DESCENT THEORY FOR QUASI-COHERENT SHEAVES

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ABSTRACT. Faithfully flat descent of quasi-coherent sheaves is discussed in this paper. The affine case is discussed in subsection 1.2, the general case is in subsection 5.3. Before the descent theory, the notions of Grothendieck topology, fibred category and stack are introduced briefly without any proofs. An almost complete proof of faithfully flat descent is given. Some applications can be found at the end.

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Here I mainly follow the explained FGA [2], but not the original SGA [1]. For a more elementary language/treatment of this topic one can refer to [3], where the idea of descent theory is written very clear. Some useful information can be also found in [5].

1. Commutative Algebra

Let A be a commutative ring with identity. The notation $\otimes$ means $\otimes_A$ if there will be no confusion.

1.1. **Faithfully Flatness.**

Definition 1.1. An A-module M(resp. A-algebra B) is called *faithfully flat* if for any sequence of A-modules $0 \to N' \to N \to N'' \to 0$, it is exact if and only if $0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$(resp.$0 \to N' \otimes B \to N \otimes B \to N'' \otimes B \to 0$) is exact.

Theorem 1.2. *For an A-module M(resp. A-algebra B), the following are equivalent:*

(1)M(resp.B) is faithfully flat;

(2)M(resp.B) is flat, and A-module $N \neq 0$ implies $N \otimes M \neq 0$(resp.$N \otimes B \neq 0$);

(3)M(resp.B) is flat, and for any maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}M \neq M$(resp.$\mathfrak{m}B \neq B$);

(4)(only for B) $Spec(B) \to Spec(A)$ is flat and surjective.

Proof. see [4]. □

Define a A-module sequence for any A-algebra B:

$$A \overset{d_0}{\to} B \overset{d_1}{\to} \ldots \overset{d_{n-1}}{\to} B^{\otimes n} \overset{d_n}{\to} \ldots \tag{1.1.1}$$

$$d_n(b_0 \otimes \ldots \otimes b_{n-1}) = 1 \otimes b_0 \otimes \ldots \otimes b_{n-1} - b_0 \otimes 1 \otimes \ldots \otimes b_{n-1} + \ldots + (-1)^n b_0 \otimes \ldots \otimes b_{n-1} \otimes 1$$

It is a complex. Tensor it with A-module M, we get a A-module complex:

$$M \overset{d_0}{\to} M \otimes B \overset{d_1}{\to} \ldots \overset{d_{n-1}}{\to} M \otimes B^{\otimes n} \overset{d_n}{\to} \ldots \tag{1.1.2}$$

Theorem 1.3. *If B is faithfully flat A algebra, then the complexes 1.1.1 and 1.1.2 above are exact.*

Proof. We only need to prove exactness 1.1.2.

First we assume that there is a section $g : B \to A$ of $f : A \to B$ (i.e. $gf = id$). Define an A-module morphism for each n,

$$s_n : M \otimes B^{\otimes n} \to M \otimes B^{\otimes(n-1)}$$

$$m \otimes b_0 \otimes \ldots \otimes b_{n-1} \mapsto g(b_0)m \otimes b_1 \otimes \ldots \otimes b_{n-1},$$

it makes sense and can be check that $sd + ds = id - 0$, hence the identity map is null chain homotopic, the complex is exact.

In general, apply $\otimes_A B$ to the A-module complex 1.1.2, we get an $A \otimes_A B \simeq B$-module complex for the pair $(B \stackrel{1\otimes id}{\rightarrow} B \otimes_A B, M \otimes_A B)$ since $(M \otimes_A B) \otimes_B (B \otimes_A B)^{\otimes_B n} \simeq M \otimes_A B^{\times_A (n+1)}$. But now, $B \stackrel{1\otimes id}{\rightarrow} B \otimes_A B$ has a section $B \otimes_A B \to B; x \otimes y \mapsto xy$, hence the new complex is exact and by faithfully flatness we obtain the desired result. □

1.2. **Descent of Modules.** Let $f : A \to B$ be a ring homomorphism, M an A-module, then $N = B \otimes M$ is a B-module, the descent problem is that in which case a B-module is of this form, here we restrict ourselves to assume f to be faithfully flat.

For an A-module M, we have a canonical A-module isomorphism $\iota_M : M \otimes B \to B \otimes M; m \otimes b \mapsto b \otimes m$, and a A-module homomorphism $\alpha_M : M \to B \otimes M; m \mapsto 1 \otimes m$.

Let N be a B-module, then $N \otimes B \otimes B$, $B \otimes N \otimes B$ and $B \otimes B \otimes N$ are $B^{\otimes 3}$-modules with scalar product $(b_1 \otimes b_2 \otimes b_3)(x_1 \otimes x_2 \otimes x_3) = (b_1 x_1 \otimes b_2 x_2 \otimes b_3 x_3)$ (different meanings in three cases), they are not isomorphic in general.

Assume that there is a morphism $\psi : N \otimes B \to B \otimes N$ of $B^{\otimes 2}$-modules. Then we get three morphisms of $B^{\otimes 3}$-modules,

$$\psi_1 = id_B \otimes \psi : B \otimes N \otimes B \to B \otimes B \otimes N,$$
$$\psi_2 = (id_B \otimes \iota_N) \circ (\psi \otimes id_B) \circ (id_N \otimes \iota_B) : N \otimes B \otimes B \to B \otimes B \otimes N,$$
$$\psi_3 = \psi \otimes id_B : N \otimes B \otimes B \to B \otimes N \otimes B,$$

in deed, it is just by inserting the identity in the first, second and third position, respectively.

We define a category $Mod_{A\to B}$. Its objects are pairs (N, ψ), where N is a B-module, $\psi : N \otimes B \to B \otimes N$ is an isomorphism of $B^{\otimes 2}$-modules such that $\psi_2 = \psi_1 \circ \psi_3$. Its morphism $\beta : (N, \psi) \to (N', \psi')$ is a B-module morphism $\beta : N \to N'$ such that the following diagram commutes.

$$\begin{tikzcd} N \otimes B \arrow[r, "\psi"] \arrow[d, "\beta\otimes id_B"'] & B \otimes N \arrow[d, "id_B \otimes \beta"] \\ N' \otimes B \arrow[r, "\psi'"] & B \otimes N' \end{tikzcd}$$

We can define a functor $F : Mod_A \to Mod_{A\to B}$, on objects an A-module M sends to $(B \otimes M, \psi_M)$ with $\psi_M = id_B \otimes \iota_M : (B \otimes M) \otimes B \to B \otimes (B \otimes M)$ an isomorphism of $B^{\otimes 2}$-modules. On morphisms, $\alpha : M \to M'$ sends to $id_B \otimes \alpha : B \otimes M \to B \otimes M'$ which can be checked to be a morphism in $Mod_{A\to B}$.

Theorem 1.4. *If B is a faithfully flat A-algebra, then $F : Mod_A \to Mod_{A\to B}$ is an equivalent of categories.*

Proof. First we define a functor $G : Mod_{A\to B} \to Mod_A$ as follows. The pair (N,ψ) sends to $GN \stackrel{def}{=} \{n \in N | 1\otimes n = \psi(n\otimes 1)\}$ which is a A-submodule of N. A morphism $\beta : (N,\psi) \to (N',\psi')$ sends to $\beta : GN \to GN'$ which can be checked to make sense.

