Hefei-Lectures 2015
Second Lesson: Photons
Part 2

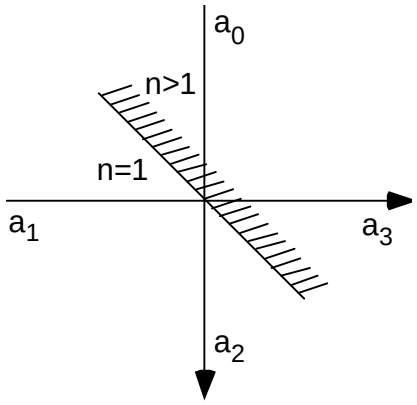
Claus Zimmermann, Eberhard-Karls-Universität Tübingen, Germany

November 1, 2015

2.4 Beam splitter and Hong-Ou-Mandel-Interferometer

- beam splitter

  As simple example for a beam splitter we consider the interface between a material with index of refraction n and vacuum. At the interface, part of the light is reflected the rest is transmitted. The index n can, in principle, be chosen such that 50% of the power is transmitted and 50% is reflected.



  We get 4 modes for the four ports of the splitter. A photon which enters through port 1 cannot be divided and will exit as whole either through port 2 or port 3 with equal probability and an yet unknown phase. So we guess that it will be in the general superposition state

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_2|0\rangle_3 + e^{i\theta}|0\rangle_2|1\rangle_3\right).$$

  We introduce only the relative phase factor $e^{i\theta}$since a global phase can be ignored. Indeed, such a state has the correct properties: The probability amplitude to find a photon in mode 2 and no photon in mode 3 is given by the projection

$$\begin{aligned} {}_2\langle 1|_3\langle 0|\varphi\rangle &= \frac{1}{\sqrt{2}}\left({}_2\langle 1|1\rangle_{23}\langle 0|0\rangle_3 + e^{i\theta}\,{}_2\langle 1|0\rangle_{23}\langle 0|1\rangle_3\right) \\ &= \frac{1}{\sqrt{2}}\left({}_2\langle 1|1\rangle_{23}\langle 0|0\rangle_3\right) \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

  The same holds for the probability amplitude to find a photon in mode 3 and no photon

in mode 2:

$$\begin{aligned} {}_2\langle 0|_3\langle 1|\varphi\rangle &= \frac{1}{\sqrt{2}}\left({}_2\langle 0|1\rangle_{23}\langle 1|0\rangle_3 + e^{i\theta}\,{}_2\langle 0|0\rangle_{23}\langle 1|1\rangle_3\right) \\ &= \frac{1}{\sqrt{2}}\left(e^{i\theta}\,{}_2\langle 0|0\rangle_{23}\langle 1|1\rangle_3\right) \\ &= \frac{1}{\sqrt{2}}e^{i\theta}. \end{aligned}$$

The probability for both cases is 1/2. Furthermore, the probability to find a photon in port 2 and a photon in port 3 is zero:

$$\begin{aligned} {}_2\langle 1|_3\langle 1|\varphi\rangle &= \frac{1}{\sqrt{2}}\left({}_2\langle 1|1\rangle_{23}\langle 1|0\rangle_3 + e^{i\theta}\,{}_2\langle 1|0\rangle_{23}\langle 1|1\rangle_3\right) \\ &= \frac{1}{\sqrt{2}}\left({}_2\langle 1|1\rangle\cdot 0 + e^{i\theta}\,\cdot 0\cdot{}_3\,\langle 1|1\rangle_3\right) \\ &= 0 \end{aligned}$$

How can we theoretically justify this state and what is the value of θ ?

- Ansatz for the beam splitter

  If we create a photon in mode 1 we can observe it with 50% probability in each of the exit modes 2 and 3. Creating a photon in mode 1 is thus equivalent to creating a photon in mode 2 and a photon in mode 3, each with equal probability. We thus try the Ansatz

$$\begin{aligned} a_2 &= ra_1 + t'a_0 \\ a_3 &= ta_1 + r'a_0. \end{aligned}$$

  with complex reflection and transmission coefficients , r, r', t, t'.

- Commutation relations

  We also require that the creation and annihilation operators between different modes commute and that they the usual commutation relations hold:

$$\begin{aligned} \left[a_i, a_j^+\right] &= \delta_{ij} \\ [a_i, a_j] &= \left[a_i^+, a_j^+\right] = 0 \\ i, j &= 0, 1, 2, 3 \end{aligned}$$

  For a hypothetical beam splitter with one input and two output port theses relations cannot be fulfilled. Such beam splitters are thus impossible to realize. The second input port is necessary which also means that the vacuum cannot be excluded from the game.

- Phase relations

  From theses relations one can derive (exercise) that

$$\begin{aligned} |r'| &= |r| \\ |t| &= |t'|\,, \end{aligned}$$

  and

$$|r|^2 + |t|^2 = 1,$$

  which makes sense. Furthermore it follows from the commutator relations that

$$r^* t' + r' t^* = 0$$

  and

$$r^* t + r' t'^* = 0.$$

  The last two relations are the same given the above relations for the the absolute values. We can see this as follows: For the first of the two equations we obtain

$$\begin{aligned} |r|\,|t|\,e^{-i\varphi_r} e^{i\varphi_{t'}} + |r|\,|t|\,e^{i\varphi_{r'}} e^{-i\varphi_t} &= 0 \\ e^{-i\varphi_r} e^{i\varphi_{t'}} &= -e^{i\varphi_{r'}} e^{-i\varphi_t} \\ -\varphi_r + \varphi_{t'} &= \varphi_{r'} - \varphi_t + \pi \end{aligned}$$

  The second equation yields the same result

$$\begin{aligned} e^{-i\varphi_r} e^{i\varphi_t} &= -e^{i\varphi_{r'}} e^{-i\varphi_{t'}} \\ -\varphi_r + \varphi_t &= \varphi_{r'} - \varphi_{t'} - \pi \\ -\varphi_r + \varphi_{t'} &= \varphi_{r'} - \varphi_t - \pi. \end{aligned}$$

- dielectric interface

  Different realizations of beam splitters have different phase relations which needs to be discussed for each individual type of beam splitter separately. For a dielectric interface we know from Fresnell-equations that

$$\begin{aligned} t &= t' = |t| \\ r &= |r| = -r' \end{aligned}$$

  There is no phase shift in transmission. In reflection the beam splitter is not symmetric: there is no phase shift for reflection at the optically thinner medium and a phase shift of π for reflection at the optically thicker medium.

