. (off S)?

A known result:

Theorem (D. Wei 2014)

Let X be a smooth toric variety such that $\bar{k}[X]^\times = \bar{k}^\times$. Then X verifies arithmetic purity for str. approx. with BM obs. off $S \neq \emptyset$.

Question

Question

Suppose that $\bar{k}[X]^\times = \bar{k}^\times$ and $\mathrm{Pic}(X_{\bar{k}})$ is finitely generated, does X verify arithmetic purity for str. approx. with BM obs. (off S)?

A known result:

Theorem (D. Wei 2014)

Let X be a smooth toric variety such that $\bar{k}[X]^\times = \bar{k}^\times$. Then X verifies arithmetic purity for str. approx. with BM obs. off $S \neq \emptyset$.

Examples: $\mathbb{A}^n$, $\mathbb{P}^n$

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
- certain homogeneous spaces

Our results

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
- certain homogeneous spaces

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
  - mentioned above:
  Abelian varieties, tori, semi-abelian varieties
- certain homogeneous spaces

Our results

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
  - mentioned above:
  Abelian varieties, tori, semi-abelian varieties
  - quasi-split semi-simple simply connected groups
- certain homogeneous spaces

Our results

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
  - mentioned above:
  Abelian varieties, tori, semi-abelian varieties
  - quasi-split semi-simple simply connected groups
  - connected linear algebraic groups with G^{sc} quasi-split
- certain homogeneous spaces

Our results

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
  - mentioned above:
  Abelian varieties, tori, semi-abelian varieties
  - quasi-split semi-simple simply connected groups
  - connected linear algebraic groups with G^{sc} quasi-split
- certain homogeneous spaces

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
  - mentioned above:
  Abelian varieties, tori, semi-abelian varieties
  - quasi-split semi-simple simply connected groups
  - connected linear algebraic groups with G^{sc} quasi-split
- certain homogeneous spaces

Theorem

Let G be a semi-simple simply connected linear algebraic group defined over a number field. Suppose that G is quasi-split. Then G verifies arithmetic purity for strong approximation off $S \neq \emptyset$.

Our results

In our paper, we study arithmetic purity for strong approximation for

- Algebraic groups
  - mentioned above:
  Abelian varieties, tori, semi-abelian varieties
  - quasi-split semi-simple simply connected groups
  - connected linear algebraic groups with G^{sc} quasi-split
- certain homogeneous spaces

Theorem

Let G be a semi-simple simply connected linear algebraic group defined over a number field. Suppose that G is quasi-split. Then G verifies arithmetic purity for strong approximation off $S \neq \emptyset$. (In this case $\mathrm{Br}(G)/\mathrm{Br}(k) = 0$.)

Sketch of proof

- G quasi-split:
  $B \subset G$ be a k-Borel, $T \subset B = T \ltimes B^{\mathrm{u}}$ a maximal torus.
- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K}|\mathrm{k}}\mathbb{G}_{\mathrm{m},\mathrm{K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- G quasi-split:
  $B \subset G$ be a k-Borel, $T \subset B = T \ltimes B^{\mathrm{u}}$ a maximal torus.
- Bruhat decomposition: the big cell $V = BwB \subset G$ is an open dense, where $w \in G(k)$ is the longest root
- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K}|\mathrm{k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- G quasi-split:
  $B \subset G$ be a k-Borel, $T \subset B = T \ltimes B^{\mathrm{u}}$ a maximal torus.
- Bruhat decomposition: the big cell $V = BwB \subset G$ is an open dense, where $w \in G(k)$ is the longest root
- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K}|\mathrm{k}}\mathbb{G}_{\mathrm{m},\mathrm{K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- G quasi-split:
  $B \subset G$ be a k-Borel, $T \subset B = T \ltimes B^{\mathrm{u}}$ a maximal torus.
- Bruhat decomposition: the big cell $V = BwB \subset G$ is an open dense, where $w \in G(k)$ is the longest root
- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- G quasi-split:
  $B \subset G$ be a k-Borel, $T \subset B = T \ltimes B^{\mathrm{u}}$ a maximal torus.
- Bruhat decomposition: the big cell $V = BwB \subset G$ is an open dense, where $w \in G(k)$ is the longest root
- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^{\times}/\bar{k}^{\times} \to \bar{k}[V]^{\times}/\bar{k}^{\times}$
- G: ss sc $\implies$ $\bar{k}[V]^{\times}/\bar{k}^{\times}$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
  - $V \subset Y \subset G$ & $\mathrm{codim}(G \setminus Y, G) \geq 2$
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
  - $V \subset Y \subset G$ & $\mathrm{codim}(G \setminus Y, G) \geq 2$
  - $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m}} \subset \mathrm{R} \subset \mathrm{Res}_{\mathrm{K|k}}\mathbb{A}^1 = \mathbb{A}^{\mathrm{d}}$ & $\mathrm{codim}(\mathbb{A}^d \setminus R, \mathbb{A}^d) \geq 2$
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\Longrightarrow$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
  - $V \subset Y \subset G$ & $\mathrm{codim}(G \setminus Y, G) \geq 2$
  - $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m}} \subset \mathrm{R} \subset \mathrm{Res}_{\mathrm{K|k}}\mathbb{A}^1 = \mathbb{A}^{\mathrm{d}}$ & $\mathrm{codim}(\mathbb{A}^d \setminus R, \mathbb{A}^d) \geq 2$
  - more or less $\phi : G \to \mathbb{A}^d$ up to some 2-codimensional things
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
  - $V \subset Y \subset G$ & $\mathrm{codim}(G \setminus Y, G) \geq 2$
  - $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m}} \subset \mathrm{R} \subset \mathrm{Res}_{\mathrm{K|k}}\mathbb{A}^1 = \mathbb{A}^{\mathrm{d}}$ & $\mathrm{codim}(\mathbb{A}^d \setminus R, \mathbb{A}^d) \geq 2$
  - more or less $\phi : G \to \mathbb{A}^d$ up to some 2-codimensional things
  - most fibres look like $B^{\mathrm{u}} \times B^{\mathrm{u}}$
- some kind of fibration argument completes the proof.

