Hefei-Lectures 2015
First Lesson: Optical Resonators

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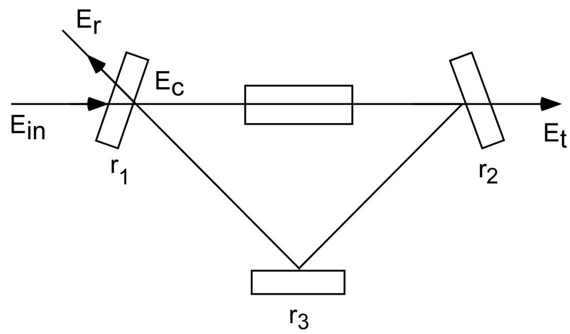
1.1 Power and Field in an Optical Resonator (steady state solutions)

The left mirror is the input coupler with a given field reflectivity r_1, the right mirror is the output coupler with a reflectivity r_2. The third mirror reflects the light back to the left mirror and should be as ideal and lossless as technically possible ($r_3 = 1$) . The resonator may contain a nonlinear crystal, a cloud of atoms, or something else.



- electric field in the resonator

  If the resonator is in a steady state, the field inside the cavity, right behind the input coupling mirror, E_c, should be the same after one round trip

  $$E_{n+1} = E_n$$

  During a round trip of length l the field accumulates a phase

  $$\varphi = kl,$$

  with k being the wave number. Furthermore the field is reduced by a factor

  $$r_m := r_1\sqrt{1-L},$$

  with the field-reflectivity of the input coupler r_1 and the power losses L of one round trip including the losses at the output coupler and the third mirror (but without the losses due to transmission back through the input coupler). Finally there is field coupled in through the input coupler

  $$t_1 E_i = \sqrt{1-r_1^2}E_i.$$

  In total one obtains

  $$E_{n+1} = r_m e^{i\varphi}E_n + t_1 E_i = E_n := E_c,$$

  or

  $$\frac{E_c}{E_i} = \frac{t_1}{1 - r_m e^{i\varphi}}$$

- intensity circulating in the resonator

  The power time averaged over one oscillation of the light field is obtained from the modulus of the field:

$$\frac{P_c}{P_i} = \frac{|E_c|^2}{|E_i|^2} = \frac{t_1^2}{\left(1 - r_m e^{i\varphi}\right)\left(1 - r_m e^{-i\varphi}\right)} = \frac{t_1^2}{1 - 2r_m \cos\varphi + r_m^2}$$

  There is a maximum for

$$\varphi = q \cdot 2\pi$$

  with q being a natural number. One obtains a series of resonances

$$\begin{aligned} \varphi &= q \cdot 2\pi = kl \\ k &= \frac{q \cdot 2\pi}{l} \\ \nu &= \frac{\omega}{2\pi} = \frac{ck}{2\pi} = \frac{c}{2\pi}\frac{q \cdot 2\pi}{l} = q \cdot \frac{c}{l} = q \cdot \nu_{FSR} \\ \nu_{FSR} &:= \frac{c}{l} \end{aligned}$$

  Close to a resonance one can expand $\cos(x) \simeq 1 - x^2/2$ and obtains

$$\frac{P_c}{P_i} \simeq \frac{t_1^2/r_m^2}{\left(1 - r_m\right)^2/r_m^2 + \varphi^2}.$$

  This is a Lorentzian line shape function in φ.

- impedance matching

  On resonance, $\varphi = 0$, one obtains

$$\frac{P_c}{P_i} = \frac{t_1^2}{\left(1 - r_m\right)^2} = \frac{1 - r_1^2}{\left(1 - r_1\sqrt{1 - L}\right)^2}$$

  With the definition of the power transmission T and the power reflectivity R

$$\begin{aligned} R_1 &:= r_1^2 \\ T_1 &:= t_1^2 = 1 - R_1 \\ R_m &:= r_m^2 = R_1\left(1 - L\right) \end{aligned}$$

  we obtain

$$\frac{P_c}{P_i} = \frac{1 - R_1}{\left(1 - \sqrt{R_m}\right)^2}.$$

Let's calculate the optimal input coupler i.e. the R_1 for which the power in the cavity is largest:

$$\begin{aligned}
\frac{d}{dR_1}\frac{1-R_1}{\left(1-\sqrt{R_1\left(1-L\right)}\right)^2} &= \frac{\sqrt{-R_1\left(L-1\right)}+L-1}{\left(-1+\sqrt{-R_1\left(L-1\right)}\right)^3\sqrt{-R_1\left(L-1\right)}}=0 \\
\sqrt{-R_1\left(L-1\right)} &= 1-L \\
R_1\left(1-L\right) &= \left(1-L\right)^2 \\
R_1 &= 1-L
\end{aligned}$$

This is the impedance matched case. Lets calculate the enhancement A :

$$A:=\frac{P_{\max}}{P_i}=\frac{1}{1-R_1}=\frac{1}{L}.$$

With 1% losses and perfect impedance matching the power in the cavity is 100 times the power incident on the input coupler.

- reflected intensity

  The reflected field consists of a part which is directly at the input coupler and a part which comes from inside the cavity:

$$\begin{aligned}
E_r &= -r_1E_i+E_c\sqrt{1-L}e^{ikl}t_1 \\
&= -r_1E_i+E_i\frac{t_1}{1-r_1\sqrt{1-L}e^{i\varphi}}\sqrt{1-L}t_1e^{i\varphi}
\end{aligned}$$

  The negative sign of the first term takes care of the time reversal symmetry of a beam splitter: Any energy conserving surface reflects the field on one side with a π phase shift relative to the reflection from the opposite side.

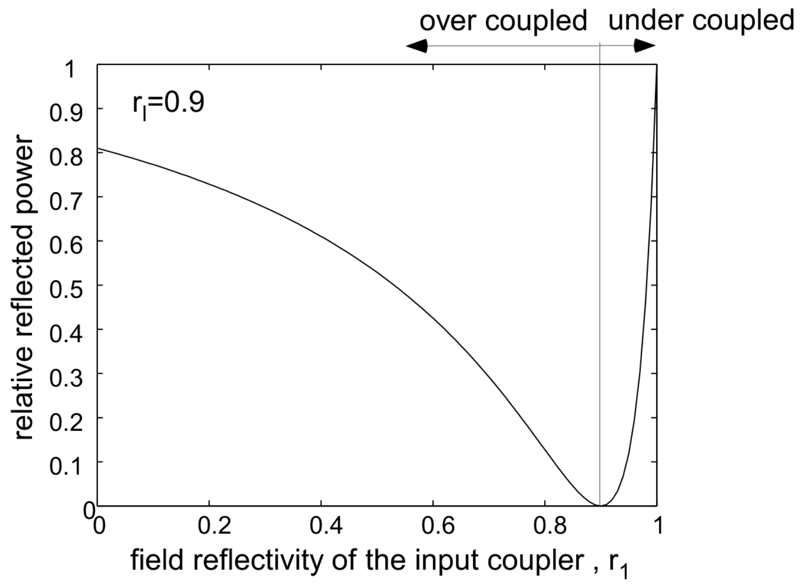
  On resonance on gets

$$\begin{aligned}
\frac{E_r}{E_i} &= -r_1+\frac{t_1^2r_l}{1-r_1r_l} \\
&= -r_1+\frac{\left(1-r_1^2\right)r_l}{1-r_1r_l} \\
&= \frac{\left(1-r_1^2\right)r_l-r_1\left(1-r_1r_l\right)}{1-r_1r_l} \\
&= \frac{r_l-r_1}{1-r_1r_l}
\end{aligned}$$

  with the abbreviation

$$r_l:=\sqrt{1-L}$$

  The power may be calculated form the field modulus as above. For perfect impedance matching ,i.e. for $r_1=r_l$, the reflection vanishes and all the incident light is coupled into the resonator.



- Question to think about

  Assume that the only round trip losses are given by the transmission of the output coupler:

  $$L = 1 - R_2 = T_2$$

  In the impedance matched case one obtains for the intensity behind the output coupler

  $$P_{trans} = P_c \cdot T_2 = P_{in} \cdot \frac{1}{T_2} \cdot T_2 = P_{in}.$$

  This holds independently of the value of T_2 , i.e. also for $T_2 = 0$. Impedance matching then requires that also $T_1 = 0$. In other words: If you build a resonator with two lossless mirrors with 100% reflectivity the transmission through the cavity on resonance will be 100% . How can that be?

1.2 Equation of motion

- differential equation for the field

  We require that the field after one round trip is the field at the beginning of the round trip corrected for the round trip losses and the round trip phase shift. In addition, one has to add the field which is coupled in during on round trip

  $$E(t + \tau) = E(t) r_m e^{i\varphi} + t_1 E_i$$

  For very short round trip times (what is the typical round trip time?) one can expand the electric field as

  $$E\left(t + \tau\right) \simeq E\left(t\right) + \frac{dE}{dt}\tau$$

and obtains

$$\begin{aligned} E\left(t\right)+\frac{dE}{dt}\tau &= E(t)r_m e^{i\varphi}+t_1E_i \\ \frac{dE}{dt} &= \frac{1}{\tau}E(t)\left(r_m e^{i\varphi}-1\right)+\frac{1}{\tau}t_1E_i \end{aligned}$$

Near resonance, $\varphi \ll 1$, we expand the exponential

$$e^{i\varphi}\simeq 1+i\varphi$$

and obtain

$$\frac{dE}{dt}=E\frac{(r_m i\varphi-(1-r_m))}{\tau}+\frac{t_1}{\tau}E_i.$$

- resonators with low losses

  We first write down the round trip transmission

$$t_m := \sqrt{1-r_m^2}=\sqrt{1+r_m}\sqrt{1-r_m}$$

then solve for

$$1-r_m=\frac{t_m^2}{1+r_m}.$$

For small losses $r_m\simeq 1$:

$$\frac{t_m^2}{1+r_m}\simeq\frac{t_m^2}{2}$$

with this one obtains

$$\begin{aligned} \frac{dE}{dt} &= \left(i\Delta-\kappa\right)E+\eta \\ \Delta &:= r_m\frac{\varphi}{\tau}\simeq\frac{\varphi}{\tau}\text{, cavity detuning} \\ \kappa &:= \frac{1-r_m}{\tau}\simeq\frac{t_m^2}{2\tau}=\frac{L}{2\tau}\text{, field decay rate} \\ \eta &:= \frac{t_1}{\tau}E_i\text{, field pump rate} \end{aligned}$$

We look for the solution on resonance of a filled cavity after the pump field has turned of:

$$\begin{aligned} \frac{dE}{dt} &= -\kappa E \\ E(t) &= E_0e^{-\kappa t}. \end{aligned}$$

measuring the decay rate is one way to determine the losses of the resonator (works only for very good cavities and fast detectors). Which decay rates do you expect for typical resonators?