- symmetric beam splitter

  A symmetric beam splitter is obtained if the phase shifts are the same for both sides:

$$\varphi_{t'} = \varphi_t$$

and

$$\varphi_{r'} = \varphi_r.$$

Inserting in the above phase relation yields

$$\frac{\pi}{2} = \varphi_r - \varphi_t.$$

A typical choice is

$$\begin{aligned} \varphi_t &= 0 \\ \varphi_r &= \frac{\pi}{2}. \end{aligned}$$

- symmetric 50/50 splitter

  With the phases chosen like this and with

$$|r| = |t| = \frac{1}{\sqrt{2}}$$

  we now have

$$\begin{aligned} t &= t' = \frac{1}{\sqrt{2}} \\ r &= r' = \frac{i}{\sqrt{2}} \end{aligned}$$

  and for the operators one obtains

$$\begin{aligned} a_2 &= \frac{1}{\sqrt{2}}\left(a_0 + ia_1\right) \\ a_3 &= \frac{1}{\sqrt{2}}\left(a_1 + ia_0\right) \\ a_0 &= \frac{1}{\sqrt{2}}\left(a_2 - ia_3\right) \\ a_1 &= \frac{1}{\sqrt{2}}\left(-ia_2 + a_3\right). \end{aligned}$$

  and

$$\begin{aligned} a_2^+ &= \frac{1}{\sqrt{2}}\left(a_0^+ - ia_1^+\right) \\ a_3^+ &= \frac{1}{\sqrt{2}}\left(a_1^+ - ia_0^+\right) \\ a_0^+ &= \frac{1}{\sqrt{2}}\left(a_2^+ + ia_3^+\right) \\ a_1^+ &= \frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right). \end{aligned}$$

- Splitting a single photon

  We look at a single photon in mode 1. The state of the four modes reads

  $$|0\rangle_0|1\rangle_1|0\rangle_2|0\rangle_3.$$

  It can be regarded as being generated from the vacuum by applying the creation operator to mode 1:

  $$|0\rangle_0|1\rangle_1|0\rangle_2|0\rangle_3 = a_1^+|0\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3.$$

  With

  $$a_1^+ = \frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right),$$

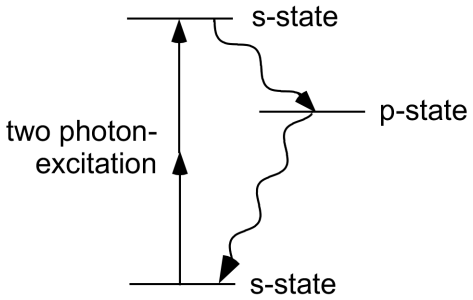
  one obtains

  $$\begin{aligned} a_1^+|0\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3 &= \frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right)|0\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3 \\ &= |0\rangle_0|0\rangle_1\frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right)|0\rangle_2|0\rangle_3 \\ &= |0\rangle_0|0\rangle_1\frac{1}{\sqrt{2}}\left(ia_2^+|0\rangle_2|0\rangle_3 + a_3^+|0\rangle_2|0\rangle_3\right) \\ &= |0\rangle_0|0\rangle_1\frac{1}{\sqrt{2}}\left(i|1\rangle_2|0\rangle_3 + |0\rangle_2|1\rangle_3\right) \end{aligned}$$

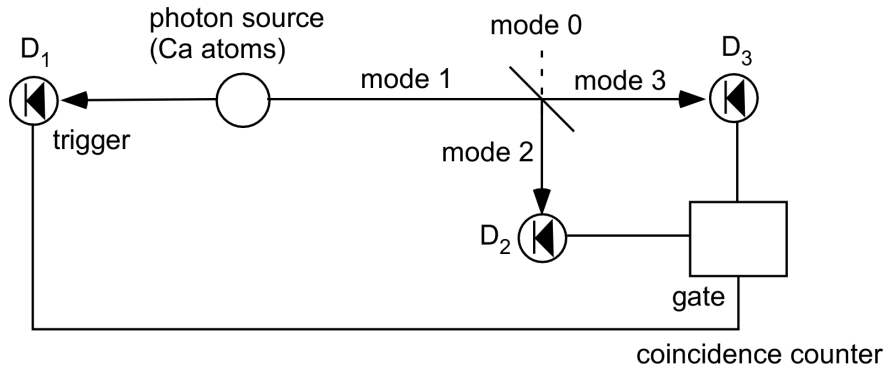
  which is the superposition state that we guessed in the beginning.

- Detecting single photon states

  Single photon states cannot be realized by simply reducing the power of the laser beam. This would still yield a coherent state with a small but nonzero probability of having two photons or more at the same time in the beam. With the following setup one can generate single photon states: Calcium atoms are excited in an s-state which decays in the s-groundstate via an intermediate p-state. Two photons are generated in modes with very different frequencies. The two modes can thus be seperated with simple color filters.



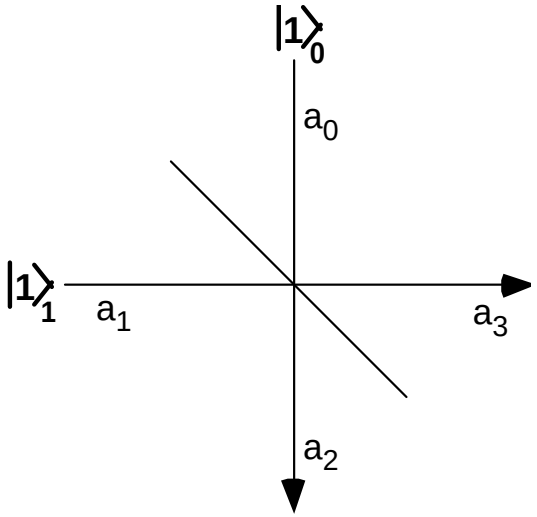
One of the generated photons is used a trigger. When it hits detector D_1 it triggers a gate of a coincidence counter. It monitors the two output ports of the beam splitter which sees the second photon at one of its input ports.



