Some Arithmetic Duality Theorems

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Outline of Part I

Galois cohomology

1. Local duality
   - Duality with respect to a class formation
   - Local duality
   - Euler-Poincaré characteristic

2. An application to Abelian varieties

3. Global duality
   - A duality theorem
   - Poitou-Tate exact sequence
   - Euler-Poincaré characteristic
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Etale cohomology

4 Local duality

5 Global cohomology
   - Some notations and calculations
   - Euler-Poincaré characteristic

6 Artin-Verdier’s theorem
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A very brief introduction

Why arithmetic duality??

- In mathematics, solving equations is always interesting.
- e.g. rational points on a variety $V(\mathbb{Q}) = ?$
- Why Galois / étale cohomology?
  - e.g. $H^1_{\text{ét}}(\text{spec}(\mathcal{O}_K), \mathbb{Z}/m\mathbb{Z})^* = \text{Cl}(K)/m\text{Cl}(K)$ for $K$ a number field
  - e.g. $H^1(\mathbb{Q}_p, E)^* = E(\mathbb{Q}_p)$ for $E/\mathbb{Q}_p$ an elliptic curve
- They give some certain obstructions of the local-global principal for the problem of rational points.
  - A famous example : $\text{III}(\mathbb{Q}, E)$ for an elliptic curve.
- Tentative conclusion : the cohomology groups contain important arithmetic information.
- Arithmetic duality theorems may help to understand the question of rational points.
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Part I

Galois cohomology
Definition

Let $G$ be a profinite group, and $C$ be a $G$-module (such that $C = \bigcup_{U \leq_o G} C^U$). We say that $(G, C)$ is a class formation if there exists an isomorphism $\text{inv}_U : H^2(U, C) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ for each open subgroup $U \leq_o G$ with the commutative diagram for $V \leq_o U \leq_o G$:

$$
\begin{array}{ccc}
H^2(U, C) & \xrightarrow{Res_{V,U}} & H^2(V, C) \\
\downarrow \text{inv}_U & & \downarrow \text{inv}_V \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{[U:V]} & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

and $H^1(U, C) = 0$. 
(G, C) = class formation, M = G-module \[\implies\] natural pairing:

\[\text{Ext}_G^r(M, C) \times H^{2-r}(G, M) \rightarrow H^2(G, C) \simeq \mathbb{Q}/\mathbb{Z},\]

\[\implies\]

\[\alpha^r(G, M) : \text{Ext}_G^r(M, C) \rightarrow H^{2-r}(G, M)^* = \text{Hom}(H^{2-r}(G, M), \mathbb{Q}/\mathbb{Z}),\]

On the other hand, (G, C) \[\implies\] the reciprocity map

\[\text{rec} : C^G \rightarrow G^{ab}.\]
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\text{Ext}_G^r(M, C) \times H^{2-r}(G, M) \rightarrow H^2(G, C) \cong \mathbb{Q}/\mathbb{Z},
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\[\sim \sim \sim \rightarrow\]

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On the other hand, (G, C) \[\sim \sim \sim \rightarrow\] the reciprocity map

\[\text{rec} : C^G \rightarrow G^{ab}.\]
Lemma

Let \((G, \mathcal{C})\) be a class formation and \(M\) be a finite \(G\)-module, then

(i) \(\alpha^r(G, M)\) is bijective for all \(r \geq 2\);

(ii) \(\alpha^1(G, M)\) is bijective if \(\alpha^1(U, \mathbb{Z}/m\mathbb{Z})\) is bijective for all \(m\) and all \(U \leq_o G\);

(iii) \(\alpha^0(G, M)\) is surjective (resp. bijective) if \(\alpha^0(U, \mathbb{Z}/m\mathbb{Z})\) is surjective (resp. bijective) for all \(m\) and all \(U \leq_o G\).
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Some Arithmetic Duality Theorems
Lemma