- general solution

  is

$$E(t) = \frac{\eta}{\kappa - i\Delta} + Ce^{(i\Delta-\kappa)t}$$

C is given by the initial conditions. After several decay times the second term decays and one obtains (compare to the steady state solution above):

$$\begin{aligned} E_c &= \frac{\eta}{\kappa - i\Delta} \\ &= \eta\frac{i\Delta + \kappa}{(\kappa - i\Delta)(\kappa + i\Delta)} \\ &= \eta\frac{i\Delta}{\Delta^2+\kappa^2} + \eta\frac{\kappa}{\Delta^2+\kappa^2} \end{aligned}$$

The field oscillates with a dispersive and an absorptive amplitude (plot the two terms, does it remind you to something ?). In fact, the cavity behaves like a driven, damped mechanical harmonic oscillator only that the phase θ between the driver and the oscillator is different. If the phase of the driving field is taken as reference, the relative phase is given by the phase of the field inside the cavity:

$$\theta = \arctan\frac{\mathrm{Im}(E_c)}{\mathrm{Re}(E_c)} = \arctan\frac{\eta\frac{i\Delta}{\Delta^2+\kappa^2}}{\eta\frac{\kappa}{\Delta^2+\kappa^2}} = \arctan\frac{\Delta}{\kappa}.$$

For $\Delta \gg \kappa$ the phase shift is π relative to the limiting case of $\Delta \ll \kappa$. This is the same for a mechanical resonator. What differs is the phase at resonance which is 90° for the mechanical oscillator and 0° for the optical resonator.

1.3 Line width, finesse, decay rate

- line width

  is defined as full width at half maximum (FWHM) of the power resonance function.

$$\frac{P_c}{P_i} = \frac{t_1^2}{1 - 2r_m\cos\varphi + r_m^2}$$

close to resonance ($\varphi \ll 1$) the expansion yields

$$\frac{P_c}{P_i} \simeq \frac{t_1^2}{1 - 2r_m\left(1 - \frac{1}{2}\varphi^2\right) + r_m^2} = \frac{t_1^2}{(1-r_m)^2 + r_m\varphi^2}$$

half maximum is obtained for

$$\sqrt{r_m}\varphi_{1/2} = 1 - r_m$$

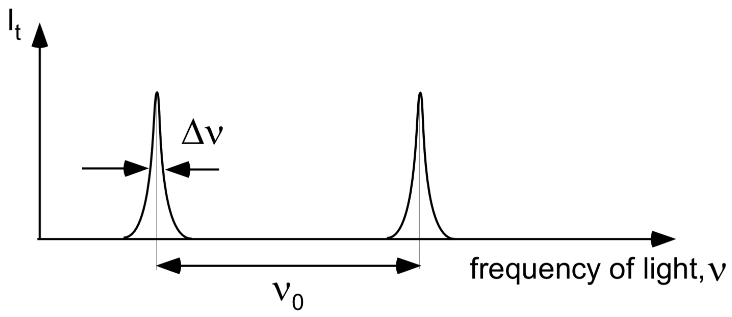
which corresponds to a detuning $\delta_{1/2}$

$$\frac{\nu_0}{2\pi} = \frac{\delta_{1/2}}{\varphi_{1/2}}$$

so

$$\delta_{FWHM} = 2\delta_{1/2} = \frac{\nu_0}{\pi}\varphi_{1/2} = \frac{\nu_0}{\pi}\frac{1-r_m}{\sqrt{r_m}}.$$

- finesse



is defined as the ratio

$$F = \frac{\nu_0}{2\delta_{1/2}} = \pi\frac{\sqrt{r_m}}{1-r_m}$$

with

$$r_m = r_1\sqrt{1-L} = \sqrt{(1-T_1)(1-L)}$$

and in the impedance matched case

$$r_1 = \sqrt{1-L},$$

one obtains

$$F = \pi\frac{\sqrt{r_1\sqrt{1-L}}}{1-r_1\sqrt{1-L}} = \pi\frac{\sqrt{1-L}}{L}.$$

and for small losses, $L \ll 1$, one finally gets

$$F \simeq \frac{\pi}{L}.$$

- decay rate

the power decay rate γ_P is twice the field decay rate

$$\gamma_P := 2\kappa = 2\frac{1-r_m}{\tau}.$$

with this, one obtains

$$\delta_{FWHM} = \frac{\nu_0 \tau}{\pi} \frac{\kappa}{r_m} = \frac{1}{\pi} \frac{\kappa}{r_m} \simeq \frac{\kappa}{\pi} = \frac{\gamma_P}{2\pi} = \frac{1}{2\pi} \frac{1}{T_{res}},$$

where T_{res} is the $1/e$-lifetime of the power in the resonator. The finesse and the life time is connected according to

$$F = \frac{\nu_0}{\delta_{FWHM}} = 2\pi \nu_0 T_{res} = \omega_0 T_{res}.$$

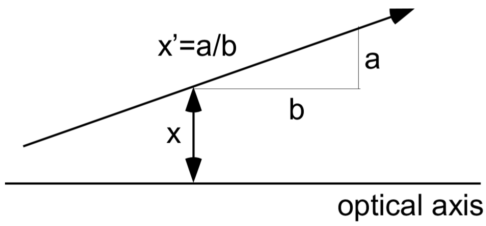
1.4 Optical Beams (Kogelnik, Li, Applied Optics 5, 1550, (1966))

To understand laser beams in resonators we first need to talk about geometric optics.

- beam vector

  we introduce a vector

$$\vec{x} := \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \text{distance to optical axis} \\ \text{slope of the beam} \end{pmatrix}$$



- ABCD-matrices

  optical elements may be described by matrices

  1) path of length d

$$M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

  2) lens

$$M_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

  (convex lenses $f > 0$, concave lenses $f < 0$)

  3) path within a material with index of refraction

$$M_{\bar{d}} = \begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix}$$

In general, the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is called ABCD-matrix.

It transforms an optical beam into a new one

$$\vec{x}_2 = M\,\vec{x}_1$$

- example 1: propagation along a distance d

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} x_1 + d \cdot x_1' \\ x_1 \end{pmatrix}$$

changes distance to optical axis but keeps the slope constant

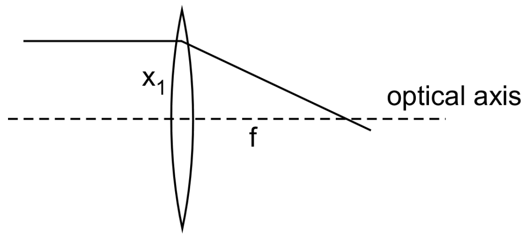
- lens with focal length f:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1' - \frac{x_1}{f} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2' \end{pmatrix}$$

changes slope but keeps the distance

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{x_1}{f} \end{pmatrix}$$

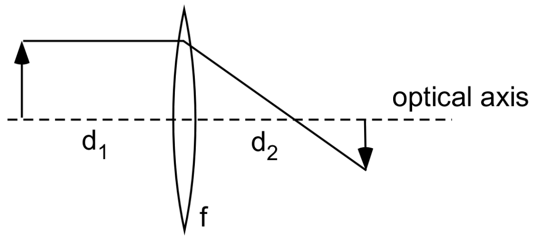
The new beam intersects the optical axis at a distance f from the lens.



- optical systems

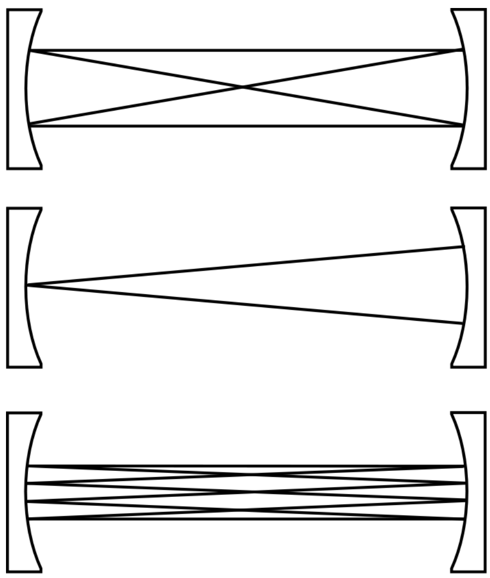
composite systems are described by the product of the corresponding matrices. Example: path d_1, lens f, path d_2

$$M = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$



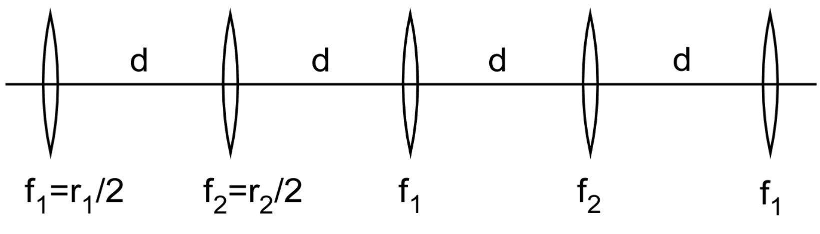
- lens conductor

  The possible trajectories between two mirrors



with radius of curvature r corresponds to a periodically repeating series of lenses with focal length

$$f = \frac{r}{2}$$



One round trip is described by the Matrix

$$M = M_d \cdot M_{f_2} \cdot M_d \cdot M_{f_1}$$

A number of n round trips are described by

$$M^n = (M_d \cdot M_{f_2} \cdot M_d \cdot M_{f_1})^n$$

- Sylvester Theorem

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = \frac{1}{\sin\theta} \begin{pmatrix} A\sin(n\cdot\theta) - \sin((n-1)\cdot\theta) & B(\sin n\cdot\theta) \\ C\sin(n\cdot\theta) & D\sin(n\cdot\theta) - \sin((n-1)\cdot\theta) \end{pmatrix}$$

with

$$\cos\theta := \frac{1}{2}(A+D)$$

periodic focusing and defocussing with the "frequency" θ.

