

Hefei-Lectures 2015  
Third Lesson: Two Level System and Optical Cooling

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November 15, 2015

### 3.1 Two level system (semiclassical treatment)

- Hamilton in energy representation

We look at a simplified version of an atom with an electron that can only be in two states  $\varphi_1(\vec{r})$  and  $\varphi_2(\vec{r})$ . They are eigen-states of the stationary Schrödinger-equation defined by the Hamilton-operator of the atom  $H$ :

$$H\varphi_i = \varepsilon_i\varphi_i$$

We change from the representation in position space to a representation in the two-dimensional energy space. For that, we have to calculate the matrix elements of the Hamilton operator.

$$H_{ij} := \int \varphi_i^*(\vec{r}) H\varphi_j(\vec{r}) d\vec{r}$$

We use the above eigen-equation such that

$$H_{ij} = \int \varphi_i^*(\vec{r}) \varepsilon_j \varphi_j(\vec{r}) d\vec{r} = \varepsilon_j \delta_{ij} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}.$$

The zero of the energy scale is set in the middle between the two levels such that

$$\varepsilon_1 + \varepsilon_2 = 0.$$

With the "Bohr frequency"  $\omega_0$  defined as

$$\hbar\omega_0 := \varepsilon_1 - \varepsilon_2$$

the Hamilton reads

$$H = \frac{1}{2}\hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- interaction Hamilton

In dipole approximation ( $\lambda > a_B$ ) the interaction between a classical light field  $\vec{E}$  and an atom is given by

$$\begin{aligned} H_{int} &= \vec{d} \cdot \vec{E} \\ \vec{E} &= E_0 \vec{\varepsilon} \cos(\omega t) \end{aligned}$$

with the optical frequency  $\omega$ , the polarization vector  $\vec{\varepsilon}$ , and the dipole operator  $\vec{d}$ . For light linearly polarized in the  $x$ -direction we get

$$H_{int} = d_x \cdot E_x = d_x E_0 \cos(\omega t)$$

The classical dipole moment of the atom is given by the charge of the electron  $-e$  times the position of the electron (the nucleus is positioned at the origin).

$$d_x = -ex$$

The quantum mechanical dipole operator is obtained by replacing the classical position  $x$  by the position operator  $\hat{x}$ . In position space, however, this operator is simply the scalar  $x$ . We calculate the matrix elements of  $-e\hat{x} = -ex$

$$d_{ij} := -e \int \varphi_i^*(\vec{r}) \cdot x \cdot \varphi_j(\vec{r}) d\vec{r}$$

and obtain a 2x2-matrix

$$d := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

We analyze the integrand: For  $i = j$  we obtain  $\varphi_i^*(\vec{r}) \varphi_i(\vec{r}) = |\varphi_i(\vec{r})|^2$  which is symmetric in  $\vec{r}$  (if the parity operator commutes with the Hamilton which is the case). The operator  $x$  is antisymmetric in  $\vec{r}$ . The integrand is thus antisymmetric and the integral vanishes. The matrix has thus no diagonal elements. We only get non diagonal elements and since  $d_{ij} = d_{ji}^*$  we obtain

$$d = \begin{pmatrix} 0 & d_{12} \\ d_{12}^* & 0 \end{pmatrix}.$$

For linearly polarized light  $d_{1,2}$  is real and positive (here without proof) and one gets

$$\begin{aligned} H_{int} &= d_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E_0 \cos(\omega t) \\ &= \hbar\Omega \cos(\omega t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \Omega &: = \frac{E_0 d_{12}}{\hbar} \end{aligned}$$

The "Rabi-frequency"  $\Omega$  depends linearly on the field amplitude and the dipole matrix element. It indicates the strength of the atom-light-coupling.

- Schrödinger-equation

The equation of motion in the Schrödinger picture is given by the Schrödinger-equation

$$\begin{aligned} i \frac{d}{dt} \vec{\psi} &= H \vec{\psi} \\ H &= H_0 + H_{int} = \frac{1}{2} \hbar \omega_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hbar \Omega \cos(\omega t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

The solution is a superposition of the eigen-states of  $H_0$ . This superposition may change in time which leads to the ansatz

$$\vec{\psi}(t) = c_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\frac{\omega_0}{2}t} + c_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\frac{\omega_0}{2}t}.$$

Inserting into the Schrödinger equation yields equations of motion for the coefficients (as exercise)

$$\begin{aligned}i\dot{c}_1 &= \Omega \cos(\omega t) e^{i\omega_0 t} c_2 \\i\dot{c}_2 &= \Omega \cos(\omega t) e^{-i\omega_0 t} c_1\end{aligned}$$

- density matrix

Damping (spontaneous decay) is usually included by means of the density matrix formalism. The density matrix is defined as

$$\rho = \begin{pmatrix} c_1 c_1^* & c_1 c_2^* \\ c_2 c_1^* & c_2 c_2^* \end{pmatrix}.$$

Obviously, the diagonal element  $\rho_{ii} = |c_i|^2$  is the probabilities to find the electron in state  $i$ . The non diagonal elements  $\rho_{ij}$  are called "coherences". Later we will see that they are proportional to the amplitude of the dipole oscillation of the electron. We can directly calculate the time derivative of the density matrix and obtain the equations (as exercise)

$$\begin{aligned}\frac{d}{dt}\rho_{11} &= \frac{d}{dt}(c_1 c_1^*) = \left(\frac{d}{dt}c_1\right)c_1^* + c_1\left(\frac{d}{dt}c_1^*\right) \\ &= i\Omega \cos(\omega t) (e^{-i\omega_0 t}\rho_{12} - e^{i\omega_0 t}\rho_{21}) \\ \frac{d}{dt}\rho_{12} &= i\Omega \cos(\omega t) e^{i\omega_0 t} (\rho_{11} - \rho_{22}) \\ \frac{d}{dt}\rho_{22} &= -\frac{d}{dt}\rho_{11} \\ \frac{d}{dt}\rho_{21} &= \left(\frac{d}{dt}\rho_{12}\right)^*\end{aligned}$$

- rotating wave approximation

In the equations terms of the form

$$\cos(\omega t) e^{\pm i\omega_0 t} = \frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) e^{\pm i\omega_0 t} = \frac{1}{2} (e^{i(\omega \pm i\omega_0)t} + e^{-i(\omega \pm i\omega_0)t})$$

appear. We neglect terms which rotate at optical frequencies and keep only terms that contain the difference of the optical frequency and the Bohr frequency. The approximation is justified if both frequencies are similar, i.e. close to resonance. The result is

$$\begin{aligned}\frac{d}{dt}\rho_{11} &= \frac{1}{2}i\Omega (e^{i\delta t}\rho_{12} - e^{-i\delta t}\rho_{21}) = -\frac{d}{dt}\rho_{22} \\ \frac{d}{dt}\rho_{12} &= \frac{1}{2}i\Omega e^{-i\delta t} (\rho_{11} - \rho_{22}) = \frac{d}{dt}\rho_{21}^*\end{aligned}$$

with the detuning defined as

$$\delta := \omega - \omega_0.$$

- damping

An excited atom cannot avoid contact with the electromagnetic vacuum. Its random fluctuations of the electric field drive a transition from the excited state to the ground state. This transition is called "spontaneous" decay. The vacuum cannot drive a transition from the ground state to the excited state since this would violate energy conservation. The atom would absorb energy from the vacuum which by definition has no extractable energy, i.e. no photons.