The detectors D_2 and D_3 never click together, at least not in the time window set by the trigger photon. This anticoincidence is the signature of a single photon state.

- Two photons

  What do we get if both input ports are occupied each with a single photon?



The input state can be generated from the vacuum as

$$|1\rangle_0|1\rangle_1|0\rangle_2|0\rangle_3 = a_0^+ a_1^+ |0\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3.$$

With

$$\begin{aligned} a_0^+ &= \frac{1}{\sqrt{2}}\left(a_2^+ + ia_3^+\right) \\ a_1^+ &= \frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right), \end{aligned}$$

one obtains

$$\begin{aligned} a_0^+ a_1^+ |0\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3 &= \frac{1}{\sqrt{2}}\left(a_2^+ + ia_3^+\right)\frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right)|0\rangle_0|0\rangle_1|0\rangle_2|0\rangle_3 \\ &= \frac{1}{2}|0\rangle_0|0\rangle_1\left(ia_2^+a_2^+ + a_2^+a_3^+ - a_3^+a_2^+ + ia_3^+a_3^+\right)|0\rangle_2|0\rangle_3 \\ &= \frac{i}{2}|0\rangle_0|0\rangle_1\left(a_2^+a_2^+ + a_3^+a_3^+\right)|0\rangle_2|0\rangle_3. \\ &= \frac{i}{2}|0\rangle_0|0\rangle_1\sqrt{2}\left(|2\rangle_2|0\rangle_3 + |0\rangle_2|2\rangle_3\right) \\ &= \frac{i}{\sqrt{2}}|0\rangle_0|0\rangle_1\left(|2\rangle_2|0\rangle_3 + |0\rangle_2|2\rangle_3\right). \end{aligned}$$

Both photons can be detected either at port 2 or at port 3. One photon at each port is impossible. The two corresponding terms mutually cancel.

- coherent state

  To understand how a coherent states behaves at a beam splitter we introduce the "displacement operator"

  $$|\alpha\rangle = \hat{D}(\alpha)|0\rangle,$$

  which generates a coherent state from a vacuum state. One can show that such an operator exists and that it can be written as

  $$\hat{D}(\alpha) = e^{\alpha a^+ - \alpha^* a}.$$

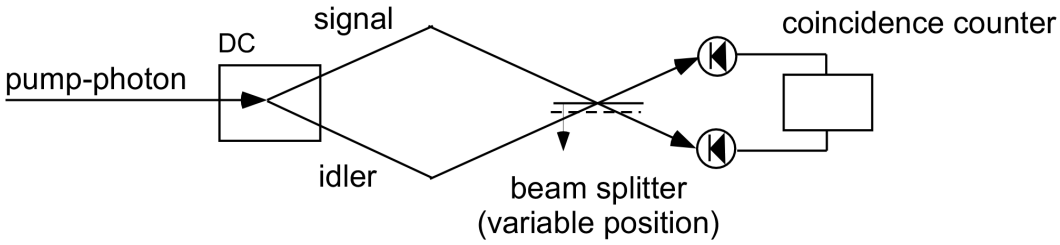
  Now we can write

  $$\begin{aligned} |0\rangle_0|\alpha\rangle_1|0\rangle_2|0\rangle_3 &= |0\rangle_0 e^{\alpha a_1^+ - \alpha^* a_1}|0\rangle_1|0\rangle_2|0\rangle_3 \\ &= |0\rangle_0 \exp(\alpha\frac{1}{\sqrt{2}}\left(ia_2^+ + a_3^+\right) - \alpha^*\frac{1}{\sqrt{2}}\left(-ia_2 + a_3\right))|0\rangle_1|0\rangle_2|0\rangle_3 \\ &= |0\rangle_0|0\rangle_1 \exp(\frac{i\alpha}{\sqrt{2}}a_2^+ - \frac{(i\alpha)^*}{\sqrt{2}}a_2)\exp(\frac{\alpha}{\sqrt{2}}a_3^+ - \alpha^*\frac{a_3}{\sqrt{2}})|0\rangle_2|0\rangle_3 \\ &= |0\rangle_0|0\rangle_1|\frac{i\alpha}{\sqrt{2}}\rangle_2|\frac{\alpha}{\sqrt{2}}\rangle_3. \end{aligned}$$

  At each output port we obtain a coherent state with half the average photon number. The reflected beam is phase shifted by 90° which agrees with the classical expectation.

- Hong Mandel Ou interferometer

In this famous experiment the beam splitter behavior is tested with a single photon state at each port.



The "down converter" DC is a nonlinear crystal which generates two modes (signal and idler) each occupied with a single photon. The frequencies of the individual modes are uncertain within a certain range $\Delta\omega$ due to the details of the setup. However, the frequency fluctuations of the two modes are related since the sum of the two frequencies is not uncertain but given by the frequency of the pump beam. (Hong et al., Physical Review Letters, 59 2044 (1987):

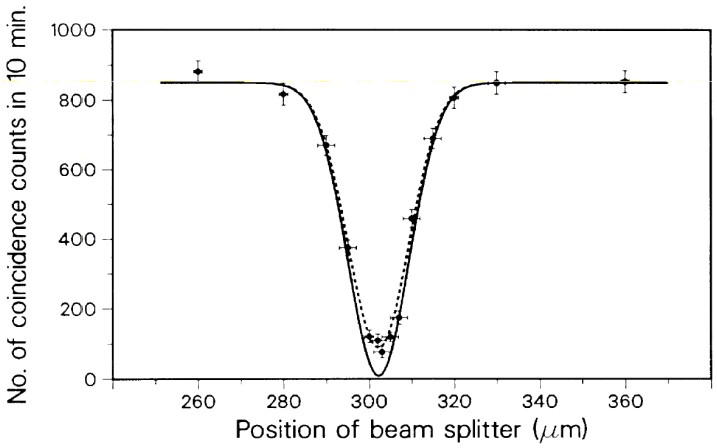


FIG. 2. The measured number of coincidences as a functior of beam-splitter displacement $c\,\delta\tau$, superimposed on the solic theoretical curve derived from Eq. (11) with $R/T=0.95$ $\Delta\omega=3\times10^{13}$ rad s^{-1}. For the dashed curve the factor $2RT/(R^2+T^2)$ in Eq. (11) was multiplied by 0.9. The verti· cal error bars correspond to one standard deviation, whereas horizontal error bars are based on estimates of the measure· ment accuracy.