Consider $e_1, e_2 : B \to B\otimes B$, $e_1(b) = b\otimes 1$ and $e_2(b) = 1\otimes b$. From 1.3, we get a exact sequence

$$0 \to M \stackrel{\alpha_M}{\to} B\otimes M \stackrel{(e_1-e_2)\otimes id_M}{\longrightarrow} B\otimes B\otimes M \tag{1.2.1}$$

Notice that

$$\begin{aligned}((e_1-e_2)\otimes id_M)(b\otimes m) &= b\otimes 1\otimes m - 1\otimes b\otimes m \\ &= \psi_M(b\otimes m\otimes 1) - 1\otimes b\otimes m\end{aligned}$$

for any $m\in M$ and $b\in B$, hence we obtain

$$((e_1-e_2)\otimes id_M)(x) = \psi_M(x\otimes 1) - 1\otimes x$$

for any $x\in M$, so $G(B\otimes M,\psi_M) = ker((e_1-e_2)\otimes id_M)$ by definition of G. From the exact sequence 1.2.1, we obtain that

$$M \mapsto im(\alpha_M) = ker((e_1-e_2)\otimes id_M) = G(B\otimes M,\psi_M) = GF(M)$$

defines a natural isomorphism from the functor GF to identity.

Conversely, starts from an object (N,ψ) in $Mod_{A\to B}$, $M = G(N,\psi)$ is an A-submodule of N which itself is a B-module, this induces a B-module homomorphism $\theta : B\otimes M \to N; b\otimes m \mapsto bm$. We want to check that θ is morphism in $Mod_{A\to B}$, that is the following diagram commutes.

$$\begin{tikzcd} B\otimes M\otimes B \arrow[r, "\theta\otimes id_B"] \arrow[d, "\psi_M = id_B\otimes\iota_M"'] & N\otimes B \arrow[d, "\psi"] \\ B\otimes B\otimes M \arrow[r, "id_B\otimes\theta"] & B\otimes N \end{tikzcd}$$

In fact,

$$\begin{aligned}\psi(\theta\otimes id_B)(b_1\otimes m\otimes b_2) &= \psi(b_1 m\otimes b_2) \\ &= (b_1\otimes b_2)\psi(m\otimes 1) \\ &= (b_1\otimes b_2)(1\otimes m) \qquad \text{(since } m\in M) \\ &= b_1\otimes b_2 m \\ &= (id_M\otimes\theta)(b_1\otimes b_2\otimes m) \\ &= (id_B\otimes\theta)(id_B\otimes\iota_M)(b_1\otimes m\otimes b_2)\end{aligned}$$

Hence θ defines a natural transformation of functors FG and id.

We have the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & M\otimes B \arrow[r, "i\otimes id_B"] \arrow[d, "\theta\circ\iota_M"] & N\otimes B \arrow[r, "(\alpha-\beta)\otimes id_B"] \arrow[d, "\psi"] & B\otimes N\otimes B \arrow[d, "\psi_1"] \\ 0 \arrow[r] & N \arrow[r, "\alpha_M"] & B\otimes N \arrow[r, "(e_1-e_2)\otimes id_N"] & B\otimes B\otimes N \end{tikzcd}$$

The first row is exact since B is flat, the second row is exact since B is faithfully flat and theorem1.3.

This diagram commutes. For the first square, $\alpha_M \theta \iota_M(m \otimes b) = 1 \otimes bm$ by definition, and $\psi(i \otimes id_B)(m \otimes b) = 1 \otimes bm$ follows from the fact that $m \in M, \psi(m \otimes 1) = 1 \otimes m$. For the second square, it is immediate to check that $\psi_1 \circ (\alpha \otimes id_B) = (e_2 \otimes id_N) \circ \psi$,and on the other hand,

$$\begin{aligned}
\psi_1(\beta \otimes id_B)(n \otimes b) &= \psi_1(\psi(n \otimes 1) \otimes b) \\
&= \psi_1\psi_3(n \otimes 1 \otimes b) \\
&= \psi_2(n \otimes 1 \otimes b) \\
&= (id_B \otimes \iota_N)(\psi \otimes id_B)(id_N \otimes \iota_B)(n \otimes 1 \otimes b) \\
&= (id_B \otimes \iota_N)(\psi \otimes id_B)(n \otimes b \otimes 1) \\
&= (id_B \otimes \iota_N)(\psi(n \otimes b) \otimes 1) \\
&= (e_1 \otimes id_N)(\psi(n \otimes b)),
\end{aligned}$$

where the last equality can be checked as follows: By linearity, we can assume that $\psi(n \otimes b) \in B \otimes N$ is of the form $a \otimes m$ with $a \in B$ and $m \in N$,

$$(id_B \otimes \iota_N)(a \otimes m \otimes 1) = a \otimes 1 \otimes m = (e_1 \otimes id_N)(a \otimes m).$$

Notice that ψ is an isomorphism and $\psi_1 = id_B \otimes \psi$ is also an isomorphism since B is flat, hence $\theta \circ \iota_M$ then θ is an isomorphism. □

2. Algebraic Geometry

Definition 2.1. A morphism of schemes is called *faithfully flat* if it is flat (i.e. the induced morphism on the stalks is flat morphism of algebras for every point) and surjective.

Proposition 2.2. *Let $f : X \to Y$ be a surjective morphism of schemes. Then the following conditions are equivalent,*

(1)Every quasi-compact open subset of Y is the image of a quasi-compact open subset of X;

(2)There exists a covering V_i of Y by open affine subschemes, such that each V_i is the image of a quasi-compact open subset of X;

(3)Given a point $x \in X$, there exists an open neighborhood U of x in X, such that the image of U is open in Y, and the restriction $U \to f(U)$ is quasi-compact;

(4)Given a point $x \in X$, there exists a quasi-compact open neighborhood U of x in X, such that the image of U is open and affine of Y.

Proof. see [2]. □

Definition 2.3. An *fpqc morphism of schemes* is a faithfully flat morphism that satisfies the equivalent conditions of Proposition 2.2.

The word "fpqc" stands for "fidèlement plat et quasi-compact".

Remark 2.4. A quasi-compact faithfully flat morphism is always fpqc.

3. Topologies and Sheaves

In this section, we assume that any category under discussing has a final object and finite fibre product exists.

3.1. **Grothendieck Topologies.**

Definition 3.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology*(or simply *topology*) on $\mathcal{C}$ is for each object U of $\mathcal{C}$ a collection of sets of morphisms $\{U_i \to U\}$ called a *coverings* of U such that

(1)if $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering of U;

(2)if $\{U_i \to U\}$ is a covering of U, then for any morphism $V \to U$, $\{U_i \times_U V \to V\}$ is covering of V;

(3)if $\{U_i \to U\}$ is a covering and for each i $\{V_{ij} \to U_i\}$ is a covering, then $\{V_{ij} \to U_i \to U\}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Example 3.2. (the global Zariski topology)

Let $\mathcal{C}$ be the category of schemes. As definition $\{U_i \to U\}$ is a covering of U if $U_i \to U$ is open immersion for every i and $\{U_i \to U\}$ is surjective (i.e. the union of images is U).

Example 3.3. (Zariski topology on a scheme) Let X be a scheme, $\mathcal{C}$ be the full subcategory of the category $\underline{X\text{-schemes}}$ containing only open immersions of X. A covering $\{U_i \to U \in Hom_X(U_i, U)\}$ of $U \to X \in \mathcal{C}$ consists of open immersions of U, and this family of morphisms is surjective. This site is denoted by X_{zar}.

