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Polymorphisms

《程序语言设计和程序分析》

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References

- PFPL
  - Chapter 16 System F of Polymorphic Types
    - System F: polymorphic typed lambda calculus
  - Chapter 17 Abstract Types
  - Chapter 18 Higher kinds
- TAPL (pdf)
  - Chapter 22 Type Reconstruction
  - Chapter 23 Universal Types
  - Chapter 24 Existential Types
- Modules in OCaml (L7-L11 in Cornell CS 3110)
- Stanford CS242 Notes

Outline

- Various polymorphisms
- Polymorphic types: $\forall(t.\tau)$
  - Procedural abstraction
- Data abstraction: existential types $\exists(t.\tau)$
  - Abstract data types, Generic abstractions
- Overloading and type classes
- Subtyping

Various Polymorphisms

Static polymorphism: binding at compile-time

- ***Parametric* polymorphism (参数化多态)**
  - **polymorphism (FPL), templates, generics (OO)**
- ***Ad-hoc* polymorphism**
  - **Overloading: function or operator overloading**
  - **Coercion: implicit type conversion**

Dynamic polymorphism: binding at run-time

- ***Subtyping* (inclusion) polymorphism**
  - **inheritance, virtual function**

(Parametric) Polymorphism

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- **Example: identity function**
  - **write different function for different type:**

$$\lambda(x : \text{int}).x \quad \lambda(x : \text{int} \to \text{int}).x$$

  - **But in OCaml, we can write the function:**

```
let id = fun x -> x
```

id : 'a -> 'a   where 'a is type variable

  - **Typed lambda calculus**
    - For any type α, $\lambda(x{:}\,\alpha).x$
    - Type application:

$$\Lambda(\alpha)\,\lambda(x{:}\,\alpha)x$$

$$(\Lambda(\alpha)\,\lambda(x{:}\,\alpha)x)[\text{int}]$$
$$\mapsto[\text{int}/\alpha]\ (\lambda(x{:}\,\alpha)x) = \lambda(x{:}\,\text{int})x$$

- **Features**
  - **Single algorithm may be given many types**
  - **Type variable may be replaced by any type**

More Examples

- Polymorphism occurs frequently in data structures

```
type 'a tree = Node of 'a tree * 'a * 'a tree | Leaf
let x : int list = [1; 2] in
let y : string list = ["a"; "b"] in
let z : int tree = Node (Leaf, 3, Node(Leaf, 2, Leaf))
```

'a tree is a polymorphic type

- **tree is not a type but a type constructor:**

  takes a type as input and returns a type

  - int tree
  - string tree
  - (int * string) tree
  - …

System F Formal Semantics

System F - Syntax

Typ	τ	$::=$	t	t	variable
			$\mathtt{arr}(\tau_1;\tau_2)$	$\tau_1 \to \tau_2$	function
			$\mathtt{all}(t.\tau)$	$\forall(t.\tau)$	polymorphic
Exp	e	$::=$	x	x	
			$\mathtt{lam}\{\tau\}(x.e)$	$\lambda\,(x:\tau)\,e$	abstraction
			$\mathtt{ap}(e_1;e_2)$	$e_1(e_2)$	application
			$\mathtt{Lam}(t.e)$	$\Lambda(t)\,e$	type abstraction
			$\mathtt{App}\{\tau\}(e)$	$e[\tau]$	type application

Statics

Δ contains **type variables** and Γ contains term variables

System F Formal Semantics

- **Syntax**

$$\text{Typ}\quad \tau \quad ::= \quad \cdots \mid t \mid \forall(t.\tau)$$
$$\text{Exp}\quad e \quad ::= \quad \cdots \mid \Lambda(t)e \mid e[t]$$

- **Statics:**

Well-formed types

$$\frac{}{\Delta, t\ \text{type} \vdash t\ \text{type}} \qquad \frac{\Delta, t\ \text{type} \vdash \tau\ \text{type}}{\Delta \vdash \forall(t.\tau)\ \text{type}}$$

Typing rules of terms

$$\frac{\Delta, t\ \text{type}\ \ \Gamma \vdash e: \tau}{\Delta\ \Gamma \vdash \Lambda(t)e: \forall(t.\tau)} \qquad \frac{\Delta\ \ \Gamma \vdash e: \forall(t.\tau') \qquad \Delta \vdash \tau\ \text{type}}{\Delta\ \Gamma \vdash e[\tau]: [\tau/t]\ \tau'}$$

