Hefei-Lectures 2015
Second Lesson: Photons
Part 1

Claus Zimmermann, Eberhard-Karls-Universität Tübingen, Germany

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2.1 Quantization of an Optical Mode

- Maxwells equations

  We start with Maxwells equations in free space

$$\begin{aligned} \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times B &= \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \\ \nabla \cdot B &= 0 \\ \nabla \cdot E &= 0 \end{aligned}$$

  and use them to derive the wave equation for the electric field (see basic physics classes)

$$\nabla^2 \vec{E}(\vec{r}, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}(\vec{r}, t)$$

- separation of variables

  We look for solutions of the form

$$\vec{E}(\vec{r}, t) = E(t) \cdot \vec{u}(\vec{r})$$

  and obtain

$$\begin{aligned} \nabla^2 \left(E(t) \cdot \vec{u}(\vec{r}) \right) &= -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(E(t) \cdot \vec{u}(\vec{r}) \right) \\ E(t) \nabla^2 \vec{u}(\vec{r}) &= \vec{u}(\vec{r}) \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(t) \\ \frac{1}{\vec{u}(\vec{r})} \nabla^2 \vec{u}(\vec{r}) &= \frac{1}{c^2} \frac{1}{E(t)} \frac{\partial^2}{\partial t^2} E(t) \end{aligned}$$

  Since the left part only depends on $\vec{r}$ and the right part only on t, a solution is possible only if the two parts of the equation are constant. As it will turn out the constant is negative and we may call it: $-k^2$. We obtain two equations

$$\begin{aligned} \frac{1}{\vec{u}(\vec{r})} \nabla^2 \vec{u}(\vec{r}) &= -k^2 \\ \frac{1}{c^2} \frac{1}{E(t)} \frac{\partial^2}{\partial t^2} E(t) &= -k^2 \end{aligned}$$

  The second equation

$$\frac{\partial^2}{\partial t^2} E(t) = -c^2 k^2 E(t)$$

  is solved by

$$E\left(t\right) = E e^{\pm i \omega t}$$

and one obtains

$$\omega^2 = k^2 c^2.$$

We call k the wave number and ω the angular frequency of the light field.

- Helmholtz equation and optical modes

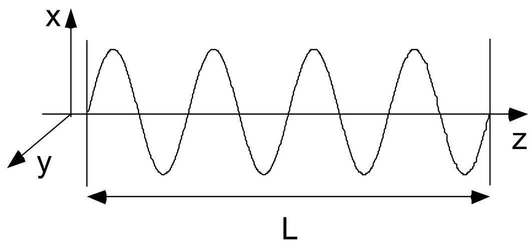
  The second equation

  $$\nabla^2 \vec{u}(\vec{r}) = -k^2 \vec{u}(\vec{r})$$

  is the Helmholtz equation. It has the form of an eigen-equation. Its solution returns not only the value of k^2 and thus the frequency of the light field but also the spacial distribution of the light field $u_k(\vec{r})$. These solutions are called "optical modes". They depend on the boundary conditions. A instructive example are the Gaussian modes of a cavity (see first lesson).

- standing waves

  To keep things simple we use linearly polarized, plane standing waves. They form a set of modes in a volume that is limited by two parallel, perfectly reflecting planes separated by a distance L. The light propagates along the z-direction and is polarized in the x-direction. At the planes the field vanishes.



We write the field as a product of the mode function $\sin(kx)$ and a prefactor.

$$E_x(z,t) = \sqrt{\frac{2\omega_m^2}{V\varepsilon_0}} \cdot q(t) \sin(k \cdot z)$$

The root has the dimensions of a field such that $q(t)$ is dimensionless. V is the Volume between the planes and $\omega_m = ck_m$ the frequency which can be derived from the boundary conditions:

$$n \cdot \frac{\lambda_m}{2} = L.$$

The wave length is $\lambda_m = 2\pi/k$. Possible wave numbers are:

$$k_m = \frac{2\pi}{\lambda_m}.$$

resulting in

$$\omega_m = c \cdot k_m = c \cdot m \cdot \frac{\pi}{L}.$$

Similarly we obtain from Maxwell's equations the magnetic field (as exercise):

$$B_y(z,t) = \left(\frac{\mu_0 \varepsilon_0}{k}\right) \sqrt{\frac{2\omega_m^2}{V\varepsilon_0}} \dot{q}(t) \cos(k \cdot z).$$

- quantization

  We know from Planck's derivation of the blackbody-radiation spectrum that the energy of the electromagnetic field can only have discrete energy values. This also means that the electric and the magnetic field amplitudes are restricted an cannot take just any value. Consequently, we have to reinterpret the normalized pre factors q and $\dot{q}$ somehow. As first step we calculate the energy of the mode:

$$H = \frac{1}{2} \int dV \left(\varepsilon_0 E_x^2(z,t) + \frac{1}{\mu_0} B_y^2(z,t) \right).$$

It can be rewritten as (exercise)

$$H = \frac{1}{2}(p^2 + \omega_m^2 q^2)$$

with

$$p := \dot{q}.$$

We now interpret q and p as operators $\hat{q}$ and $\hat{p}$ which act in a Hilbert space that is still to be constructed. If we work with operators we need to know their commutation relation. We choose

$$[\hat{q}, \hat{p}] = i\hbar.$$

With this step, the fields and the energy also become operators:

$$\begin{aligned}
H &\rightarrow \hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2) \\
E &\rightarrow \hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \hat{q}(t) \sin(k \cdot z) \\
B &\rightarrow \hat{B}_y(z,t) = \frac{\mu_0 \varepsilon_0}{k} \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \hat{p}(t) \cos(k \cdot z)
\end{aligned}$$