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2. \((ii)\alpha^1(G, M)\) is bijective if \(\alpha^1(U, \mathbb{Z}/m\mathbb{Z})\) is bijective for all \(m\) and all \(U \trianglelefteq G\);
3. \((iii)\alpha^0(G, M)\) is surjective (resp. bijective) if \(\alpha^0(U, \mathbb{Z}/m\mathbb{Z})\) is surjective (resp. bijective) for all \(m\) and all \(U \trianglelefteq G\).
Duality with respect to a class formation

Remark

$P = \text{a set of prime numbers}$

Considering only the $P$-primary part, a $P$-class formation will give us a similar lemma.
Notations

- $K = \text{non-Archimedean local field}$
- $k = \text{residue field, } \text{char}(k) = p$
- $G = \text{Gal}(K^s/K)$
- $(G, K^{s*})$ is a class formation by LCFT
Notations

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Local duality

**Theorem**

Let $M$ be a finite $G$-module, then

$$\alpha^r(G, M) : \text{Ext}^r_G(M, K^{s*}) \to H^{2-r}(G, M)^*$$

is an isomorphism for all $r$. If $\text{char}(K) \nmid \#M$, then $\text{Ext}^r_G(M, K^{s*})$ and $H^r(G, M)$ are finite.

**Corollary**

If $\text{char}(K) \nmid \#M$, then there exists a perfect pairing of finite groups (where $M^D = \text{Hom}(M, K^{s*})$)

$$H^r(G, M^D) \times H^{2-r}(G, M) \to H^2(G, K^{s*}) \simeq \mathbb{Q}/\mathbb{Z}.$$
Sketch of proof

- LCFT \( \rightsquigarrow \) info. of \( \text{rec} : K^* \to G^{ab} \),
- \( \alpha^1(G, \mathbb{Z}/m\mathbb{Z}) = \text{rec}^m : K^*/K^{*m} \to (G^{ab})^m \),
- commutative diagram

\[
\begin{align*}
\mu_m(K) & \xrightarrow{\alpha^0(G, \mathbb{Z}/m\mathbb{Z})} H^2(G, \mathbb{Z}/m\mathbb{Z})^* \\
& \xrightarrow{\psi} (G^{ab})_m
\end{align*}
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Sketch of proof (continued).

- In general, $\psi$: NOT a bijection, BUT in our case $scd(G) = 2 \leadsto H^3(G, \mathbb{Z}) = 0 \leadsto \psi$: isomorphism,

- info. of rec $\leadsto$ info. of $\begin{cases} \alpha^0(G, \mathbb{Z}/m\mathbb{Z}) \\ \alpha^1(G, \mathbb{Z}/m\mathbb{Z}) \end{cases}$

- Apply the previous lemma $\Rightarrow$ the statement, spectral sequence $\leadsto$ finiteness.

- some simple calculations

- For the corollary, $char(K) \nmid \#M \leadsto$ identify $Ext^r_G(M, K^{s*})$ and $H^r(G, M^D)$ by spectral sequence.

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- Apply the previous lemma \( \Rightarrow \) the statement,

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- For the corollary, \( char(K) \nmid \#M \leadsto \) identify \( Ext'_G(M, K^{s*}) \) and \( H^r(G, M^D) \) by spectral sequence.
We define the Euler-Poincaré characteristic
\[ \chi(G, M) = \frac{\#H^0(G, M) \cdot \#H^2(G, M)}{\#H^1(G, M)} \],
and we have the following formula

**Theorem**

*For M finite of order m such that char(K) ∤ m, then*

\[ \chi(G, M) = |m|_K. \]
As an application of the local duality theorem, we get

**Theorem (Tate)**

Let $K$ be a non-Archimedean local field of characteristic 0, and $A$ be an Abelian variety over $K$ with dual $A^t$, then there exists a perfect pairing

$$A^t(K) \times H^1(K, A) \to \mathbb{Q}/\mathbb{Z}.$$
Sketch of proof.