- **Lemma (Substitution)**
  - **If** $\Delta, t\ \text{type} \vdash \tau'\ \text{type}$**, and** $\Delta \vdash \tau\ \text{type}$**, then** $\Delta \vdash [\tau/t]\tau'\ \text{type}$
  - **If** $\Delta, t\ \text{type} \vdash e': \tau'$**, and** $\Delta \vdash \tau\ \text{type}$ **, then** $\Delta\ [\tau/t]\Gamma \vdash [\tau/t]e': [\tau/t]\tau'$
  - **If** $\Delta\ \Gamma, x: \tau \vdash e': \tau'$**, and** $\Delta\ \Gamma \vdash e: \tau$ **, then** $\Delta\ \Gamma \vdash [e/x]e': \tau'$

Formal Semantics

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$$\text{Typ } \tau ::= \cdots \mid t \mid \forall(t.\tau)$$
$$\text{Exp } e ::= \cdots \mid \Lambda(t)e \mid e[t]$$

- Syntax
- Statics
- Dynamics

$$\frac{}{\Lambda(t)e \ \text{val}} \qquad \frac{e \mapsto e'}{e[\tau] \mapsto e'[\tau]} \qquad \frac{}{(\Lambda(t)e)[\tau] \mapsto [\tau/t]e}$$

- **Lemma (Canonical Forms)**

  If $e{:}\,\tau$ and e val

  - If $\tau = \tau_1 \to \tau_2$, then $e = \lambda(x{:}\,\tau_1)e_2$ with $x{:}\,\tau_1 \vdash e_2{:}\,\tau_2$
  - If $\tau = \forall(t.\tau')$, then $e = \Lambda(t)\, e'$ with t type $\vdash e'{:}\,\tau'$

- **Theorem (Safety)**
  - Preservation: If $e{:}\,\tau$ and $e \to e'$, then $e'{:}\,\tau$
  - Progress: If $e{:}\,\tau$, then either e val or there exists e' such that $e \to e'$

Example

- Polymorphic composition function

- Polymorphic composition function type

Example

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Polymorphic composition function

$$\Lambda(t_1)\Lambda(t_2)\Lambda(t_3)\lambda(f: t_2 \rightarrow t_3)\lambda(g: t_1 \rightarrow t_2)\lambda(x: t_1)\ f(g(x))$$

Polymorphic composition function type

$$\forall(t_1.\forall(t_2.\forall(t_3.(t_2 \rightarrow t_3) \rightarrow (t_1 \rightarrow t_2) \rightarrow t_1 \rightarrow t_3)))$$
$$= \forall(t_1.\forall(t_2.\forall(t_3.(t_2 \rightarrow t_3) \rightarrow (t_1 \rightarrow t_2) \rightarrow (t_1 \rightarrow t_3))))$$

Data Abstraction

- Interface
  - A contract between the client and the implementor
- Implementations
  - Satisfy the contract
  - One implementation can be replaced by another without affecting the behavior of the client

Data abstraction is formalized by

extending System F with *existential types*

- Interfaces: *existential types* that provide a collection of operations acting on abstract type
- Implementations: packages, the introduction form of existential types

Modules in OCaml

Different implementations of Counter

```
module IntCounter = struct
    type t = int
    let make (n : int) : t = n
    let incr (ctr : t) (n : int) : t = ctr + n
    let get (ctr : t) : int = ctr
end
```

```
let ctr : IntCounter.t = IntCounter.make 3 in

let ctr : IntCounter.t = IntCounter.incr ctr 5 in

assert((IntCounter.get ctr) = 8);

assert(ctr = 8)
```

IntCounter.t = int
make: int → int
incr:int → int → int
get: int → int

```
module RecordCounter = struct
    type t = { x: int }
    let make (n : int) : t = {x = n}
    let incr (ctr : t) (n : int) : t = {x = ctr.x + n}
    let get (ctr : t) : int = ctr.x
end
```

RecorderCounter.t={x:int}
make: int → {x:int}
incr:{x:int} → int → {x:int}
get:{x:int}→ int

Modules in OCaml

Interfaces

- **module signature**

```
module type Counter =
sig
    type t
    val make : int -> t
    val incr : t -> int -> t
    val get : t -> int
end
```

Implementations

- **modules**

```
module IntCounter : Counter = struct
    type t = int
    let make (n : int) : t = n
    let incr (ctr : t) (n : int) : t = ctr + n
    let get (ctr : t) : int = ctr
end
```

```
module RecordCounter  : Counter= struct
    type t = { x: int }
    let make (n : int) : t = {x = n}
    let incr (ctr : t) (n : int) : t = {x = ctr.x + n}
    let get (ctr : t) : int = ctr.x
end
```