- stability

unstable solutions are obtained if θ is undefined i.e. $\left|\frac{1}{2}(A+D)\right| > 1$. Stability range is thus

$$-1 < \frac{1}{2}(A+D) < 1$$

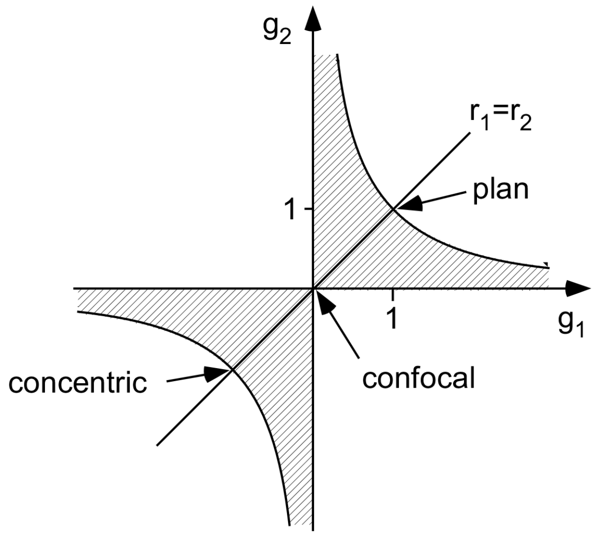
standing wave resonator:

$$0 < \underbrace{\left(1 - \frac{d}{r_1}\right)}_{:=g_1}\underbrace{\left(1 - \frac{d}{r_2}\right)}_{:=g_2} < 1$$

$$0 < g_1 g_2 < 1$$

- stability diagram

Since $g_1 g_2 > 0$ the point $(g_1 g_2)$ is in the first and third quadrant. The stability boundaries fulfill $g_1 g_2 = 1$, i.e. they form the functions $g_2 = \frac{1}{g_1}$:



- boundary at $g_1 g_2 = 1$ for resonators with equal mirrors

  for $r_1 = r_2 = r$, i.e.

$$\begin{aligned} g_1 g_2 &= 1 \\ \left(1 - \frac{d}{r}\right)\left(1 - \frac{d}{r}\right) &= 1 \\ 1 - \frac{d}{r} &= \pm 1 \\ \frac{d}{r} &= 0 \text{ or } \frac{d}{r} = 2 \end{aligned}$$

- plane resonator

$$\frac{d}{r} = 0$$

  r is infinite and the resonator consists of two plane mirrors. The resonator is unstable against an infinitesimal tilt of the beam.

- concentric resonator

$$\frac{d}{r} = 2$$

  The radius is half the distance and the mirror planes lie on a sphere. The resonator is unstable against an infinitesimal transversal shift of the beam.

- confocal resonator, boundary at $g_1 g_2 = 0$

  for equal mirrors one obtains

$$\begin{aligned} 1 - \frac{d}{r} &= 0 \\ d &= r. \end{aligned}$$

  The mirrors have their focal point at the same position. The resonator is stable. However an infinitesimal difference in the radius makes the resonator unstable.

- Questions to think about

  After how many round trips does the beam comes back to the starting point in a confocal resonator if the beam is injected not along the optical axis? What is the free spectral range in this case? If the cavity has two equal mirrors and no losses, is the cavity still impedance matched?

1.5 Gaussian beams (Kogelnik, Li, Applied Optics 5, 1550, (1966))

- paraxial wave equation

  In vacuum, Maxwells equations can be used to derive the Helmholtz equation.

$$\nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = 0$$

  with

$$k = \frac{2\pi}{\lambda}.$$

  Ansatz for the solution.

$$u(\vec{r}) = \Psi(x, y, z) \cdot e^{-ikz}$$

  Ψ is slowly varying in space. The Helmholtz equation then becomes the "paraxial wave equation".

$$\frac{\partial^2}{\partial x^2}\Psi + \frac{\partial}{\partial y^2}\Psi - 2ik\frac{\partial \Psi}{\partial z} = 0.$$

  The term $\sim \frac{\partial^2 \Psi}{\partial^2 z}$ has been neglected. It would by responsible for a fast spatial modulation which is already taken care of in the Ansatz. The remaining changes of Ψ in the z-direction should stay linear. This is the paraxial approximation.

- solution

  The most simple solution (fundamental mode) has the form

$$\Psi(r, z) = e^{-i\left(P(z) + \frac{k}{2q(z)} \cdot r^2\right)}$$

  with the functions $q(z)$ and $P(z)$ obeying the differential equations

$$\frac{dq(z)}{dz} = 1$$

  and

$$\frac{dP(z)}{dz} = -\frac{i}{q(z)}$$

  also

$$r^2 := x^2 + y^2.$$

  From

$$\frac{dq}{dz} = 1$$

  an gets

$$q = q_0 + z$$

  with q_0 can be complex.

We choose the origin such that the real part of q_0 vanishes. q_0 is then complex and we can write

$$q_0 = iz_0$$

with the real valued "Rayleigh-length" z_0. This is the most important parameter of a laser beam

- beam waist, Rayleigh length, confocal parameter

  with $q = iz_0 + z$ the solution Ψ writes

$$\begin{aligned}
\Psi &= e^{-iP(z)} \cdot e^{-i\frac{k \cdot q^*}{2q \cdot q^*} r^2} \\
\Psi &= e^{-iP(z)} \cdot e^{-i\frac{k}{2} \cdot \frac{z+iz_0}{z^2+z_0^2} r^2} \\
\Psi &= e^{-iP(z)-i\frac{k}{2}\frac{r^2 z}{z^2+z_0^2}} \cdot e^{-\frac{r^2}{w^2}}
\end{aligned}$$

with

$$w^2 := \frac{2}{k} \cdot \frac{z^2+z_0^2}{z_0} = \frac{2z_0}{k} \cdot \left(1 + \left(\frac{z}{z_0}\right)^2\right).$$

The field amplitude has an envelope according to a Gaussian $e^{-\frac{r^2}{w^2}}$ with a $\frac{1}{e}$-beam radius w of

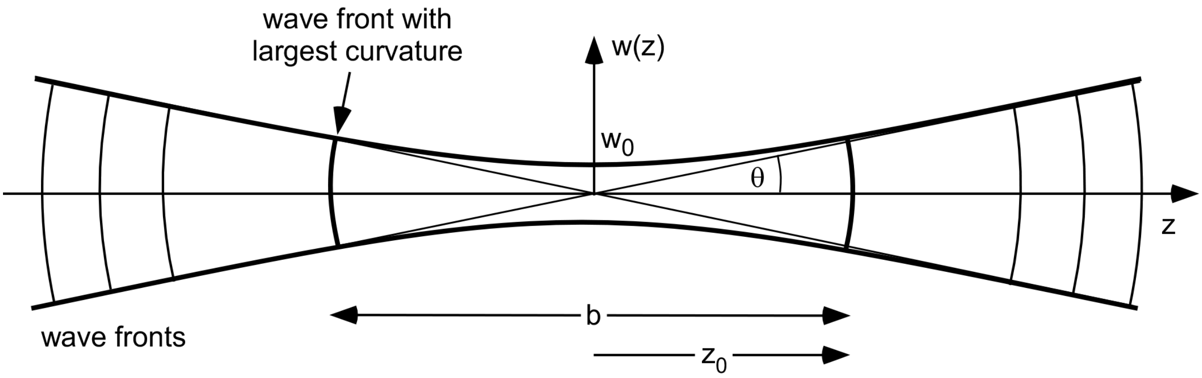
$$w = w_0 \cdot \sqrt{1 + \left(\frac{z}{z_0}\right)^2}.$$

The beam waist w_0 is the radius at $z = 0$:

$$w_0^2 := \frac{2z_0}{k} = \frac{b}{k}$$

The "confocal parameter" is defined as twice the Rayleigh length

$$b := 2z_0.$$



- far field angle

  in the limit $|z| \gg |z_0|$ one gets

$$\begin{aligned} w &\simeq w_0 \cdot \frac{z}{z_0} \\ \frac{w}{z} &= \frac{w_0}{z_0} = \tan\theta \end{aligned}$$

θ is the far field angle

$$\tan\theta = \frac{w_0}{z_0} = \sqrt{\frac{2}{kz_0}} = \sqrt{\frac{\lambda}{\pi \cdot z_0}} = \frac{\lambda}{\pi \cdot w_0}$$