- We are going to study the $\Ext^r_K(-, \mathbb{G}_m)$ sequence and $H^r(K, -)$ sequence of $0 \to A_n \to A \xrightarrow{n} A \to 0$,
- The local duality $\rightsquigarrow$ info. of $\alpha^r(K, A_n)$,
- info. of $\alpha^r(K, A_n)$ $\rightsquigarrow$ info. of $\{ \alpha^r(K, A)_n, \alpha^r(K, A)^{(n)} \}$
- Take the limit on $n$, get the info. on $\alpha^r(K, A)$: iso.,
- Finally, Barsotti-Weil formula: $A^t(K) = \Ext^1_K(A, \mathbb{G}_m)$
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Notations

- $K$ = a global field
- $S \neq \emptyset$ a set of places containing all the archimedean places
- $K_S$ = the maximal extension of $K$ unramified outside $S$
- $G_S = \text{Gal}(K_S/K)$
- $\mathcal{O}_{F,S} = \{x \in F; w(x) \geq 0, \forall w \notin S\}$, $S$-integers for $K \subseteq F \subseteq K_S$ with $F/K$ finite (Galois) extension.
- $J_{F,S} = \prod_{w \in S_F} F_w^*$, $S$-idèles
- $E_{F,S} = \mathcal{O}_{F,S}^\times = \{x \in F; w(x) = 0, \forall w \notin S\}$, $S$-units
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**Theorem**

Let $M$ be a finite $G_S$-module, then for any prime number $p \in P$, 

$$
\alpha^r(G_S, M)(p) : \text{Ext}^r_{G_S}(M, C_S)(p) \xrightarrow{\sim} H^{2-r}(G_S, M)^*(p)
$$

is an isomorphism for $r \geq 1$. Moreover, if $K$ is a function field then the statement is also true for $r = 0$, in which case $P$ is all the prime numbers.

**Proof.**

The proof: similar to the local case, BUT in case $K = \text{number field}$, NOT necessary that $\text{scd}(G_S) = 2$, GCFT $\sim \Rightarrow$ info. of $\text{rec} \not\Rightarrow$ info. of $\alpha^0(G_S, \mathbb{Z}/p^s\mathbb{Z})$, that is why the statement is only for $r \geq 1$ in this case.
Notations

- $M^D = \text{Hom}(M, K_S^*)$
- $G_v = \text{Gal}(K_v^s/K_v) \rightarrow g_v = \text{Gal}(k(v)^s/k(v))$
- $H^r(K_v, M) = \left\{ \begin{array}{ll} H^r_T(G_v, M), & v \in S_\infty \\ H^r(G_v, M), & v \text{ non-Archimedean} \end{array} \right.$
- $H^r_{un}(K_v, M) = \text{im}(H^r(g_v, M) \rightarrow H^r(G_v, M))$ for $v \notin S_\infty$
- $P^r_S(K, M) = \prod'_{v \in S} H^r(K_v, M)$ restrict prod. wrt. $H^r_{un}(K_v, M)$

Lemma

The image of the homomorphism $H^r(G_S, M) \rightarrow \prod_{v \in S} H^r(K_v, M)$ is contained in $P^r_S(K, M)$.

- $\beta^r_S(K, M) : H^r(G_S, M) \rightarrow P^r_S(K, M)$
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**Theorem (Poitou-Tate)**

Let $M$ be a finite $G_S$-module of order $m$ satisfying $m\mathcal{O}_{K,S} = \mathcal{O}_{K,S}$, then