As a basis for the new Hilbert space we use the eigen-states of the Hamilton operator $\hat{H}$. These states are called "Fock-states" $|n\rangle$ and they obey the eigen equation

$$\hat{H}|n\rangle = \varepsilon_n|n\rangle.$$

As eigen-states of an Hermitian operator they form a orthonormal basis set with

$$\langle n'|n\rangle = \delta_{nn'}$$

and

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = 1$$

- creation and annihilation operators

  $\hat{H}$ has the same form as the Hamilton-operator of the harmonic oscillator. We can thus use the same method for finding the eigen-values and eigen-states. We thus define the creation operator

$$\hat{a}^{+} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

and the annihilation operator

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}),$$

with

$$[a, a^{+}] = 1.$$

The field operators now read

$$\begin{aligned}\hat{E}_x(z,t) &= E_0(\hat{a} + \hat{a}^{+})\sin(k\cdot z)\\ \hat{B}_y(z,t) &= -iB_0(\hat{a} - \hat{a}^{+})\cos(k\cdot z)\end{aligned}$$

with

$$E_0 := \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}},\ B_0 := \frac{\mu_0}{k}\sqrt{\frac{\varepsilon_0\hbar\omega^3}{V}},$$

and the Hamilton operator may be written as (exercise):

$$\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2}).$$

For simplicity we drop the hat $\hat{}$ on a and a^{+}.

- time dependence

  The temporal evolution of a and $a^+$is given by the Heisenberg equations

$$\begin{aligned}
\frac{da}{dt} &= \frac{i}{\hbar}[\hat{H}, a] \\
&= \frac{i}{\hbar}[\hbar\omega(a^+a + \frac{1}{2}), a] \\
&= i\omega(a^+aa - aa^+a) \\
&= -i\omega[a, a^+]a \\
&= -i\omega a
\end{aligned}$$

  with the solution

$$a(t) = a(0) \cdot e^{-i\omega t}.$$

  With the same argument we obtain

$$a^+(t) = a^+(0) \cdot e^{i\omega t}$$

- constructing the spectrum

  We take the eigen-equation for $\hat{H}$

$$\hat{H}|n\rangle = \hbar\omega(a^+a + \frac{1}{2})|n\rangle = \varepsilon_n|n\rangle,$$

  and apply a^+ from the left

$$\hbar\omega(a^+a^+a + \frac{1}{2}a^+)|n\rangle = a^+\varepsilon_n|n\rangle.$$

  using

$$\left[a, a^+\right] = 1 \Rightarrow a^+a = aa^+ - 1$$

  we obtain

$$\begin{aligned}
\hbar\omega(a^+aa^+ - a^+ + \frac{1}{2}a^+)|n\rangle &= \varepsilon_n a^+|n\rangle \\
\hbar\omega\left(a^+a - 1 + \frac{1}{2}\right)a^+|n\rangle &= \varepsilon_n a^+|n\rangle \\
\hbar\omega\left(a^+a + \frac{1}{2}\right)a^+|n\rangle &= (\varepsilon_n + \hbar\omega)a^+|n\rangle \\
Ha^+|n\rangle &= (\varepsilon_n + \hbar\omega)a^+|n\rangle
\end{aligned}$$

  If $|n\rangle$ is an eigen-state with the energy ε_n also $a^+|n\rangle$ is an eigen-state with the energy $\varepsilon_n + \hbar\omega$. The operator a^+ creates a state with an energy increment $\hbar\omega$. It creates energy.

By the same way we also find that

$$Ha|n\rangle = (\varepsilon_n - \hbar\omega)a|n\rangle.$$

The annihilation operator creates a new state with an energy smaller by $\hbar\omega$. It annihilates energy.

- ground state

  There must be a state $|0\rangle$ with smallest energy. We define this state by requiring that

  $$a|0\rangle = 0.$$

  The ground state is called the vacuum state. It has the energy

  $$\begin{aligned}\langle 0|\, H|0\rangle &= \langle 0|\, \hbar\omega(a^+a + \frac{1}{2})|0\rangle = \langle 0|\, \hbar\omega a^+ \overbrace{a|0\rangle}^{0} + \langle 0|\, \hbar\omega\frac{1}{2}|0\rangle \\ &= \frac{1}{2}\hbar\omega\end{aligned}$$

  The total spectrum now is

  $$\varepsilon_n = \hbar\omega(n + \frac{1}{2}) \quad n = 0, 1, 2, ...$$

- number operator

  We now know the spectrum and can write down the eigen-equation as

  $$\begin{aligned}H|n\rangle &= \varepsilon_n|n\rangle \\ \hbar\omega(a^+a + \frac{1}{2})|n\rangle &= \hbar\omega(n + \frac{1}{2})|n\rangle \\ \hbar\omega a^+a|n\rangle &= \hbar\omega n \cdot |n\rangle \\ a^+a|n\rangle &= n \cdot |n\rangle\end{aligned}$$

  With the definition of the "number operator" $\hat{n} := a^+a$ one gets

  $$\hat{n}|n\rangle = n \cdot |n\rangle.$$

  The number operator is an observable and represents the number of photons in the mode.