- wave fronts

  The solution of

$$\frac{dP}{dz} = -\frac{i}{q} = -\frac{i}{z + iz_0}$$

  is

$$iP(z) = \ln(1 - i\frac{z}{z_0}),$$

  (check it by taking the derivative). Decomposition it in real and imaginary part yields

$$\begin{aligned} \mathrm{Re}\,(iP(z)) &= \frac{1}{2}\left(\ln(1 - i\frac{z}{z_0}) + \ln(1 + i\frac{z}{z_0})\right) \\ &= \frac{1}{2}\ln\left((1 - i\frac{z}{z_0})(1 + i\frac{z}{z_0})\right) \\ &= \frac{1}{2}\ln\left(1 + (\frac{z}{z_0})^2\right) = \ln\sqrt{1 + (\frac{z}{z_0})^2} \end{aligned}$$

  and

$$\begin{aligned} \mathrm{Im}(iP(z)) &= \frac{1}{2i}\left(\ln(1 - i\frac{z}{z_0}) - \ln(1 + i\frac{z}{z_0})\right) \\ &= \frac{1}{2i}\ln\left(\frac{1 - i\frac{z}{z_0}}{1 + i\frac{z}{z_0}}\right) \\ &= -i\ln\left(\sqrt{\frac{1 - i\frac{z}{z_0}}{1 + i\frac{z}{z_0}}}\right) \\ &= -\arctan\frac{z}{z_0} \end{aligned}$$

  (note that $\arctan x = i\ln\sqrt{\frac{1-ix}{1+ix}}$). As result one obtains

$$iP(z) = \ln\sqrt{1 + (\frac{z}{z_0})^2} - i\arctan(\frac{z}{z_0}).$$

With

$$-i\left(P(z)+\frac{k}{2}\frac{r^2z}{z^2+z_0^2}\right) = -\ln\sqrt{1+(\frac{z}{z_0})^2} +i\left(\arctan(\frac{z}{z_0})-\frac{k}{2}\frac{r^2z}{z^2+z_0^2}\right).$$

the total field

$$u(\vec{r}) = \Psi(x,y,z)\cdot e^{-ikz}$$

can now be written

$$\begin{aligned} u(\vec{r}) &= \Psi(x,y,z)\cdot e^{-ikz} \\ &= \Psi_0 \cdot \underbrace{\frac{1}{\sqrt{1+(\frac{z}{z_0})^2}}}_{\text{conservation of total power of } u^2} \cdot \underbrace{e^{-(\frac{r}{w(z)})^2}}_{\text{envelope}} \cdot \underbrace{e^{-i\left(kz-\arctan(\frac{z}{z_0})\right)}}_{\text{phase factor}} \cdot \underbrace{e^{-i\frac{k}{2}\cdot\frac{r^2}{R(z)}}}_{\text{curved wave fronts}} \end{aligned}$$

With

$$R(z) := z\cdot\left(1+(\frac{z_0}{z})^2\right)$$

We now discuss the different terms:

- wave front curvature

  $R(z)$ is the radius of the wave fronts which can be seen by looking at the position of constant phase

$$\frac{k}{2}\cdot\frac{r^2}{R(z)}+kz+\arctan(\frac{z}{z_0}) = const.$$

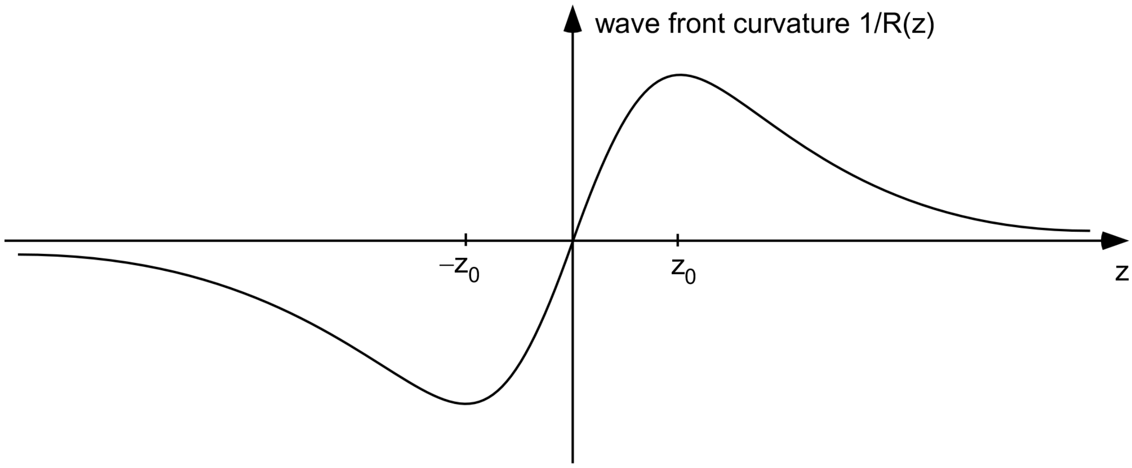
  $\arctan(\frac{z}{z_0})$ depends only slowly on z and can be neglected. One gets

$$z(r) = \frac{const}{k}-\frac{1}{2}\cdot\frac{r^2}{R(z)}.$$

  The wave front runs along a parabola $z(r)\sim r^2$ with a curvature

$$\frac{d^2}{dr^2}z(r) = \frac{1}{R(z)}.$$

  Plotting $1/R\,(z)$ yields:



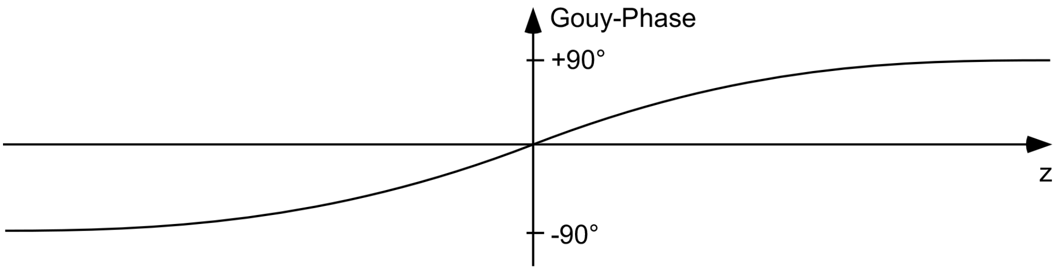
Strongest curvature is at $\pm z_0$.

- Gouy phase

  finally the phase

$$\arctan(\frac{z}{z_0})$$

  is called Gouy phase. While going through the focus there is an overall phase shift of π.



  This phase is fundamentally responsible for the absorption of light by a point like scatterer as for instance an atom. (see also optical theorem)

- transverse modes

  Other solutions are obtained with the Ansatz

$$\Psi = g(\frac{x}{w}) \cdot h(\frac{y}{w}) e^{-i\left(P(z)+\frac{k}{2q(z)}(x^2+y^2)\right)}$$

  g and h are real functions with real variables. The Helmholtz equation is solved if g is one of the functions

$$N_m \cdot H_m\left(\sqrt{2}\frac{x}{w}\right)$$

with

$$\frac{d^2}{dx^2}H_m - 2x\frac{d}{dx}H_m + 2mH_m = 0.$$

and

$$N_m = \frac{1}{\sqrt{\sqrt{\frac{2}{\pi}}w \cdot 2^m m!}}$$

The same holds for h accordingly.

Thus

$$g_m \cdot h_n = N_m \cdot H_m\left(\sqrt{2}\frac{x}{w}\right) \cdot N_n \cdot H_n\left(\sqrt{2}\frac{y}{w}\right).$$

The first few Hermitian polynomials are

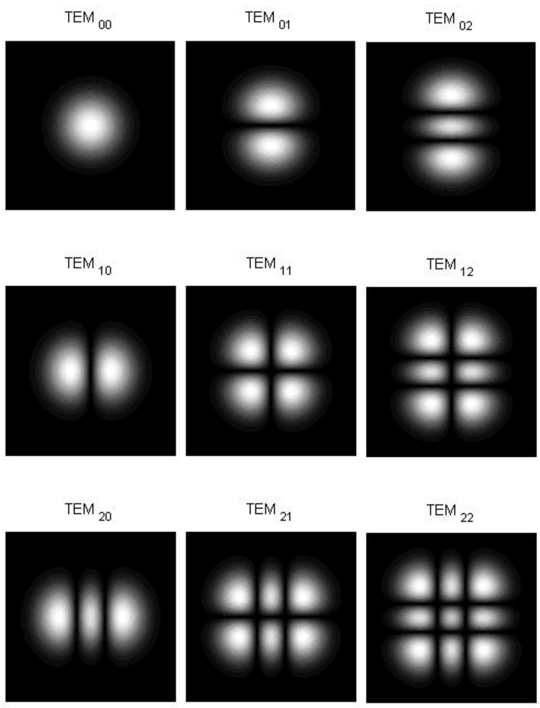
$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \end{aligned}$$

The Gouy phase now reads:

$$\phi(m, n, z) = (m + n + 1)\arctan(\frac{z}{z_0}).$$

Theses solutions are called "transverse electrical modes (TEM$_{nm}$)".

- intensity profiles



- transformation of a Gaussian beam and ABCD-law

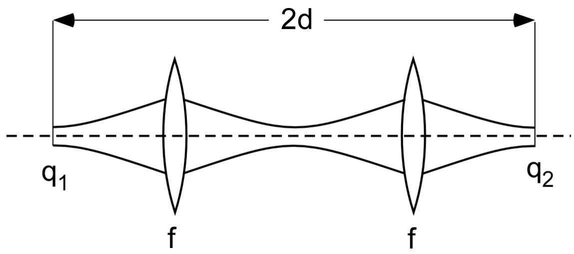
  The q-parameters of a Gaussian beam before (q_1) and after (q_2) an optical system is connected by the ABCD-law

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

  A, B, C, D are the Elements of the above ray matrix M of geometrical optics. A proof is found in A. Siegman: "Laser", University Science Books, 1986, chapter 20.

1.6 Modes of a standing wave resonator

- self-consistency of q.



After one round trip q should reproduce:

$$q = \frac{Aq + B}{Cq + D}$$

with

$$M = \begin{pmatrix} A & B \\ D & C \end{pmatrix}$$

describing the optical system of one round trip. In general q is a complex number

$$q = z + iz_0$$

The real part gives the position of the beam waist, the imaginary part is the Rayleigh length of the beam.

$$z_0 = b/2 = z_0 = \frac{1}{2}kw_0^2$$

The condition for self-consistency yield

$$\begin{aligned} z &= \frac{1}{2}\frac{A-D}{C} \\ z_0 &= \sqrt{-\left(\frac{1}{2}\frac{A-D}{C}\right)^2 - \frac{B}{C}} \\ z_0 &= \sqrt{-z^2 - \frac{B}{C}} \end{aligned}$$

with A, B, C, D containing the geometrical variables such as $r = 2f$ and d.