Technically, the spontaneous decay is correctly included if we add decay terms for the population of the upper state and for the coherences. The population of the lower state is given by the conservation of the total probability to be in one of the states.

$$\begin{aligned}\frac{d}{dt}\rho_{11} &= \frac{1}{2}i\Omega (e^{i\delta t}\rho_{12} - e^{-i\delta t}\rho_{21}) - \rho_{11}\gamma_{\parallel} \\ \frac{d}{dt}\rho_{12} &= \frac{1}{2}i\Omega e^{-i\delta t}(\rho_{11} - \rho_{22}) - \rho_{12}\gamma_{\perp} \\ \rho_{11} + \rho_{22} &= 1 \\ \frac{d}{dt}\rho_{21} &= \frac{d}{dt}\rho_{12}^*\end{aligned}$$

A strict derivation of the decay terms is possible (reservoir theory) which we don't do here. If damping is only caused by the electromagnetic vacuum the two decay constants are not independent but

$$\gamma := \gamma_{\parallel} = 2\gamma_{\perp}.$$

The 1/e-life time of the population of the excited state obviously is

$$\tau = \frac{1}{\gamma}.$$

- optical Bloch equations

We can eliminate the explicit time dependence on the right side by using  $\tilde{\rho}_{12}$  which is defined by

$$\rho_{12} = e^{-i\delta t}\tilde{\rho}_{12}.$$

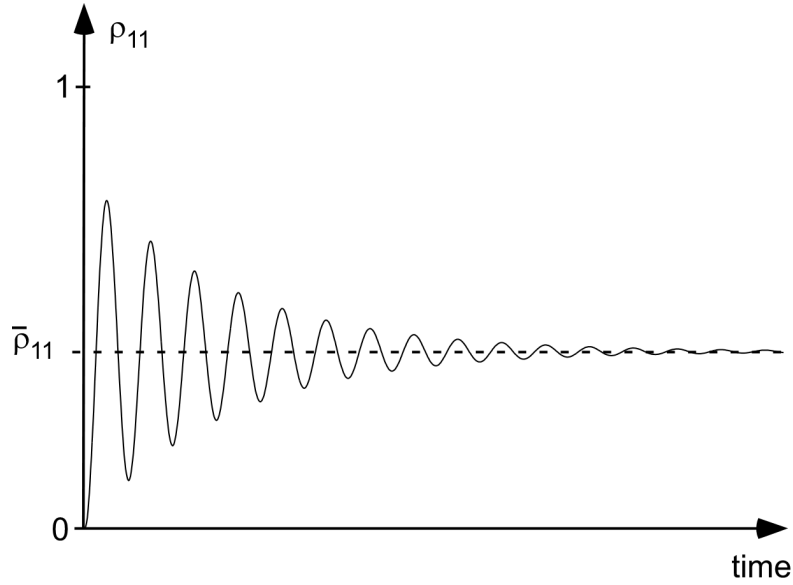
Now the equations of motions read (as exercise)

$$\begin{aligned}\frac{d}{dt}\rho_{11} &= \frac{1}{2}i\Omega (\tilde{\rho}_{12} - c.c.) - \gamma\rho_{11} \\ \frac{d}{dt}\tilde{\rho}_{12} &= -\left(i\delta + \frac{1}{2}\gamma\right)\tilde{\rho}_{12} + \frac{1}{2}i\Omega(\rho_{11} - \rho_{22}) \\ \rho_{11} + \rho_{22} &= 1 \\ \frac{d}{dt}\tilde{\rho}_{21} &= \frac{d}{dt}\tilde{\rho}_{12}^*\end{aligned}$$

(Note: the second equation has the same form as the equation we got for the field amplitude in a cavity with the pumping field  $\eta \leftrightarrow \frac{1}{2}i\Omega(\rho_{11} - \rho_{22})$  and the decay constant  $\kappa \leftrightarrow \gamma/2$ .)

- solutions

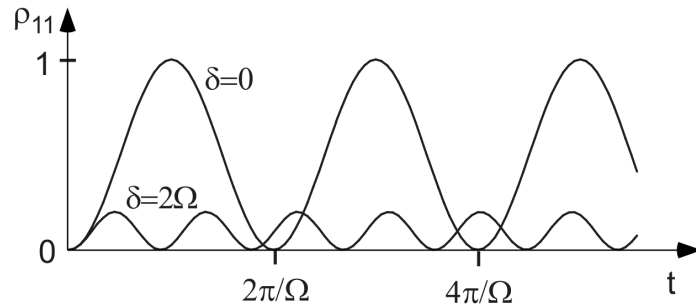
One can easily simulate the equations with the computer and you should definitely do it to get a feeling for the solutions. If we start in the ground state ( $\psi_2$ ) we observe an oscillation of the population which eventually decays into a steady state.



There is also an analytic expression for the solution which for  $\gamma = 0$  (no damping) has a simple form:

$$\rho_{11}(t) = \left(\frac{\Omega}{\tilde{\Omega}}\right)^2 \sin^2\left(\frac{1}{2}\tilde{\Omega}t\right)$$

$$\tilde{\Omega} : = \sqrt{\Omega^2 + \delta^2}$$



On resonance ( $\delta = 0$ ) the population oscillates between 0 and 1 at the Rabi-frequency  $\Omega$ . For  $\delta \neq 0$  the modulation amplitude is reduced and one no more gets full excitation. Also the oscillation is now faster.

- Rabi cycle

What happens during such a Rabi-cycle? Since an energy eigen-state cannot develop a dipole moment, at first a little (infinitesimal) bit of the excited state is mixed to the ground state. This superposition now develops a dipole. The dipole oscillation is  $90^\circ$  phase delayed relative to the incident electric field such that the field further drives the dipole oscillation. Energy flows into the atom and consequently the probability of being in the excited state grows. This also enhances the dipole amplitude and the energy transfer becomes more efficient. For equal population the dipole has the maximum value. The energy flows at a maximum rate which then reduces until the atom is fully excited and the dipole has disappeared. Now the process starts again but with the excited state as initial condition. A little bit of the ground state is mixed to the excited state and a dipole forms, however, with a phase that is  $90^\circ$  advanced relative to the exciting field. The energy now flows from the atom into the light field until the atom is in the ground state again.