Modules in OCaml

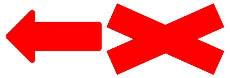
- Interfaces
  - module signature

```
module type Counter = sig
    type t
    val make : int -> t
    val incr : t -> int -> t
    val get : t -> int
end
```

- Implementations
  - modules

```
module IntCounter  : Counter= struct
    type t = int
    let make (n : int) : t = n
    let incr (ctr : t) (n : int) : t = ctr + n
    let get (ctr : t) : int = ctr
end
```

```
let ctr : Counter.t = IntCounter.make 3 in
let ctr : Counter.t = IntCounter.incr ctr 5 in
assert((IntCounter.get ctr) = 8);
assert(ctr = 8)
```



Packing of a Package

IntCounter can be represented as

$\textbf{pack}\ \ \text{int}\ \ \textbf{with}$

$\langle \text{make} \hookrightarrow \lambda(n{:}\,\text{int})n, \text{incr} \hookrightarrow \lambda(c{:}\,\text{int})\ \lambda(n{:}\,\text{int})c + n, \text{get} \hookrightarrow \lambda(c{:}\,\text{int})c \rangle$

$\textbf{as}\ \exists t. \langle \text{make} \hookrightarrow \text{int} \rightarrow t, \text{incr} \hookrightarrow t \rightarrow \text{int} \rightarrow t, \text{get} \hookrightarrow t \rightarrow \text{int} \rangle$

***packing* of a “package” -- introduction form**

- **Package**
  - **Implementation: the second term in the curly braces**

    $\langle \text{make} \hookrightarrow \lambda(n{:}\,\text{int})n, \text{incr} \hookrightarrow \lambda(c{:}\,\text{int})\ \lambda(n{:}\,\text{int})c + n, \text{get} \hookrightarrow \lambda(c{:}\,\text{int})c \rangle$
  - **Interface: the type after as keyword**

    $\exists t. \langle \text{make} \hookrightarrow \text{int} \rightarrow t, \text{incr} \hookrightarrow t \rightarrow \text{int} \rightarrow t, \text{get} \hookrightarrow t \rightarrow \text{int} \rangle$
  - **Abstracted type: the first term in the curly braces, i.e. int**

Unpacking: eliminating a package

Unpack: enable a client to use a package

$$\textbf{open}$$
$$(\textbf{pack}\ \ \text{int}\ \ \textbf{with}$$
$$\langle \text{make} \hookrightarrow \lambda(n{:}\,\text{int})n, \text{incr} \hookrightarrow \lambda(c{:}\,\text{int})\ \lambda(n{:}\,\text{int})c + n, \text{get} \hookrightarrow \lambda(c{:}\,\text{int})c\rangle$$
$$\textbf{as}\ \exists t.\langle \text{make} \hookrightarrow \text{int} \to t, \text{incr} \hookrightarrow t \to \text{int} \to t, \text{get} \hookrightarrow t \to \text{int}\rangle)$$
$$\textbf{as}\ t\ \textbf{with}\ x{:}\,\tau$$
$$\textbf{in}\ \ \text{let}\ c{:}\,t = x.\text{make}\ 3\ \ \text{in let}\ c{:}\,t = x.\text{incr}\ c\ 3\ \text{in}\ \ x.\text{get}\ c$$

$\textbf{open}\ e_1\ \textbf{as}\ t\ \textbf{with}\ x{:}\,\tau\ \textbf{in}\ e_2$

打开一个包e_1，将其表示类型int绑定到t，将其实现绑定到x，以便在客户端 e_2中使用

```
let ctr : IntCounter.t = IntCounter.make 3 in
let ctr : IntCounter.t = IntCounter.incr ctr 5 in
IntCounter.get ctr
```

Syntax

Typ	τ	$::=$	$\mathtt{some}(t.\tau)$	$\exists(t.\tau)$	interface
Exp	e	$::=$	$\mathtt{pack}\{t.\tau\}\{\rho\}(e)$	$\mathtt{pack}\,\rho\,\mathtt{with}\,e\,\mathtt{as}\,\exists(t.\tau)$	implementation
			$\mathtt{open}\{t.\tau\}\{\rho\}(e_1;t,x.e_2)$	$\mathtt{open}\,e_1\,\mathtt{as}\,t\,\mathtt{with}\,x{:}\tau\,\mathtt{in}\,e_2$	client

pack ρ **with** e **as** $\exists(t.\tau)$

把表示类型ρ 和实现e打成满足接口$\exists(t.\tau)$的一个包，t为抽象类型

open e_1 **as** t **with** $x:\tau$ **in** e_2

打开一个包e_1，将其表示类型int绑定到t，将其实现绑定到x，以便在客户端 e_2中使用

FE $\lambda^{\rightarrow,\forall,\exists}$ Formal Semantics

☐ Syntax

$\text{some}(t.\tau)$	$\exists(t.\tau)$
$\text{pack}\{t.\tau\}\{\rho\}(e)$	**pack** ρ **with** e **as** $\exists(t.\tau)$
$\text{open}\{t.\tau\}\{\tau_2\}(e_1;t;x.e_2)$	**open** e_1 **as** t **with** $x:\tau$ **in** e_2

