HILBERT-SERRE THEOREM ON REGULAR NOETHERIAN LOCAL RINGS

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Abstract. The aim of this work is to prove a theorem of Serre(Noetherian local ring is regular iff global dimension is finite). The proof follows that one given by Nagata simplified by Grothendieck without using Koszul complex.

1. Regular Rings

In this section, basic definitions related to regular rings and some basic properties which are needed in the proof of the main theorem are given. Proofs can be found in [3] or [2].

Let A be a ring and M be an A-module.

Definition 1.1. The set $Ass(M)$ of *associated prime ideals of* M contains prime ideals satisfy the following two equivalent conditions:

(i)there exists an element $x \in M$ with $ann(x) = \mathfrak{p}$;

(ii)M contains a submodule isomorphic to $A/\mathfrak{p}$.

Definition 1.2. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension d, it is called *regular* if it satisfies the following equivalent conditions:

(i)$Gr_{\mathfrak{m}}(A) \cong k[T_1, \cdots, T_d]$;

(ii)$dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$;

(iii)$\mathfrak{m}$ can be generated by d elements.

In this case, the d generators of $\mathfrak{m}$ is called a *regular system of parameters* of A.

(see [3])

Remark 1.3. It is always true that $dim(A) \leq dim_k(\mathfrak{m}/\mathfrak{m}^2)$, equality holds if and only if A is regular.(see [3])

Proposition 1.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, $t \in \mathfrak{m}$, then the following conditions are equivalent:*

(i)A/tA is regular and t is not zero divisor of A;

(ii)A is regular and $t \notin \mathfrak{m}^2$.

(see [1, EGA IV chapter 0, 17.1.8] *)*

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Definition 1.5. An element $a \in A$ is said to be *M-regular* if $M \to M; m \mapsto am$ is injective.

Definition 1.6. Let $a_1, \cdots, a_r$ be a sequence of elements of A. $a_1, \cdots, a_r$ is called an *M-regular sequence* if

(1)for each $1 \leq i \leq r$, a_i is $(M/\sum_{s=1}^{i-1} a_s M)$-regular;

(2)$M \neq \sum_{s=1}^{r} a_s M$.

Proposition 1.7. *Let A be a regular local ring and $x_1, \cdots, x_n$ a regular system of parameters, then $x_1, \cdots, x_n$ is an A-regular sequence.*

(see [2, chaper 7, section 17]*)*

2. Homological Dimension

In this section, basic properties of homological dimension without proof are given, for details see [2].

Let A be a ring.

Definition 2.1. The *projective (resp. injective) dimension* of an A-module M is defined to be the length of a shortest projective (resp. injective) resolution of M, denoted by $proj.dim_A(M)$ (resp. $inj.dim_A(M)$).

Proposition 2.2. *Let n be a non-negative integer, then the following conditions are equivalent:*

(i)$proj.dim_A(M) \leq n$ for any A-module M;

(ii)$inj.dim_A(M) \leq n$ for any A-module M;

(iii)$Ext_A^{n+1}(M, N) = 0$ for any A-module M and N.

(see [2, chaper 7, section 18]*.)*

Definition 2.3. *Global dimension $gl.dim(A)$* of a ring A is defined to be the number $sup_M(proj.dim(M)) = sup_M(inj.dim(M))$ (not necessary finite).

Proposition 2.4. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Then*

(1)If M is a finitely generated A-module, then $proj.dim(M) \leq n$ if and only if $Tor_{n+1}^A(M, k) = 0$;

(2)$gl.dim(A) \leq n$ if and only if $Tor_{n+1}^A(k, k) = 0$, moreover in this case, $Tor_m^A(k, k) = 0$ for any $m > n$.

(see [2, chaper 7, section 18]*.)*

Proposition 2.5. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring an M a finite generated A-module. If $proj.dim_A(M) = r$ and if x is an M-regular element in $\mathfrak{m}$, then $proj.dim_A(M/xM) = r + 1$.*

(see [2, chaper 7, section 18]*.)*

Proposition 2.6. *Let A be a Noetherian ring, then $gl.dim(A) = sup_{\mathfrak{m}}(gl.dim(A_{\mathfrak{m}}))$ where $\mathfrak{m}$ runs through all the maximal (or prime) ideals of A.*

(see [1, EGA IV chapter 0, 17.2.10] *or* [2, chaper 7, section 18]*)*

3. Hilbert-Serre Theorem

In this section, the statement of Hilbert-Serre Theorem is given, and we give the full proof.