- steady state solution

We obtain the steady state solution if we set  $\frac{d}{dt}\rho_{11} = \frac{d}{dt}\rho_{12} = 0$ :

$$\begin{aligned}\bar{\rho}_{11} &= \frac{1}{4}\Omega^2 \frac{1}{\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{1}{2}\Omega^2} \\ \bar{\rho}_{12} &= \frac{1}{2}\Omega \frac{\delta - i\frac{\gamma}{2}}{\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{1}{2}\Omega^2}\end{aligned}$$

- saturation intensity

It is very useful to introduce the "saturation intensity". Since  $\Omega$  is proportional to the electric field (see above),  $\Omega^2$  is proportional to the light intensity  $I$ . We define  $I_{sat}$  via

$$\frac{I}{I_{sat}} = \frac{2\Omega^2}{\gamma^2}$$

We can explicitly calculate the saturation intensity by using the definition of the Rabi-frequency

$$\Omega = \frac{E_0 d_{1,2}}{\hbar}$$

and the connection between the electric field and the intensity (averages over one optical cycle)

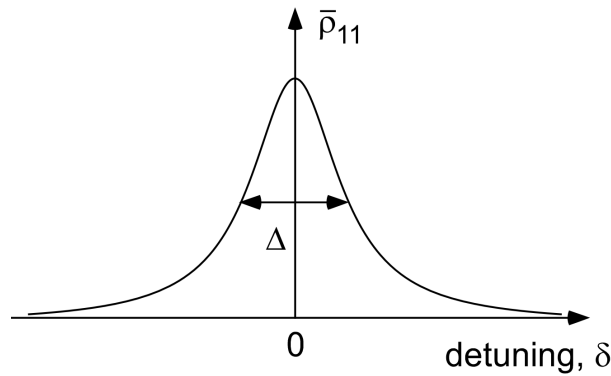
$$I = \frac{1}{2}c\varepsilon_0 E_0^2$$

Then

$$I_{sat} = \frac{\gamma^2}{2\Omega^2} I = \frac{\gamma^2}{2\left(\frac{E_0 d_{1,2}}{\hbar}\right)^2} \frac{1}{2}c\varepsilon_0 E_0^2 = \frac{\varepsilon_0 c \hbar^2}{4} \left(\frac{\gamma}{d_{1,2}}\right)^2.$$

The population now can be written as

$$\begin{aligned}
 \bar{\rho}_{11} &= \frac{1}{4} \Omega^2 \frac{1}{\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{1}{2} \Omega^2} \\
 &= \frac{1}{4} \frac{4\Omega^2}{\gamma^2 \left(\frac{2\delta}{\gamma}\right)^2 + 1 + 2\frac{\Omega^2}{\gamma^2}} \\
 &= \frac{1}{2} \frac{I}{I_{sat}} \frac{1}{\left(\frac{2\delta}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}}
 \end{aligned}$$



The occupation of the excited state is given by a Lorentzian resonance line with the full width at half maximum of

$$\frac{\Delta_{FWHM}}{\gamma} := \sqrt{1 + \frac{I}{I_{sat}}}$$

For small intensity we can neglect the second term in the root and obtain the so called "natural linewidth" of the transition which in Hz is

$$\Delta\nu = \frac{\gamma}{2\pi} = \frac{1}{2\pi\tau}$$

The increase of the line width for larger intensities is called "saturation broadening". For large intensities the occupation of the excited state grows but the steady state value can never exceed 1/2.

- saturation

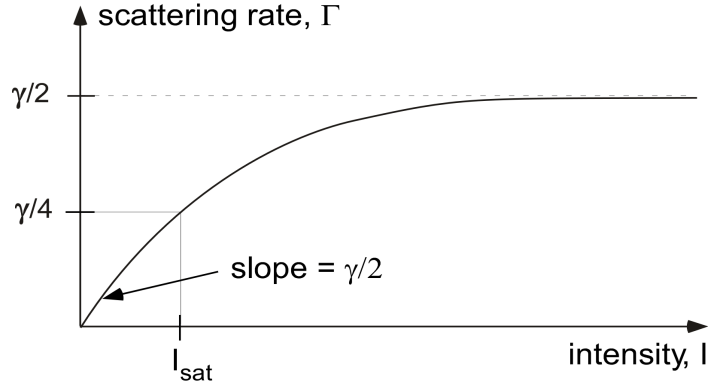
The scattering rate is simply given by the occupation of the excited state times the decay rate

$$\Gamma = \bar{\rho}_{11}\gamma = \frac{\gamma}{2} \frac{I}{I_{sat}} \frac{1}{\left(\frac{2\delta}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}}$$



On resonance,  $\delta = 0$ , the scattering rate saturates with increasing light intensity at a value  $\Gamma = \gamma/2$ .

$$\Gamma = \frac{\gamma}{2} \frac{I}{I_{sat}} \frac{1}{1 + \frac{I}{I_{sat}}}$$



If we admit detuning, the maximum scattering rate is still  $\gamma/2$ , however the transition now saturates at larger intensities.

- dipole moment

the expectation value of the dipole moment is

$$\langle d \rangle = -e \left\langle \vec{\psi}^*(t) \left| d \right| \vec{\psi}(t) \right\rangle$$

We insert the dipole matrix  $d$  and the state of the system  $\vec{\psi}(t)$  from above and obtain

$$\begin{aligned} \langle d \rangle &= d_{12} (c_1 c_2^* e^{-i\omega_0 t} + c_1^* c_2 e^{i\omega_0 t}) \\ &= 2d_{12} \text{Re}(\rho_{12}) \end{aligned}$$

The expectation value of the dipole moment oscillates as the coherences. We can calculate  $\rho_{12}$  for the steady state with the definition of  $\tilde{\rho}_{12}$  from above

$$\rho_{12} = e^{-i\delta t} \tilde{\rho}_{12}$$

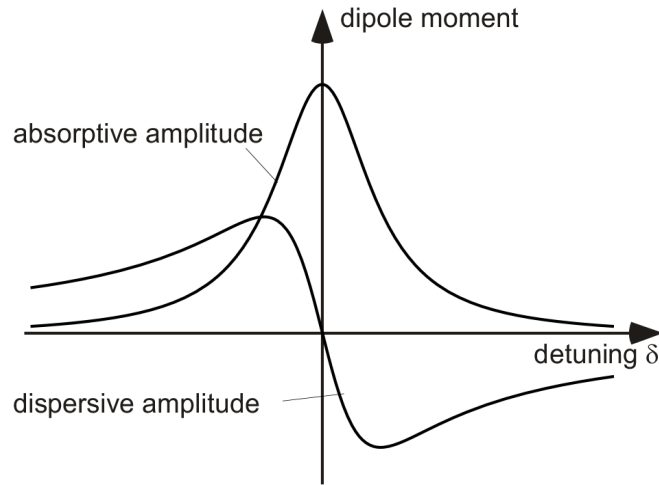
and by using the steady state value

$$\tilde{\rho}_{12} = \bar{\rho}_{12}.$$

We write the result in terms of quadrature components and obtain (as exercise)

$$\begin{aligned} \langle d \rangle &= d_{12} \Omega \left( \frac{\frac{\gamma}{2}}{\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{1}{2}\Omega^2} \sin(\omega t) + \frac{-\delta}{\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{1}{2}\Omega^2} \cos(\omega t) \right) \\ &= d_a \sin(\omega t) + d_d \cos(\omega t) \end{aligned}$$

with the "absorptive amplitude"  $d_a$  and the "dispersive amplitude"  $d_d$ .



Qualitatively, the two level atom behaves exactly as a classical driven damped harmonic oscillator. For large intensities  $I \gg I_{sat}$ , the dipole is proportional to  $\langle d \rangle \sim 1/\Omega$  and disappears on average. This can be understood intuitively if we remember the details of the Rabi cycle. The dipole moment changes its phase by  $180^\circ$  after half a Rabi cycle. If the Rabi cycle becomes increasingly faster, the dipole moment averages out.