- constructing the Fock states

  We compare

  $$\hat{H}a|n\rangle = (\varepsilon_n - \hbar\omega)a|n\rangle$$

with

$$\hat{H}|n-1\rangle = \varepsilon_{n-1}|n-1\rangle = (\varepsilon_n - \hbar\omega)|n-1\rangle$$

and obtain

$$a|n\rangle = C_n|n-1\rangle$$

with an unknown C_n. It can be calculated:

$$\begin{aligned}
\langle n|\hat{n}|n\rangle &= n \\
\langle n|a^+ a|n\rangle &= n \\
\langle an|an\rangle &= n \\
C_n^* \langle n-1|n-1\rangle C_n &= n \\
|C_n|^2 &= n \quad \text{or} \quad C_n = \sqrt{n}.
\end{aligned}$$

The same way we get

$$a^+|n\rangle = \sqrt{n+1}|n+1\rangle$$

All Fock states can be now be constructed from the vacuum state by applying a^+ repetitively:

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

- matrix

  Written as a matrix the creation and annihilation operator have elements only in the upper and lower off diagonal.

$$\begin{aligned}
\langle n-1|a|n\rangle &= \sqrt{n} \\
\langle n+1|a^+|n\rangle &= \sqrt{n+1}
\end{aligned}$$

- expectation value of the fields

  The expectation value of the electric field is

$$\begin{aligned}
\langle n|\hat{E}_x(z,t)|n\rangle &= E_0 \sin(k\cdot z)(\langle n|a|n\rangle + \langle n|a^+|n\rangle) \\
&= E_0 \sin(k\cdot z)(C_n\langle n|n-1\rangle + C_n^*\langle n-1|n\rangle \\
&= 0
\end{aligned}$$

and of the magnetic field

$$\begin{aligned}
\langle n|\hat{B}_x(z,t)|n\rangle &= \langle n| - iB_0(a-a^+)\cos(k\cdot z)|n\rangle \\
&= -iB_0\cos(k\cdot z)((C_n\langle n|n-1\rangle - C_n^*\langle n-1|n\rangle = 0
\end{aligned}$$

For Fock states, the expectations values of the fields vanish! Where is the energy?

- field fluctuations

  The energy is in the fluctuations. To calculate them we have to look at the variance

$$\langle n|(\Delta \hat{E}_x(z,t))^2|n\rangle = \langle n|\hat{E}_x^2(z,t)|n\rangle - \underbrace{\langle n|\hat{E}_x(z,t)|n\rangle^2}_{0}$$

  and evaluate

$$\langle n|\hat{E}_x^2|n\rangle = E_0^2 \sin^2(k\cdot z)\langle n|(a+a^+)^2|n\rangle$$

  with

$$\begin{aligned}
\langle n|(a+a^+)^2|n\rangle &= \langle n|a^+a^+ + aa + a^+a + aa^+|n\rangle \\
&= \langle n|a^+a^+|n\rangle + \langle n|aa|n\rangle + \langle n|a^+a|n\rangle + \langle n|aa^+|n\rangle \\
&= \langle n|a^+a|n\rangle + \langle n|\left([a,a^+] + a^+a\right)|n\rangle \\
&= 2\langle n|\hat{n}|n\rangle + \langle n|1|n\rangle \\
&= 2n+1
\end{aligned}$$

  one obtains

$$\langle n|\hat{E}_x^2|n\rangle = 2E_0^2 \sin^2(k\cdot z)(n+\frac{1}{2}).$$

  The standard deviation is

$$\Delta E = \sqrt{\langle \Delta \hat{E}_x^2\rangle} = \sqrt{2}E_0\sqrt{n+\frac{1}{2}}\cdot|\sin(k\cdot z)|$$

  For $n \gg 1$ the fluctuations increase with $\sqrt{n}$.

  The fluctuations of the magnetic field are calculated fully analogous. The result is

$$\Delta B = \sqrt{\langle \Delta \hat{B}_x\rangle} = \sqrt{2}B_0\sqrt{n+\frac{1}{2}}\cdot|\cos(k\cdot z)|$$

- vacuum fluctuations

  The energy of the vacuum of $\hbar\omega$ is due to fluctuations. The vacuum is not empty but rather noisy. This noise is responsible for the spontaneous decay of electronically excited atoms, for van der Waals forces and for the lamb shift.

- getting used to photons

  Lets discuss some consequences of the mode model by looking at a gedanken-experiment. Consider a very long standing wave resonator with one mirror placed in Hefei and one in Shanghai (by the way: even longer resonators are discussed for detection of gravitational waves in space). A special device placed close to the mirror in Hefei injects a photon into the fundamental mode. What is the earliest time the photon can be detected with a suitable detector close to the mirror in Shanghai?