- (i) The map $\beta_1^1(K, M)$ is proper, in particular $\Omega_1^1(K, M)$ is finite.
- (ii) There exists a perfect pairing of finite groups

$$\Omega_1^1(K, M) \times \Omega_2^1(K, M^D) \to \mathbb{Q}/\mathbb{Z}.$$ 

- (iii) For $r \geq 3$, $\beta_r^r(K, M) : H^r(G_S, M) \cong \prod_{v \in S^\mathbb{R}} H^r(K_v, M)$ is an isomorphism.
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$$\Xi^1_S(K, M) \times \Xi^2_S(K, M^D) \rightarrow \mathbb{Q}/\mathbb{Z}.$$ 

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(ii) There exists a perfect pairing of finite groups

$$\Sha^1_S(K, M) \times \Sha^2_S(K, M^D) \to \mathbb{Q}/\mathbb{Z}.$$ 

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Theorem (Poitou-Tate)

(iv) There is an exact sequence

\[ 0 \rightarrow H^0(G_S, M) \xrightarrow{\beta^0_S} P^0_S(K, M) \xrightarrow{\gamma^0_S} H^2(G_S, M^D)^* \]

\[ H^1(G_S, M^D)^* \xleftarrow{\gamma^1_S} P^1_S(K, M) \xleftarrow{\beta^1_S} H^1(G_S, M) \]

\[ H^2(G_S, M) \xrightarrow{\beta^2_S} P^2_S(K, M) \xrightarrow{\gamma^2_S} H^0(G_S, M^D)^* \rightarrow 0. \]
Sketch of proof

(i) Properness of $\beta_S^1(K, M)$: Spectral sequence $\Longrightarrow$ reduction to simple case,

- Direct calculations for the simple case,
  finiteness of class group $\Rightarrow$ properness of $\beta_S^1(K, M)$.
- Poitou-Tate sequence $\Rightarrow$ (ii) perfect pairing of III.
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(iii) & (iv): Local duality \[ \gamma^r_S(K, M^D) : P^r_S(K, M^D) \to H^{2-r}(G_S, M)^* \] is the dual of \[ \beta^{2-r}_S(K, M) : H^{2-r}(G_S, M) \to P^{2-r}_S(K, M), \]

Symmetry \(\Rightarrow\) only need to proof the second half of the sequence,

\[ Ext^r_{G_S}(M^D, -), 0 \to E_S \to J_S \to C_S \to 0 \] long exact sequence,

Complicated calculations \(\Rightarrow\) \[ Ext^r_{G_S}(M^D, E_S) = H^r(G_S, M) \] and \[ Ext^r_{G_S}(M^D, J_S) = P^r_S(K, M) \] for any \(r\),

Previous duality theorem \(\Rightarrow\) \[ Ext^r_{G_S}(M^D, C_S) = H^r(G_S, M^D)^* \] for \(r \geq 1\) (the last six terms of the Poitou-Tate sequence).
Sketch of proof (continued).

(iii) & (iv): Local duality $\sim \Rightarrow$

$\gamma^r_S(K, M^D) : P^r_S(K, M^D) \to H^{2-r}(G_S, M)^*$ is the dual of

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LIANG, Yong Qi

Some Arithmetic Duality Theorems
Sketch of proof (continued).

(iii)&(iv): Local duality \( \leadsto \)
\[
\gamma^r_S(K, M^D) : P^r_S(K, M^D) \to H^{2-r}(G_S, M)^* \text{ is the dual of } \\
\beta^{2-r}_S(K, M) : H^{2-r}(G_S, M) \to P^{2-r}_S(K, M),
\]

Symmetry \( \Rightarrow \) only need to proof the second half of the sequence,

\[
Ext^r_{G_S}(M^D, -), \ 0 \to E_S \to J_S \to C_S \to 0 \leadsto \text{long exact sequence},
\]

Complicated calculations \( \Rightarrow \)
\[
Ext^r_{G_S}(M^D, E_S) = H^r(G_S, M) \text{ and } \\
Ext^r_{G_S}(M^D, J_S) = P^r_S(K, M) \text{ for any } r,
\]

Previous duality theorem \( \Rightarrow \)
\[
Ext^r_{G_S}(M^D, C_S) = H^r(G_S, M^{D})^* \text{ for } r \geq 1 \text{ (the last six terms of the Poitou-Tate sequence)}. \]
Sketch of proof (continued).