- phase shift

The dipole oscillates with the laser frequency but is shifted in phase depending on the detuning. By using

$$a \cos(x) + b \sin(x) = \sqrt{a^2 + b^2} \cos\left(x - \arctan\left(\frac{b}{a}\right)\right)$$

we can calculate this phase shift:

$$\begin{aligned} \varphi &= \arctan\left(\frac{b}{a}\right) = \arctan\left(-\frac{1}{2} \frac{\gamma}{\delta}\right). \\ \tan(\varphi) &= -\frac{1}{2} \frac{\gamma}{\delta} \end{aligned}$$

To bring the detuning in the counter we take the inverse

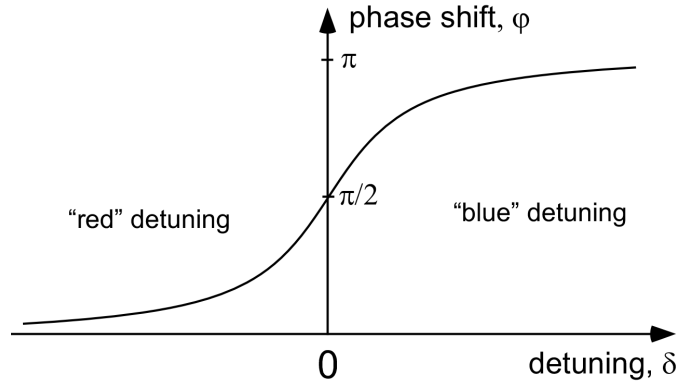
$$\cot(\varphi) = -\frac{2\delta}{\gamma}$$

and use

$$\cot(x) = \tan\left(\frac{\pi}{2} - x\right).$$

Thus

$$\varphi = \arctan\left(\frac{2\delta}{\gamma}\right) + \frac{\pi}{2}$$



### 3.2 Absorption and dispersion

- absorption

The absorptive oscillation is  $90^\circ$  retarded relative to the incident electric field that drives the atom. An additional phase of  $90^\circ$  is obtained if we regard many atoms which are arranged in a plane parallel to the phase fronts of the incident field. Such a 2D-sheet of synchronously oscillating dipoles generate a plane wave that is  $90^\circ$  retarded relative to the oscillation of the single dipoles in the sheet. This wave is thus  $180^\circ$  shifted relative to the incident wave and interferes with it destructively. This reduction of the transmitted field is called "absorption". In a gas one can think of the atoms being arranged in many parallel sheets. Each sheet absorbs a little bit of light and reduces the field by a constant value  $\Delta E$ .

$$\frac{dE}{dz} \simeq \Delta E \sim \langle d \rangle \sim E$$

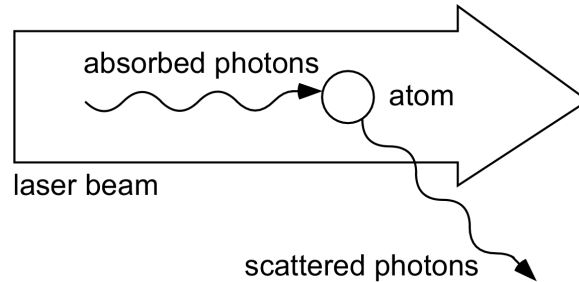
The field decays exponentially along the propagation axis.

- scattering

The absorbed energy can be stored in the atom which increases the probability to be in the excited state. An excited atom decays spontaneously within its natural life time. During this decay light is emitted in a random direction and with a random phase (relative to the incident light). This type of scattering is called incoherent scattering or resonance fluorescence if  $\delta = 0$ . The absorbed energy can also be reemitted immediately without being stored in the atom. This emission is identical to the emission of a classical oscillating dipole. The emitted intensity forms a dipole pattern and the phase of the light is locked to the phase of the dipole oscillation (in the far field its the same phase). This type of scattering is called coherent scattering or Rayleigh scattering.

- radiation pressure

If an atom gets excited by the light field it absorbs a photon. But with it, the atom not only absorbs the energy  $\hbar\omega$  but also the momentum  $\hbar k$ : The atom gets a kick along the direction of light propagation. Later the atom reemits the photon by spontaneous decay and again the atom feels a recoil kick, however now in a random direction. On average there is no momentum transferred to the atom. Since the decay is stimulated by the vacuum fluctuations, the phase between the absorbed and the emitted light wave is also random and the two processes can be regarded independently.



We calculate the force that is related to the change in momentum during absorption. Since photons cannot accumulate in the atom, the photon absorption rate must be the same as the photon emission rate  $\Gamma$ . The photon momentum  $\hbar k$  is thus also absorbed at this rate and the force is

$$F = \frac{dp}{dt} = \hbar k \Gamma = \hbar k \frac{\gamma}{2} \frac{I}{I_{sat}} \frac{1}{\left(\frac{2\delta}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}}.$$

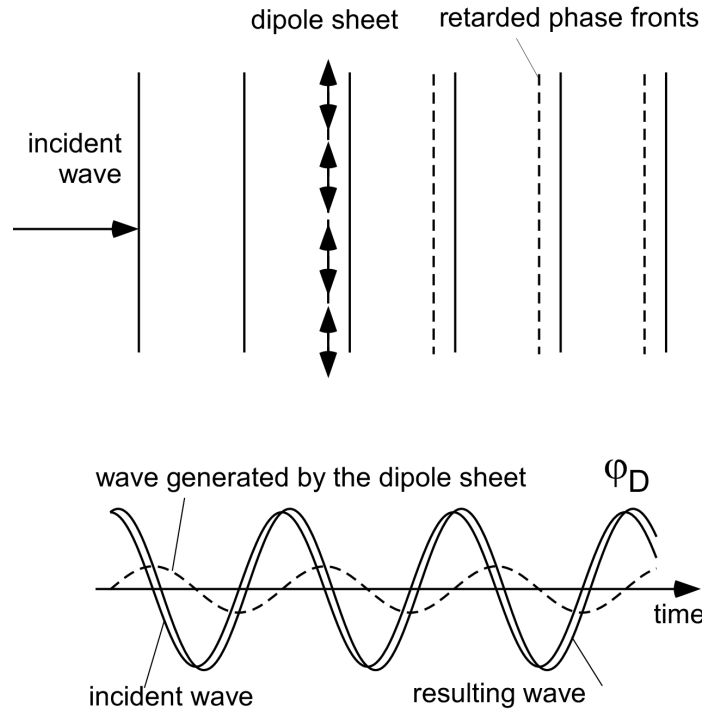
On resonance and for large intensity the force saturates at a value

$$F_{\max} = \frac{1}{2} \hbar k \gamma.$$

The atomic properties only enter via the natural decay rate  $\gamma$ . Atoms with a large line width can be pushed around with a larger force. For alkalis the acceleration can be as high as  $10^6 g$ .

- Lorentz-model for the index of refraction

The dispersive amplitude oscillates in phase with the incident field such that a dipole sheet emits a plane wave that is retarded relative to the incident wave only by  $90^\circ$ . It thus cannot reduce or enhance the incident wave (to first order). However, it leads to a phase shift of the incident plane wave.



