Widening and Narrowing

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Most content comes from http://cs.au.dk/~amoeller/spa/

Interval Analysis

- Compute upper and lower bounds for integers
- Possible applications:
  - array bounds checking
  - integer representation
  - …
- Lattice of intervals:

$$\mathit{Interval} = \mathit{lift}\,(\{\ [\mathrm{l},\mathrm{h}]\ |\ \mathrm{l},\mathrm{h} \in N\ \wedge \mathrm{l}\ \leq \mathrm{h}\ \})$$

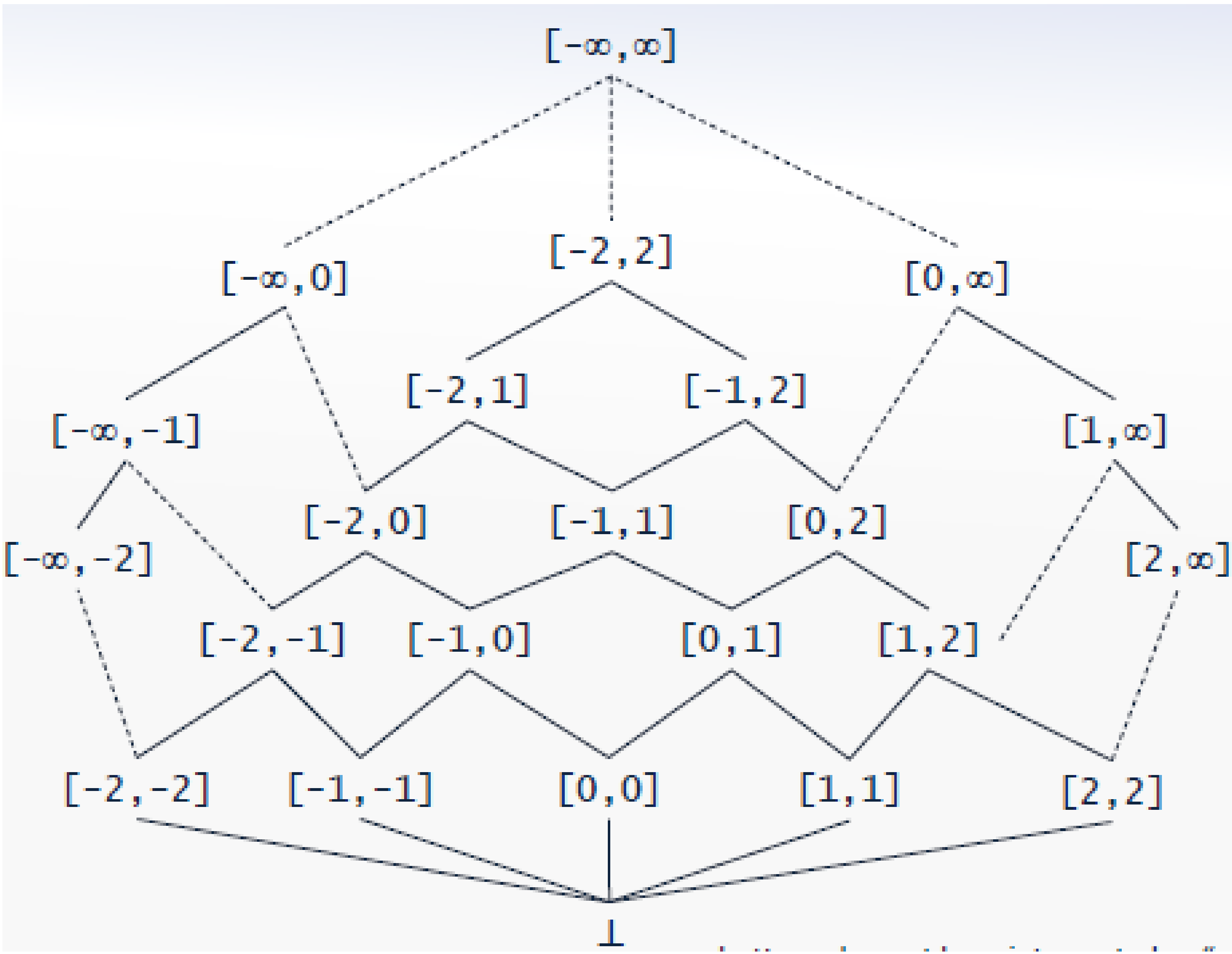
where

N= {-∝, ..., -2, -1, 0, 1, 2, ..., ∝}

and intervals are ordered by inclusion:

$$[l_1,\mathrm{h}_1] \sqsubseteq [l_2,\mathrm{h}_2] \text{ iff } l_2 \leq l_1 \wedge \mathrm{h}_1 \leq \mathrm{h}_2]$$