The anti-coincidence goes away if the path difference is larger than a value Δs, given by the frequency uncertainty of the two modes

$$\Delta s = \frac{c}{\Delta\omega}$$

This is about the spatial width of a pulse that can be formed by waves with a frequency spread of $\Delta\omega$.

2.5 Jaynes-Cummings-Model

We look at a two level system interacting with a single mode.

- Interaction

  The interaction is given by the product of the field operator and the dipole operator of the atom

  $$\hat{H}_I = \overrightarrow{\hat{d}} \cdot \overrightarrow{\hat{E}}$$

  We take a standing wave polarized along $\vec{e}$:

  $$\overrightarrow{\hat{E}} = \vec{e} \cdot \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}(a + a^+) \sin kz.$$

  and obtain

  $$\hat{H}'_I = -\hat{d} \cdot E_0(z) \cdot (a + a^+),$$

  with the component of the dipole along the polarization

  $$\hat{d} := \overrightarrow{\hat{d}} \cdot \overrightarrow{e}$$

  and

  $$E_0(z) := \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz.$$

- Dipole-operator

  In a two level system with the ground state $|g\rangle$ and an excited state $|e\rangle$ the dipole operator has the matrix elements

  $$\begin{pmatrix} \langle e|\hat{d}|e\rangle & \langle e|\hat{d}|g\rangle \\ \langle g|\hat{d}|e\rangle & \langle g|\hat{d}|g\rangle \end{pmatrix} = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix}$$

  The diagonal elements vanish since energy eigen-states have no dipole moments. The only remaining number d can be restricted to real values without loss of generality.

- Excitation and deexcitation operators

  The operators

$$\begin{aligned}\hat{\sigma}_+ &= |e\rangle\langle g| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \hat{\sigma}_- &= |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

  create and annihilate an excitation of the two level atom. We also introduce the "inversion operator"

$$\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  The operators obey the commutation relations known from the Pauli matrices

$$\begin{aligned}[\sigma_+, \sigma_-] &= \sigma_3 \\ [\sigma_3, \sigma_\pm] &= 2\sigma_\pm.\end{aligned}$$

  A two level system is thus often called "pseudo spin system". The dipole operator now reads :

$$\begin{aligned}\hat{d} &= d|g\rangle\langle e| + d^+|e\rangle\langle g| = d\hat{\sigma}_- + d^*\hat{\sigma}_+ \\ &= d(\hat{\sigma}_+ + \hat{\sigma}_-)\end{aligned}$$

  The factor d is a complex number which indicates the strength of the dipole.

- Hamilton operator

  With this definitions the interaction Hamilton has the form

$$\hat{H}_I = \hbar\Omega_0(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^+)$$

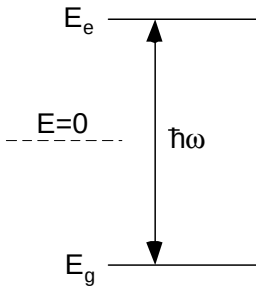
  with

$$\Omega_0 := d \cdot E_0(z)/\hbar$$

  being the "single photon Rabi-frequency". The maximum of the single photon Rabi frequency at the antinodes of the standing wave is called "coupling constant" g_0.

$$g_0 := d \cdot E_{0,\max}/\hbar = \sqrt{\frac{d^2\omega}{\hbar\varepsilon_0 V}}$$

  The Hamilton of the atom is diagonal and with the energy scale fixed at the middle between the two states



it reads:

$$\hat{H}_A = \frac{1}{2}(E_e - E_g) \cdot \hat{\sigma}_3 = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3.$$

The total Hamilton

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}'$$

thus is

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega(a^+a + \frac{1}{2}) + \hbar\Omega_0(\hat{\sigma}_+ + \hat{\sigma}_-)(a + a^+).$$

The term $\frac{1}{2}\hbar\omega$ can be discarded since it just shifts the energy scale.

- Rotating wave approximation

  The pure terms in the interaction part are dropped because they describe a double excitation/ annihilation: exciting the atom and creating a photon at the same time violates energy conservation. The same holds for deexciting the atom and annihilating photon. These double excitation/deexcitation terms rotate at a much higher frequency compared to the mixed terms:

$$\begin{aligned} \hat{\sigma}_+ a &\sim e^{i(\omega_o - \omega)t} \\ \hat{\sigma}_- a^+ &\sim e^{-i(\omega_o - \omega)t} \\ \hat{\sigma}_+ a^+ &\sim e^{i(\omega_o + \omega)t} \\ \hat{\sigma}_- a &\sim e^{-i(\omega_o + \omega)t} \end{aligned}$$

  On short time scales the time energy uncertainty allows energy violation, however we are interested in time scales much longer than an optical cycle such that energy should be conserved. The RWA results in the Hamilton of the so called Jaynes-Cummings-Model:

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega a^+a + \hbar\Omega_0(\hat{\sigma}_+ a + \hat{\sigma}_- a^+)$$

- Constants of motion

We can write the Hamiltonian as a sum of two parts

$$\begin{aligned} H &= H_I + H_{II} \\ H_I &= \hbar\omega N_e + \hbar(\frac{\omega_0}{2} - \omega)P_E \\ H_{II} &= -\hbar\Delta + \hbar\Omega_0(\sigma_+ a + \sigma_- a^+), \end{aligned}$$

with the detuning

$$\Delta := (\omega_0 - \omega) \cdot |g\rangle\langle g|.$$

The so called "electron number operator"

$$\begin{aligned} P_E &= |e\rangle\langle e| + |g\rangle\langle g| \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \end{aligned}$$

obviously commutes with the Hamilton

$$[P_E, H] = 0,$$

just as the excitation number operator

$$N_e = a^+ a + |e\rangle\langle e|$$

$$[N_e, H] = 0.$$

N_e and P_E are thus constants of motion. The dynamics of the system is thus determined only be H_{II}.

