Local-global principle: Rational points *vs.* Degree zero Chow groups on rationally connected varieties

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The rationals

- Let $\mathbb{Q}$ be the set of rational numbers.
- $\mathbb{Q}$ is endowed with a topology defined by the usual distance :
- the absolute value $\forall a, b \in \mathbb{Q}$, $|a-b|_\infty$
- passing to the completion: we get $\mathbb{R}$
- $\mathbb{Q} \subset \mathbb{R}$ dense
- all Cauchy sequences converge in $\mathbb{R}$, we can do analysis on $\mathbb{R}$
- Other (non trivial) topologies on $\mathbb{Q}$?

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p-adic numbers

- $p =$ a prime number
- $\forall a \in \mathbb{Z}$ define $|a|_p = p^{-v_p(a)}$ where $n = v_p(a)$ is an integer such that $p^n | a$ but $p^{n+1} \nmid a$
- $\forall r = \frac{a}{b} \in \mathbb{Q}$ define $|r|_p = |\frac{a}{b}|_p = p^{-(v_p(a)-v_p(b))}$
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p-adic numbers

- under the usual topology 75 is smaller than 324
- Examples of p-adic topology:
- $p_1 = 3$ then $|0|_3 = 0$, $|75|_3 = \frac{1}{3}$, $|324|_3 = \frac{1}{81}$
- under the 3-adic topology, 324 is much smaller than 75
- however, for $p_2 = 5$, $|75|_5 = \frac{1}{25}$, $|324|_5 = 1$
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- Conclusion:
- for different p we get inequivalent topologies on $\mathbb{Q}$
- none of these is equivalent to the usual topology induced by $\mathbb{Q} \subset \mathbb{R}$

Theorem (Ostrowski)

These are all possible (inequivalent and non-trivial) distances on $\mathbb{Q}$.

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- passing to the completion with respect to $|\cdot|_p$, we get $\mathbb{Q} \subset \mathbb{Q}_p$ dense
- as in $\mathbb{R}$, we can also do analysis on $\mathbb{Q}_p$
- $\mathbb{Q}_p$ — the field of p-adic numbers
- k = a number field = a finite field extension of $\mathbb{Q}$
- v either a prime ideal of $\mathcal{O}_k$ — the ring of integers of k
- $k \subset k_v$ the completion of k with respect to the v-adic topology (k_v is a finite extension of a certain $\mathbb{Q}_p$)
- or an inclusion with dense image $v : k \hookrightarrow \mathbb{R}$ or $v : k \hookrightarrow \mathbb{C}$
- $\mathbb{Q}$, k: global fields; $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, k_v local fields.

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Algebraic varieties

- “Algebraic variety” = algebraic version of “manifold”
- can be defined over any fields (not only over $\mathbb{R}$ or $\mathbb{C}$)
- Algebraic variety = (locally) defined by polynomials
- examples:
- a circle $x^2 + y^2 = 1$ is an algebraic variety over $\mathbb{Q}$
- a parabola $y = x^2 + 6x + 1$ is an algebraic variety over $\mathbb{Q}$
- however, $y = e^x$ does not define an algebraic variety : $exp(x)$ is not a polynomial

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- $X \subset \mathbb{P}^n$ defined by finitely many (homogeneous) polynomials $\in k[x_0, \ldots, x_n]$, is call a *projective algebraic variety* over k
- any compact Riemann surface is a projective algebraic curve (variety of dimension 1) over $\mathbb{C}$
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Rational points

- the variety X defined over $\mathbb{Q}$ by $x^2 + y^2 = -1$
- $X(\mathbb{Q}) = \emptyset$, $X(\mathbb{R}) = \emptyset$, but $X(\mathbb{C}) \neq \emptyset$

Theorem (A. Wiles 1995: Fermat's last theorem)

For $n \geq 3$, define X by $x^n + y^n = z^n$. If $(x, y, z) \in X(\mathbb{Q})$ then $xyz = 0$.

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- In general, for an algebraic variety X defined over a number field k, to study the set $X(k)$ of rational points is a very important and very difficult question in number theory and in arithmetic algebraic geometry.