For many dipole sheets arranged at distances  $a$  along the optical axis ( $z$ -axis) the temporal delay  $\varphi_D$  of a single dipole sheet adds up to a total phase delay  $\Delta\varphi$

$$\Delta\varphi = \varphi_D \frac{z}{a}.$$

To this phase we add the phase  $k \cdot z$  which is accumulated along the optical axis anyway also in vacuum without atoms:

$$\varphi_{tot} = k \cdot z + \Delta\varphi,$$

with  $k$  being the wave number of vacuum. This is the total phase a light field accumulates when it passes a cloud of atoms. We express this total phase shift by introducing the effective wave number  $n \cdot k$  and call  $n$  the "index of refraction":

$$\begin{aligned} n \cdot k \cdot z &= \varphi_{tot} = k \cdot z + \varphi_D \frac{z}{a} \\ n &= 1 + \frac{1}{k} \frac{\varphi_D}{a} \end{aligned}$$

At least for thin gases, the phase  $\varphi_D$  increases linearly with the number of oscillating atomic dipoles in a single dipole sheet. This number is proportional to the areal atomic density  $1/a^2$ .

$$\varphi_D \sim 1/a^2.$$

The index of refraction thus grows linearly with the atomic density

$$\rho := 1/a_3.$$

We thus write

$$n = 1 + \frac{K}{k^3} \cdot \rho$$

with a dimensionless constant  $K$  that contains the phase shift of the single atom dipole oscillation and the details about how it determines  $\varphi_D$ . We don't discuss these details here but we will use this picture later in the context of Bose-Einstein condensates. The factor  $k^{-2}$  has been introduced to make  $K$  dimensionless.

- optical potentials

We now consider large detuning such that the atom is in the ground state with a high probability. However it can still have a dipole moment which interacts with the electric field. The energy of the atom-light system is then mainly given by the expectation value of the interaction Hamilton

$$\langle H_{int} \rangle = \langle \vec{\psi}(t) | H_{int} | \vec{\psi}(t) \rangle$$

with

$$\vec{\psi}(t) = c_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\frac{\omega_0}{2}t} + c_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\frac{\omega_0}{2}t}$$

and

$$H_{int} = \hbar\Omega \cos(\omega t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Only the mixed term are non zero and with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

one obtains

$$\begin{aligned} \langle H_{int} \rangle &= \hbar\Omega \cos(\omega t) \left( c_1^*(t) e^{i\frac{\omega_0}{2}t} c_2(t) e^{i\frac{\omega_0}{2}t} + c_2^*(t) e^{-i\frac{\omega_0}{2}t} c_1(t) e^{-i\frac{\omega_0}{2}t} \right) \\ &= \hbar\Omega \cos(\omega t) (\rho_{12}^* e^{i\omega_0 t} + \rho_{12} e^{-i\omega_0 t}) \end{aligned}$$

In the rotating wave approximation we drop the fast terms and with  $\delta = \omega - \omega_0$  we get

$$\langle H_{int} \rangle = \hbar\Omega (\rho_{12}^* e^{-i\delta t} + \rho_{12} e^{i\delta t}).$$

With

$$\rho_{12} = e^{-i\delta t} \tilde{\rho}_{12}$$

and for the steady state

$$\tilde{\rho}_{12} = \bar{\rho}_{12} = \frac{1}{2} \Omega \frac{\delta - i\frac{\gamma}{2}}{\delta^2 + (\frac{\gamma}{2})^2 + \frac{1}{2}\Omega^2}$$

the energy is

$$\begin{aligned} \langle H_{int} \rangle &= \hbar \Omega \operatorname{Re}(\tilde{\rho}_{12}) = \frac{1}{2} \hbar \Omega^2 \frac{\delta}{\delta^2 + (\frac{\gamma}{2})^2 + \frac{1}{2}\Omega^2} \\ &= \frac{1}{4} \hbar \gamma \frac{2\Omega^2}{\gamma^2} \frac{\delta/\gamma}{\left(\frac{\delta}{\gamma}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{1}{4} \frac{2\Omega^2}{\gamma}} \\ &= \hbar \gamma \frac{I}{I_{sat}} \frac{\delta/\gamma}{\left(\frac{2\delta}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}}. \end{aligned}$$

For an inhomogeneous intensity distribution  $I(\vec{r})$  also the energy changes in space and one gets an optical potential

$$U(\vec{r}) = \langle H_{int} \rangle \sim I(\vec{r}).$$

For negative detuning the atom is drawn into regions of high intensity. The potential depth is largest near resonance where the dipole amplitude is large. However, so close to resonance the absorption is still high and radiation pressure will push away the atoms. For traps with a long life time we require an absorption rate in the range of seconds which is possible only at large detuning. Fortunately, dispersion drops as  $\delta^{-1}$  with detuning whereas absorption drops as  $\delta^{-2}$ . One thus can suppress scattering stronger than the trap depth. With sufficient laser power one can afford very large detunings and finds a realistic parameter range with little scattering but still relevant trap depth.

### 3.3 Optical Doppler cooling

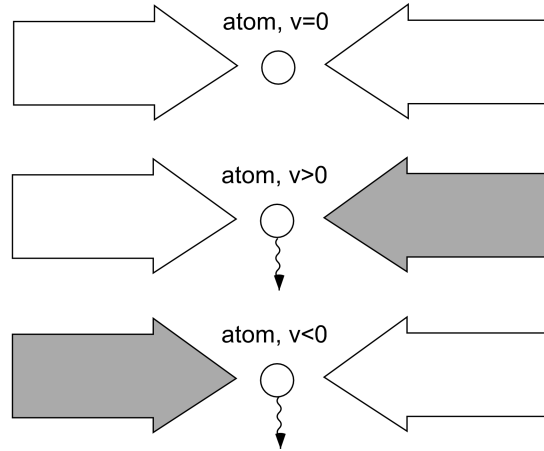
- friction force

Two counter propagating laser beam, red detuned relative to the atomic resonance may generate a friction force. The laser beams are red detuned such that an atom at rest does not scatter photons. If the atom moves with a velocity  $v$  the Doppler shift

$$\Delta\omega = -\vec{k} \cdot \vec{v}$$

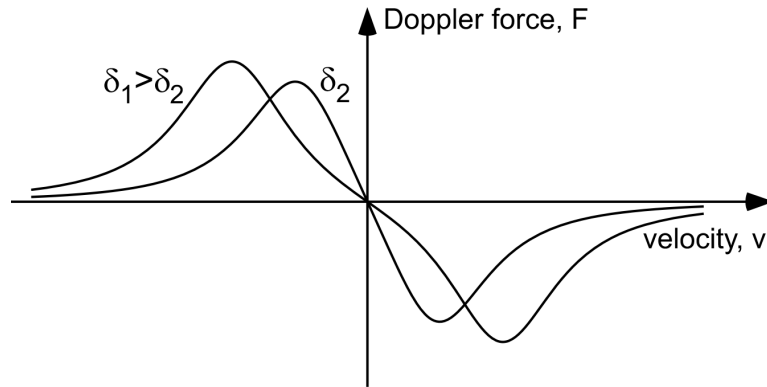
tunes the laser frequency in resonance if the atom moves against the direction of the beam propagation. It starts to scatter atoms and the radiation pressure slows the atom down. This works for both directions.

red detuned laser beams



We can easily calculate the resulting friction force. The Doppler shift enters the game via the frequency dependence of the radiation pressure force  $F(\delta)$ . The total radiation pressure is the sum of the two forces exerted on a moving atom by the two beams. For one beam the detuning is positive for the other negative:

$$F = \hbar k \frac{\gamma}{2} \frac{I}{I_{sat}} \left( \frac{1}{\left(\frac{2(\delta - kv)}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}} - \frac{1}{\left(\frac{2(\delta + kv)}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}} \right).$$



For small velocities we can expand the force and obtain a Stokesian friction force

$$\vec{F} = R\vec{v}$$



with a friction coefficient

$$\begin{aligned}
R &: = -\frac{\hbar k^2 \gamma \Omega^2 \delta}{\left(\delta^2 + \left(\frac{\gamma}{2}\right)^2 + \frac{\Omega^2}{2}\right)^2} \\
&= -4\hbar k^2 \cdot \frac{2\delta}{\gamma} \cdot \frac{\frac{I}{I_{sat}}}{\left(\left(\frac{2\delta}{\gamma}\right)^2 + 1 + \frac{I}{I_{sat}}\right)^2} \\
&= -32\hbar k^2 \cdot \frac{\delta}{\gamma} \cdot \frac{I_{sat}}{I} \bar{\rho}_{11}
\end{aligned}$$

By increasing the detuning the velocity capture range can be enlarged but the slope near the origin is reduced. The slope determines the friction coefficient.