The Interval Lattice



Interval Analysis Lattice

- The total lattice for a program point is

$$L = Vars \rightarrow Interval$$

that provides bounds for each (integer) variable

- If using the worklist solver that initializes the worklist with only the *entry*node, use the lattice *lift*(L)
  - bottom value of *lift*(L) represents “unreachable program point”
  - bottom value of L represents “maybe reachable, but all variables are non-integers”

- This lattice has *infinite height*, since the chain

$$[0,0] \sqsubseteq [0,1] \sqsubseteq [0,2] \sqsubseteq [0,3] \sqsubseteq [0,4] \ldots$$

occurs in *Interval*

Interval Constraints

- For assignments:

$$[\![x = E]\!] = JOIN(v)[x \rightarrow eval(JOIN(v), E)]$$

- For all other nodes:

$$[\![v]\!] = JOIN(v)$$

Least upper bound

where $$JOIN(v) = \bigsqcup_{w \in pred(v)} [\![w]\!]$$

Evaluating Intervals

- The *eval* functionis an *abstract evaluation*:
  - *eval*(σ, *x*) = σ(*x*)
  - *eval*(σ, *intconst*) = [*intconst*,*intconst*]
  - *eval*(σ, E_1 op E_2) = op(*eval*(σ,E_1),*eval*(σ,E_2))
- Abstract arithmetic operators:

$$\overline{\mathrm{op}}([\, l_1, h_1\,],[\, l_2, h_2\,])$$

$$= [\min_{x\in[l_1,h_1],y\in[l_2,h_2]} x \text{ op } y\,,\ \max_{x\in[l_1,h_1],y\in[l_2,h_2]} x \text{ op } y]$$

Not trivial to implement

- Abstract comparison operators (could be improved):

$$\overline{\mathrm{op}}([\, l_1, h_1\,],[\, l_2, h_2\,]) = [0,1]$$

Fixed-point Problems

- The lattice has infinite height, so the fixed-point algorithm does not work ☹
- In L^n, the sequence of approximants

$$f^i(\bot, \bot, ..., \bot)$$

is not guaranteed to converge

- (Exercise: give an example of a program where this happens)

- Restricting to 32 bit integers is not a practical solution
- *Widening* gives a useful solution …

Widening

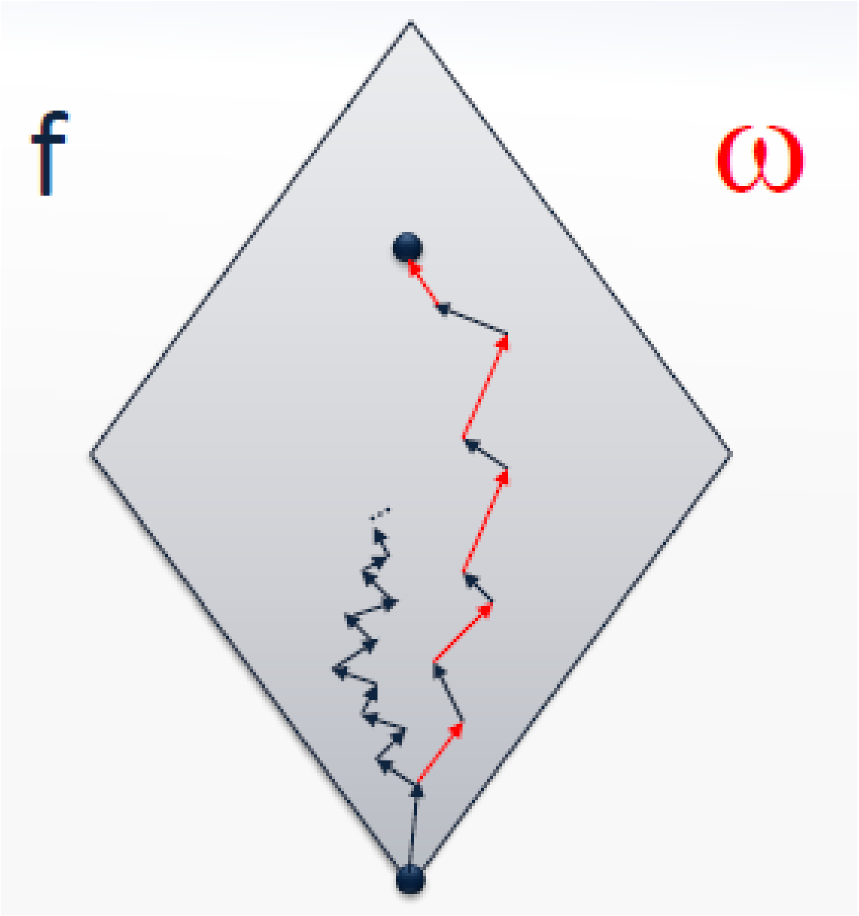
- Introduce a *widening* function ω: $L^n \rightarrow L^n$ so that