Example 3.4. (the global small étale topology)

Example 3.5. (small étale topology on a scheme) Let X be a scheme, $\mathcal{C}$ be the full subcategory of the category $\underline{X\text{-scheme}}$ containing only locally of finite presentation morphisms $U \to X$. A set of morphisms $\{U_i \to U \in Hom_X(U_i, U)\}$ is a covering of $U \to X \in \mathcal{C}$ if it is surjective and the induced morphism $\coprod U_i \to U$ is a locally of finite presentation étale morphism. This site on X is denoted by X_t.

Example 3.6. (the fpqc topology on a scheme) Let X be a scheme, $\mathcal{C}$ be the category $\underline{X\text{-scheme}}$ containing morphisms $U \to X$. A set of morphisms $\{U_i \to U \in Hom_X(U_i, U)\}$ is a covering of $U \to X \in \mathcal{C}$ if it is surjective and the induced morphism $\coprod U_i \to U$ is a fpqc morphism (see Definition 2.3).

Remark 3.7.

(1)The fpqc topology on X is finer than the étale topology, and the étale topology is finer than the Zariski topology.

(2)If we define fpqc topology by using the word "faithfully flat quasi-compact" instead of "fpqc", then the topology cannot be compared with the Zariski topology.

3.2. **Canonical Topology of a Category.**

Definition 3.8. A family of morphisms $\{U_i \to U\}$ in $\mathcal{C}$ is called *epimorphism* if $Hom(U, Z) \to \prod_i Hom(U_i, Z)$ is injective for any object $Z \in \mathcal{C}$.

It is called*effective epimorphism* if

$$0 \to Hom(U, Z) \to \prod_i Hom(U_i, Z) \rightrightarrows \prod_{i,j} Hom(U_i \times_U U_j, Z)$$

is exact for any object $Z \in \mathcal{C}$.

It is called *universal effective epimorphism* if $\{U_i \times_U V \to V\}$ is effective epimorphism for any $V \to U$.

Define a topology on a given category $\mathcal{C}$ as follows. For any object U of $\mathcal{C}$, $\{U_i \to U\}$ is a covering of U if it is universal effective epimorphism. It's easy to check that this is a topology on $\mathcal{C}$, called *canonical topology* on $\mathcal{C}$.

3.3. **Sheaves on Topologies.**

Definition 3.9. Let $\mathcal{C}$ be a site, a functor $F : \mathcal{C}^o \to \underline{set}$ is called a *presheaf* on the site $\mathcal{C}$.

It is called a *sheaf* if the following sequence is exact for any object U in $\mathcal{C}$ and any covering $\{U_i \to U\}$ of U:

$$0 \to FU \to \prod_i FU_i \overset{pr_1^*, pr_2^*}{\rightrightarrows} \prod_{i,j} F(U_i \times_U U_j)$$

If the sequence is only exact at the first position, then the presheaf is called *separated.*

Remark 3.10. It may happen that for different site $\mathcal{C}_1$ and $\mathcal{C}_2$ with the same underlining category, the categories $\underline{\text{sheaf on } \mathcal{C}_1}$ and $\underline{\text{sheaf on } \mathcal{C}_2}$ may be the same (see the next subsections).

Not all representable presheaves are sheaves. If we put the canonical topology on a category $\mathcal{C}$, then all representable presheaves on $\mathcal{C}$ are sheaves. It is the finest topology that we can put on a category such that this property holds.

On X_{zar}, all representable presheaves are sheaves, Zariski topology is weaker than the canonical topology. In fact, this exactly means that we can glue morphisms together in zariski topology. But in other topologies, it is not trivial at all.

Theorem 3.11 (Grothendieck)**.** *A representable presheaf on $\underline{X\text{-scheme}}$ is a sheaf in the fpqc topology. Consequently, it is also a sheaf on étale topology.*

Proof. see [2], procedure is similar to the proof of descent theory of quasi-coherent sheaves, using the following reduction lemma 3.12. □

Lemma 3.12 (Reduction Lemma). *Let S be a scheme, $F : \underline{S - scheme}^o \to \underline{set}$ a presheaf. Suppose that F satisfies the following conditions, then F is a sheaf in the fpqc topology.*

(1)F is a sheaf in the global Zariski topology.

(2)Whenever $V \to U$ is a faithfully flat morphism of S-schemes which are both affine, the following sequence is exact.

$$0 \to FU \to FV \rightrightarrows F(V \times_U V)$$

Proof. The proof is similar to the stack version version reduction lemma 5.14, but much more easy. For details see [2]. □

3.4. **Sieves.**

Definition 3.13. Let U be an object of a category $\mathcal{C}$. A *sieve* on U is a subfunctor of $h_U = Hom(\cdot, U) : \mathcal{C}^o \to \underline{set}$.

Given any set of morphisms $\mathcal{U} = \{U_i \to U\}$ (not necessary a covering of U), we can define a subfunctor $h_{\mathcal{U}} \subseteq h_U$, by taking $h_{\mathcal{U}}(T)$ to be the set of morphisms $T \to U$ that factors through some $U_i \to U$ in $\mathcal{U}$.

Let $\mathcal{U} = \{U_i \to U\}$ be a covering of U, $F : \mathcal{C}^o \to \underline{set}$ be a functor. We define $F\mathcal{U}$ to be the subset of $\prod_i FU_i$ containing elements whose images in $\prod_{i,j} F(U_i \times_U U_j)$ are equal. Then the restriction maps induce a function $FU \to F\mathcal{U}$, then F is a sheaf (resp. separated) if and only if this map is bijective(resp. injective).

Definition 3.14. Let $\mathcal{T}$ be a topology on a category $\mathcal{C}$. A sieve $S \subseteq h_U$ on an object U of $\mathcal{C}$ is said to *belong to* $\mathcal{T}$ if there exists a covering $\mathcal{U}$ of U such that $h_{\mathcal{U}} \subseteq S$.

Definition 3.15. Let $\mathcal{C}$ be a category, $\{U_i \to U\}_{i \in I}$ a set of morphisms. A *refinement* $\{V_a \to U\}_{a \in A}$ is a set of morphisms such that for each index $a \in A$ there is some index $i \in I$ such that $V_a \to U$ factors through $U_i \to U$.

Proposition 3.16. *Given two sets $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{V} = \{V_a \to U\}$ in $\mathcal{C}$, then $\mathcal{V}$ is a refinement if and only if $h_{\mathcal{V}} \subseteq h_{\mathcal{U}}$.*

Proof. Check by definition. □

Definition 3.17. Let $\mathcal{C}$ be a category, $\mathcal{T}$ and $\mathcal{T}'$ two topologies on $\mathcal{C}$. We say that $\mathcal{T}$ is *subordinate* to $\mathcal{T}'$, and denoted $\mathcal{T} \prec \mathcal{T}'$, if every covering in $\mathcal{T}$ has a refinement that is a covering in $\mathcal{T}'$.

If $\mathcal{T}$ and $\mathcal{T}'$ are subordinate to each other, then we say that they are *equivalent*, denoted $\mathcal{T} \equiv \mathcal{T}'$

Proposition 3.18. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on $\mathcal{C}$. Then $\mathcal{T} \prec \mathcal{T}'$ if and only if every sieve belonging to $\mathcal{T}$ also belongs to $\mathcal{T}'$.*

In particular, two topologies are equivalent if and only if they have the same sieves.