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Local-global principle

- An easy observation: If a polynomial has solutions in $\mathbb{Q} \Rightarrow$ it has solutions in all extensions of $\mathbb{Q}$, in particular in $\mathbb{R}$ and in all $\mathbb{Q}_p$
- for an algebraic variety X, $X(\mathbb{Q}) \neq \emptyset \ \Rightarrow X(\mathbb{R}) \neq \emptyset$ and $X(\mathbb{Q}_p) \neq \emptyset$
- it is relatively easy to decide if $X(\mathbb{R}) = \emptyset$: real analysis
- also "easy" to decide if $X(\mathbb{Q}_p) = \emptyset$: p-adic analysis
- p-adic analysis on $X \iff$ the defining polynomials of X have common integer solutions $\mod p^n$ for all $n \in \mathbb{N}$

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- k = a number field
- similarly $X(k) \neq \emptyset \Rightarrow X(k_v) \neq \emptyset (\forall v \in \Omega_k)$ and $X(k) \subset \prod_{v \in \Omega} X(k_v)$
- Hasse principle: if the inverse is also true $X(k_v) \neq \emptyset \ (\forall v \in \Omega) \Rightarrow X(k) \neq \emptyset$

Theorem (Hasse-Minkowski)

Let X be defined by a quadratic form with coefficients in k. Then the Hasse principle is true.

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- Selmer: counter-example over $\mathbb{Q}$, $X: \ 3x^3 + 4x^3 + 5z^3 = 0$
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Theorem (Hasse-Minkowski)

Let X be defined by a quadratic form with coefficients in k. Then the Hasse principle is true.

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- Weak approximation: if $X(k)$ is dense in $\prod_{v\in\Omega} X(k_v)$
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- Example (Colliot-Thélène, Sansuc, Swinnerton-Dyer 1987): $k=\mathbb{Q}$, Châtelet surface $x^2+y^2=P(z)$, $P(z)\in\mathbb{Q}[z]$ irreducible of degree 4
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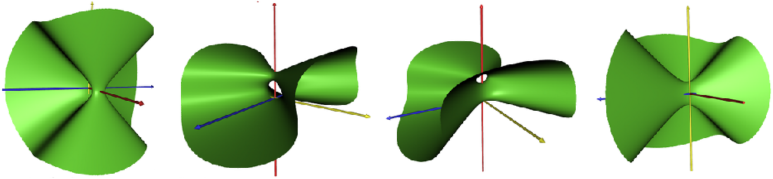

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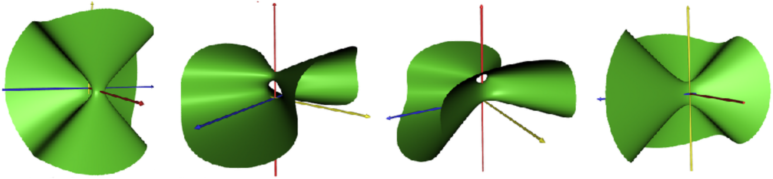

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- Different behaviours between $X_1 : \ x^2 + y^2 = P(z)$ (P irreducible) and $X_2 : \ x^2 + y^2 = -(z^2 - 2)(z^2 - 3)$
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$$[\prod_{v\in\Omega} X(k_v)] \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$
$$(\{x_v\}_{v\in\Omega}, \beta) \mapsto \langle \{x_v\}_v, \beta\rangle := \sum_{v\in\Omega} inv_v(\beta(x_v))$$
- local class field theory: $inv_v : Br(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$
- $[\prod_{v\in\Omega} X(k_v)]^{Br} = \{\{x_v\}_v; \{x_v\}_v \perp Br(X)\}$ Brauer-Manin set
- Fact. $X(k) \subseteq \overline{X(k)} \subseteq [\prod_{v\in\Omega} X(k_v)]^{Br} \subseteq \prod_{v\in\Omega} X(k_v)$
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- Obstruction: if $\left[\prod_{v\in\Omega} X(k_v)\right]^{Br} \subsetneq \prod_{v\in\Omega} X(k_v)$, weak approximation never happens
- This explains the differences between the above example and the counter-example
- If $\left[\prod_{v\in\Omega} X(k_v)\right]^{Br} \neq \emptyset \Rightarrow X(k) \neq \emptyset$, we say that Brauer-Manin obstruction is the only obstruction to Hasse principle
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Zero-cycles and Chow groups