$$(\omega \circ f)^i(\bot, \bot, ..., \bot)$$

converges on a fixed-point that is a safe approximation of each $f^i(\bot, \bot, ..., \bot)$

- i.e. the function ω coarsens the information

Turbo Charging the Iterations



Widening for Intervals

- The function ω is defined pointwise on L^n
- Parameterized with a fixed finite subset $B \subset \mathcal{N}$
  - must contain $-\infty$ and ∞ (to retain the T element)
  - typically seeded with all integer constants occurring in the given program
- Idea: Find the nearest enclosing allowed interval
- On single elements from *Interval* :

$$\omega([a, b]) = [\, max\{i \in B \mid i \leq a\},\ min\{i \in B \mid b \leq i\} \,]$$

$$\omega(\perp) = \perp$$

Divergence in Action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [8,8], y \to [0,1]]$
$[x \to [8,8], y \to [0,2]]$
$[x \to [8,8], y \to [0,3]]$
...

在while循环之后的程序点的状态

Divergence in Action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$

8

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of Widening

- Widening works when:
  - ω is an *extensive* and *monotone* function, and
  - $\omega(L)$ is a *finite-height* lattice

- Safety: $\forall i: f^i(\bot, \bot, ..., \bot) \sqsubseteq (\omega \circ f)^i(\bot, \bot, ..., \bot)$
since f is monotone and ω is extensive
- $\omega \circ f$ is a monotone function $\omega(L) \rightarrow \omega(L)$
so the fixed-point exists

Exercise 4.16: A function $f: L \rightarrow L$ where L is a lattice is *extensive* when $\forall x \in L: x \sqsubseteq f(x)$. Assume L is the powerset lattice $2^{\{0,1,2,3,4\}}$ Give examples

- Almost "correct by definition"!
- When used in the worklist algorithm, it suffices to apply widening on back-edges in the CFG

Narrowing

- Widening generally shoots over the target
- *Narrowing* may improve the result by applying f
- Define:

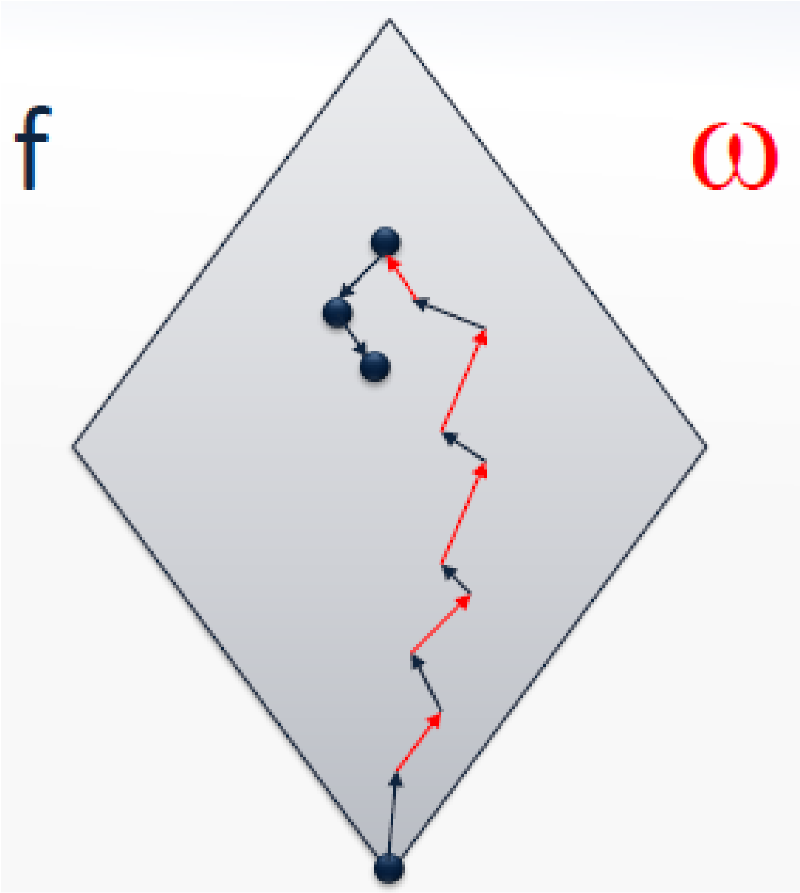
  $fix = \bigsqcup f^{i}(\bot, \bot, ..., \bot)$ $\quad$ $fix\omega = \bigsqcup(\omega \circ f)^{i}(\bot, \bot, ..., \bot)$