(iii) & (iv): Local duality $\sim \rightarrow$

$$\gamma^r_S(K, M^D) : P^r_S(K, M^D) \rightarrow H^{2-r}(G_S, M)^\ast$$

is the dual of

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Symmetry $\Rightarrow$ only need to proof the second half of the sequence,

$$\text{Ext}^r_{G_S}(M^D, -), \ 0 \rightarrow E_S \rightarrow J_S \rightarrow C_S \rightarrow 0 \sim \rightarrow$$

long exact sequence,

Complicated calculations $\Rightarrow$ $\text{Ext}^r_{G_S}(M^D, E_S) = H^r(G_S, M)$ and

$$\text{Ext}^r_{G_S}(M^D, J_S) = P^r_S(K, M)$$

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for $r \geq 1$ (the last six terms of the Poitou-Tate sequence).
If \( m = \#M \) such that \( m\mathcal{O}_K, S = \mathcal{O}_K, S \), and if \( S \) is finite, then \( H^r(G_S, M) \) is finite, we define

\[
\chi(G_S, M) = \frac{\#H^0(G_S, M) \cdot \#H^2(G_S, M)}{\#H^1(G_S, M)},
\]

we have the following formula

\[
\chi(G_S, M) = \prod_{v \in S_\infty} \frac{\#H^0(G_v, M)}{|m|_v}.
\]
Part II

Etale cohomology
From now on, all the cohomology groups = étale cohomology
groups, "sheaf" = étale sheaf of abelian groups

- $R$: Henselian DVR, $K = \text{Frac}(R)$, $k = R/m$ residue field
- $X = \text{spec}(R) = \{u, x\}$ where
  - $j : u = \text{spec}(K) \to X$ is the generic point
  - $i : x = \text{spec}(k) \to X$ is the closed point
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Suppose that $k$ is a finite field. Let $\mathcal{F}$ be a constructible sheaf on $X$, if one of the following conditions holds (1) $K$ is complete, (2) $\text{char}(K) = 0$, (3) $\text{char}(K) = p$ and $p\mathcal{F} = \mathcal{F}$, then we have a perfect pairing:

$$
\text{Ext}^r_X(\mathcal{F}, \mathbb{G}_m) \times H^{3-r}_X(X, \mathcal{F}) \to H^3_X(X, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}.
$$

Corollary

Suppose that $k$ is finite of characteristic $p$, for a locally constant constructible sheaf $\mathcal{F}$ on $X$ such that $p\mathcal{F} = \mathcal{F}$, then we have a perfect pairing (where $\mathcal{F}^D = \text{Hom}_X(\mathcal{F}, \mathbb{G}_m)$)

$$
H^r(X, \mathcal{F}^D) \times H^{3-r}_X(X, \mathcal{F}) \to \mathbb{Q}/\mathbb{Z}.
$$
Sketch of proof.

1. For sheaves of the form $j_!\mathcal{F}$, we identify the pairing with the local duality of Galois cohomology,

2. For sheaves of the form $i_*\mathcal{F}$, we identify the pairing with the duality of the class formation $(\text{Gal}(k^s/k), \mathbb{Z})$,

3. Finally, for general $\mathcal{F}$ we take the cohomology sequence and Ext sequence of

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

and combine the first two cases.

4. For the corollary, $p\mathcal{F} = \mathcal{F} \leadsto$ identify $\text{Ext}^r_X(\mathcal{F}, \mathbb{G}_m)$ and $H^r(X, \mathcal{F}^D)$ by the local-global Ext spectral sequence.
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Local duality
Global cohomology
Artin-Verdier’s theorem

Proof of the theorem

Sketch of proof.