Theorem 3.1 (Hilbert-Serre)**.** *Let A be a Noetherian local ring, then A is regular if and only if A has finite global dimension. In this case, $gl.dim(A) = dim(A)$*

Remark 3.2. The proof of the theorem given after the corollaries will follow the one given by Nagata simplified by Grothendieck without using Koszul complex. For the one using Koszul complex, see [2, section 18 chapter 7].

Corollary 3.3 (Hilbert's Syzygy Theorem)**.** *Let $A = k[T_1, \cdots, T_d]$ be a polynomial ring over a field k, then $gl.dim(A) = d$.*

Proof. A is regular of Krull dimension d, localize it with applying 2.6, then the statement follows from Hilbert-Serre. □

Corollary 3.4. *Let A be a Noetherian regular local ring, then $A_{\mathfrak{p}}$ is regular for every prime ideal $\mathfrak{p}$ of A.*

Proof. By 2.6 one obtains $gl.dim(A_{\mathfrak{p}}) \leq gl.dim(A)$, the assertion follows immediately from Hilbert-Serre. □

Proof. (proof of Hilbert-Serre, necessity) Let $x_1, \cdots, x_n$ be a regular system of parameters, Then the sequence $x_1, \cdots, x_n$ is an A-regular sequence and $k = A/\mathfrak{m} = A/\sum_{i=1}^{n} x_i A$, hence we have $proj.dim_A(k) = n$ by 2.5(apply induction), 2.4 implies that $gl.dim(A) \leq n < +\infty$. On the other hand, $gl.dim(A) \geq proj.dim_A(k) = n$ by definition, hence $gl.dim(A) = dim_{Krull}(A)$. □

For completing the proof of Hilbert-Serre theorem, we need several lemmas:

Lemma 3.5 (Nagata)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring, if all the elements of $\mathfrak{m} \setminus \mathfrak{m}^2$ are zero divisors of A, then there exists $c \neq 0$ in A such that $c\mathfrak{m} = 0$.(in other words, $\mathfrak{m} \in Ass(A)$)*

Proof. One can assume that $\mathfrak{m} \neq 0$, then $\mathfrak{m} \neq \mathfrak{m}^2$ by Nakayama, the hypothesis implies $\mathfrak{m} \setminus \mathfrak{m}^2 \subseteq \bigcup_{\mathfrak{p} \in Ass(A)} \mathfrak{p}$ (see [4]), then $\mathfrak{m} \subseteq \bigcup_{\mathfrak{p} \in Ass(A)} \mathfrak{p} \cup \mathfrak{m}^2$, this implies $\mathfrak{m} = \mathfrak{p}$ for some $\mathfrak{p} \in Ass(A)$ since $\mathfrak{m} \nsubseteq \mathfrak{m}^2$(only one of the union is not a prime, see [4]). □

Lemma 3.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, if $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ then $\mathfrak{m}/aA$ is isomorphic to a direct summand of $\mathfrak{m}/a\mathfrak{m}$.*

Proof. Since $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $(A, \mathfrak{m})$ is a Noetherian local ring, there exists a minimal system of generators $\{a, x_1, \cdots, x_s\}$ of $\mathfrak{m}$, with reduction modulo $\mathfrak{m}^2$ becoming a $k = A/\mathfrak{m}$ basis of $\mathfrak{m}/\mathfrak{m}^2$. Set $\mathfrak{b} = x_1 A + \cdots + x_s A \subseteq \mathfrak{m}$. By reduction modulo $\mathfrak{m}^2$, it is easy to see that for any $x \in A$, $xa \in \mathfrak{b}$ implies $x \in \mathfrak{m}$, hence $\mathfrak{b} \cap aA \subseteq a\mathfrak{m}$. Therefore the inclusion $\mathfrak{b} \subseteq \mathfrak{m}$ induces the injection $\mathfrak{b}/\mathfrak{b} \cap aA \hookrightarrow \mathfrak{m}/a\mathfrak{m}$, then $\mathfrak{m}/aA = \mathfrak{b} + aA/aA \cong \mathfrak{b}/\mathfrak{b} \cap aA \hookrightarrow \mathfrak{m}/a\mathfrak{m} \to \mathfrak{m}/aA$ is identity, therefore $\mathfrak{m}/aA$ is a direct summand of $\mathfrak{m}/a\mathfrak{m}$ as A-module. □