Proof. Check directly by definition. ∎

Proposition 3.19. *Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on $\mathcal{C}$. If $\mathcal{T} \prec \mathcal{T}'$, then every sheaf in $\mathcal{T}'$ is also a sheaf in $\mathcal{T}$.*

In particular, two equivalent topologies have the same sheaves.

Proof. see [2]. ∎

This means sheaf theory does not depend on the topology, but depends on which sieves are belonging to the topology.

4. Categorical Language

4.1. **Fibred Categories.** Fix a category $\mathcal{C}$, we do not need any topology in this section. $\mathcal{F}$ is a category with a functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$.. We draw the following diagram to mean that for object $p_{\mathcal{F}}\xi = U$ and for morphism $p_{\mathcal{F}}\phi = f$.

$$\begin{tikzcd}
\xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r, "f"] & V
\end{tikzcd}$$

Definition 4.1. Let $\mathcal{F}$ be a category over $\mathcal{C}$. An morphism $\phi : \xi \to \eta$ of $\mathcal{F}$ is *cartesian* if for any morphism $\psi : \zeta \to \eta$ in $\mathcal{F}$ and any morphism $h : p_{\mathcal{F}}\zeta \to p_{\mathcal{F}}\xi$ in $\mathcal{C}$ with $p_{\mathcal{F}}\phi \circ h = p_{\mathcal{F}}\psi$, there exists a unique morphism $\theta : \zeta \to \xi$ with $p_{\mathcal{F}}\theta = h$ and $\phi \circ \theta = \psi$, as in the commutative diagram

$$\begin{tikzcd}
\zeta \arrow[dr, dashed, "\theta"'] \arrow[drr, bend left, "\psi"] \arrow[ddd, mapsto] & & \\
& \xi \arrow[r, "\phi"] \arrow[ddd, mapsto] & \eta \arrow[ddd, mapsto] \\
& & \\
p_{\mathcal{F}}\zeta \arrow[dr, "h"'] \arrow[drr, bend left] & & \\
& p_{\mathcal{F}}\xi \arrow[r] & p_{\mathcal{F}}\eta
\end{tikzcd}$$

If $\xi \to \eta$ is a cartesian morphism of $\mathcal{F}$ mapping to an morphism $U \to V$ of $\mathcal{C}$, we also say that ξ is a *pullback* of η to U. It is unique up to a unique isomorphism.

Definition 4.2. A *fibred category over* $\mathcal{C}$ is a category $\mathcal{F}$ over $\mathcal{C}$, such that given a morphism $f : U \to V$ in $\mathcal{C}$ and an object η of $\mathcal{F}$ mapping to V, there is a cartesian morphism $\phi : \xi \to \eta$ with $p_{\mathcal{F}}\phi = f$.

Definition 4.3. If $\mathcal{F}$ and $\mathcal{G}$ are two fibred categories over $\mathcal{C}$, then a *morphism* of fibred categories $F : \mathcal{F} \to \mathcal{G}$ is a functor such that:

(1)F is *base-preserving*, that is, $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$;

(2)F sends cartesian morphisms to cartesian morphisms.

Definition 4.4. Let $\mathcal{F}$ be a fibred categories over $\mathcal{C}$. Given an object U of $\mathcal{C}$, the *fibre* $\mathcal{F}(U)$ of $\mathcal{F}$ over U is the subcategory of $\mathcal{F}$ whose objects are the objects ξ of $\mathcal{F}$ that are mapping to U, and whose morphisms are morphisms in $\mathcal{F}$ that are mapping to id_U.

If $F : \mathcal{F} \to \mathcal{G}$ is a morphism of fibred categories over $\mathcal{C}$, and U is an object of $\mathcal{C}$, the functor F sends $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so we have a restriction functor $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

Definition 4.5. A *cleavage* of a fibred category $\mathcal{F} \to \mathcal{C}$ consists of a class K of cartesian morphisms in $\mathcal{F}$ such that for each morphism $f : U \to V$ in $\mathcal{C}$ and each object η in $\mathcal{F}(V)$ there exists a unique morphism in K with target η mapping to f in $\mathcal{C}$.

A cleavage is called a *splitting* if it contains all the identities, and it is closed under composition. A fibred category endowed with a splitting is called *split*

By the axiom of choice, every fibred category $\mathcal{F} \to \mathcal{C}$ has a cleavage. In fact, it is unique up to a unique isomorphism. But it is not necessary that the fibred category has a splitting.

Proposition 4.6. *Every fibred category is equivalent to a canonically defined split fibred category.*

Proof. See [2]. □

Suppose that $\mathcal{V} \to \mathcal{C}$ is a fibred category with a chosen cleavage, S an object of $\mathcal{C}$, $\mathcal{C}/S$ the subcategory of S-objects in $\mathcal{C}$. Let ξ and η be two objects in $\mathcal{F}(U)$. Given $u : U \to S$ in $\mathcal{C}/S$, $u^*\xi$ and $u^*\eta$ are pullbacks in the chosen cleavage of ξ and η. Define $\underline{Hom}_S(\xi, \eta)(U)$ to be the set $Hom_{\mathcal{F}(U)}(u^*\xi, u^*\eta)$. If $f : U_1 \to U_2$ is a morphism in $\mathcal{C}/S$, denote $\xi_i = u_i^*\xi$ and $\eta_i = u_i^*\eta$, they are objects in $\mathcal{F}(U_i)$. Then there is a unique morphism α_f (resp. β_f), which is again cartesian, making the following diagram commute.

$$\begin{tikzcd}
\xi_1 \arrow[rr, dashed, "\alpha_f"] \arrow[dr] & & \xi_2 \arrow[dl] \\
& \xi &
\end{tikzcd}$$

. Therefore from the universal property of cartesianess, there exists a unique morphism $f^*\phi$ for any $\phi \in Hom_{\mathcal{F}(U_2)}(\xi_2, \eta_2)$ satisfying the following diagram.

$$\begin{tikzcd}
 & \xi_1 \arrow[rr, dashed, "f^*\phi"] \arrow[dl, "\alpha_f"'] \arrow[dd] & & \eta_1 \arrow[dl, "\beta_f"'] \arrow[dd] \\
\xi_2 \arrow[rr, "\phi"] \arrow[dd] & & \eta_2 \arrow[dd] & \\
 & U_1 \arrow[rr, equal] \arrow[dl, "f"'] & & U_1 \arrow[dl, "f"'] \\
U_2 \arrow[rr, equal] & & U_2 &
\end{tikzcd}$$

, hence we have defined a pullback function

$$f^* : Hom_{\mathcal{F}(U_2)}(\xi_2, \eta_2) \to Hom_{\mathcal{F}(U_1)}(\xi_1, \eta_1)$$

It can be check that this gives a functor $\underline{Hom}_S(\xi, \eta) : (\mathcal{C}/S)^o \to \underline{set}$ since the pullback in a chosen cleavage is unique, sending U to $Hom_{\mathcal{F}(U)}(u^*\xi, u^*\eta_2)$ and f to f^*. This functor does not depend on the choice of the cleavage in the sense that different cleavages give functors which are canonically isomorphic.

4.2. **Pseudo-functor.**

Definition 4.7. A *pseudo-functor* Φ on $\mathcal{C}$ consists of the following data.

