ZERO-CYCLES ON VARIETIES FIBERED OVER CURVES BY CHÂTELET SURFACES

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Notation

k : a number field. Ω_k : set of places of k. $k_v(v \in \Omega_k)$: local field at v.

$X_{/k}$: a smooth projective algebraic variety (separated scheme of finite type), geometrically integral over k.

$X_v = X \otimes_k k_v$.

$\mathrm{Br}(X) := \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ the cohomological Brauer group of X.

For any Abelian group A and $m \in \mathbb{Z}$ we write $A/m = \mathrm{Coker}(A \xrightarrow{.m} A)$.

1. **Question.**
Does there exist a 0-cycle of degree 1 on X? Does local-global principle hold for 0-cycles on X?

The Brauer group $\mathrm{Br}(X)$ gives an obstruction to the Hasse principle (HP) and weak approximation (WA) for 0-cycles.

2. **Manin pairing.**

$$\begin{array}{rcccl} \langle\cdot,\cdot\rangle : \prod_{v\in\Omega_k} Z_0(X_v) & \times & \mathrm{Br}(X) & \to & \mathbb{Q}/\mathbb{Z}, \\ (\ \{z_v\}_{v\in\Omega_k} = \{\sum n_{P_v}P_v\}_v & , & b \ ) & \mapsto & \sum_{v\in\Omega_k} \mathrm{inv}_v(\sum n_{P_v} \mathrm{cores}_{k_v(P_v)/k_v}(b(P_v))), \end{array}$$

where $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant at v. This pairing factorizes through $\prod_v \mathrm{CH}(X_v)$, and further the product of modified local Chow groups $\prod_v \mathrm{CH}'(X_v)$ where

$$CH'_0(X_v) = \begin{cases} \mathrm{CH}_0(X_v), & v \text{ is non archimedean} \\ \mathrm{CH}_0(X_v)/N_{\mathbb{C}|\mathbb{R}}\mathrm{CH}_0(X_{v\mathbb{C}}), & v \in \Omega^{\mathbb{R}} \\ 0, & v \in \Omega^{\mathbb{C}}. \end{cases}$$

This indices a complex

$$\mathrm{CH}_0(X) \to \prod_{v\in\Omega} \mathrm{CH}'_0(X_v) \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

and

$$\mathrm{CH}_0(X)/m \to \prod_{v\in\Omega} \mathrm{CH}'_0(X_v)/m \to \mathrm{Hom}(\mathrm{Br}(X)[m], \mathbb{Q}/\mathbb{Z}),$$

$$\text{(E)} \qquad \varprojlim_m \mathrm{CH}_0(X)/m \to \prod_{v\in\Omega} \varprojlim_m \mathrm{CH}'_0(X_v)/m \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}),$$

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24 July 2013.

Summer Workshop on Arithmetic Geometry, Mathematical Sciences Center, Tsinghua University.

similarly,

$$(\mathrm{E}_0) \qquad \varprojlim_m \mathrm{A}_0(X)/m \to \prod_{v\in\Omega} \varprojlim_m \mathrm{A}_0(X_v)/m \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}),$$

where $\mathrm{A}_0(-) = \mathrm{Ker}[\deg : \mathrm{CH}_0(-) \to \mathbb{Z}]$.

Conjecture (Colliot-Thélène–Sansuc, Kato–Saito)**.** *The sequences* (E) *and* (E_0) *are exact for all smooth proper varieties defined over number fields.*

Remark. The exactness of (E) implies the following statements.

- (E_0) is exact;
- the Brauer–Manin obstruction is the only obstruction to the HP for 0-cycles of degree 1, *i.e.* the existence of $\{z_v\}\perp\mathrm{Br}(X)$ with $\deg(z_v) = 1$ implies the existence of a global 0-cycle of degree 1;
- the Brauer–Manin obstruction is the only obstruction to WA for 0-cycles of degree $\delta\in\mathbb{Z}$, *i.e.* for any integer δ, any integer $m\geq 0$, any finite subset $S\subset\Omega_k$, for all $\{z_v\}\perp\mathrm{Br}(X)$ with $\deg(z_v)=\delta$ there exists $z = z_{m,S}\in\mathrm{CH}_0(X)$ such that $\deg(z)=\delta$ and $z = z_v \in \mathrm{CH}_0(X_v)/m$ for every $v\in S$.