- Standing wave resonator with two curved mirrors

  The matrix is

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{2} \\ 0 & 1 \end{pmatrix}.$$

  From this one can calculate the matrix elements which are plucked in the general solution above. For two equal mirrors ($f_1 = f_2$) one obtains:

$$\begin{aligned} z_0 &= \frac{1}{2}\sqrt{d(2r-d)} \quad \text{Rayleigh-length} \\ z &= 0 \quad \text{position of the waist} \end{aligned}$$

- stability range

  with the definition

$$g := 1 - \frac{d}{r}$$

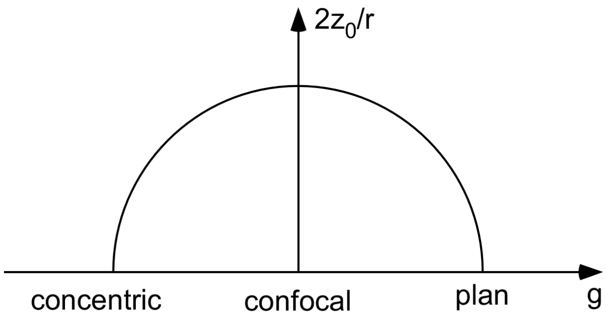
  one may write

$$\begin{aligned} g^2 &= \left(1 - \frac{d}{r}\right)^2 = 1 + \left(\frac{d}{r}\right)^2 - 2\frac{d}{r} \\ &= 1 - \frac{1}{r^2}\left(2rd - d^2\right) \\ &= 1 - \frac{(2z_0)^2}{r^2} \end{aligned}$$

  or

$$g^2 + \left(\frac{2z_0}{r}\right)^2 = 1$$

  g the Rayleigh-length z_0 normalized to the only length scale of the system, $f = r/2$, form half a circle.



- spectrum

  The resonance frequencies are given by the phase that is collected by propagating from one mirror to the other including the Gouy phase

$$\varphi = k \cdot d - \underbrace{2(m+n+1)\arctan(\frac{d/2}{z_0})}_{\text{Gouy-phase}}$$

The phase of a complete round trip, 2φ, must be a multiple of 2π.

$$\begin{aligned} 2\varphi &= k \cdot 2d - 4(m+n+1)\arctan(\frac{d/2}{\frac{1}{2}\sqrt{d(2r-d)}}) = 2\pi q \\ k &= \frac{\pi}{d}q + \frac{2}{d}(m+n+1)\arctan(\frac{d/2}{\frac{1}{2}\sqrt{d(2r-d)}}) \end{aligned}$$

The corresponding frequency in units of the free spectral range $\nu_0 = \frac{c}{2d}$ then reads

$$\frac{\nu(q,m,n)}{\nu_0} = q + \frac{1}{\pi}(m+n+1) \cdot 2\arctan(\frac{1}{\sqrt{\frac{2r}{d} - 1}}).$$

and since

$$\arccos(1-x) = 2\arctan(\frac{1}{\sqrt{\frac{2}{x} - 1}})$$

one finally gets:

$$\frac{\nu(q,m,n)}{\nu_0} = (q+1) + \frac{1}{\pi}(m+n+1)\arccos(1 - \frac{d}{r}).$$

The natural number q counts the longitudinal modes. The numbers m and n count the transverse modes. The additional 1 in the brackets, $(q+1)$, is a matter of convention: The mode with the lowest frequency is here counted as the zeroth longitudinal mode.

By introducing the transverse oscillation frequency

$$\nu_t := \frac{1}{\pi} \arccos(1 - \frac{d}{r})$$

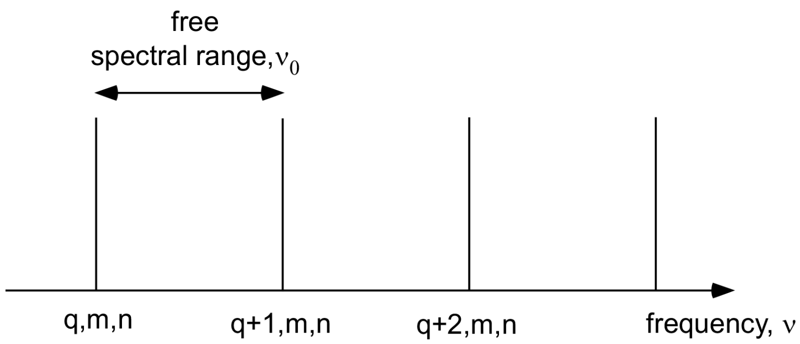
the spectrum has the very simple form

$$\nu\,(q,m,n) = (q+1)\,\nu_0 + (m+n+1)\nu_t.$$

- plane cavity

$$\begin{aligned} \frac{d}{r} &= 0 \\ \arccos(1) &= 0 \\ \frac{\nu(q,m,n)}{\nu_0} &= q+1 \\ \nu_t &= 0 \end{aligned}$$

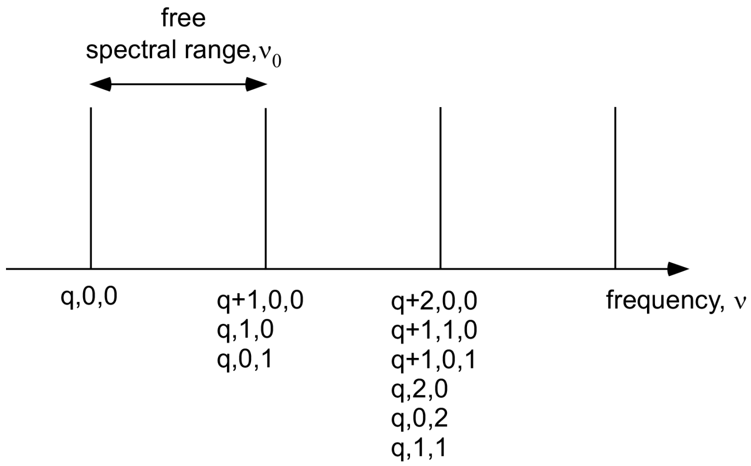
The mode consists only of the near field part of the Gaussian beam. The Gouy phase can be neglected. The transverse frequency is zero. All transverse modes are degenerate.



- concentric resonator

$$\begin{aligned} \frac{d}{r} &= 2 \\ \arccos(-1) &= \pi \\ \frac{\nu(q,m,n)}{\nu_0} &= (q+1) + (m+n+1) \\ \nu_t &= \nu_0 \end{aligned}$$

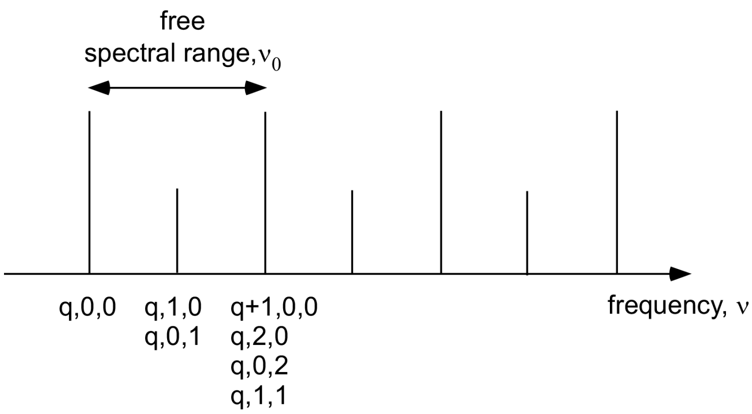
Each transverse mode is degenerate with a longitudinal mode. As in the plane cavity, all modes are degenerate.



- confocal resonator

$$\begin{aligned}
\frac{d}{r} &= 1 \\
\arccos(0) &= \frac{\pi}{2} \\
\frac{\nu(q,m,n)}{\nu_0} &= (q+1) + \frac{1}{2}(m+n+1) \\
\nu_t &= \frac{\nu_0}{2}
\end{aligned}$$

Between two longitudinal modes there is always a transverse mode. In addition, half of the transverse modes are degenerate with a longitudinal mode.



- Interpretation

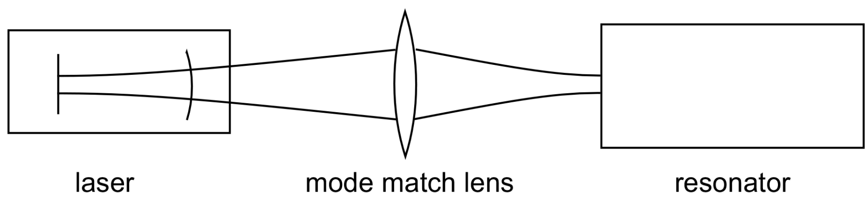
For a fixed q, the transverse modes for the spectrum of a harmonic oscillator of frequency ν_t. The light swings around the optical axis with a frequency that depends on the curvature of the mirrors. Small curvature results in a low frequency, no curvature in a vanishing frequency of the degenerate plane cavity. In the confocal cavity the light bounces twice, i.e. it needs two round trips before it comes back to the initial position: the transverse frequency is twice as large as the free spectral range which gives raise to a semi degenerate spectrum. The concentric resonator both frequencies are the same and the spectrum is fully degenerate.

The mode profile is given by Hermitian polynomials since this is the solution of the quantum harmonic oscillator. For the photons in the resonator Helmholtz equation plays the same role as Schrödinger's equation for a massive particle in a harmonic potential. The resonator is a harmonic trap for the transverse motion of the photon.