We must distinguish two cases. In the first case the photon is injected very slowly. The energy of the photon is then pretty well defined. According to the energy time uncertainty the energy uncertainty of the photon is about

$$\Delta E \simeq \hbar \frac{1}{\tau}$$

with τ being the time it takes to get the photon into the cavity. If the spread is smaller that the free spectral range,

$$\Delta E < \hbar\omega_{FSR}$$

only one longitudinal mode is involved and occupied by the photon. If the photon is in the mode it is at all positions inside the mode immediately with equal probability. It doesn't need time to "travel" to Shanghai. Once it occupies the mode it can be detected immediately at any position inside the mode volume. Lets calculate the minimum injection time which does not violate the above condition: With

$$\hbar\omega_{FSR} = \hbar \cdot 2\pi \frac{c}{L}$$

we obtain

$$\begin{aligned} \Delta E &< \hbar\omega_{FSR} \\ \hbar \frac{1}{\tau} &< \hbar \cdot 2\pi \frac{c}{2L} \\ \frac{1}{\pi} \frac{L}{c} &< \tau \end{aligned}$$

The injection time must not be shorter than the time of flight from Hefei to Shanghai at the speed of light (except for the factor π which is not annoying for such a crude estimation)! No superluminal transport!

In the second case we inject the photon very fast. The energy spread is large and many longitudinal modes are occupied simultaneously according to a certain probability distribution. All the modes interfere and form a pulse that propagates at the speed of light. The pulse carries the photon to Shanghai. This corresponds to the classical picture only that the energy in the pulse is quantized. Don't mix the photon and the pulse! The pulse is only a superposition of modes and a classical object. The photon is the particle that occupies this superposition state.

We summarize: A photon in a single mode is not a localized particle. The transport inside the mode cannot be imagined as something travelling along a classical trajectory. The mode is a static object. To shape a propagating pulse a superposition of modes is necessary.

- BB84 Protocol.

Photons cannot be divided in parts. They are elementary. This property can be used to exchange keys for cryptography in a way such that any possible spy who has hacked the connection will be discovered. We use the polarization of the photon to transmit the bits from Alice to Bob: horizontal polarization means 1 and vertical polarization means 0. Eve (for eavesdropper) intercepts the photon, detects its polarization, and sends a new photon to Bob with the same polarization as the one she took from the fiber. Bob would not notice this manipulation. But now comes the trick: Alice and Bob use two types of photons. The first type is polarized in the horizontal/vertical Basis, the second type is polarized in a diagonal Basis. Alice and Bob change their basis randomly. After transmitting a few thousand photons they use the telephone (or any other non secure classical communication channel) to find out which photon was send and received in the same basis. These are the good photons. They don't reveal the value of the bit only the basis! The specific polarization of the good photons is still a secret! The non secure line is only used to identify the useful photons. The other photons are ignored. Now Eve has a problem because during the transmission of the photons she cannot know the basis of the polarization which later turn out to be the one of the good photons. All she can do is randomly select a basis, measure the polarization of the intercepted photon, and then replace the detected photon by one which has the same polarization as the result of her measurement. In 50% of the cases she is wrong and sends the wrong photon. Alice and Bob reveal this by taking a certain subset of the good photons and compare them via the classical channel. If Eve sits in the line, on average half of photons are detected by Bob with a polarization that is different to the polarization Alice chose. If this is not the case the remaining non revealed photons form the key. This method was proposed by Charles Bennett and Gilles Brassard in 1984 and is now developed to an extend that is can be used routinely. It also demonstrates the necessity of developing efficient and reliable single photon sources.

2.2 Coherent states

Fock states are very non classical. How can we construct states with oscillating expectation value of the fields?

- coherent states

  the coherent state $|\alpha\rangle$ is defined as eigen-states of the annihilation operator a:

  $$a|\alpha\rangle = \alpha|\alpha\rangle$$

  By comparing

  $$\langle\alpha|a|\alpha\rangle = \langle\alpha|\alpha|\alpha\rangle = \alpha$$

  and

  $$\langle\alpha|a|\alpha\rangle = \left\langle\alpha|a^{+}|\alpha\right\rangle^{*}$$

  we obtain

  $$\left\langle\alpha|a^{+}|\alpha\right\rangle = \alpha^{*}$$

and finally

$$\langle\alpha|a^+ = \alpha^*\langle\alpha|.$$

- Fock state expansion

  We expand $|\alpha\rangle$ in the Fock Basis

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$$

and determine the coefficients c_n by inserting the sum into the eigen-equation

$$a\sum_{n=0}^{\infty} c_n|n\rangle = \alpha\cdot\sum_{n=0}^{\infty} c_n|n\rangle.$$

The right side can be written as

$$\alpha\cdot\sum_{n=0}^{\infty} c_n|n\rangle = \alpha\cdot\sum_{n=1}^{\infty} c_{n-1}|n-1\rangle = \alpha\cdot\sum_{n=0}^{\infty} c_{n-1}|n-1\rangle.$$

while the left side can be transformed into

$$a\sum_{n=0}^{\infty} c_n|n\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n}|n-1\rangle$$

Comparing both sums yields

$$c_n\sqrt{n} = \alpha\cdot c_{n-1}$$

and

$$c_n = \frac{\alpha}{\sqrt{n}}c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}c_{n-2} = ... = \frac{\alpha^n}{\sqrt{n!}}c_0$$

resulting in

$$|\alpha\rangle = c_0\cdot\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

The first coefficient c_0 is fixed by the normalization condition

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= 1 = |c_0|^2\sum_n\sum_{n'}\frac{{\alpha^*}^n\alpha^{n'}}{\sqrt{n!n'!}}\underbrace{\langle n|n'\rangle}_{\delta nn'} \\
&= |c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |c_0|^2\cdot e^{|\alpha|^2} \\
&\Rightarrow c_0 = e^{-\frac{1}{2}|\alpha|^2}.
\end{aligned}$$