  then $fix \sqsubseteq fix\ \omega$
- But we also have that

  $fix \sqsubseteq f(fix\omega) \sqsubseteq fix\ \omega$

  so applying f again may improve the result and remain sound!
- This can be iterated arbitrarily many times
  - may diverge, but safe to stop anytime

Backing up



Narrowing in Action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$

$[x \to [7, \infty], y \to [0, 1]]$

$[x \to [7, \infty], y \to [0, 7]]$

$[x \to [7, \infty], y \to [0, \infty]]$

...

$[x \to [8, 8], y \to [0, \infty]]$

$$B = \{-\infty, 0, 1, 7, \infty\}$$

Correctnessof (Repeated) Narrowing

- $f(\mathit{fix}\omega) \sqsubseteq \omega(f(\mathit{fix}\omega)) = (\omega \circ f)(\mathit{fix}\omega) = \mathit{fix}\omega$ since ω is extensive
  - by induction we also have, for all i:

    $f^{i+1}(\mathit{fix}\omega) \sqsubseteq f^{i}(\mathit{fix}\omega) \sqsubseteq \mathit{fix}\omega$
  - i.e. $f^{i+1}(\mathit{fix}\omega)$ is at least as precise as $f^{i}(\mathit{fix}\omega)$
- $\mathit{fix} \sqsubseteq \mathit{fix}\omega$ hence $f(\mathit{fix}) = \mathit{fix} \sqsubseteq f(\mathit{fix}\omega)$ by monotonicity of f
  - by induction we also have, for all i:

    $\mathit{fix} \sqsubseteq f^{i}(\mathit{fix}\omega)$
  - i.e. $f^{i}(\mathit{fix}\omega)$ is a sound approximation of *fix*

More Powerful Widening

- Defining the widening function based on constants occurring in the given program may not work

```
f(x) { // "McCarthy's 91 function"
  var r;
  if (x > 100) {
    r = x - 10;
  } else {
    r = f(f(x + 11));
  }
  return r;
}
```

https://en.wikipedia.org/wiki/McCarthy_91_function

- Note: this example requires interprocedural analysis…

More Powerful Widening

- A *widening* is a function ∇: $L \times L \rightarrow L$ that is extensive in both arguments and satisfies the following property:

  for all increasing chains $z_0 \sqsubseteq z_1 \sqsubseteq \ldots$,

  the sequence $y_0 = z_0, \ldots, y_{i+1} = y_i \nabla z_{i+1}, \ldots$ converges

  (i.e. stabilizes after a finite number of steps)

- Now replace the basic fixed point solver by computing $x_0 = \bot, \ldots, x_{i+1} = x_i \nabla F(x_i), \ldots$ until convergence

More Powerful Widening for Interval Analysis

- Extrapolates unstable bounds to B:

$$\bot \nabla y = y$$

$$x \nabla \bot = x$$

$$[a_1, b_1] \nabla [a_2, b_2] =$$

$$[\text{if } a_1 \le a_2 \text{ then } a_1 \text{ else } max\{i \in B \mid i \le a_2\},$$

$$\text{if } b_2 \le b_1 \text{ then } b_1 \text{ else } min\{i \in B \mid b_2 \le i\}]$$

The ∇ operator on L is then defined pointwise down individual intervals

For the small example program, we now get the same result as with simple widening plus narrowing (but now without using narrowing)