1.7 Mode Matching

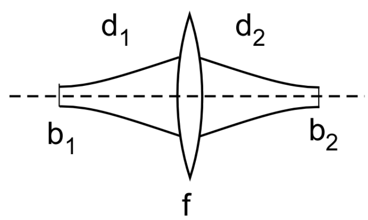
- mode matching

  How can the output mode of a laser (or other source) be transformed such that is matches with the mode of the resonator (or other device)?



- Lens

  Transformation of a Gaussian beam by a lens.



The matrix of the lens

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

the beam parameter of the input beam

$$q_1 = z + iz_0 = z + i\frac{b_1}{2}$$

and that of the output beam

$$q_2 = z_2 + iz_{0,2} = z_2 + i\frac{b_2}{2}$$

are connected by the ABCD-law

$$\begin{aligned}
q_2 &= z_2 + i\frac{b_2}{2} = \frac{Aq_1 + B}{Cq_1 + D} \\
&= \frac{1 \cdot q_1 + 0}{-\frac{1}{f}q_1 + 1} = \frac{z + iz_0}{-\frac{z}{f} - \frac{iz_0}{f} + 1} \\
&= \frac{(z + iz_0)(1 - \frac{z}{f} + i\frac{z_0}{f})}{(1 - \frac{z}{f})^2 + (\frac{z_0}{f})^2} \\
&= \frac{z(1 - \frac{z}{f}) - \frac{z_0^2}{f}}{(1 - \frac{z}{f})^2 + (\frac{z_0}{f})^2} + i\frac{z_0(1 - \frac{z}{f}) + \frac{z_0 z}{f}}{(1 - \frac{z}{f})^2 + (\frac{z_0}{f})^2}.
\end{aligned}$$

This yields an equation for the real and the imaginary part.

$$z_2 = \frac{z(1 - \frac{z}{f}) - \frac{z_0^2}{f}}{(1 - \frac{z}{f})^2 + (\frac{z_0}{f})^2}.$$

$$\frac{b_2}{2} = \frac{z_0(1 - \frac{z}{f}) + \frac{z_0 z}{f}}{(1 - \frac{z}{f})^2 + (\frac{z_0}{f})^2}.$$

- position of the new waist

  We use dimensionless quantities.

$$\begin{aligned}
\tilde{z} &:= \frac{z}{f}, \\
\tilde{q} &:= \frac{q}{f}, \\
\tilde{z}_0 &:= \frac{z_0}{f} \\
\tilde{d}_1 &:= \frac{d_1}{f} \\
\tilde{d}_2 &:= \frac{d_2}{f}
\end{aligned}$$

and obtain

$$\begin{aligned}
\tilde{z}_2 &= \frac{\tilde{z}(1 - \tilde{z}) - \tilde{z}_0^2 + 1 - 1 + \tilde{z} - \tilde{z}}{(1 - \tilde{z})^2 + \tilde{z}_0^2} \\
&= \frac{-(1 - \tilde{z})^2 - \tilde{z}_0^2 + 1 - \tilde{z}}{(1 - \tilde{z})^2 + \tilde{z}_0^2} \\
&= -1 + \frac{1 - \tilde{z}}{(1 - \tilde{z})^2 + \tilde{z}_0^2}.
\end{aligned}$$

- diagram

  As seen from the input beam the lens is at a distance d_1 from the waist.

$$z = d_1$$

As seen from the output beam the lens is at a distance $-d_2$ from the waist.

$$z_2 = -d_2$$

This yields

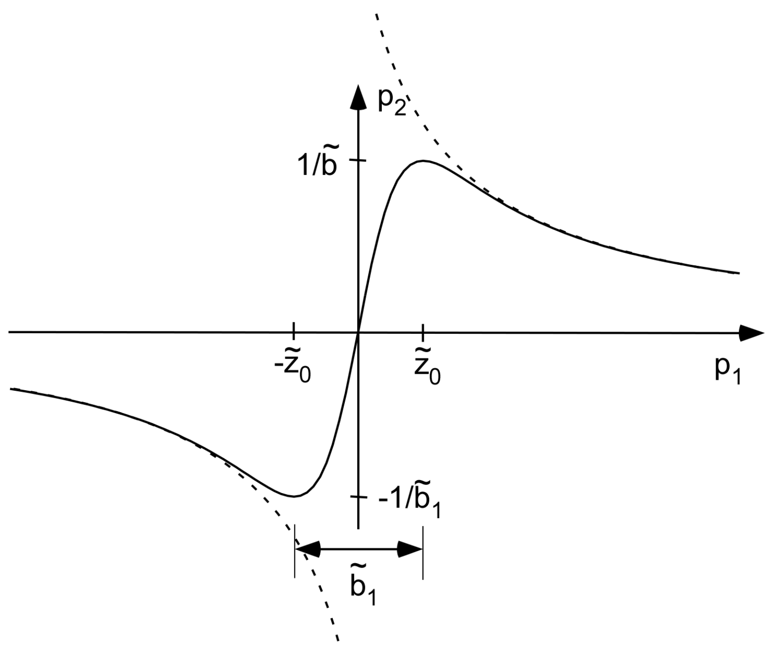
$$1 - \tilde{d}_2 = \frac{1 - \tilde{d}_1}{(1 - \tilde{d}_1)^2 + \tilde{z}_0^2}.$$

Introducing

$$\begin{aligned} p_1 &:= 1 - \tilde{d}_1 \\ p_2 &:= 1 - \tilde{d}_2 \end{aligned}$$

one obtains a dispersive Lorentzian curve with a width $\tilde{b}_1$

$$p_2 = \frac{p_1}{p_1^2 + \tilde{z}_0^2}.$$



At the origin of the diagram $p_1 = 0$ and $p_2 = 0$, i.e. $d_1 = f$ and $d_2 = f$: a waist of the input beam at a distance f from a lens is mapped to a waist of the output beam at the same distance f from the lens

In contrary, geometric optics images an object at f to an image at infinite distance (dashed line)!

The largest possible distance from the output waist to the lens is $\frac{1}{\tilde{b}_1}$.

For a collimated input beam ($p_1 = \infty$) the output is $p_2 = 0$ i.e. $d_2 = f$. This agrees with geometric optics.

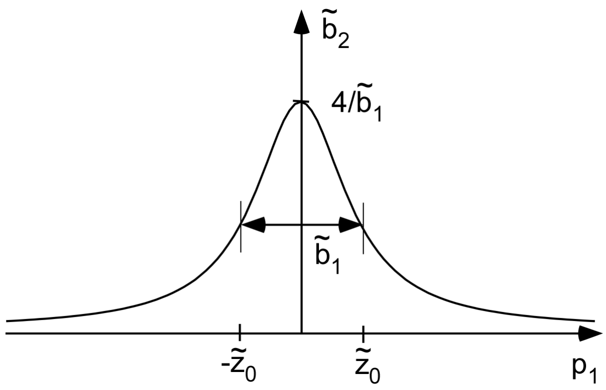
For small waists the transformation approaches the classical limit.

- waist of the output beam

  As above one obtains

$$\tilde{b}_2 = \frac{\tilde{b}_1}{p_1^2 + \tilde{z}_1^2}$$

  That is an absorptive Lorentzian:

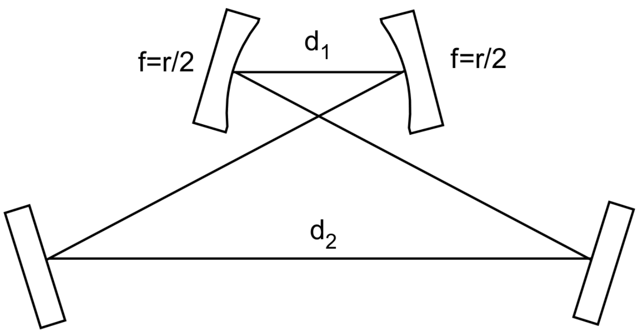


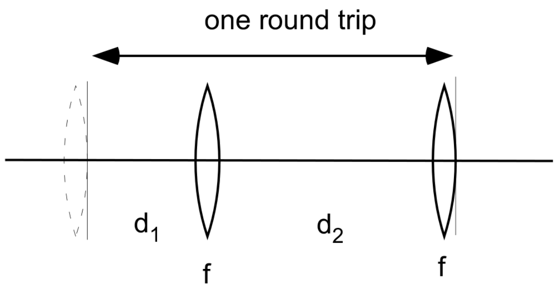
  The largest b_2 (collimated output)) is obtained for $p_1 = 0$ i.e. $d_1 = f$. The value is $b_{2,\max} = \frac{4}{\tilde{b}_1}$.

1.8 Ring resonators

- Bowtie resonator

  Bowtie-resonators are often used for single mode lasers and frequency doublers



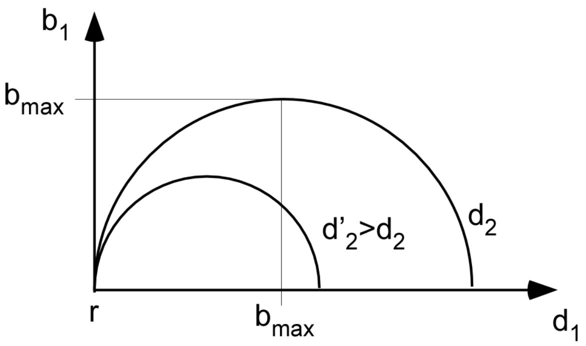


With the definitions

$$g_1 := 1 - \frac{d_1}{r} \qquad g_2 := 1 - \frac{d_2}{r}$$

the solution may be written as (calculation is a very good exercise and not very difficult)

$$\left(\frac{b_1}{r}\right)^2 = -g_1^2 + \frac{g_1}{g_2} \qquad \left(\frac{b_2}{r}\right)^2 = -g_2^2 + \frac{g_2}{g_1}.$$



Again one obtains a semi circle with a radius

$$b_{1\,\max} = \frac{1}{2}\frac{r^2}{d_2 - r}.$$

A strongly focused beam is obtained for $d_2 \gg d_1$. This is useful for applications where a high intensity is needed (laser, nonlinear optics). Similarly focussed beams may be obtained with standing wave cavities only if they are very small, which is often unpractical.