- cooling power

From classical mechanics we know that a particle moving with velocity  $v$  against a force  $F$  changes its energy at a rate

$$\frac{dE}{dt} = P = \vec{v} \cdot \vec{F}.$$

Thus the cooling power is

$$P_c = \vec{v} \cdot \vec{F} = \vec{v} \cdot R\vec{v} = Rv^2$$

- heating during absorption

After a time  $t$  the atom has absorbed an average photon number

$$\bar{N} = \bar{\rho}_{11} \gamma t$$

Since the incoming photon number fluctuates according to a Poissonian statistics also the number of absorbed photons fluctuates with a Poissonian statistics

$$\Delta N = \sqrt{\bar{N}} = \sqrt{\bar{\rho}_{11} \gamma t}$$

Due to these number fluctuations also the radiation pressure force fluctuates around an average value. The random force fluctuations kick the atom and result in a random atomic motion. The related kinetic energy can be interpreted as heat. We calculate the heating rate. The momentum transfer to the atom has a random component

$$\Delta p = \hbar k \Delta N = \hbar k \sqrt{\bar{\rho}_{11} \gamma t}$$

The related energy is

$$E = \frac{\Delta p^2}{2m} = \frac{\hbar^2 k^2 \bar{\rho}_{11} \gamma t}{2m}$$

and thus the heating power is

$$P_a = \frac{dE}{dt} = \frac{\hbar^2 k^2}{2m} \bar{\rho}_{11} \gamma$$

- heating during emission

The emission occurs in random directions. On average the momentum transfer is zero but the fluctuations again generate heat. The total momentum transferred to the atom after  $N$  emission events is

$$\vec{p}(t) = \sum_{i=1}^N \hbar \vec{k}_i$$

As above,  $N = \bar{\rho}_{11} \gamma t$ . The transferred kinetic energy is

$$E = \frac{\vec{p}^2}{2m} = \frac{\hbar^2}{2m} \left( \sum_{i=1}^N \vec{k}_i \right)^2 = \frac{\hbar^2}{2m} \left( \sum_{i=1}^N k_i^2 + \underbrace{\sum_{i=1}^N \sum_{j=1}^N \alpha_{ij} \vec{k}_i \cdot \vec{k}_j}_0 \right)$$

The last terms vanishes for large  $N$  because of the random relative orientation of the  $\vec{k}$ -vectors. We get

$$E = \frac{\hbar^2 k^2}{2m} N = \frac{\hbar^2 k^2}{2m} \bar{\rho}_{11} \gamma t$$

and for the heating power

$$P_e = \frac{dE}{dt} = \frac{\hbar^2 k^2}{2m} \bar{\rho}_{11} \gamma = P_a$$

- Doppler temperature

In equilibrium heating balances cooling. We have two beams which heat the atoms and we thus get

$$P_c = 2P_e + 2P_a = 4P_e$$

Using the expression for the cooling and heating powers we get

$$\begin{aligned} Rv^2 &= 2 \frac{\hbar^2 k^2}{m} \bar{\rho}_{11} \gamma \\ \frac{1}{2} m v^2 &= \frac{\hbar^2 k^2}{R} \bar{\rho}_{11} \gamma \end{aligned}$$

With

$$\frac{1}{2} m v^2 = E_{heat} = k_B T$$

the resulting temperature is

$$k_B T = \frac{\hbar \gamma}{2} \cdot \frac{\gamma}{4\delta} \left( 1 + \frac{I}{I_{sat}} + \left( \frac{2\delta}{\gamma} \right)^2 \right).$$

To calculate the optimal detuning we use  $dt/d\delta = 0$  and obtained

$$\delta_{\min} = \frac{\gamma}{2} \sqrt{1 + \frac{I}{I_{\text{sat}}}}.$$

The temperature then amounts to

$$T_{\min} = \frac{\hbar\gamma}{2} \sqrt{1 + \frac{I}{I_{\text{sat}}}}.$$

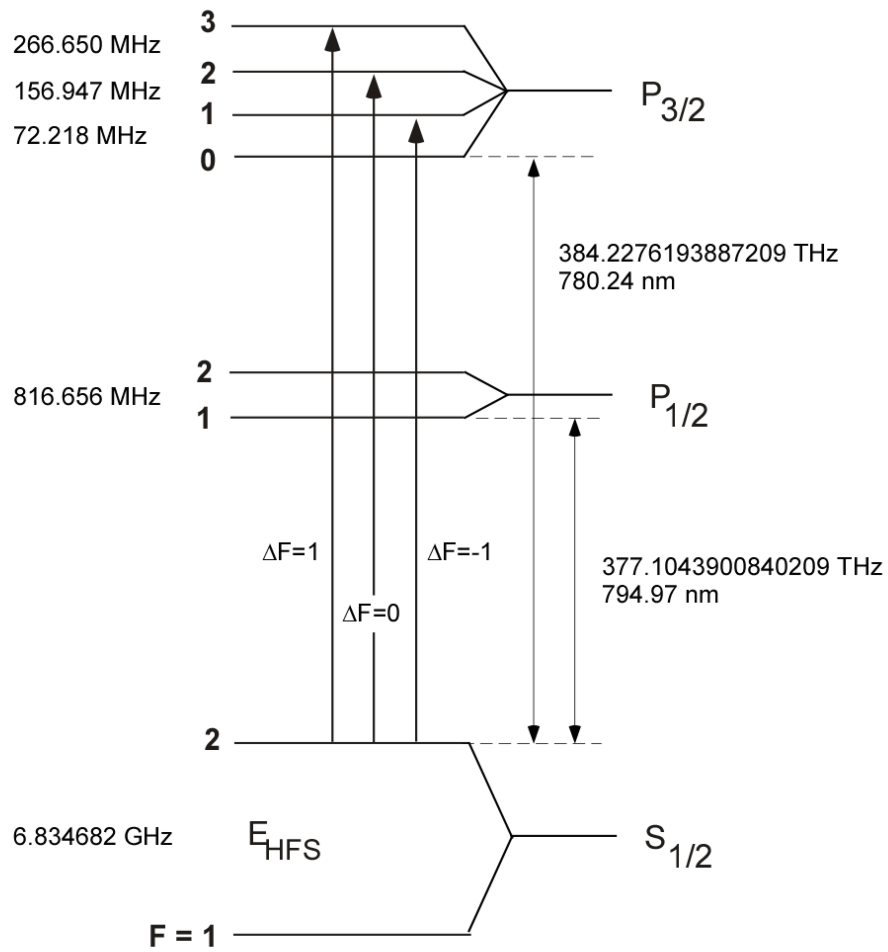
For small intensities the temperature approaches the "Doppler temperature"