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Notations

- \( K \): a global field
- \( X \)
  - \( X = \text{spec}(\mathcal{O}_K) \) if \( K \) is a number field
  - \( X \) the unique complete smooth curve with function field \( K \)

- Usually, for open subschemes \( V \subset U \subset X \),
  \( j : V \to U = \) the open immersion
  \( i : U \setminus V = Z \to U = \) the (reduced) closed immersion

- For a closed point \( v \) of \( X \), \( \mathcal{O}_v^h = \) Henselization of the stalk of \( \mathcal{O}_X \) at \( v \), \( K_v = \text{Frac}(\mathcal{O}_v^h) \)

- For an Archimedean place \( v \), we set \( K_v = \mathbb{R} \) or \( \mathbb{C} \)
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Some calculations

- We can compute $H^r(U, \mathbb{G}_m)$, they are related to the ideal class group (or $Pic(U)$) and the group of unites.

- We can define $H^r_c(U, \mathcal{F}) = "cohomology with compact support"
  - in case $K =$ function field, $H^r_c(U, \mathcal{F}) \simeq H^r(X, j_! \mathcal{F})$ is the cohomology with compact support in the classic sense;
  - if $K =$ number field, $H^r_c(U, \mathcal{F})$ is NOT the classic one, but it will give the perfect pairing in the future.

- The important point: $H^r_c(U, \mathcal{F})$ is fixed into a long exact sequence

$$
\cdots \to H^r_c(U, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \bigoplus_{v \notin U} H^r(K_v, \mathcal{F}_v) \to \cdots
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- Then we can also compute $H^r_c(U, \mathbb{G}_m)$, $H^3_c(U, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$. 

\begin{flushright}
LIANG, Yong Qi
Some Arithmetic Duality Theorems
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For an open subscheme $U$ of $X$, $\mathcal{F} \in Sh(U)$ constructible sheaf s.t. $\exists m \in \mathbb{Z}$ satisfying $m\mathcal{F} = 0$ and $m$ invertible on $U$ (i.e. $m\mathcal{O}_v = \mathcal{O}_v$ for all closed point $v \in U$), then $H^r(U, \mathcal{F})$ and $H^r_c(U, \mathcal{F})$ are finite, we define

- $\chi(U, \mathcal{F}) = \frac{\#H^0(U, \mathcal{F}) \cdot \#H^2(U, \mathcal{F})}{\#H^1(U, \mathcal{F}) \cdot \#H^3(U, \mathcal{F})}$
- $\chi_c(U, \mathcal{F}) = \frac{\#H^0_c(U, \mathcal{F}) \cdot \#H^2_c(U, \mathcal{F})}{\#H^1_c(U, \mathcal{F}) \cdot \#H^3_c(U, \mathcal{F})}$
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\end{align*}
Theorem

Let $\mathcal{F}$ a constructible sheaf on $U$ such that $m\mathcal{F} = 0$ for a certain integer $m$ invertible on $U$, then we have the formulae

1. $\chi(U, \mathcal{F}) = \prod_{v \in S_{\infty}} \frac{\# \mathcal{F}(K_v)}{\# H^0(K_v, \mathcal{F}) \cdot \# \mathcal{F}(K^s)_v}$
2. $\chi_c(U, \mathcal{F}) = \prod_{v \in S_{\infty}} \# \mathcal{F}(K_v)$

Sketch of proof.

- First, relate $\chi(U, \mathcal{F})$ with $\chi(V, \mathcal{F}|_V)$
- Take a small $V$ s.t. $\mathcal{F}$ is locally constant on $V$, identify $H^r(V, \mathcal{F})$ with Galois cohomology, and apply the $\chi$ global formula for Galois cohomology.
Theorem

Let $\mathcal{F}$ a constructible sheaf on $U$ such that $m\mathcal{F} = 0$ for a certain integer $m$ invertible on $U$, then we have the formulae

\begin{itemize}
  \item[(i)] $\chi(U, \mathcal{F}) = \prod_{v \in S_\infty} \frac{\#\mathcal{F}(K_v)}{\#H^0(K_v, \mathcal{F}) \cdot \#\mathcal{F}(K^s)_v}$,
  \item[(ii)] $\chi_c(U, \mathcal{F}) = \prod_{v \in S_\infty} \#\mathcal{F}(K_v)$.
\end{itemize}

Sketch of proof.