- astigmatism in a ring resonator

  Hitting the mirror under an angle leads to different focal length in x and y direction.

$$2f_x = r_x = \frac{r}{\cos(\theta)}.$$

$$2f_y = r_y = r \cdot \cos(\theta),$$

  with the angle of incidence θ. One may write a Gaussian beam as the product of two Gaussian functions

$$I = I_0 \exp(-2x^2/w_x^2) \exp(-2y^2/w_y^2),$$

  which describe the profile for the two transverse directions. The waists w_x and w_y may be calculated independently as above taking into account the two different effective radii (see also Jenkins and White, Fundamentals of Optics, McGraw-Hill, New York, 1957, page 95).

- Spectrum

  By properly taking into account the Gouy phase one obtains (after some lengthy but simple calculation)

$$\begin{aligned}
\nu(q,m,n) &= q\nu_0 + \left(\frac{1}{2}+m\right)\nu_{tx} + \left(\frac{1}{2}+n\right)\nu_{ty} \\
\nu_{tx} &:= \nu_0 \cdot \frac{\arctan\left(\frac{d_1/2}{z_{1x}}\right) + \arctan\left(\frac{d_2/2}{z_{2x}}\right)}{4\pi} \\
\nu_{ty} &:= \nu_0 \cdot \frac{\arctan\left(\frac{d_1/2}{z_{1y}}\right) + \arctan\left(\frac{d_2/2}{z_{2y}}\right)}{4\pi} \\
\nu_0 &:= \frac{c}{L}
\end{aligned}$$

  with the free spectral range

$$\nu_0 := \frac{c}{L}$$

- geometric phase for odd number of mirrors: transverse modes

  the above spectrum only hold for resonators with a even number of mirrors.

  Each mirror creates the mirror image of the function which describes the profile in the plane of incidence

$$H_n(x) \to H_n(-x).$$

Symmetric transverse mode functions have an even index. They are left unchanged by the mirror.

$$H_{2n}(x) \to H_{2n}(-x) = H_{2n}(x).$$

Mode functions with odd index are antisymmetric and thus change sign.

$$H_{2n+1}(x) \to H_{2n+1}(-x) = -H_{2n+1}(x).$$

The sign change corresponds to an additional phase of π collected during one round trip and thus the spectrum of the odd modes is shifted by half a free spectral range.

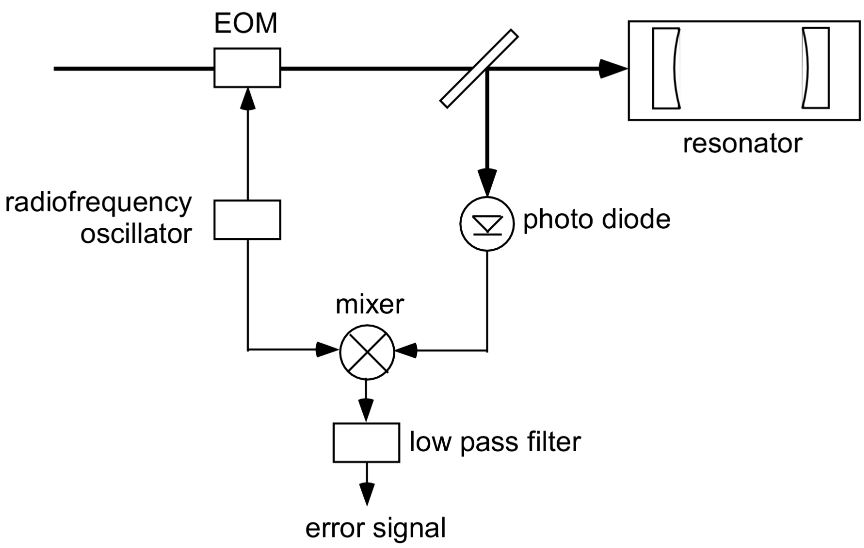
- geometric phase for odd number of mirrors: polarization

  Similarly, light which is linearly polarized in the plane of the resonator suffers a sign change at each reflection. For cavities with an odd number of mirrors the spectrum for this polarization is shifted by half a free spectral range. Such cavities are very good polarization filters.

1.9 Locking a laser to a cavity

- Generating an error signal, setup

  The method of Pound, Drever and Hall (1983) is based on detecting the phase shift of the light reflected from the resonator.



A radio frequency generator drives an electro-optical modulator (EOM) which modulates the phase of the laser light with the frequency Ω $(20 - 100\text{MHz})$. The reflected light is recorded with a photodiode. The electric signal is sent to one of the inputs of a mixer. The output of mixer is proportional to the product of the two input signals. After time averaging with a low pass filter one obtains the desired error signal.

- Generating an error signal, qualitative explanation

  Qualitative explanation: the EOM generates two sidebands frequency shifted from the carrier frequency ω by $\pm\Omega$. If the resonator is replaced by a mirror one cannot observe the modulation with the photo diode. The beat signal between the carrier and the left side band would be 180 $^\circ$ out of phase with the beat signal between the carrier and the right side band and both beat signals exactly cancel. If we replace the resonator and tune the carrier close to resonance the phase of the carrier is shifted by the resonator. The two side bands are far detuned from resonance and their phases stay unchanged. Now the two beat signals do not cancel any more and one obtains an electric signal. The phase of the signal is analyzed by first mix it with the driving oscillator and than filter out harmonics. For a quantitative analysis we first have to understand the spectrum of the light behind the EOM.

- spectrum of a phase modulated light field

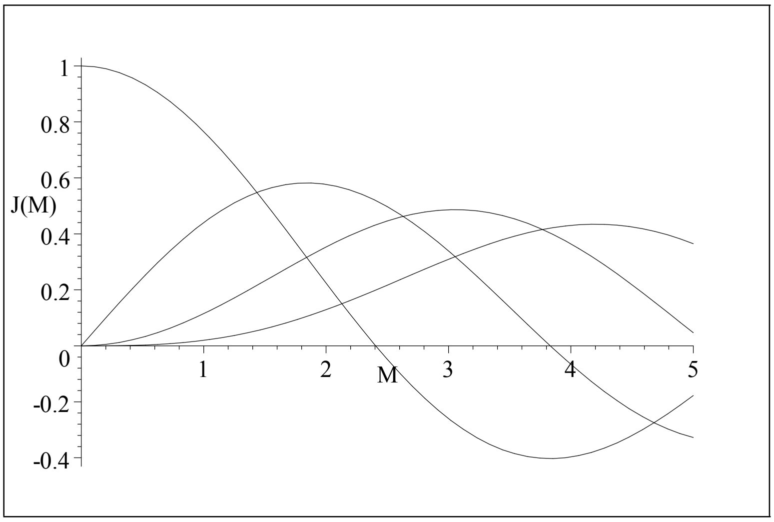
  The modulated field is given in complex writing by

$$E(t) = E_0 \cdot e^{i(\omega t + M\cos\Omega t)}$$

  with the modulation index M. We use the so called Jacobi–Anger identity:

$$e^{i(\omega t + M\cos\Omega t)} = J_0\left(M\right)e^{i\omega t} + 2\sum_{n=1}^{+\infty} i^n \cdot J_n(M) \cdot \cos\left(n\Omega t\right) \cdot e^{i\omega t}.$$

  with the Bessel- functions of first kind $J_n(M)$ (http://mathworld.wolfram.com/BesselFunctionofth



  The physical electric field is the real part:

$$\mathrm{Re}\left(E\left(t\right)\right) = E_0 \cdot \left(J_0(M) \cdot \cos\omega t + \sum_{n=1}^{+\infty} J_n(M)\,\mathrm{Re}\left(e^{i\omega t + in\frac{\pi}{2}}\cos(n\Omega t)\right)\right)$$

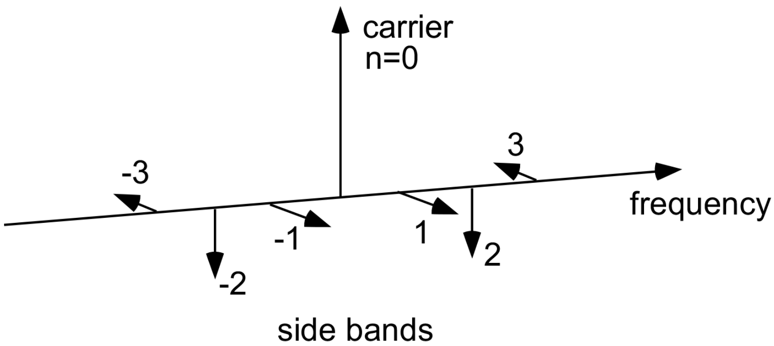
With

$$\begin{aligned}
\mathrm{Re}\left(e^{i\omega t+in\frac{\pi}{2}}\cos(n\Omega t)\right) &= \mathrm{Re}\left(e^{i\omega t+in\frac{\pi}{2}}\frac{1}{2}\left(e^{in\Omega t}+e^{-in\Omega t}\right)\right)\\
&= \frac{1}{2}\mathrm{Re}\left(e^{i(\omega+n\Omega)t+n\frac{\pi}{2}}+e^{i(\omega-n\Omega)t+n\frac{\pi}{2}}\right)\\
&= \frac{1}{2}\cos\left((\omega+n\Omega)\,t+n\frac{\pi}{2}\right)+\frac{1}{2}\cos\left((\omega-n\Omega)\,t+n\frac{\pi}{2}\right)
\end{aligned}$$

on obtains

$$E_0\cdot\left(J_0(M)\cdot\cos\omega t+\frac{1}{2}\sum_{n=1}^{\infty}J_n(M)\left(\cos\left((\omega+n\Omega)\,t+n\frac{\pi}{2}\right)+\cos\left((\omega-n\Omega)\,t+n\frac{\pi}{2}\right)\right)\right)$$

The first term is the carrier. It has the intensity $J_0^2(M)$. The sum contains pairs of side bands of equal intensity $J_n^2(M)$. Their frequency is shifted relative to the carrier by $\pm n\Omega$. The n-th side bands oscillate in phase but shifted relative to the carrier by $n\cdot 90°$.