- the group of zero-cycles:
- $Z_0(X) = \bigoplus_{P \in X} \mathbb{Z} \cdot P =$ free Abelian group generated by closed points on X
- the -group of zero-cycles:
- $CH_0(X) = Z_0(X)/\sim$ rational equivalence
- rational equivalence : a zero-cycle can be obtained from the other zero-cycle by a certain deformation
- example: $CH_0(\mathbb{P}^n) = \mathbb{Z}$
- $\deg : Z_0(X) \to \mathbb{Z}$, $\deg(\sum_P n_P P) = \sum n_P[k(P) : k]$
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- (Colliot-Thélène) Similarly, Brauer-Manin pairing

$$[\prod_{v\in\Omega} CH_0'(X_v)] \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

- The modified Chow group:

$$CH_0'(X_v) = \begin{cases} CH_0(X_v), & v \text{ is p-adic} \\ CH_0(X_v)/N_{\mathbb{C}|\mathbb{R}}CH_0(\overline{X}_v), & v \text{ is real} \\ 0, & v \text{ is complex} \end{cases}$$

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- $M\widehat{\ } := \varprojlim_n M/nM$ for any abelian group M
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Exactness of $(E) \Longrightarrow$

- Exactness of (E_0)
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- (Cassels-Tate) (E_0) is exact if $X = E$ is an elliptic curve (with finiteness of $Ш(E)$ supposed).
- (Kato-Saito) (E) is exact if $X = C$ is a smooth curve (with finiteness of $Ш(Jac(C))$ supposed).

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Poonen's 3-folds

- fibration $X \to C$
  - base: C curve $C(k) \neq \emptyset$ finite and $Ш(Jac(C)) < \infty$
  - fibers: Châtelet surfaces
- Poonen 2010: $\emptyset = X(k) \subset \left[\prod_{v\in\Omega} X(k_v)\right]^{Br} \neq \emptyset$
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Rationally connectedness

Definition

$X_{/k}$ is called *rationally connected*,
if for any $P, Q \in X(\mathbb{C})$, there exists a $\mathbb{C}$-morphism $f : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}}$ such that $f(0) = P$ and $f(\infty) = Q$.

- Example:
  - A homogeneous space of a connected linear algebraic group is rationally connected.
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  - An abelian variety is *never* rationally connected.
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Relation between rational points and 0-cycles

Theorem (Liang)

Let X be a smooth (projective) rationally connected variety defined over a number field k.

Assume that the Brauer-Manin obstruction is the only obstruction to weak approximation for rational points on $X \otimes_k K$, for any finite extension K/k.

Then, the complex (E) and (E_0) are exact for X.

An application

- *Recall* : a result of Borovoi (1996).
  $G_{/k}$: connected linear algebraic group.
  Y : homogeneous space of G with connected stabilizer (or with abelian stabilizer if G is simply connected).
  X : smooth compactification of Y.
  Then the Brauer-Manin obstruction is the only obstruction to weak approximation for rational points on X.

Corollary

The complex $(E), (E_0)$ are exact for smooth compactifications of any homogeneous space of any connected linear algebraic group with connected stabilizer (or with abelian stabilizer if the group is simply connected).

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(Outline of) Proof.

- BM obstruction is the only obs. to weak approx. for rational points on X_K, $\forall K/k$ finite.

$\Longrightarrow$ (key: fibration method applied to $X \times \mathbb{P}^1 \to \mathbb{P}^1$)

- BM obstruction is the only obs. to "weak approx." for zero-cycles of degree 1 on X_K, $\forall K/k$ finite.

$\Longrightarrow$ (key: generalized Hilbertian subset)

- $\forall d \in \mathbb{Z}$, BM obstruction is the only obs. to "weak approx." for zero-cycles of degree d on $(X \times \mathbb{P}^1)_K$, $\forall K/k$ finite.

$\Longrightarrow$ (key: Theorem of Kollár-Szabó (X is RC) + an argument of Wittenberg)

- Exactness of (E) for $X \times \mathbb{P}^1$.

$\Longrightarrow$(key: homotopic invariance)

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Thank you for your attention !

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