(1)For each object U of $\mathcal{C}$ a category ΦU;

(2)For each morphism $f : U \to V$ a functor $f^* : \Phi V \to \Phi U$;

(3)For each object U of $\mathcal{C}$ an isomorphism $\epsilon_U : id_U^* \simeq id_{\Phi U}$ of functors $\Phi U \to \Phi U$;

(4)For each pair of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ an isomorphism

$$\alpha_{f,g} : f^*g^* \simeq (gf)^* : \Phi W \to \Phi U$$

of functors $\Phi W \to \Phi U$;

These data are required to satisfy the following conditions:

(a)If $f : U \to V$ is an morphism in $\mathcal{C}$ and η is an object of ΦV, we have

$$\alpha_{id_U, f}(\eta) = \epsilon_U(f^*\eta) : id_U^* f^*\eta \to f^*\eta$$

and

$$\alpha_{f, id_V}(\eta) = f^*\epsilon_V(\eta) : f^* id_V^* \eta \to f^*\eta;$$

(b)Whenever we have morphisms $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$ and an object θ of $\mathcal{F}(T)$, the diagram following commutes.

$$\begin{tikzcd}
f^*g^*h^*\theta \arrow[r, "\alpha_{f,g}(h^*\theta)"] \arrow[d, "f^*\alpha_{g,h}(\theta)"'] & (gf)^*h^*\theta \arrow[d, "\alpha_{gh,f}(\theta)"] \\
f^*(hg)^*\theta \arrow[r, "\alpha_{f,hg}(\theta)"] & (hgf)^*\theta
\end{tikzcd}$$

Proposition 4.8. *A fibred category over $\mathcal{C}$ with a cleavage defines a pseudo-functor on $\mathcal{C}$.*

Proof. As seen in definition 4.4, from a fibred category $\mathcal{F} \to \mathcal{C}$ we nearly get a functor $U \mapsto \mathcal{F}(U)$ the fibred over any object U of $\mathcal{C}$. Once we choose a cleavage, we will get the $\alpha's$ from the uniqueness of cartesian morphism with respect to this cleavage, then every condition can be checked. For details, see [2]. □

Proposition 4.9. *From a pseudo-functor on $\mathcal{C}$, a fibred category over $\mathcal{C}$ can be defined. Moreover, these two procedures are inverse to each other(up to an isomorphism of fibred categories).*

Proof. It is easy to construct a fibred category naturally, but one needs to check everything which is confusing and boring. For details, see [2]. □

Therefore, to study a pseudo-functor is equivalent to study a fibred category with a cleavage.

4.3. **Examples.** In fact, only one example will be given here: the fibred category of quasi-coherent sheaves.

Let $\mathcal{C} = \underline{X - scheme}$, for $U \to X$, we define $QCoh(U)$ to be the category of quasi-coherent sheaves over U. For any X-morphism $f : U \to V$, the pull back of any quasi-coherent sheaf is quasi-coherent, so we get a functor $f^* : QCoh(V) \to QCoh(U)$. However, in general for $U \xrightarrow{f} V \xrightarrow{g} W$, $(gf)^* \neq f^*g^*$, so $U \mapsto QCoh(U)$ is not a functor. But $(gf)^*$ and f^*g^* are canonically isomorphic since $(gf)_* = f_*g_*$ and f^* is left adjoin to f_*, Yoneda lemma induces the canonical isomorphism between functors $(gf)^*$ and f^*g^*. One can also check that the isomorphisms above satisfy the conditions in Definition 4.7, so we get a pseudo-functor, hence a fibred category $\underline{QCoh/X} \to \underline{X - scheme}$ by Proposition 4.9. For details, see [2].

5. Descent Theory for Quasi-coherent Sheaves

5.1. **Descent Data.** Let $\mathcal{C}$ be a site, $\mathcal{F}$ a fibred category over $\mathcal{C}$, we fix a cleavage. Given a covering $\mathcal{U} = \{\sigma_i : U_i \to U\}$, set $U_{ij} = U_i \times_U U_j$ and $U_{ijk} = U_i \times_U U_j \times_U U_k$ (in fact they depend on the "restriction" map), sometimes they are denoted simply by U_α, where the index α stands for i, ij or ijk etc.

Definition 5.1. Let $\mathcal{U} = \{\sigma_i : U_i \to U\}$ be a covering in $\mathcal{C}$. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$, is a colletion of objects ξ_i in $\mathcal{F}(U_i)$, together with isomorphisms $\phi_{ij} : pr_2^*\xi_j \simeq pr_1^*\xi_i$ in $\mathcal{F}(U_{ij})$, such that the following cocycle condition is satisfied.

For any triple of indices i, j and k, we have the equality

$$pr_{13}^*\phi_{ik} = pr_{12}^*\phi_{ij} \circ pr_{23}^*\phi_{jk} : pr_3^*\xi_k \to pr_1^*\xi_i$$

The isomorphisms ϕ_{ij} are called *transition isomorphisms* of the object with descent data.

An morphism between objects with descent data

$$\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of morphisms $\alpha_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$ such that the following diagram commutes for each pair of indices i, j.

$$\begin{array}{ccc} pr_2^*\xi_j & \xrightarrow{pr_2^*\alpha_j} & pr_2^*\eta_j \\ \phi_{ij}\downarrow & & \downarrow\psi_{ij} \\ pr_1^*\xi_i & \xrightarrow{pr_1^*\alpha_i} & pr_1^*\eta_i \end{array}$$

It can be check that we have defined a category with objects with descent data as objects, and morphism as above, this category is denoted by $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\{U_i \to U\})$

Remark 5.2. This category does not depend on the choice of fibred product U_{ij} and U_{ijk}, from different choices we get isomorphic categories.

Until now, we didn't use the fixed cleavage.

For each object ξ of $\mathcal{F}(U)$ we can construct a object with descent data on the covering $\mathcal{U} = \{\sigma_i : U_i \to U\}$. First set $\xi_i = \sigma_i^*\xi$. $pr_1^*\xi_i$ is the unique pull back of ξ_i along $pr_1 : U_{ij} \to U_i$ in the fixed cleavage, $pr_2^*\xi_j$ respectively, then there is a unique isomorphism $\phi_{ij} : pr_2^*\xi_j \to pr_i^*\xi_i$. It is easy to check that this gives a object with descent data on $\mathcal{U}$. Similarly for morphisms in $\mathcal{F}(U)$. We have defined a functor $\mathcal{F}(U) \to \mathcal{F}(U_i \to U)$.

Remark 5.3. This functor does not depend on the choice of the cleavage on $\mathcal{C}$ up to a canonical isomorphism of functors. There are other ways to define the category $\mathcal{F}(\mathcal{U})$ which do not need cleavage at all.

For better understanding of this most important concept, I would like to give another definition of descent data as follows.

First of all, we define an object with descent data to be a triple of sets

$$(\{\xi_i\}_{i\in I}\{\xi_{ij}\}_{i,j\in I}\{\xi_ijk\}_{i,j,k\in I}),$$

with a commutative diagram–the first one below, where each ξ_α is an object of $\mathcal{F}(U_\alpha)$, and in the diagram each arrow is cartesian and when applying the functor $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ one get part of the second commutative diagram with all faces cartesian(notice that it is not necessary always commutes).

$$
\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] & & & U_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & U_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_{j} & & \overset{p_{\mathcal{F}}}{\longmapsto} & U_{ij} \arrow[rr] \arrow[dd] & & U_{j} \arrow[dd] & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_{k} & & & U_{ik} \arrow[rr] \arrow[dl] & & U_{k} \arrow[dl] \\
\xi_{i} & & & & & U_{i} \arrow[rr] & & U &
\end{tikzcd}
$$

These form the objects of a category $\mathcal{F}_{desc}(\{U_i \to U\})$.