- Solution

  We look for eigen-states of H_{II}

  $$H_{II}\Psi = E\Psi$$

  for resonant interaction, $\Delta = 0$. The interaction couples the initial state

  $$|i\rangle := |e\rangle|n\rangle$$

  with energy

  $$E_i = \frac{1}{2}\hbar\omega + n\hbar\omega$$

  and the final states

  $$|f\rangle := |g\rangle|n+1\rangle$$

  with the energy

  $$E_f = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega.$$

  The energy of both states are the same $E_i = E_f$.

- Rabi oscillation

  For the general state
  $$|\Psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle.$$
  the Schrödinger equation
  $$\hbar\frac{d}{dt}|\Psi\rangle = H_{II}|\Psi\rangle,$$
  yields
  $$\begin{aligned}\dot{C}_i &= -i\Omega_0\sqrt{n+1}C_f \\ \dot{C}_f &= -i\Omega_0\sqrt{n+1}C_i\end{aligned}$$
  and
  $$\ddot{C}_i + \Omega_0^2\,(n+1)\,C_i = 0$$
  With the initial conditions $C_i(0) = 1$ and $C_f(0) = 0$ the solution is
  $$\begin{aligned}C_i(t) &= \cos(\Omega_0\sqrt{n+1}\cdot t) \\ C_f(t) &= -i\sin(\Omega_0\sqrt{n+1}\cdot t).\end{aligned}$$
  The population probabilities and the inversion is
  $$\begin{aligned}P_i(t) &= |C_i(t)|^2 = \cos^2(\frac{\Omega}{2}t) \\ P_f(t) &= |C_f(t)|^2 = \sin^2(\frac{\Omega}{2}t) \\ W(t) &= \langle\sigma_3\rangle = P_i(t) - P_f(t) = \cos(\Omega t)\end{aligned}$$
  with the "Rabi-frequency":
  $$\Omega(n) := 2\Omega_0\sqrt{n+1}.$$
  It is proportional to $\sqrt{n}$ (for $n \gg 1$).

  For $n = 0$ we obtain $\Omega = 2\Omega_0 > 0$. The vacuum drives the system!

- pulses

  If the atom is in one of the two states and interacts with the light for a time interval of length
  $$t_\pi = \frac{\pi}{\Omega}$$
  the atom has changed its state. This interaction is called "π-pulse". If the interaction is stopped after a time
  $$t_{\pi/2} = \frac{\pi}{2\Omega}$$

the atom is in a superposition state with equal probability to be in one of the two states ($\pi/2$-pulse). Now the atom has a dipole moment which oscillates with the Bohr frequency

$$\omega_0 = \frac{E_e - E_g}{\hbar}.$$

The application of a second $\pi/2$-pulse brings the atom either back to the initial state and completes the transition. This depends on the relative phase of the dipole oscillation and the oscillation of the electric field during the second pulse. The length of the π- and the $\pi/2$-pulses depends on the number of photons in the mode. Vice versa, by detecting the population after a pulse one can learn something about the number of photons in the mode.

- Bloch sphere

  The general state of a two level system at a given time

  $$\psi = \cos(\theta)\,|i\rangle + e^{i\varphi}\sin(\theta)\,|f\rangle$$

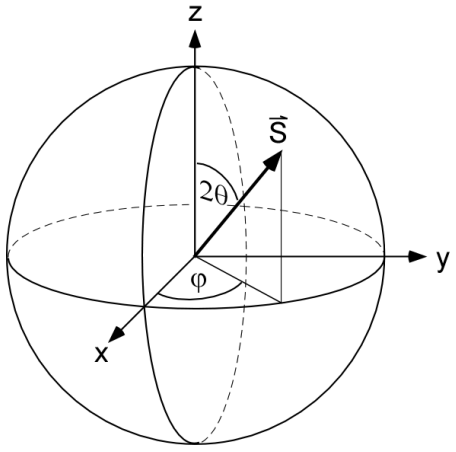
  can be visualized by plotting the components of

  $$\vec{S} = \begin{pmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{pmatrix}$$

  with the Pauli spin matrices

  $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  The vector $\vec{S}$ is proportional to the expectation value of the spin operator. It has a fixed length and thus lies on a three-dimensional sphere, the so called Bloch sphere. A specific state of the system can thus be visualized as a point on the Bloch-sphere.



The direction of the vector is given by two angles: the mixing angle $\theta \in \{0, 90°\}$ and the azimuthal angle φ. After a $\pi/2$-pulse the initial state (north pole) is tilted into the equatorial plane and rotates in the plane with a frequency given by energy separation of the two states. If the state has rotated in the equatorial plane by 180° before a second $\pi/2$-pulse is applied it is turned back to the north pole. If it has rotated by 90° nothing happens. In general: a $\pi/2$-pulse rotates the vector around a fixed axis in the equatorial plane.

2.6 Fock States in the lab: The Micro Maser

The research group of Serge Haroche at the Ecole Normale Superieure (ENS) in Paris developed an apparatus which is able to study quantum optics with only few photons and atoms in great detail. For this work Serge Haroche received the Nobel prize in 2012. We will have a brief look at this famous machine. See also: Christine Guerlin, et al., Nature 448, 889 (2007).

- Linear Rydberg-states

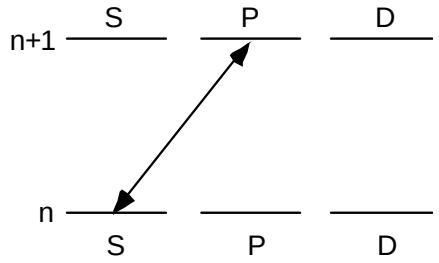
  The alkali atom ^{87}Rb is optically excited to states with a very high principal quantum number n. The atom in such states is very similar to hydrogen and has energy levels approximately given by

  $$E_n \simeq \frac{1}{2} mc^2 \frac{\alpha^2}{n^2}.$$

  The energy separation between two levels is

  $$\omega_0 = \frac{2Ry}{\hbar n^3}$$

  with $Ry = 13,6eV$ (Rydberg).