The final result is

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\cdot|n\rangle$$

- field of a running wave

  The operator of a running wave is (superposition of two standing waves).

$$\begin{aligned} \hat{E}_x(\vec{r},t) &= iE_r \left(a e^{i(kr-\omega t)} - a^+ e^{-i(kr-\omega t)} \right) \\ E_r &= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \end{aligned}$$

  We calculate the expectation value:

$$\langle \alpha | \hat{E}_x(r,t) | \alpha \rangle = iE_r \left(\alpha e^{i(kr-\omega t)} - \alpha^* e^{-i(kr-\omega t)} \right).$$

  By writing $\alpha = |\alpha| \cdot e^{i\theta}$ we obtain

$$\langle \alpha | \hat{E}_x(r,t) | \alpha \rangle = 2|\alpha| E_r \sin(\omega t - kr - \theta).$$

  This corresponds to a classical wave with an amplitude proportional to $|\alpha|$ and a phase θ at $t = 0$ and $z = 0$.

- fluctuations

  To calculate the variance

$$\langle \Delta \hat{E}^2 \rangle = \langle \hat{E}^2 \rangle - \langle \hat{E} \rangle^2.$$

  we need

$$\langle \alpha | \hat{E}^2(r,t) | \alpha \rangle = E_r^2 \left(1 + 4|\alpha|^2 \sin^2(\omega t - kr - \theta) \right).$$

  with this we get

$$\langle \Delta \hat{E}^2 \rangle = E_r^2 \left(1 + 4|\alpha|^2 \sin^2(\omega t - kr - \theta) \right) - 4|\alpha|^2 E_r^2 \sin^2(\omega t - kr - \theta) = E_r^2$$

  and for the standard deviation

$$\Delta E = \sqrt{\langle \Delta \hat{E}^2 \rangle} = E_r = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} = \frac{1}{\sqrt{2}} E_0.$$

- number operator

  The expectations value for the number operator is

$$\begin{aligned} \langle \alpha | n | \alpha \rangle &= \langle \alpha | a^+ a | \alpha \rangle = \alpha^* \langle \alpha | \alpha | \alpha \rangle \\ &= |\alpha|^2 \langle \alpha | \alpha \rangle = |\alpha|^2 =: \bar{n} \end{aligned}$$

  $|\alpha|^2$ is the average photon number of the coherent state $|\alpha\rangle$.

- number fluctuations

  With

$$
\begin{aligned}
\langle\alpha|n^2|\alpha\rangle &= \langle\alpha|a^+aa^+\hat{a}|\alpha\rangle \\
&= \langle\alpha|a^+a^+aa + a^+a|\alpha\rangle \\
&= |\alpha|^4 + |\alpha|^2 = \bar{n}^2 + \bar{n}
\end{aligned}
$$

  we get

$$\Delta n = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2} = \sqrt{\bar{n}^2 + \bar{n} - \bar{n}^2} = \sqrt{\bar{n}}$$

- Poisson-distribution

  The probability to observe the photon number n in a coherent state $|\alpha\rangle$ is given by the projection to the Fock state $|n\rangle$:

$$
\begin{aligned}
P_n &= |\langle n|\alpha\rangle|^2 = |\langle n|e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^{n'}}{\sqrt{n'|}}|n'\rangle|^2 \\
&= |\sum_{n'=0}^{\infty}e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^{n'}}{\sqrt{n'!}}\underbrace{\langle n|n'\rangle}_{\delta nn'}|^2 \\
&= |e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \\
&= e^{-\bar{n}}\frac{\bar{n}^n}{n!} = P_{\bar{n}}(n).
\end{aligned}
$$

  This is a Poissonian distribution with a mean value $\bar{n}$. The relative number uncertainty is:

$$\frac{\Delta n}{\bar{n}} = \frac{\sqrt{\bar{n}}}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}.$$

  Same statistics as rain collecting in a set of equal cups. Each cup is filled according to a Poissonian distribution.

- shot noise

  Since we usually deal with large numbers of photons the shot noise is small. Laser beam: $1\,\mathrm{mW} \sim 10^{16}$ photons/s. Atom: lifetime of an excited state $\tau \sim 20 = 2\cdot 10^{-8}\,\mathrm{s}$. At saturation intensity about $2\cdot 10^8$ photons are scattered per second. If we collect the emitted photons during one second the signal has a shot noise of only.

$$\frac{\Delta n}{n} \approx 10^{-4} = 0.01\%.$$

- Quadratur-components

We decompose the field in two components, one which oscillates with a sin-function and one with the cos-function: For this purpose we define the operators for the quadrature amplitudes:

$$\begin{aligned} x_1 &:= \frac{1}{2}(a + a^+) \\ x_2 &:= \frac{1}{2i}(a - a^+). \end{aligned}$$

The electric field

$$\hat{E}_x(t) = E_0(a\, e^{-i\omega t} + a^+ e^{i\omega t}) \sin(kz)$$

now reads

$$\hat{E}_x(t) = 2E_0 \sin(kz)\, (x_1 \cos\omega t + x_2 \sin\omega t)$$

x_1 are x_2 the Amplitudes of a sine and a cosine oscillation.