$$T_D = \frac{\hbar\gamma}{2}.$$

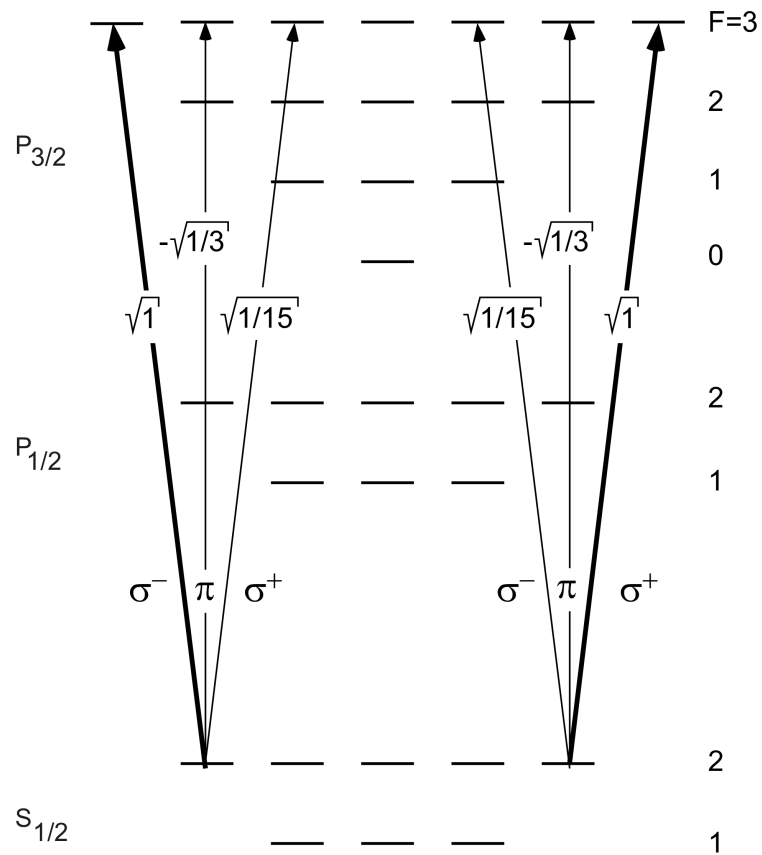
For alkalis it is typically about  $130\mu\text{K}$ .

### 3.4 Magneto-optical trap

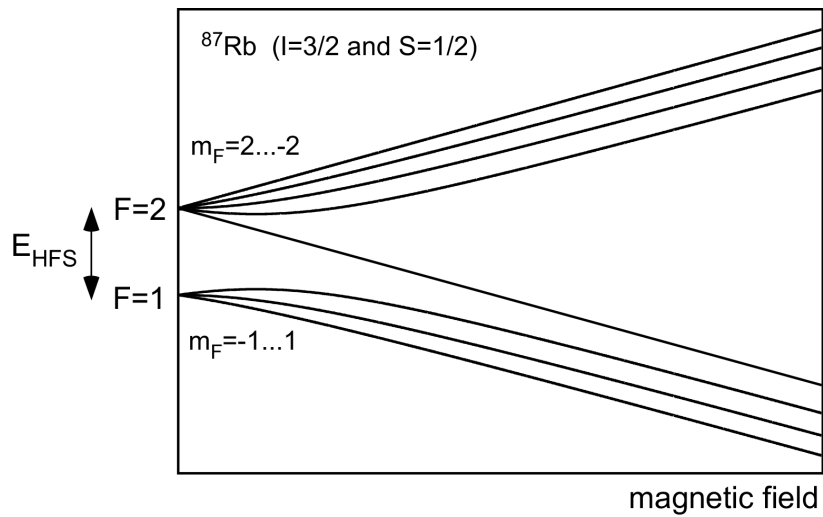
- For trapping atoms with radiation pressure the Doppler force must drive the atoms back into the trap center where the force vanishes. We thus need a mechanism that makes the Doppler force spatially dependent. With a two level system this is not possible. We need at least 3 states. We have a look at the energy levels of Rubidium.



We are interested in the transition from  $5p_{1/2}(F = 2, m_F = 2)$  to  $5p_{3/2}(F = 3, m_F = 3)$ . Its a closed transition which means that the excited state can only decay into the initial state. The transition is excited with an electric field that rotates around the quantization axis with a right hand sense of rotation (counterclockwise). Similarly, the transition from  $5p_{1/2}(F = 2, m_F = -2)$  to  $5p_{3/2}(F = 3, m_F = -3)$  is also a closed transition which can be excited with a clockwise rotating electric field vector.



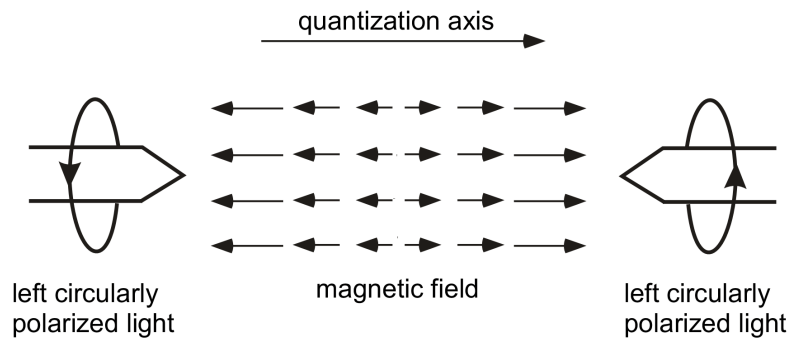
In a magnetic field the states are shifted in energy by the Zeeman-effect. The levels of the ground state  $5p_{1/2}$  are, for instance, shifted like this:



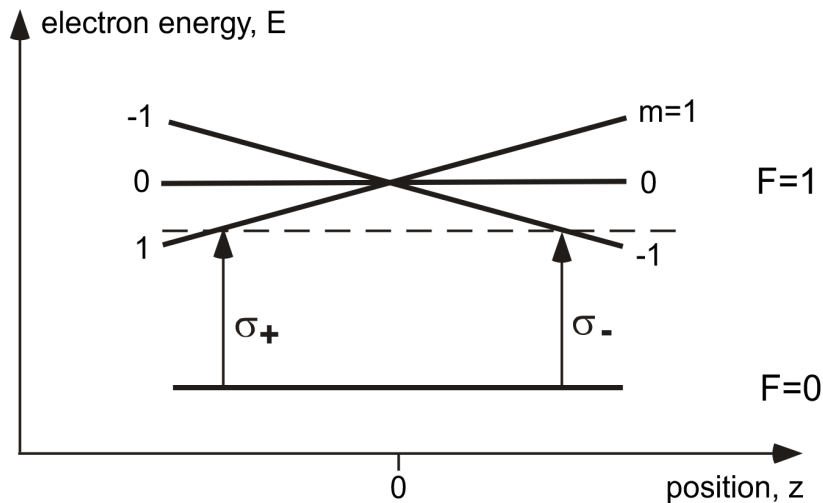
Qualitatively, the scheme also applies to the  $5p_{1/2}$  state. The  $5p_{3/2}$  state has a similar structure with more states. The point is that the detuning of a particular transition can be controlled with a magnetic field by the Zeeman effect.