  Electric dipole transitions between Rydberg states have a very large dipole moment which scales with n like

  $$d \simeq n^2 \cdot e \cdot a_0$$

with unit charge e and Bohr radius a_0. The decay rate between Rydberg states however is quite small. With the standard spontaneous decay rate

$$\Gamma = \frac{d^2\omega_0^3}{3\pi\varepsilon_0\hbar c^3}$$

and

$$d^2 \sim n^4$$

and

$$\omega_0^3 \sim \frac{1}{n^9}$$

the decay rate scales with n as

$$\Gamma \sim \Gamma_0 \cdot n^{-5},$$

with

$$\Gamma_0 = \frac{c\alpha^4}{a_0} \sim 10^9 \frac{1}{s}$$

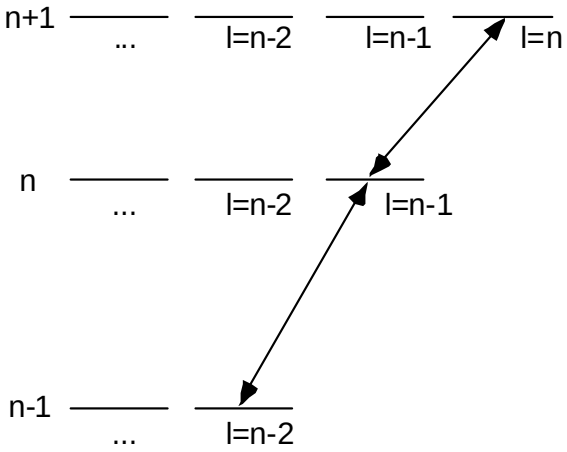
(α is the fine structure constant). For $n = 50$ the spontaneous transition time between Rydberg states is

$$\tau = \frac{1}{\Gamma} \sim 0.1s.$$

Because of transitions into low lying states the life time of Rydberg states is on the order of μs.

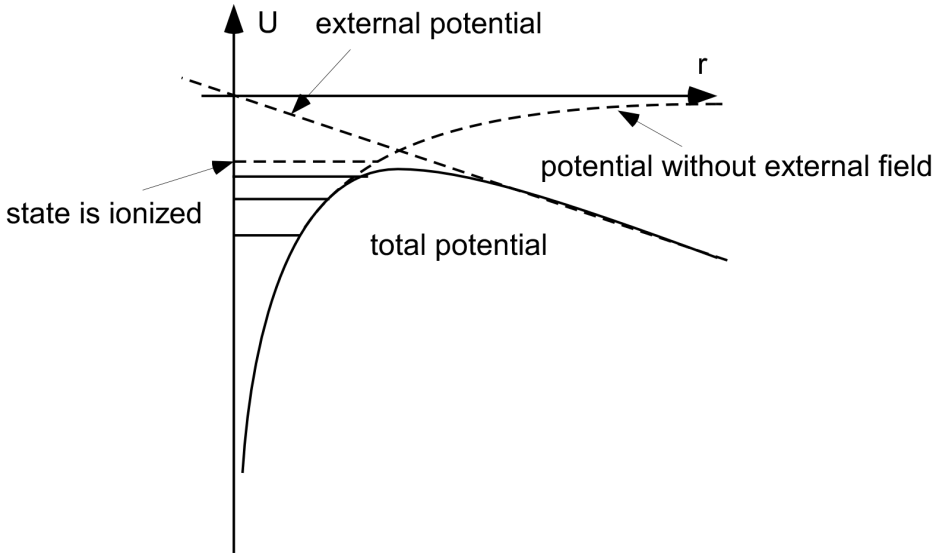
- circular Rydberg states

  Rydberg states with $l = n - 1$are called circular states since they have the maximum value of the angular momentum. They can only decay to the next lower circular Rydberg state with $n' = n - 1$ and $l' = l - 1$. Circular states thus live on the time scale of ms.

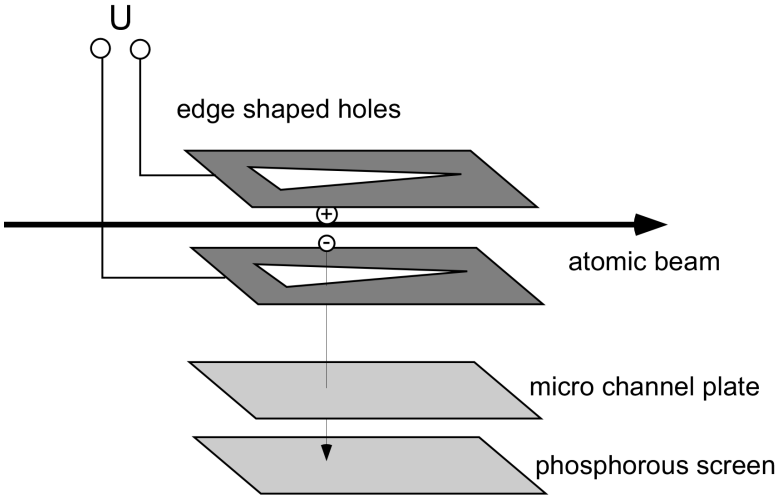


- Field ionization

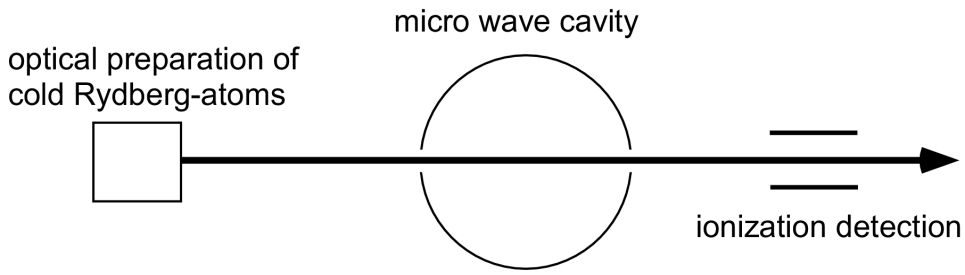
  Single Rydberg atoms can be detected by ionizing them in an external field and detect the emitted electron.



  State selective detection is possible in an inhomogenous field such that first the high lying states are ionized and then the lower lying states. The position where the electron is emitted is recorded with a micro channel plate.