3. **Some known results.**

3.1. $\dim(X) = 0$. In this case $X = \mathrm{Spec}(k)$, (E) is the dual of the exact sequence $\mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ coming from the global class field theory.

3.2. $\dim(X) = 1$. We consider a smooth projective curve X, if $Ш(k, \mathrm{Jac}(X))$ is finite, the exactness of (E) has been proved by S. Saito 89, and has been simplified by Colliot-Thélène 99.

In particular, if $X = E$ is an elliptic curve, the sequence (E_0) is part of the Cassels–Tate exact sequence

$$0 \to \overline{E(k)} \to \prod_v E(k_v)' \to \mathrm{H}^1(k,E)^* \to Ш(k,E)^* \to 0$$

3.3. $\dim(X) > 1$. There are mainly two types of results, on fibrations and on homogeneous spaces. In this talk, only results on fibrations will be concerned.

Theorem (Wittenberg 09)**.** *Let $X\to C$ be a fibration over a smooth projective curve C such that $Ш(k, \mathrm{Jac}(C))$ is finite.*

- *all fibers are abelian-split (e.g. the fibers are all geometrically integral);*
- *all but finitely many fibers satisfy WA for rational points (or for 0-cycles of degree 1).*

Then (E) *is exact for X.*

4. **Poonen's example.**
Consider fibrations $X \to C$ over a curve, suppose that

- $Ш(k, \mathrm{Jac}(C))$ is finite;
- $C(k) \neq \emptyset$ and $C(k)$ is a finite set;
- all fibers are geometrically integral;
- the generic fiber is a Châtelet surface, *i.e.* is defined by $y^2 - az^2 = P(x)$ where $P(x)$ is a polynomial (irreducible in Poonen's example) of degree 4 and where a is a non square constant.
- for each $\theta \in C(k)$ the fiber X_θ is smooth, and it has k_v-rational points for all v but it possesses no k-rational point.

It is easy to see that for this family of varieties $X(k) = \emptyset$ but $X(k_v) \neq \emptyset$ for all v. Moreover, Poonen constructed certain varieties in this family of varieties such that there exist local rational points $\{x_v\} \perp \mathrm{Br}(X)$. Hence the Brauer–Manin obstruction is not the only obstruction to HP (to WA neither) for rational points on Poonen's 3-folds. *A fortiori*, X possesses a family of local 0-cycles of degree 1 orthogonal to the Brauer group, the conjecture above predict that there exists a global 0-cycle of degree 1.

Theorem (Colliot-Thélène 10)**.** *There exists a global 0-cycle of degree* 1 *on Poonen's* 3*-folds.*

Attention One can not apply Wittenberg's theorem above to Poonen's 3-folds. For Châtelet surfaces, the Brauer–Manin obstruction is the only obstruction to HP and to WA for rational points (or for 0-cycles of degree 1), hence HP (or WA) may probably fail when the Brauer group is not trivial, this can be the case for many closed fibers of the fibration.

In order to deal with this problem, we need to introduce the following definition.

Definition. *Let V be a geometrically integral variety defined over a field k. A subset Hil of closed points of V is called a* generalised Hilbertian subset *if there exists a finite étale morphism ρ mapping from an integral k-variety Z to a certain non empty open subset U of V*

$$Z \xrightarrow{\rho} U \subset V$$

such that $\mathsf{Hil} = \{\theta \in U; \theta$ *is a closed point of V such that $\rho^{-1}(\theta)$ is connected*$\}$.

For example, for a fixed Poonen's 3-fold

$$\mathsf{Hil} = \{\theta \in C; \theta \text{ is a closed point of } C,\ P_\theta(x) \in k(\theta)[x] \text{ is irreducible over } k(\theta)\}$$

is a generalised Hilbertian subset of C.

Theorem (Liang 10)**.** *Let $X \to C$ be a fibration over a curve whose fibers are all geometrically integral. Suppose that*

- $Ш(k, \mathrm{Jac}(C))$ *is finite;*
- *there exists a generalised Hilbertian subset Hil of C, such that for all $\theta \in \mathsf{Hil}$ the fibers X_θ satisfy WA (resp. HP) for $k(\theta)$-rational points (or for 0-cycles of degree* 1*).*

Then (E) *is exact for* X *(resp. the Brauer–Manin obstruction is the only obstruction to HP for 0-cycles of degree* 1 *on* X*).*

Note that when $P_\theta(x)$ is irreducible over the residual field $k(\theta)$, the Brauer group of the fiber X_θ is trivial modulo constant, and the fiber X_θ satisfies both HP and WA.