Sketch of proof

- $\phi : V \simeq B^{\mathrm{u}} \times B \to T$ induces an isomorphism of Galois module $\bar{k}[T]^\times/\bar{k}^\times \to \bar{k}[V]^\times/\bar{k}^\times$
- G: ss sc $\implies$ $\bar{k}[V]^\times/\bar{k}^\times$ is a permutation Galois module.
- T is quasi-trivial: $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m,K}}$ for a certain finite étale k-algebra K.
- In such a case ϕ extends to a smooth morphism $\phi : Y \to R$ with non-empty geometrically integral fibres, where
  - $V \subset Y \subset G$ & $\mathrm{codim}(G \setminus Y, G) \geq 2$
  - $T = \mathrm{Res}_{\mathrm{K|k}}\mathbb{G}_{\mathrm{m}} \subset \mathrm{R} \subset \mathrm{Res}_{\mathrm{K|k}}\mathbb{A}^1 = \mathbb{A}^{\mathrm{d}}$ & $\mathrm{codim}(\mathbb{A}^d \setminus R, \mathbb{A}^d) \geq 2$
  - more or less $\phi : G \to \mathbb{A}^d$ up to some 2-codimensional things
  - most fibres look like $B^{\mathrm{u}} \times B^{\mathrm{u}}$
- some kind of fibration argument completes the proof.

Corollary

SL_n verifies arithmetic purity for strong approximation off $S \neq \emptyset$:
For any Zariski closed subset Z such that $\mathrm{codim}(Z, SL_n) \geq 2$, $SL_n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

Corollary

SL_n verifies arithmetic purity for strong approximation off $S \neq \emptyset$: For any Zariski closed subset Z such that $\mathrm{codim}(Z, SL_n) \geq 2$, $SL_n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

Corollary

Over any number field k, GL_n verifies 3-codimensional arithmetic purity for str. approx. with BM obs. off ∞_k.

SL_n and GL_n

Corollary

SL_n verifies arithmetic purity for strong approximation off $S \neq \emptyset$:
For any Zariski closed subset Z such that $\mathrm{codim}(Z, SL_n) \geq 2$, $SL_n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

Corollary

Over any number field k, GL_n verifies 3-codimensional arithmetic purity for str. approx. with BM obs. off ∞_k.

Remark. $\bar{k}[GL_n]^\times \neq \bar{k}^\times$

SL_n and GL_n

Corollary

SL_n verifies arithmetic purity for strong approximation off $S \neq \emptyset$: For any Zariski closed subset Z such that $\mathrm{codim}(Z, SL_n) \geq 2$, $SL_n \setminus Z$ satisfies strong approximation off $S \neq \emptyset$.

Corollary

Over any number field k, GL_n verifies 3-codimensional arithmetic purity for str. approx. with BM obs. off ∞_k.

Remark. $\bar{k}[GL_n]^\times \neq \bar{k}^\times$
In our previous definition, we discuss codimension 2 arithmetic purity. The additional $1(= 3 - 2)$ dimension comes from $\mathbb{G}_m$. To be more precise...