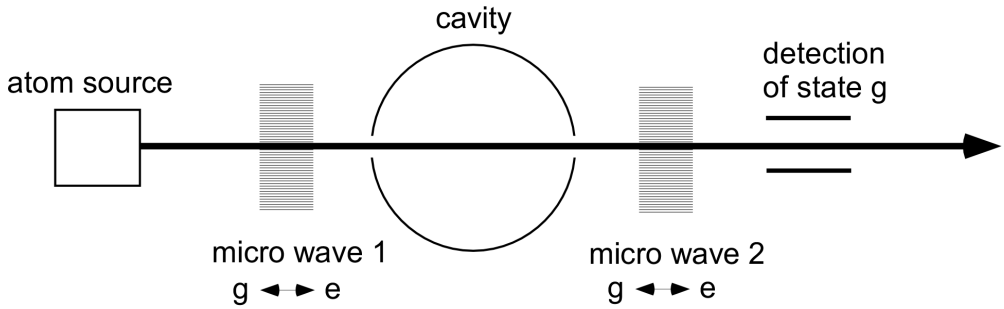
- Atom-photon-interaction in the micro maser



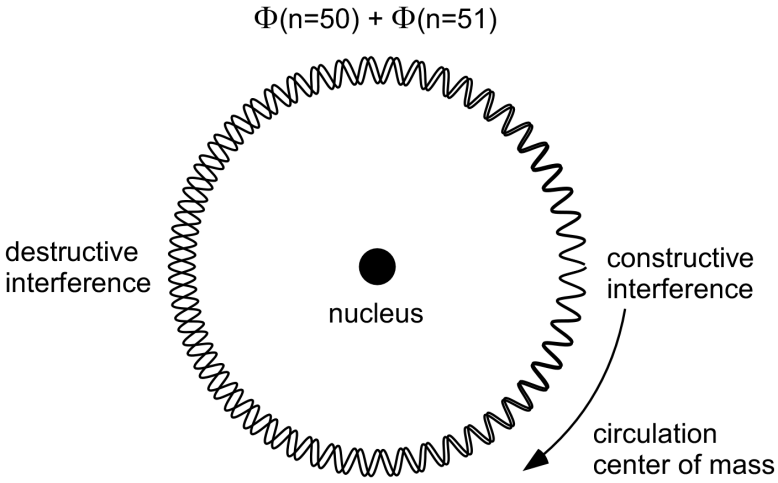
Atoms are optically prepared in the state with $n = 50$ ("ground state") or $n = 51$("excited state"). A transition between the two states requires a micro wave at $\omega_0 = 2\pi \cdot 51,1GHz$. The atoms enter a micro wave cavity tuned to this frequency one by one and their state is detected after they have left the cavity. Due to the large dipole matrix element of Rydberg states the atom and a photon in the cavity are strongly coupled and energy can be exchanged between the atom and the cavity within the time of flight through the cavity (about 10 cm long). Black body radiation is avoided by cooling the cavity to millikelvin temperature with a delution refrigerator. Spontaneous decay can be neglected for circular states.

- superposition states

  A micro wave field before the cavity can excite the atoms in a superposition of the two states by means of a $\pi/2$ pulse.



In the superposition state the atom has a dipole moment which rotates with a frequency which corresponds to the energy separation of the two states .



A second micro wave field behind the cavity applies a second $\pi/2$ pulse. Its phase is synchronized with the first micro wave and the relative phase θ between the two microwaves can be varied in a controlled way. If we would like to add up the two $\pi/2$-pulses to act as a π-pulse the adjustable phase θ can be used to compensate for any phase the dipole collects on its way from the first to the second microwave interaction zone. Since this flight takes some time and since the dipole oscillates there is already such a phaseshift which obviously depends on the velocity of the atoms. But there is also a phase shift due to the cavity.

- Quantum non demolition measurement

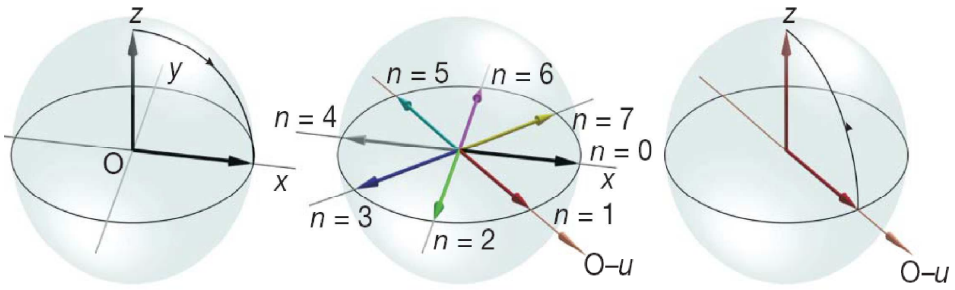
  If the light in the cavity is red detuned relative to the atomic transition, the dipole moment of the atom interacts with the electric field. As a result the energy of the "ground state" is reduced and the energy of the excited state is increased ("light shift"). The energy separation between the two atomic states is now larger and the dipole rotates at a higher frequency. The dipole is slightly out of phase now with the second micro wave and the transition is completed only with a reduced probability. By sending many atoms through the resonator and looking at the average transition probability one can measure the light intensity in the cavity without changing it. In other word: The number of photons in the cavity can be detected without destroying the photons. One can also adjust θ to compensate for the light shift effect.

- Preparing number states by QND-measurements

  Here the intial state points to the north pole of the Bloch sphere. We call ist the ground state. The excited state points to the south pole. The first microwave turns the atomic state from the north pole into the equatorial plane. The atom now has a dipole moment. The time of flight of the atom through the cavity is adjusted such that a Fock state with 8 photons would shift the phase of the dipole by an additional value

of 2π. A state $|n\rangle$ with n photons will then shift the phase by

$$\varphi_n = n \cdot \frac{2\pi}{8}$$



The relative phase θ_n between the second and the first microwave can be adjusted such that the state $|n\rangle$ will shift the dipole such that the second micro wave deexcites the atom back to the ground state with 100% probability. With this choice of θ_n, the "opposite" photon state $|n+4\rangle$ would shift the dipole moment by $\varphi_n + \pi$ such that the second micro wave will complete the transition. For instance, for $n = 1$ and $\theta_1 = \frac{5}{4}\pi$ the atom is deexcited with 100% probability. With the same phase θ_1 a state with $n = 5$ photons in the cavity would complete the excitation and the atom will be detected in the excited state with 100% probability. The states with $n = 3$ or $n = 7$ photons in the cavity would shift the phase of the dipole moment such that the second micro wave can not transfer energy to the atom which thus leaves the cavity in a superposition state with equal probability to be detected in either the ground state or the excited state.