- properties of the quadrature components

$$[x_1, x_2] = \frac{i}{2}$$

$$\langle(\Delta x_1)^2\rangle\langle(\Delta x_2)^2\rangle \geqslant \frac{1}{16}$$

$$\langle n|x_1^2|n\rangle = \frac{1}{4}(2n+1)$$

$$\langle n|x_2^2|n\rangle = \frac{1}{4}(2n+1)$$

$$\langle(\Delta x_1)^2\rangle_{vac} = \frac{1}{4} = \langle(\Delta x_2)^2\rangle_{vac}$$

- quadratures of coherent states

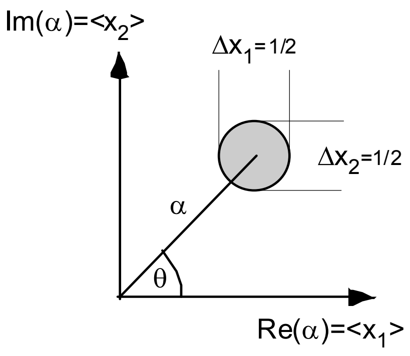
  We calculate the expectation values

$$\begin{aligned} \langle\alpha|x_1|\alpha\rangle &= \frac{1}{2}\langle\alpha|\hat{a} + \hat{a}^+|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*) = \mathrm{Re}(\alpha) \\ \langle\alpha|x_2|\alpha\rangle &= \frac{1}{2i}\langle\alpha|\hat{a} - \hat{a}^+|\alpha\rangle = \frac{1}{2i}(\alpha - \alpha^*) = \mathrm{Im}(\alpha) \end{aligned}$$

  and the fluctuations

$$\begin{aligned} \langle(\Delta x_1)^2\rangle_\alpha &= \langle(\Delta x_2)^2\rangle_\alpha = \frac{1}{4} \\ \Delta x_1 &= \frac{1}{2} = \Delta x_2 \end{aligned}$$

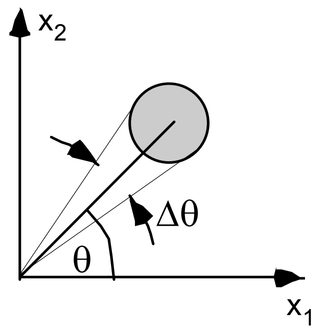
- representation of coherent states in phase space



The decomposition in quadrature components corresponds to the representation of α in the complex plane. The circle indicates the fluctuations. They are independent of the photon number and the same for vacuum and coherent states with large photon number.

- phase uncertainty

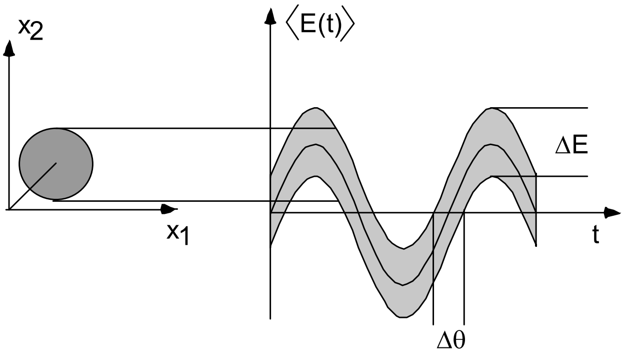
  The phase also fluctuates. The uncertainty follows from the geometry.



  It decreases with increasing photon number.

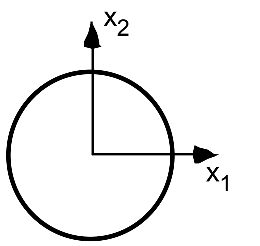
- temporal behavior

  The mean values of the two quadrature components oscillate as $\langle\hat{x}_1\rangle = \alpha\cos\omega t$ and $\langle\hat{x}_2\rangle = \alpha\sin\omega t$. The circle of uncertainty itself moves on a circle around the origin with an angular frequency ω. The actual field is the projection to one of the axes.



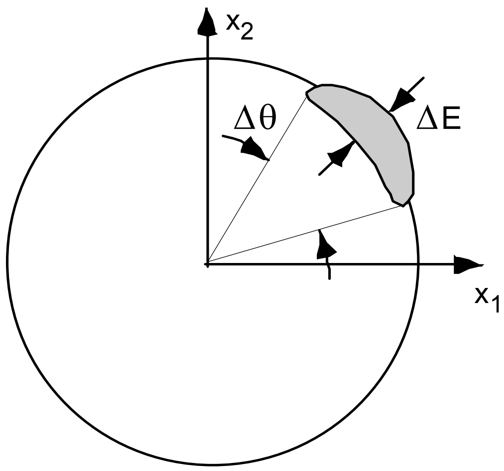
- Fock state

  the amplitude does not fluctuate but the phase is completely uncertain leading to a vanishing expectation value for the field.



- squeezed state

  in general all kinds of shapes in phase space are possible.



It turns out that there is an uncertainty relation for the photon number and the phase which must be fulfilled:

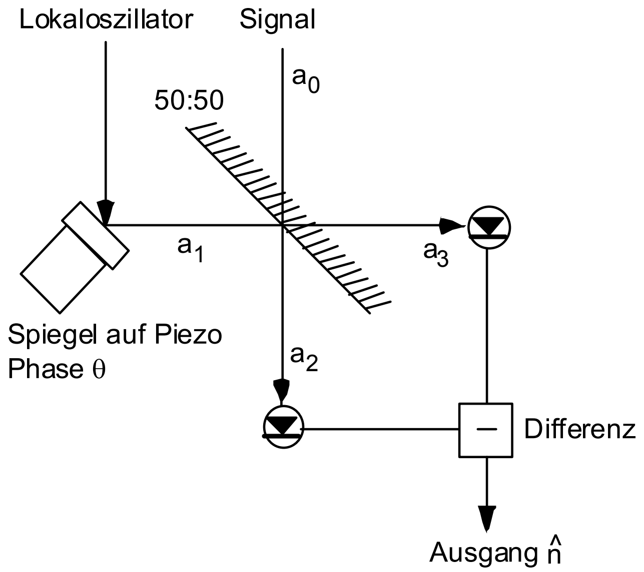
$$\Delta n \cdot \Delta\theta \geqslant 1$$

Photon number and phase cannot be sharp simultaneously.