Definition

We say that X satisfies Zariski open strong approximation with Brauer-Manin obstruction off S, if for any non-empty Zariski open $U \subset X$, $U(k)$ is dense in $pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$.

Definition

We say that X satisfies Zariski open strong approximation with Brauer-Manin obstruction off S, if for any non-empty Zariski open $U \subset X$, $U(k)$ is dense in $pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$.

Lemma (Dirichlet's unit theorem)

The k-torus $\mathbb{G}_m$ satisfies Zariski open str. approx. with BM obs. if and only if the number field k is neither $\mathbb{Q}$ nor an imaginary quadratic field.

SL_n and GL_n

Definition

We say that X satisfies Zariski open strong approximation with Brauer-Manin obstruction off S, if for any non-empty Zariski open $U \subset X$, $U(k)$ is dense in $pr^S(X(\mathbf{A}_k)^{\mathrm{Br}}) \subset X(\mathbf{A}_k^S)$.

Lemma (Dirichlet's unit theorem)

The k-torus $\mathbb{G}_m$ satisfies Zariski open str. approx. with BM obs. if and only if the number field k is neither $\mathbb{Q}$ nor an imaginary quadratic field.

Corollary

GL_n verifies arithmetic purity (codim 2) for str. approx. with BM obs. off ∞_k if and only if the number field k is neither $\mathbb{Q}$ nor an imaginary quadratic field.

Linear algebraic groups

$GL_n \rightsquigarrow$ most general setting

- G connected linear algebraic group
- $G^{\mathrm{red}} = G/G^{\mathrm{u}}$, $G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$

$GL_n \rightsquigarrow$ most general setting

- G connected linear algebraic group
- $G^{\mathrm{red}} = G/G^{\mathrm{u}}$, $G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$

$GL_n \rightsquigarrow$ most general setting

- G connected linear algebraic group
- $G^{\mathrm{red}} = G/G^{\mathrm{u}}$, $G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$

Theorem

Suppose that G^{sc} verifies arithmetic purity for str. approx. off ∞_k (in particular when it is quasi-split).
(1) G verifies arithmetic purity of codimension $(2 + \dim G^{\mathrm{tor}})$ for str. approx. with BM obs. off ∞_k.

$GL_n \rightsquigarrow$ most general setting

- G connected linear algebraic group
- $G^{\mathrm{red}} = G/G^{\mathrm{u}}$, $G^{\mathrm{ss}} = [G^{\mathrm{red}}, G^{\mathrm{red}}]$, $G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$, $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$

Theorem

Suppose that G^{sc} verifies arithmetic purity for str. approx. off ∞_k (in particular when it is quasi-split).
(1) G verifies arithmetic purity of codimension $(2 + \dim G^{\mathrm{tor}})$ for str. approx. with BM obs. off ∞_k.
(2) if furthermore $G^{\mathrm{tor}} \neq 1$ satisfies Zariski open str. approx. with BM obs., then G verifies arithmetic purity of codimension $(1 + \dim G^{\mathrm{tor}})$ for str. approx. with BM obs. off ∞_k.

Theorem

Let G be a connected linear group and $H \subset G$ be a connected closed subgroup. Let X be a G-variety containing G/H as a Zariski open dense G-orbit. Assume that $\bar{k}[X]^\times = \bar{k}^\times$. If G^{sc} verifies arithmetic purity (in particular when it is quasi-split), then X satisfies arithmetic purity for str. approx. with BM obs. off $S \neq \emptyset$.

- Example: $X \subset \mathbb{A}^4$ defined by $x_1x_2 + x_3x_4 = c$ where $c \in k^\times$, then X verifies arithmetic purity.

Theorem

Let G be a connected linear group and $H \subset G$ be a connected closed subgroup. Let X be a G-variety containing G/H as a Zariski open dense G-orbit. Assume that $\bar{k}[X]^\times = \bar{k}^\times$. If G^{sc} verifies arithmetic purity (in particular when it is quasi-split), then X satisfies arithmetic purity for str. approx. with BM obs. off $S \neq \emptyset$.

- Example: $X \subset \mathbb{A}^4$ defined by $x_1x_2 + x_3x_4 = c$ where $c \in k^\times$, then X verifies arithmetic purity.

The following question is still open.

Question

What about arithmetic purity for semi-simple simply connected groups (not necessarily quasi-split)?

An open question

The following question is still open.

Question

What about arithmetic purity for semi-simple simply connected groups (not necessarily quasi-split)?

Merci de votre attention!