A morphism

$\{\phi_i\}_{i\in I} : (\{\xi_i\}_{i\in I}\{\xi_{ij}\}_{i,j\in I}\{\xi_i jk\}_{i,j,k\in I}) \to (\{\eta_i\}_{i\eta I}\{\eta_{ij}\}_{i,j\in I}\{\eta_i jk\}_{i,j,k\in I})$

consists of set of morphisms $\phi_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$, such that for every pair of indices i and j we have $pr_1^*\phi_i = pr_2^*\phi_j : \xi_{ij} \to \eta_{ij}$.

Similarly, we can define a category whose object is triples $(\{\xi_i\}_{i\in I}\{\xi_{ij}\}_{i,j\in I}\{\xi_i jk\}_{i,j,k\in I})$ with commutative diagram in $\mathcal{F}$:

$$
\begin{tikzcd}
& \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\
\xi_{ij} \arrow[rr] \arrow[dd] & & \xi_{j} \arrow[dd] & \\
& \xi_{ik} \arrow[rr] \arrow[dl] & & \xi_{k} \arrow[dl] \\
\xi_{i} \arrow[rr] & & \xi &
\end{tikzcd}
$$

in which all arrows are cartesian and when applying $p_{\mathcal{F}}$ we get the commutative diagram of U_α in $\mathcal{C}$ as above. An morphism of $\mathcal{F}_{comp}(\{U_i \to U\})$ is just an morphism $\phi : \xi \to \eta$ in $\mathcal{F}(U)$.

We claim without proof that $\mathcal{F}_{comp}(\{U_i \to U\})$ is equivalent to $\mathcal{F}(U)$ by forgetting everything except ξ. And $\mathcal{F}_{desc}(\{U_i \to U\})$ is equivalent to $\mathcal{F}(\{U_i \to U\})$. Moreover the functor $\mathcal{F}_{comp}(\{U_i \to U\}) \to \mathcal{F}_{desc}(\{U_i \to U\})$ by forgetting ξ corresponds to the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$.

This is similar to the several equivalent but different definitions of vector bundles in geometry, in fact essentially they are one thing since a vector bundle can be view as a sheaf which is a particular case of stack.

Remark 5.4. Since sheaf theory is determined by sieves on a topology, in the language of sieves the results can be stated more neatly, but I'm not going to give the details at this moment, for the some discussion one can refer to [2].

5.2. **Stacks.**

Definition 5.5. Let $\mathcal{U}$ be a fibred category over a site $\mathcal{C}$.

(1)$\mathcal{F}$ is a *prestack* if for each covering $\mathcal{U} = \{U_i \to U\}$ of U, the functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is fully faithful.

(2)$\mathcal{F}$ is a *stack* if for each covering $\mathcal{U} = \{U_i \to U\}$ of U, the functor $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is an equivalence.

Definition 5.6. An object with descent data in $\mathcal{F}(\{U_i \to U\})$ is *effective* if it is isomorphic to the image of an object of $\mathcal{F}(U)$.

Concretely, for fibred category $\mathcal{F}$ over a site $\mathcal{C}$ to be a prestack means the following. For any object U in $\mathcal{C}$ and any covering $\{U_i \to U\}$, two objects ξ and η in $\mathcal{F}(U)$, ξ_i, ξ_{ij}, η_i, η_{ij} are some pullbacks to U_i and U_{ij}. Suppose that there are morphisms $\alpha_i : \xi_i \to \eta_i$ in $\mathcal{F}(U_i)$, such that $pr_1^*\alpha_i = pr_2^*\alpha_j : \xi_{ij} \to \eta ij$ for all i, j. Then there is a unique morphism $\alpha : \xi \to \eta$ in $\mathcal{F}(U)$, whose pullback to $\xi_i \to \eta_i$ is α_i for all i. By translating the language, we obtain:

Proposition 5.7. *Let $\mathcal{F}$ be a fibred category over a site $\mathcal{C}$. Then $\mathcal{F}$ is a prestack if and only if for any object S of $\mathcal{C}$ and any two objects ξ and η in $\mathcal{F}(S)$, the functor (see subsection 4.1) $\underline{Hom}_S(\xi, \eta) : (\mathcal{C}/S)^o \to \underline{set}$ is a sheaf with respect to the induced topology of $\mathcal{C}/S$ from $\mathcal{C}$.*

Proof. See [2]. □

Remark 5.8. For understanding, the notion of stack is similar to the notion of sheaf. If we can glue the sections (one kind of local data) together then what we get is a sheaf, here if we can glue the descent data (another kind of local data) together then what we get is a stack.

Example 5.9. In the classical Zariski topology, to give a sheaf on X, we can give it on the covering of X, but we require that out local data on the covering are compatible (the cocycle condition), then the standard argument shows that there exists a unique sheaf on X such that after restricting to the covering we get the local data given at the beginning. This means that the fibred category $\underline{sheaf/X_{zar}} \to X_{zar}$ is a stack.

Proposition 5.10. *Let $\mathcal{C}$ be a site, F a presheaf of sets, $F(U)$ is a set which can be viewed as a category and F is a pseudo-functor, so we get a fibred category $F \to \mathcal{C}$. Then F is a prestack (resp. stack) if and only if it is a separated presheaf (resp. sheaf).*

Proof. Omitted, see [2]. □

Proposition 5.11. *If two fibred categories over a site are equivalent, then the fact that one of them is a stack (resp. prestack) implies the other.*

Proof. See [2]. □

Lemma 5.12. *If $\mathcal{F}$ is a prestack on a site, $\mathcal{U}$ and $\mathcal{V}$ two covering of and object U of $\mathcal{C}$, with $\mathcal{V}$ a refinement of $\mathcal{U}$, and $\mathcal{F}(U) \to \mathcal{F}(\mathcal{V})$ is an equivalence, then $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ is also an equivalence.*

Proof. See [2]. □

5.3. **Descent Theory.**

Theorem 5.13. *Let $V \to U$ be a flat surjective morphism of affine schemes, then $QCoh(U) \to QCoh(V \to U)$ is an equivalence of categories.*