2.3 Squeezed states in the lab

- measuring squeezed states

  The squeezed light is overlapped with a reference laser of the same frequency (local oscillator) on a 50:50 beam splitter. The reference laser is in a coherent state and its phase θ can be shifted with a mirror on a piezo actuator that shifts the mirror by a few wavelengths in a controlled way. Two photodiodes detect the two output beams and their signal is electronically subtracted from each other. This signal is recorded for different θ.



- Observable

  What is the operator $\hat{n}$ that we measure? We will later see that the output ports of a beam splitter can be described by the two operators. They create the photons in the output beams.

$$a_2 = \frac{1}{\sqrt{2}}(a_0 + ia_1)$$
$$a_3 = \frac{1}{\sqrt{2}}(a_1 + ia_0)$$

Out recorded signal is proportional to the difference of the intensities in the output ports:

$$
\begin{aligned}
\hat{n} &= I_2 - I_3 = a_2^+ a_2 - a_3^+ a_3 \\
&= \frac{1}{\sqrt{2}} \left(a_0^+ - i a_1^+ \right) \frac{1}{\sqrt{2}} \left(a_0 + i a_1 \right) - \frac{1}{\sqrt{2}} \left(a_1^+ - i a_0^+ \right) \frac{1}{\sqrt{2}} \left(a_1 + i a_0 \right) \\
&= \frac{1}{2} \left(a_0^+ a_0 + i a_0^+ a_1 - i a_1^+ a_0 + a_1^+ a_1 - a_1^+ a_1 - i a_1^+ a_0 + i a_0^+ a_1 - a_0^+ a_0 \right) \\
&= i \left(a_0^+ a_1 - a_1^+ a_0 \right)
\end{aligned}
$$

The terms proportional to the intensity equally cancel for a 50:50 beam splitter and only the mixed terms remain. Technical noise is suppressed for a well balanced beam splitter.

- coherent local oscillator

  we calculate the expectation value for a local oscillator in a coherent state $|\alpha\rangle$ at port 1:

$$
\begin{aligned}
\langle \hat{n} \rangle &= \langle \alpha | \, i \left(a_0^+ a_1 - a_1^+ a_0 \right) |\alpha\rangle \\
&= i \langle \alpha | \, a_1 \, |\alpha\rangle \, a_0^+ - i \langle \alpha | \, a_1^+ \, |\alpha\rangle \, a_0 \\
&= i \alpha a_0^+ - i \alpha^* a_0
\end{aligned}
$$

  With

$$\alpha = |\alpha| \, e^{-i\phi}$$

  and

$$\theta := \phi + \frac{\pi}{2}$$

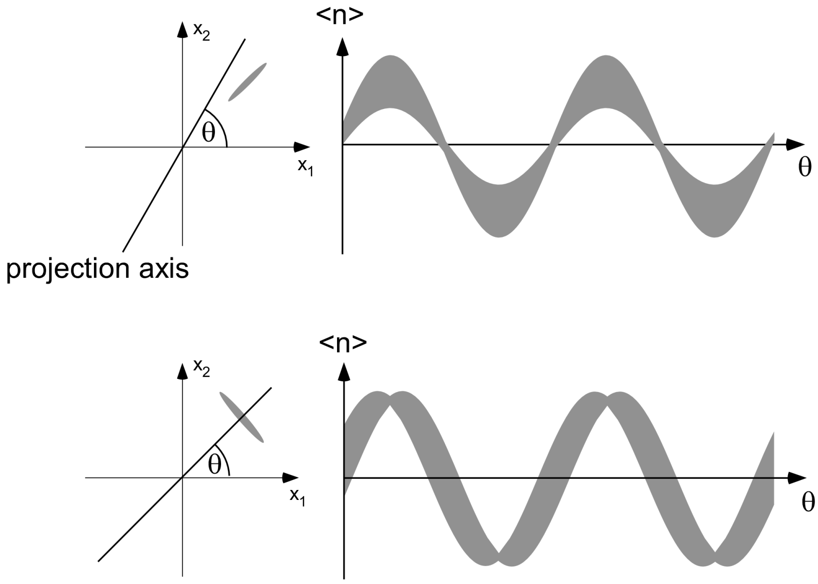
  we get

$$
\begin{aligned}
\langle \hat{n} \rangle &= i \, |\alpha| \, e^{i\phi} a_0^+ - i \, |\alpha| \, e^{-i\phi} a_0 \\
&= |\alpha| \left(e^{-i\theta} a_0 + e^{i\theta} a_0^+ \right) \\
&= |\alpha| \left(\cos\theta \left(a_0 + a_0^+ \right) - i \sin\theta \left(a_0 - a_0^+ \right) \right) \\
&= 2 \, |\alpha| \left(\hat{x}_1 \cos\theta + \hat{x}_2 \sin\theta \right)
\end{aligned}
$$

  For $\theta = 0, \pi/2$ the signal is proportional the the projection of the squeezed light at port 0 to either of its two quadrature components.