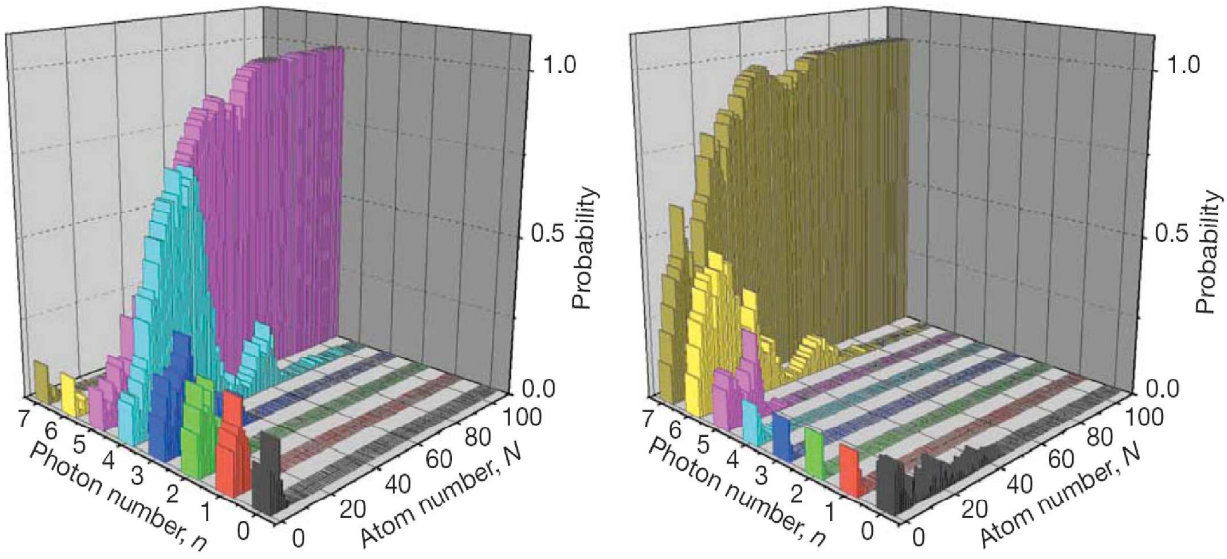
- Projection to a number state

  The cavity is initially filled with a coherent state and an average photon number around $\bar{n} = 4$. Every time an atom that leaves the cavity is detected one learns something about the photon state in the cavity. Lets say, for instance, the angle θ is chosen such that a cavity in state $|n\rangle$ would restore the initial (ground) state of the atom with 100%. If now the atom is detected in the excited state the cavity cannot be in state $|n\rangle$. The intital coherent state has lost its component of state $|n\rangle$ and is not a coherent state any more. By testing other angles θ one can, step by step, exclude other Fock states. In the experiment, the angles θ_n are statistically chosen, for instance $\theta \in \{\theta_0, \theta_1, \theta_6, \theta_7\}$, and the state of the atoms leaving the cavity is detected. The angles are denoted a,b,c,d and each detection returns a value $j \in \{0, 1\}$. Here are some of the original data

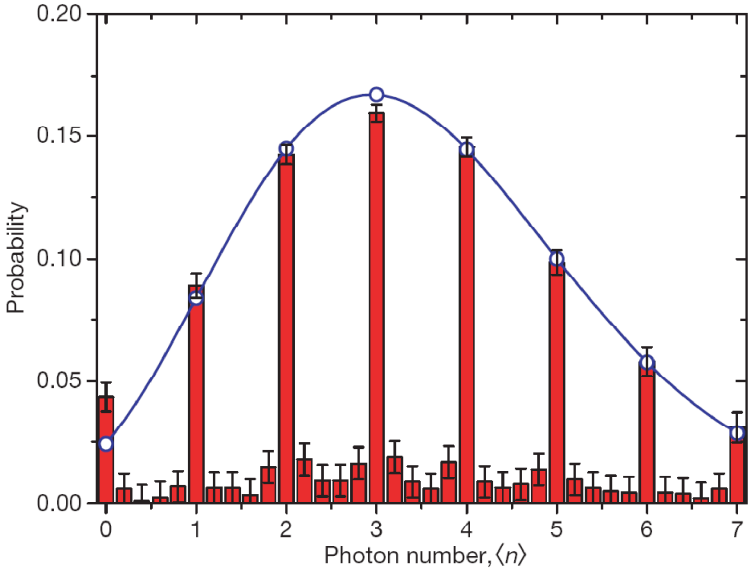
j	11011111111110011101101111
i	*ddcbccabcdaadaabadddbadbc*
j	01010011010101101101111
i	*dababbaacbccdadccdcbaaacc*

j	00010001101100000010101110
i	*ddcaddabbccdccbcdaabbccab*
j	00010101000001000111011101
i	*bcdaddaabbbbdbdcdccadaada*

Each box shows 50 detection which combine to a single measurement. After each detection one can calculate the probability for the state $|n\rangle$ to be present in the actual linear combination of Fock states. Details are found in the original paper: Christine Guerlin, et al., Nature 448, 889 (2007). The two following figures show how these probabilities evolve during the measurement (here the number of detections go up to 100).



The coherent state collapses to a Fock state $|n = 5\rangle$ in the first measurement and $|n = 7\rangle$ in the second measurement. For the same starting conditions each measurement may lead to a different outcome. The experiment realizes an energy measurement on the system that collapses the coherent state with uncertain energy to an energy eigenstate. By repeating many of those measurement one can determine the probability for a certain outcome $|n\rangle$. It is no surprise that the probability equals the Poissonian statistics of the initial coherent state.


0.20
0.15
0.10
0.05
0.00
Probability
0
1
2
3
4
5
6
7
Photon number, ⟨n⟩