- generating an error signal, quantitative analysis

  We look at the beat signal between the carrier and the left side band. The phase of the carrier is shifted due to the resonator by φ. The beat signal is proportional to

$$\begin{aligned}
\cos(\omega t+\varphi)\cdot\cos(\omega-\Omega+\pi/2)t &= -\cos(\omega t+\varphi)\cdot\sin(\omega-\Omega)t\\
&= \underbrace{-\frac{1}{2}\sin\left(2\omega t+\varphi-t\Omega\right)}_{\text{vanishes by averaging over one optical cycle}}+\frac{1}{2}\sin\left(\varphi+t\Omega\right)\\
&\simeq \frac{1}{2}\sin(\Omega t+\varphi).
\end{aligned}$$

  The beat signal with the right side band is obtained by replacing $\Omega\to-\Omega$:

$$\begin{aligned}
&-\cos(\omega t+\varphi)\cdot\sin(\omega+\Omega)t\\
\simeq\ &\frac{1}{2}\sin(-\Omega t+\varphi)
\end{aligned}$$

Adding both signals results in

$$\frac{1}{2}\left(\sin(-\Omega t+\varphi)+\sin(\Omega t+\varphi)\right)=\cos(\Omega t)\cdot\sin\varphi$$

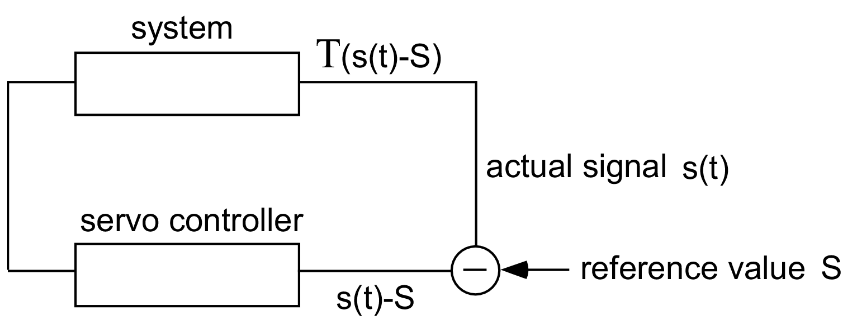
The electric signal from the diode is mixed (i.e. multiplied with $\cos(\Omega t)$) and averages by the low pass filter:

$$\begin{aligned} U &\simeq \frac{1}{T}\int_0^T \cos(\Omega t)\cdot\cos(\Omega t)\cdot\sin\varphi \\ &= \frac{1}{2}\sin\varphi \simeq \frac{1}{2}\varphi \end{aligned}$$

Close to the resonance φ is small and the sin can be approximated by its argument. The signal is proportional to the phase shift of the carried which is antisymmetric with detuning and may thus serve as an error signal.

- servo loops

  A actual time depending signal $s(t)$ is compared to a reference value S and then fed into a an electronic servo controller. Its output acts on the system such that the actual signal equals the reference value. The actual value of the system is measured and gives a new signal $T(s(t)-S)$. In general, the transfer function T can be an operator that acts on the function $s(t)$ and describes the properties of the servo and the system.



  In the steady state one requires that

$$T\cdot(s(t)-S)=s(t).$$

- proportional controller

  The minimum requirement is a negative feed back. The transfer-oparator thus inverts and amplifies the signal, i.e. multiplies it with a factor $-G$:

$$T\cdot s(t)=-G\cdot s(t)$$

The steady state condition now reads:

$$-G \cdot (s(t) - S) = s(t)$$

or

$$s(t) = \frac{G}{1+G} \cdot S.$$

The actual value s equals the reference value S only for infinite gain:

$$\lim_{G \to \infty} \frac{G}{1+G} = 1.$$

- Integral controller

  The transfer operator is now an integrator. It transforms then signal according to:

$$T \cdot s\,(t) = -G \int_0^t s\,(t')\,dt'$$

  with a gain factor G .

  The steady state condition yields

$$s(t) = T(s(t) - S) = -G \cdot \int_0^t (s(t') - S) dt'$$

  or

$$\dot{s} = -G \cdot (s - S).$$

  If the signal is constant in time, $\dot{s} = 0$, it matches the reference value $s = S$ also for finite gain G.

- noise

  If we add noise R to the signal one obtains

$$s(t) = -G \int_0^t (s(t') - S) dt' + R$$

  and

$$\dot{s} = -G(s(t) - S) + \dot{R}$$

  In steady state one obtains

$$s = \frac{\dot{R}}{G} + S.$$

We look at noise with a given frequency ω :

$$\begin{aligned} R(t) &= R(\omega) \cdot e^{i\omega t} \\ \dot{R} &= i\omega \cdot R(\omega) \cdot e^{i\omega t}, \end{aligned}$$

inserting into the steady state solution

$$s(\omega) = i\frac{\omega}{G}R(\omega) + S,$$

i.e. small frequencies are more suppressed than larger frequencies. (1/f-noise).

- complex transfer-function

  We now expand the signal in the base of harmonic oscillations $e^{i\omega t}$

$$s\left(t\right) = \int s\left(\omega\right) e^{i\omega t} d\omega.$$

  We also require that the transfer-operator T is orthogonal in this base such that it obeys an eigenequation:

$$T \cdot e^{i\omega t} = T\left(\omega\right) e^{i\omega t}.$$

  The complex function $T\left(\omega\right)$ is called the "spectrum" of the operator. One can now describe the action of the operator T on the function $s\left(t\right)$ by an integral:

$$\begin{aligned} T \cdot s(t) &= T \cdot \int s\left(\omega\right) e^{i\omega t} d\omega \\ &= \int s\left(\omega\right) T \cdot e^{i\omega t} d\omega \\ &= \int s\left(\omega\right) T\left(\omega\right) e^{i\omega t} d\omega \end{aligned}$$

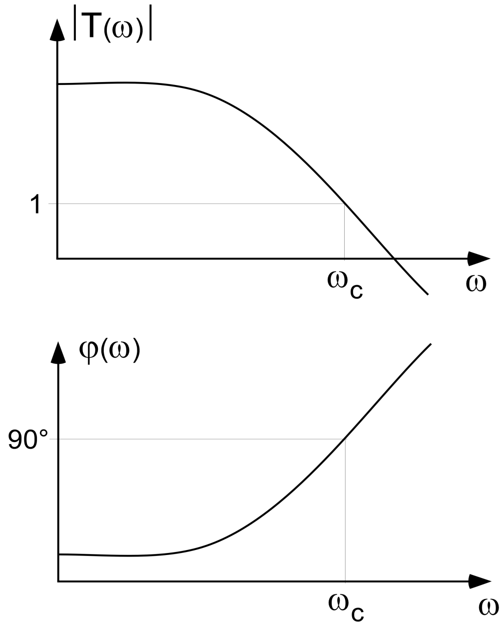
  A linear servo-loop is completely described by its spectrum, i.e. its transfer-function.

- question to think about

  What is the spectrum of an integrator, a proportional controller, or a differentiator? Plot them in a log-log-plot. Determine the slopes. In an ideal servo loop perturbations at all frequencies are compensated with the maximum gain What would be the best controller?

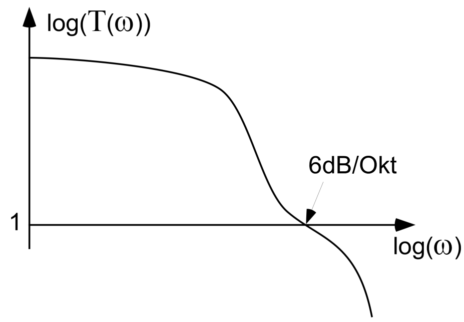
- stability

  A servo controller cannot follow arbitrarily fast perturbations and allways reacts only with some time delay. Each controller thus can be characterized by its critical frequency ω_c where the phase delay exceeds $90°$: $\varphi\left(\omega_c\right) = 90°$. The sign of the real part of the transfer function changes from negative to positive. A periodic perturbation with a frequency above ω_c cannot be compensated any more but will be amplified. This can be avoided if the gain is smaller than 1 for $\omega > \omega_c$: $|T(\omega)| < 1$. The "unity gain" -frequency must be smaller than the critical frequency.

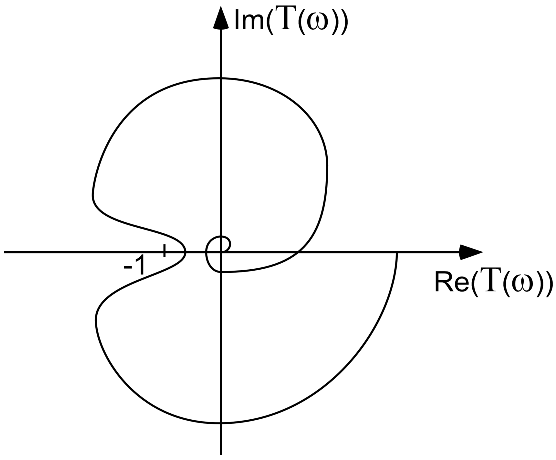


- Nyquist-criterium

  One can show from complex integral analysis that oscillations can be avoided if the slope at unity gain is smaller than $6\frac{dB}{Oct}$.



  The transfer function $T(\omega)$ can be represented by a curve in the complex plain parameterize by the frequency ω. The servo loop is stable if the the point -1 lies outside this curve.



- Question to think about

  A race car is kept on track by the driver. What is the controller and what is the system? What is the actual signal and what is the reference value? Which part of the loop determins the critical frequency? Start thinking about the problem by assuming a race track that resembles a sin curve. Think about strategies to improve the servo loop.