- controlling the radiation pressure

To keep things simple we consider an atom with one ground state that is not affected by the magnetic field and the three excited state of a level with  $F = 1$ . For small magnetic field the states with  $m = \pm 1$  are linearly shifted with the magnetic field. We place the atom in an inhomogeneous magnetic field that vanishes at the center and increases with distance to the center.



Two laser beams are red detuned and left handed circularly polarized (counterclockwise if you look into the laser beam ).

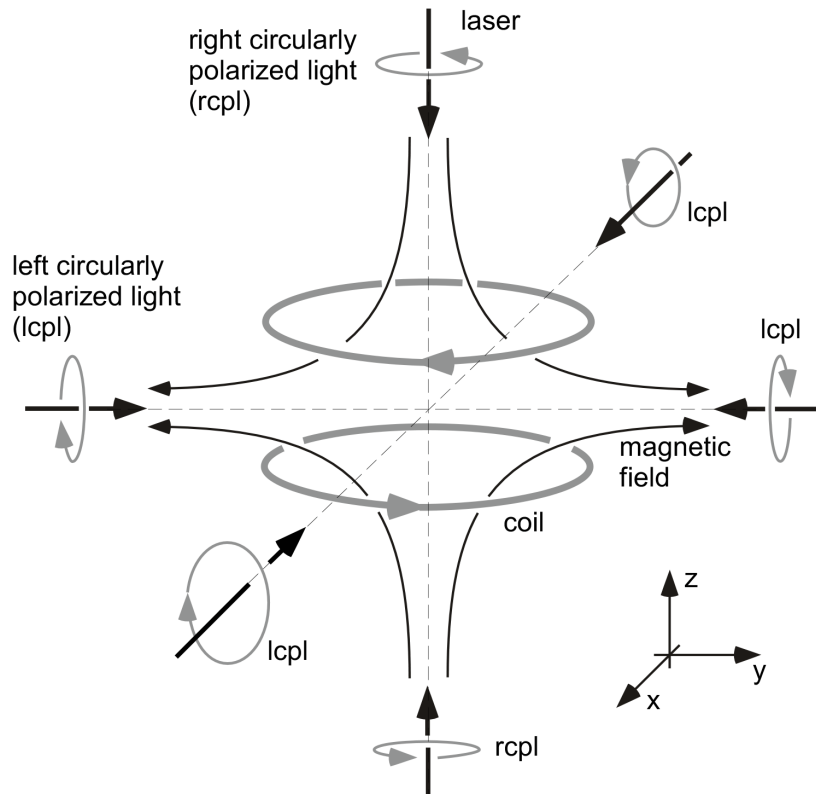


Since the laser beams are red detuned they barely excite atoms at the center where the magnetic field vanishes. If the atom moves away from the center the Zeeman-effect tunes it in resonance with that beam that pushes the atom back to the center.

The other beam is even further detuned and affects the atom even less. We obtain a spatially dependent force which point to the position where the magnetic field is zero (if both beams have the same intensity). This scheme is compatible with Doppler-cooling. The atoms are not only collected but also cooled.

- Magneto-optical trap (MOT)

This scheme can be extended to 3D:



This method is surprisingly robust and also works for atoms with a more complicated hyperfine structure. However, a realistic theory of the MOT is complicated. Here are some aspects which have to be taken into account:

- optical pumping
- hyperfine pumping
- dipole forces and optical lattices
- further cooling and heating effects ("polarization gradient cooling")
- absorption and radiation trapping
- "optical collisions" between the atoms

g) thermodynamics and diffusion

Although these individual aspects are very well understood theoretically, a closed theory of the MOT which combines all these aspects seems impossible. The setup is already too complicated.

Here, we just note that the cooling works much better than expected. The temperatures observed in the first MOTs have been as small as  $30 \mu\text{K}$  which is four times below Doppler temperature. Obviously, there were additional cooling effects at work. In fact, optical cooling turned out to be a very rich research topic and, after understanding all the details, it was eventually possible to cool atomic clouds down to temperatures of  $4 \text{ nK}$  (however only at very low densities). The different aspects of optical cooling have been extensively studied in the 90ies leading to Nobel prizes for C. Cohen Tannoudji, W. Phillips, and S. Chu in 1997.

Today the MOT is used in almost every quantum optics lab as the standard work horse to prepare cold and dense atomic clouds. Typical temperatures are  $150 \mu\text{K}$  at maximum densities around  $10^{11}/\text{cm}^3$ . At higher densities the cloud becomes optically thick i.e. non transparent such that there is no more light left at the center of the cloud. After collecting the atoms in the MOT one usually turns off the magnetic field and increases the detuning for a few milliseconds. This change of parameters optimizes a further cooling method, the so called polarization gradient cooling ("optical molasses") which further reduces the temperature by a factor 2 to 3. The cloud is then trapped in a magnetic trap or in an optical dipole trap. This is the starting point of evaporative cooling.

- evaporative cooling

If the trap depth is slowly reduced atoms start to leave the trap. Since the escaping atoms have to climb up all the entire trapping potential they carry away more than the average energy of a trapped atom. The system is now not in thermal equilibrium but if we give the system enough time to rethermalize, it will do so at a smaller temperature. By adiabatically ramping down the trap depth we can cool the cloud to very low temperatures without losing density. The number of atoms is drastically reduced but the remaining atoms are so cold that they cannot climb the potential any more and are squeezed to the trap center where they form a cold cloud with a high density. The number of atoms lost during the process depends on the time which is invested: If we keep the trap depth high and just wait, there is a very small but nevertheless finite probability that all atoms are at rest and the total energy is carried by a single atom alone. If this atom escapes, the cloud would be cooled to an arbitrarily small temperature by losing just a single atom. Of course this scenario is completely unrealistic but illustrates the physics. The relative temperature change and the relative loss of particles is proportional:

$$\frac{dT}{T} = \alpha \frac{dN}{N}.$$



The parameter  $\alpha$  is the "efficiency" of the cooling. For a three dimensional harmonic trap one can show that

$$\alpha_{HO} = \frac{1}{3} \frac{E_t}{k_B T} - 1.$$

with the trap depth  $E_t$  and the temperature of the gas  $T$ . To keep  $\alpha$  constant the trap depth has to be adjusted according to the actual temperature. By integration we obtain

$$\ln \left( \frac{T}{T_0} \right) = \alpha \ln \left( \frac{N}{N_0} \right)$$

which can be transformed into

$$\frac{T}{T_0} = \exp^{\alpha \ln \left( \frac{N}{N_0} \right)} = \left( \frac{N}{N_0} \right)^\alpha.$$

If the ratio  $E_t/k_B T$  is large the trap is deep and the escape rate is slow but the cooling is efficient since  $\alpha$  is large. The total cooling time which one can invest is determined by the trap life time. It is eventually limited by collisions with rest gas. In the best possible vacuum the life time amount to about 100 s.

In the experiment one typically chooses  $\alpha \simeq 1$  and loses atoms by about the same factor as the temperature is reduced. For cooling from  $100\mu\text{K}$  to  $1\mu\text{K}$  one loses 99% of the atoms. The cooling takes typically 10 – 30 s. By this method one can generate gases with a phase space density below 1. What this means is explained in the next lesson.