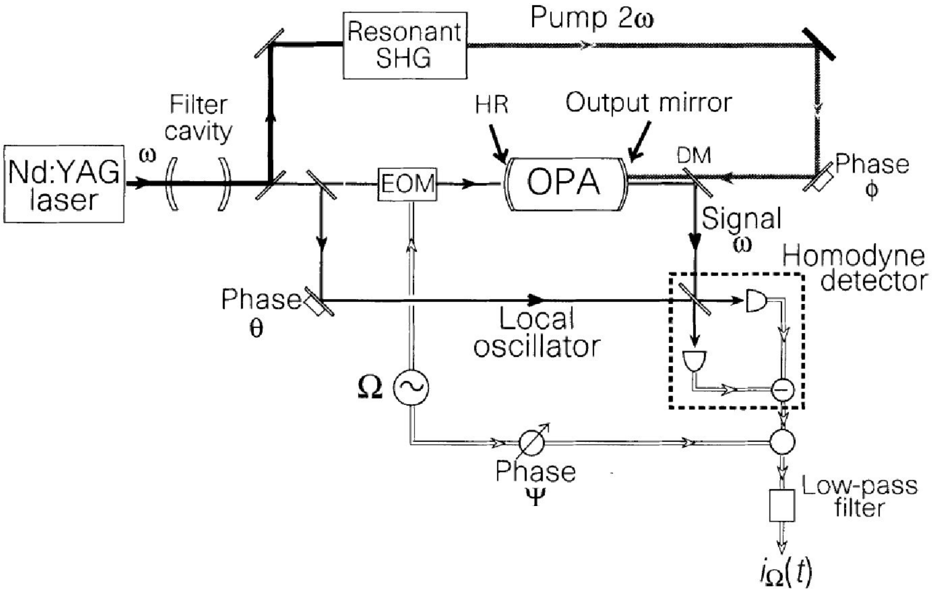
- tomography

  For other values of θ the squeezed state can be projected to any rotated reference axis. In analogy to x-ray tomography the method is also called state-tomography. As examples we look at a phase squeezed state and a number squeezed state and their projections to a rotating reference axis:



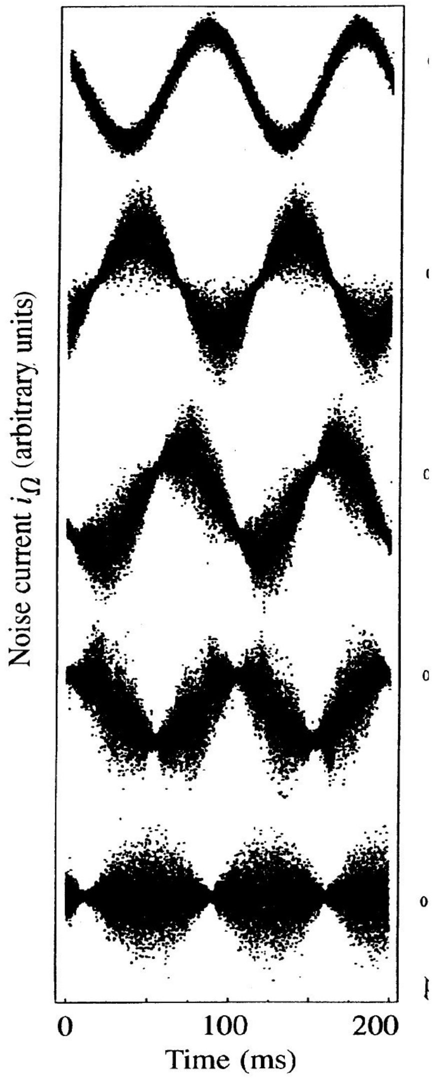
- experiment of Breitenbach, Schiller and Mlynek (Nature 387, 471 (1998)), setup

  Center piece is a monolithic, degenerate, optical parametric amplifier (OPA). This is a nonlinear crystal with coated end facets that form an optical cavity. The cavity supports a pump mode at frequency 2ω and a signal mode at frequency ω. The pump light is generated by a laser which is frequency doubled in a nonlinear crystal. The pump light enters the OPA from the right and the seed light from the left. Signal light from the right output port is analyzed by a squeezing detector with some of the seed light used as local oscillator. A phase modulation detection scheme, similar to the Pound Drever Hall method, reduces the noise of the observed signal. Seed light is amplified by the OPA. Inside the OPA a seed photon and a pump photon is transformed into three seed photons which leave the OPA in the signal beam. The signal light cannot have the same Poissonian photon statistics as the pump beam or the seed beam since two additional signal photons appear now in the signal mode with the same probability as one photon appears in the pump mode. As a consequence the signal light is squeezed. The shape of the squeezed light in phase space depends on the relative phase ϕ between the seed light and the pump light.



- observations

  The following curves have been recorded (from top to bottom): coherent state, phase squeezed state, squeezed state with $\phi = 48°$, number squeezed state, squeezed vacuum. The phase θ of the local oscillator varies linearly in time.



The coherent state is observed with the pump beam blocked. One only observes the seed beam transmitted through the OPA without getting squeezed. Squeezed vacuum is observed by blocking the seed laser.