Proof. We need only to translate the Theorem 1.4. Let $U = Spec(A)$ and $V = Spec(B)$ with $V \to U$ induced by $A \to B$ which is faithfully flat by Theorem 1.2. We have already known that the categories $QCoh(U)$ (resp. $Qcoh(V)$) and Mod_A (resp. Mod_B) are equivalent, with respect to $M \mapsto \widetilde{M}$. To describe an object (i.e. object with descent data) in $QCoh(V \to U)$, first there is a quasi-coherent sheaf $\mathcal{M}$ over V, $\mathcal{M} = \widetilde{M}$ with a B-module M. In our covering $\{\sigma : V \to U\}$ there is only one map, pull back along pr_1^* and pr_2^* from V to $V \times_U V$, we get quasi-coherent sheaves $pr_1^*\mathcal{M} = \widetilde{M \otimes_A B}$ and $pr_2^*\mathcal{M} = \widetilde{B \otimes_A M}$ on $V \times_U V$, an isomorphism $\phi : pr_2^*\mathcal{M} \to pr_1^*\mathcal{M}$ as sheaves on $V \times_U V$ corresponds to an isomorphism $\psi : B \otimes_A M \to M \otimes_A B$ as $B \otimes B$-modules (unfortunately, I'm so careless that in this equivalence the foot-indices of one of the definitions $QCoh(V \to U)$ and $Mod_{A\to B}$ should be reversed, the ψ here should be ψ^{-1} in the definition of $Mod_{A\to B}$, even [2] made this mistake!). The cocycle condition for ϕ holds if and only if $\psi_1\psi_3 = \psi_2$. And also the morphisms of objects with descent data on $V \to U$ correspond to morphisms in $Mod_{A\to B}$. So we get the equivalence $Mod_{A\to B} \simeq QCoh(V \to U)$. And the functors $QCoh(U) \to QCoh(V \to U)$ and $Mod_A \to Mod_{A\to B}$ are corresponding. Therefore we have translated the Theorem 1.4 to this theorem. □

Lemma 5.14 (Reduction Lemma)**.** *Let S be a scheme, $\mathcal{F}$ be a fibred category over the category $\underline{S - scheme}$. Suppose that the following conditions are satisfied, then $\mathcal{F}$ is a stack with respect to the fpqc topology.*

(1)$\mathcal{F}$ is a stack with respect to the Zariski topology;

(2)Whenever $V \to U$ is a flat surjective morphism of S-schemes which are both affine, the functor $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is an equivalence of categories.

Remark 5.15. This type of lemma appears frequently in any theory considering Grothendieck topology, it reduces the global fact to commutative algebra and basic algebraic geometry in Zariski topology.

Theorem 5.16 (descent of quasi-coherent sheaves)**.** *Let S be a scheme. The fibred category $\underline{QCoh/S}$ over $\underline{S - scheme}$ is a stack with respect to the fpqc topology.*

Proof. It is a standard fact that $QCoh$ is a stack in Zariski topology, see example 5.9. For the affine case, it is Theorem 5.13. This is enough by the reduction Lemma 5.14. □

Proof of reduction lemma. This is just a sketch, for a complete proof see [2].

According to Proposition 4.6 and 5.11, we can assume that $\mathcal{F}$ is split fibred category.

The proof divides to several steps.

Step 1, $\mathcal{F}$ is a prestack. Given an S-scheme $T \to S$ and two objects ξ and η of $\mathcal{F}(T)$, from the definition of descent data, we see immediately that the functor $\underline{Hom}_T(\xi, \eta) : \underline{T - scheme}^o \to \underline{set}$ satisfies the two conditions in Lemma 3.12, so it is a sheaf in the fpqc topology, then by 5.7 $\mathcal{F}$ is a prestack in the fpqc topology.

Step 2, reduction to the case of a single morphism.

First notice that if U is a disjoint union of open subschemes U_i, then the functor $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$ given by restrictions is an equivalence of categories.

Given any covering $\{U_i \to U\}$ in fpqc topology, set $V = \coprod_i U_i$ with induced morphism $V \to U$, this is again a covering in fpqc topology by definiton. We claim that the functor $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ is an equivalence if and only if $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is. In fact, we will show that there is an equivalence $\mathcal{F}(V \to U) \to \mathcal{F}(\{U_i \to U\})$ commuting with the two functors above.

It is easy to see that $V \times_U V \simeq \coprod i, j(U_i \times_U U_j)$. Hence we get the equivalences $\mathcal{F}(V) \to \prod \mathcal{F}(U_i)$ and $\mathcal{F}(V \times_U V) \to \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$. Let (η, ϕ) be an object in $\mathcal{F}(V \to U)$, this means η is an object of $\mathcal{F}(V)$ and $\phi : pr_2^* \eta \to pr_1^* \eta$ an isomorphism in $\mathcal{F}(V \times_U V)$ with cocycle condition. Let η_i be the restriction of η to U_i and ϕ_{ij} be the pull back of ϕ to $U_i \times_U U_j$, the cocycle condition of ϕ give those of ϕ_{ij}. Check that $(\eta, \phi) \mapsto (\{\eta_i\}, \{\phi_{ij}\})$ is an equivalence $\mathcal{F}(V \to U) \to \mathcal{F}(\{U_i \to U\})$.

Step 3, the case of a quasi-compact morphism with affine target

Let $V \to U$ be a faithfully flat quasi-compact morphism with U affine, it is a fpqc morphism by Remark 2.4. Then V is quasi-compact, we can take a Zariski covering of V consisting of finite many open affines V_i. Let $V' = \coprod V_i$ with induced $V' \to U$ which is again a faithfully flat quasi-compact hence fpqc morphism. And the covering $V' \to U$ is a refinement of $V \to U$ in the fpqc topology, and $\mathcal{F}$ is already a prestack by step 1. But now V' and U are both affine, by hypothesis (2) $\mathcal{F}(U) \to \mathcal{F}(V' \to U)$ is an equivalence, hence so is $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ by Lemma 5.12.

Step 4, the case of a morphism with affine target.

Now let $V \to U$ be any fpqc morphism morphism with U affine. Then by Proposition 2.2(1), there is a quasi-compact open subscheme W of V maps onto U, for each $x \in V$ take an affine open neighborhood V_x. Then $W \cup V_x$ is a Zariski open covering of V, each of which is quasi-compact, and each $W \cup V_x \to U$ is fpqc by Proposition 2.2(2). What we get is a Zariski open covering $\{V_i\}$ of V by quasi-compact open subschemes, with $V_i \to U$ a fpqc covering for all i.

Now choose an i, it follows from the previous step that $\mathcal{F}(U) \to \mathcal{F}(V_i \to U)$ is an equivalence. Consider the following diagram of functors,

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[rr] \arrow[dr] & & \mathcal{F}(V \to U) \arrow[dl] \\
& \mathcal{F}(V_i \to U) &
\end{tikzcd}$$

notice that the functors are defined through the split cleavage, in the definition for any object η in $\mathcal{F}(U)$, pulling it back two times to $\mathcal{F}(V \to U)$ then to $\mathcal{F}(V_i \to U)$ is not necessary the same with pulling it back directly to $\mathcal{F}(V_i \to U)$ (it makes sense if we fix a cleavage), but here it really equals since split cleavage is closed under composition and the pull back chosen in a fixed cleavage is unique, so the diagram commutes. Now the left edge is an equivalence and the top edge is fully faithful by step 1, in this step we need to show that the top edge is also essentially surjective, it follows from the diagram that we only need to show that the right edge $\mathcal{F}(V \to U) \to \mathcal{F}(V_i \to U)$ is fully faithful.

$\mathcal{F}(U) \to \mathcal{F}(V_i \to U)$ is an equivalence hence is full, this implies $\mathcal{F}(V \to U) \to \mathcal{F}(V_i \to U)$ is full, so it is enough to show that it is faithful. Consider the following commutative (the same reason as before) diagram involving another index j,

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[rr] \arrow[dr] & & \mathcal{F}(V_i \cup V_j \to U) \arrow[dl] \\
& \mathcal{F}(V_i \to U) &
\end{tikzcd}$$

step 3 shows that two edges are equivalences, hence so is the third. We get the equivalence $\mathcal{F}(V_i \cup V_j \to U) \to \mathcal{F}(V_i \to U)$, similarly $\mathcal{F}(V_i \cup V_j \to U) \to \mathcal{F}(V_j \to U)$, hence an equivalence $\mathcal{F}(V_j \to U) \to \mathcal{F}(V_i \to U)$ for any j. If two morphisms in $\mathcal{F}(V \to U)$ map to the same morphism in $\mathcal{F}(V_i \to U)$, then they map to the same in $\mathcal{F}(V_j \to U)$ for any j, so they are the same in $\mathcal{F}(V \to U)$ since $\mathcal{F}$ is separated by step 1.