Corollary. *The sequence* (E) *is exact for Poonen's* 3*-folds.*

Sketch of proof. Recall the fibration method for proving similar statements for rational points.

Difficulties:

1, pass from 0-cycles to rational points (with residual fields extensions of high degrees over the base field).

2, find $\theta \in \mathsf{Hil}$ instead of simply $\theta \in C$.

Why the Brauer–Manin obstruction to HP (or WA) can be the only obstruction for 0-cycles but not for rational points? The difference is that there are not so much rational points on C but lots of 0-cycles on C.

(1)good places:

Since all fibers are geometrically integral, one can apply "uniformly" the Lang–Weil estimation to obtain rational points after the reduction mod v, then Hensel's lemma permits to lift these points to local points. Hence there exists a finite subset $S \subset \Omega_k$ such that for any closed point $\theta \in C$ and for each $w \in \Omega_{k(\theta)}$ outside S we have $X_\theta(k(\theta)_w) \neq \emptyset$.

(2)bad places:

Apply a (relative) moving lemma for 0-cycles to reduce the problem to effective 0-cycles in good position, which are not far from rational points (of high degrees). Over local fields we apply also the implicit function theorem.

In order to conclude, it remains to answer "How to approximate effective 0-cycles by a closed point $\theta \in \mathsf{Hil}$?" This is completed by the following key lemma. ☐

Lemma (Hilbert–Ekedahl–Liang)**.** *Let* C *be a smooth projective curve defined over a number field* k*. Denote by* $g = g(C)$ *the genus of* C*. Let* Hil *be a generalized Hilbertian subset of* C*. Let* $y \in Z_0(C)$ *be an effective 0-cycle of degree* $d > 2g$*, and let* $S \subset \Omega$ *be a finite subset. For all* $v \in S$*, let* $z_v \in Z_0(C_v)$ *be a separable effective 0-cycle of degree* d *such that* $supp(z_v) \cap supp(y) = \emptyset$ *and* $z_v \sim y$ *on* C_v*.*

Then, there exist a closed point θ *of* C *such that*

(1) $\theta \in \mathsf{Hil}$;

(2) $\theta \sim y$ *on* C;

(3) θ *is sufficiently close to* z_v *for all* $v \in S$*.*

Proof. For $v \in S$, we can write $z_v - y = div_{C_v}(f_v)$ with certain rational functions $f_v \in k_v(C_v)^*/k_v^*$. As $deg(y) = d > 2g$, the Riemann-Roch theorem implies that $\Gamma(C, \mathcal{O}_C(y))$ is a vector space of dimension $r = d + 1 - g > g + 1$. Weak approximation for $\mathbb{P}^{r-1}$ implies that there exists a function $f \in k(C)^*/k^*$ such that

(i) f is sufficiently close to de $f_v (v \in S)$,

(ii) $div_C(f) = y' - y$ where y' is an effective 0-cycle such that $supp(y') \cap supp(y) = \emptyset$.

Then y' is sufficiently close to z_v, $y' \approx z_v$ for $v \in S$.

The function f defines a k-morphism $\psi : C \to \mathbb{P}^1$ such that $\psi^*(\infty) = y$ and $\psi^*(0) = y'$.

Suppose that the generalized Hilbertian subset Hil is defined by $Z \to U \subset C$, then its composition with ψ defines a generalized Hilbertian subset $\mathsf{Hil}' \subset \mathbb{P}^1$. By deleting some closed points, one may assume that $\theta' \in \mathsf{Hil}'$ implies $\theta = \psi^{-1}(\theta') \in \mathsf{Hil}$. Hilbert's irreducibility theorem (effective version by Ekedahl) says that $\mathsf{Hil}' \cap \mathbb{P}^1(k)$ is dense in $\prod_{v \in S} \mathbb{P}^1(k_v)$. There exists $\theta' \in \mathsf{Hil}' \cap \mathbb{P}^1(k)$ sufficiently closed to $0 \in \mathbb{P}^1(k_v)(v \in S)$. Therefore $\theta = \psi^{-1}(\theta') \in \mathsf{Hil}$ is sufficiently closed to $\psi^*(0) = y' \times_k k_v \approx z_v$ for all $v \in S$. We know also that $\theta \sim y$ on C. □

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