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  \item First, relate $\chi(U, \mathcal{F})$ with $\chi(V, \mathcal{F}|_V)$
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Artin-Verdier’s theorem

Theorem (Artin-Verdier)

Let \( \mathcal{F} \) be a constructible sheaf on \( U \), then we have the following perfect pairing of finite groups

\[
\text{Ext}^r_U(\mathcal{F}, \mathbb{G}_m) \times H^{3-r}_c(U, \mathcal{F}) \to H^3_c(U, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.
\]

Corollary

Let \( \mathcal{F} \) be a locally constant constructible sheaf on \( U \) such that \( m\mathcal{F} = 0 \) for a certain integer \( m \) invertible on \( U \), then we have the following perfect pairing of finite groups

(\text{where } \mathcal{F}^D = \text{Hom}_U(\mathcal{F}, \mathbb{G}_m))

\[
H^r(U, \mathcal{F}^D) \times H^{3-r}_c(U, \mathcal{F}) \to H^3_c(U, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.
\]
Sketch of proof of Artin-Verdier

- Proof the theorem with assumption $supp(\mathcal{F}) \subseteq Z \subsetneq X$;
- Show that we can replace $U$ by a smaller $V$, then we can assume $\mathcal{F}$ to be locally constant, killed by $m$ invertible on $V$;
- Show that we can replace $(U, \mathcal{F})$ by $(U', \mathcal{F}|_{U'})$ with a finite étale covering $U' \to U$, then we can consider only the constant sheaves and assume that $K$ is totally imaginary;
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With the above assumptions, develop a machine for doing induction on \( r \);

- Show that \( \text{Ext}^r_U \) and \( H^r_c \) vanish if \( r \) is large enough or small enough;

- Finally, complete the proof with a supplement argument of Artin-Schreier for the case \( \text{char}(K) = p \).

- For the corollary, under the assumptions, we identify \( \text{Ext}^r_U(\mathcal{F}, \mathbb{G}_m) \) and \( H^r(U, \mathcal{F}^D) \) by spectral sequence.
Sketch of proof (continued).

- With the above assumptions, develop a machine for doing induction on $r$;
- Show that $\text{Ext}_U^r$ and $H^r_c$ vanish if $r$ is large enough or small enough;
- Finally, complete the proof with a supplement argument of Artin-Schreier for the case $\text{char}(K) = p$.
- For the corollary, under the assumptions, we identify $\text{Ext}_U^r(F, \mathbb{G}_m)$ and $H^r(U, F^D)$ by spectral sequence.
Sketch of proof (continued).

- With the above assumptions, develop a machine for doing induction on $r$;
- Show that $\text{Ext}_U^r$ and $H_c^r$ vanish if $r$ is large enough or small enough;
- Finally, complete the proof with a supplement argument of Artin-Schreier for the case $\text{char}(K) = p$.
- For the corollary, under the assumptions, we identify $\text{Ext}_U^r(\mathcal{F}, \mathbb{G}_m)$ and $H^r(U, \mathcal{F}^D)$ by spectral sequence.
Sketch of proof (continued).

- With the above assumptions, develop a machine for doing induction on $r$;
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- For the corollary, under the assumptions, we identify $\text{Ext}^r_U(\mathcal{F}, \mathbb{G}_m)$ and $H^r(U, \mathcal{F}^D)$ by spectral sequence.
The End.

- Thank you very much!!
- Grazie mille!
- Merci beaucoup!

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