Step 5, the general case

Let $f : V \to U$ be a fpqc morphism, by Proposition 2.2, U can be written as a union of affine opens U_i such that $V_i = f^{-1}(U_i) \to U_i$ is a fpqc morphism which is denoted by f_i. We have seen that $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is fully faithful, need to show that it is essentially surjective. In the following, the pull back along fpqc morphism will be called pull back, but along Zariski open subset will be called restriction and denoted by "|".

For each open subscheme U' of U, we have a fully faithful functor $\Phi_{U'}$: $\mathcal{F}(U') \to \mathcal{F}(f^{-1}U' \to U')$ such that the following diagrams of functors commute for any $U'' \subseteq U'$ (this use the split cleavage).

$$
\begin{tikzcd}
\mathcal{F}(U') \arrow[r] \arrow[d, "\Phi_{U'}"'] & \mathcal{F}(U'') \arrow[d, "\Phi_{U''}"] \\
\mathcal{F}(f^{-1}U' \to U') \arrow[r] & \mathcal{F}(f^{-1}U'' \to U'')
\end{tikzcd}
\tag{5.3.1}
$$

where the rows is just the natural restrictions.

Now take a object with descent data (η, ϕ) in $\mathcal{F}(V \to U)$, want to find a object ξ in $\mathcal{F}(U)$ such that $\Phi_U(\xi) \simeq (\eta, \phi)$.

Let (η_i, ϕ_i) be restriction of (η, ϕ) to V_i. Since $\mathcal{F}(U_i) \to \mathcal{F}(V_i \to U_i)$ is an equivalence by step 4, we get an object ξ_i in $\mathcal{F}(U_i)$ and an isomorphism $\alpha_i : \Phi_{U_i}\xi_i \simeq (\eta_i, \phi_i)$ in $\mathcal{F}(V_i \to U_i)$. We want to glue it in Zariski covering $\{U_i \to U\}$ to get ξ, we need some descent data $\phi_{ij} : \xi_j|_{U_{ij}} \simeq \xi_j|_{U_{ij}}$, which will be construct as follows with the cocycle conditions.

It is easy to see that $V_{ij} = V_i \cap V_j = f^{-1}(U_{ij})$, then $\alpha_i : \Phi_{U_i}\xi_i \simeq (\eta_i, \phi_i)$ restrict to V_{ij} we get an isomorphism $\Phi_{U_{ij}}(\xi_i|_{U_{ij}}) = (\Phi_{U_i}\xi_i)|_{V_{ij}} \xrightarrow{\alpha_i|_{V_{ij}}} (\eta, \phi)|_{V_{ij}}$, where the equality comes from the diagram 5.3.1 which tells us that Φ commutes with restriction, it is still isomorphic since descent data are functorial in everything that you can thought. Therefore, we obtain the isomorphism $\alpha_i^{-1}\alpha_j : \Phi_{U_{ij}}(\xi_j|_{U_{ij}}) \simeq \Phi_{U_{ij}}(\xi_i|_{U_{ij}})$, but $\Phi_{U_{ij}}$ is an equivalence, so there exists a unique isomorphism $\phi_{ij} : \xi_j|_{U_{ij}} \simeq \xi_i|_{U_{ij}}$ such that $\Phi_{U_{ij}}\phi_{ij} = \alpha_i^{-1}\alpha_j$. By applying $\Phi_{U_{ijk}}$ to these isomorphisms we get the cocycle conditions $\phi_{ik} = \phi_{ij}\phi_{jk}$ since $\alpha_i^{-1}\alpha_j$'s do and $\Phi_{U_{ijk}}$ is at least fully faithful. By hypothesis (1), we glue together the ξ_i's to get ξ in $\mathcal{F}(U)$ with $t_i : \xi|_{U_i} \simeq \xi_i$ where $t_i|_{U_{ij}} = \phi_{ij} \circ t_j|_{U_{ij}}$. At last we need to check that under the functor $\mathcal{F}(U) \to \mathcal{F}(U \to V)$ it is sent to something isomorphic to (η, ϕ).

Since $t_i : \xi|_{U_i} \simeq \xi_i$, $\Phi_{U_i}(\xi|_{U_i}) \overset{\Phi_{U_i}(t_i)}{\simeq} \Phi_{U_i}\xi_i \overset{\alpha_i}{\simeq} (\eta_i, \phi_i) = (\eta, \phi)|_{V_i}$. On the other hand, the left hand side equals to $\Phi_U(\xi)|_{V_i}$ since the diagram 5.3.1 means Φ commutes with restriction. Hence we obtain an isomorphism (for each i) $\alpha_i \circ \Phi_{U_i}(t_i) : \Phi_U(\xi)|_{V_i} \simeq (\eta, \phi)|_{V_i}$. Combine $\Phi_{U_{ij}}\phi_{ij} = \alpha_i^{-1}\alpha_j$, $t_i|_{U_{ij}} = \phi_{ij} \circ t_j|_{U_{ij}}$ and the fact that restriction commutes with Φ, we can see when restrict further to V_{ij}, the isomorphisms $\alpha_i \circ \Phi_{U_i}(t_i)$ coincide. With respect to the Zariski covering $\{V_i \to V\}$ of V, we glue the isomorphism together to get an isomorphism $f^*\xi \simeq \eta$ whose pullback is $\alpha_i : f^*\xi_i \simeq \eta_i$ (see the discuss after Definition 5.5), this gives an isomorphism of objects with descent data $\Phi(U) \simeq (\eta, \phi)$ (we need some more commutative diagrams which can be checked by the uniqueness of Zariski gluing, but boring).

This completes the proof of reduction lemma. $\square$

6. Application

I just want to make the theory look like richer. I'm not going to make these things into precise, only some facts as application are given here. One can refer to [6] for some information, which do not contain all details either.

For a finite Galois extension of fields L/K, $Spec(L) \to Spec(K)$ is faithful flat. To give an affine group schemes over L (resp. K) is equivalent to give a Hopf algebra over L (resp. K), that is an L-vector space with some algebraic structures. These structures can be define as some mapping between L(resp. K)-vector spaces. A twisted K-form of an affine group scheme split by L is given by some descent data, two twisted form are isomorphic over K if and only if they are isomorphic over L and this isomorphism commutes with the descent data (i.e. morphism in the category of descent data). And the isomorphic classes of twisted forms are classified by the Galois cohomology $H^1(Gal(L/K), G)$ where G is the group of L-automorphisms of the affine group scheme. For the most trivial case $\mathbb{G}_a^n$, $G = Gl_n$, descent theory of faithfully flat modules tells us that two descent data are isomorphic if and only if the two vector spaces are isomorphic, but isomorphic classes of vector spaces are determined by the dimension, so we obtain the following.

Corollary 6.1 (general Hilbert's Theorem 90)**.** $H^1(Gal(L/K), \mathbf{Gl_n}) = 0$.

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