Lemma 3.7. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, E be a finite generated A-module of finite projective dimension. If $a \in \mathfrak{m}$ is A-regular and E-regular, then E/aE is a A/aA-module of finite projective dimension at most $proj.dim_A(E)$.*

Proof. Apply induction on $h = proj.dim_A(E)$, for $h = 0$, E is projective A-module, therefore E/aE is projective A/aA-module, $proj.dim_{A/aA}(E/aE) = 0$. For $proj.dim_A(E) = h$, we have an exact sequence of A-module $0 \to N \to L \to E \to 0$ where L is free and N is finite generated with $proj.dim_A(N) = h - 1$ (see [4]). Tensor with A/aA, one gets exact sequence of A/aA-module $N/aN \to L/aL \to E/aE \to 0$. Moreover, in this case, one obtains $0 \to N/aN \to L/aL \to E/aE \to 0$. In fact, $ker(N \to L/aL) = N \cap aL$, if $n = al \in N \cap aL$, the image in E is $0 = \phi(n) = a\phi(l)$, then $\phi(l) = 0$ since a is E-regular, in other words $l \in N$, $ker(N \to L/aL) = N \cap aL = aN$, $N/aN \to L/aL$ is injective. Since a is A-regular and L is free, a is L-regular, N is submodule of L, hence a is N-regular. So by induction $proj.dim_{A/aA}(N/aN) \leq h - 1$, therefore $proj.dim_{A/aA}(E/aE) \leq h$ since L/aL is free A/aA-module. □

Proof. (proof of Hilbert-Serre, sufficiency) Assume that $(A, \mathfrak{m}, k)$ is Noetherian local ring with finite global dimension, hence $proj.dim_A(\mathfrak{m})$ is finite. One applies induction on $n = dim_k(\mathfrak{m}/\mathfrak{m}^2)$ (finite by Noetherianity) to prove that "$proj.dim_A(\mathfrak{m}) < \infty$ implies A is regula", for $n = 0$, one has $\mathfrak{m} = 0$, the assertion is trivial. Now consider the following two cases (with assumption that $\mathfrak{m} \neq 0$):

(i)Suppose that all the elements of $\mathfrak{m} \setminus \mathfrak{m}^2$ are zero divisors of A, then by 3.5, there exists $0 \neq c \in A$ with $c\mathfrak{m} = 0$. A is local, $\mathfrak{m}$ is not projective (otherwise free, $c\mathfrak{m} = 0$ implies $\mathfrak{m} = 0$), so $t = gl.dim(A) \geq 1$. On the other hand, $c\mathfrak{m} = 0$ with $\mathfrak{m}$ maximal implies $\mathfrak{m} \in Ass(A)$. Then the homomorphism $A \to A; a \mapsto ac$ induces the A-module exact sequence $0 \to k \to A \to A \to 0$. Hence for $t \geq 1$ one gets $0 \to Tor^A_{t+1}(A, k) \to Tor^A_t(k, k) \to 0$, by 2.4, one has $Tor^A_t(k, k) \neq 0$ and $Tor^A_{t+1}(A, k) = 0$ at the same time, contradiction.

(ii)There exists $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is A-regular(i.e. not a zero divisor), hence $\mathfrak{m}$-regular. Consider $A' = A/aA$ and the maximal ideal $\mathfrak{m}' = \mathfrak{m}/aA$. It is clear that $dim_k(\mathfrak{m}'/\mathfrak{m}'^2) = dim_k(\mathfrak{m}/\mathfrak{m}^2) - 1 = n - 1$. We have shown that $proj.dim_A(\mathfrak{m})$ is finite, apply 3.7 with $E = \mathfrak{m}$, one obtains the fact that $proj.dim_{A'}(\mathfrak{m}/a\mathfrak{m})$ is also finite, and so is $proj.dim_{A'}(\mathfrak{m}')$ since $\mathfrak{m}' = \mathfrak{m}/aA$ is a direct summand of $\mathfrak{m}/a\mathfrak{m}$ by 3.6. By induction, A' is regular, then A is regular by the choice of a and 1.4. □

References

[1] A.Grothendieck. *Éléments de Géométrie Algébrique.* IV. IHÉS, 1964.

[2] H.Matsumura. *Commutative Algebra.* The Benjamin/Cummings Publishing Company, 1980.

[3] I.G.Macdonald M.F.Atiyah. *Introduction to Commutative Algebra.* Addison-Wesley, 1969.

[4] N.Bourbaki. *Commutative Algebra.* Hermann, 1972.

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