☐ Statics

Well-formed types

$$\frac{\Delta, t\ \text{type} \vdash \tau\ \text{type}}{\Delta \vdash \text{some}(t.\tau)\ \text{type}}$$

Typing rules of terms

- $$\frac{\Delta \vdash \rho\ \text{type} \quad \Delta, t\ \text{type} \vdash \tau\ \text{type} \quad \Delta\Gamma \vdash e: [\rho/t]\tau}{\Delta\ \Gamma \vdash \text{pack}\{t.\tau\}\{\rho\}(e): \text{some}(t.\tau)}$$

$$\frac{\Delta\ \ \Gamma \vdash e_1: \text{some}(t.\tau) \quad \Delta, t\ \text{type}\ \ \Gamma, x:\tau \vdash e_2:\tau_2 \quad \Delta \vdash \tau_2\ \text{type}}{\Delta\ \Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_1;t;x.e_2):\tau_2}$$

τ_2是client代码的结果类型

FE $\lambda^{\rightarrow,\forall,\exists}$ Formal Semantics

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- Syntax

$\text{some}(t.\tau)$	$\exists(t.\tau)$
$\text{pack}\{t.\tau\}\{\rho\}(e)$	**pack** ρ **with** e **as** $\exists(t.\tau)$
$\text{open}\{t.\tau\}\{\tau_2\}(e_1; t; x.e_2)$	**open** e_1 **as** t **with** $x{:}\tau$ **in** e_2

- Statics
- Dynamics

$$\frac{[e\ \text{val}]}{\text{pack}\{t.\tau\}\{\rho\}(e)\ \text{val}}$$

$$\left[\frac{e \mapsto e'}{\text{pack}\{t.\tau\}\{\rho\}(e) \mapsto \text{pack}\{t.\tau\}\{\rho\}(e')}\right]$$

$$\frac{e_1 \mapsto e_1'}{\text{open}\{t.\tau\}\{\tau_2\}(e_1; t; x.e_2) \mapsto \text{open}\{t.\tau\}\{\tau_2\}(e_1'; t; x.e_2)}$$

$$\frac{[e\ \text{val}]}{\text{open}\{t.\tau\}\{\tau_2\}(\text{pack}\{t.\tau\}\{\rho\}(e); t; x.e_2) \mapsto [\rho, e/t, x]e_2}$$

*Definability of Existential Types

- 引入存在类型的原因：对数据抽象进行建模
- 存在类型可由多态类型（全称类型）定义
  - 如下的client代码 e_2 是以表示类型 X 为参数的多态函数

$$\text{open}\{t.\tau\}\{\tau_2\}(e_1; t; x.e_2)$$

$$\frac{\Delta\ \Gamma \vdash e_1 : \text{some}(t.\tau) \quad \Delta, t \text{ type}\ \ \Gamma, x:\tau \vdash e_2 : \tau_2 \quad \Delta \vdash \tau_2 \text{ type}}{\Delta\ \Gamma \vdash \text{open}\{t.\tau\}\{\tau_2\}(e_1; t; x.e_2) : \tau_2}$$

    - $e_1 : \exists(t.\tau)$ 是一个package (具体实现)，$e_2 : \tau_2$是client代码
    - Client代码本质上是类型为$\forall(t.\tau \to \tau_2)$ 的多态函数， t 可能出现在 τ 中，但是不会出现在τ_2中
  - 存在类型是一个多态函数类型

$$\exists(t.\tau) = \forall(u.\forall(t.\tau \to u) \to u)$$

*Definability of Existential Types

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□ 存在类型可由多态类型（全称类型）定义

■ 存在类型是一个多态函数类型$\exists(t.\tau) = \forall(u.\forall(t.\tau \to u) \to u)$

打包

pack ρ **with** e **as** $\exists(t.\tau)$相当于$\Lambda(u)\lambda(x{:}\forall(t.\tau \to u))x[\rho](e)$

由表示类型 ρ 和实现 t 组成的包是一个多态函数，

该函数在给定结果类型 u 和 client代码 x 时，用 表示类型 ρ 实例化 t ，再将实现 e传递到其中（$x[\rho]$）.

解包

open e_1 **as** t **with** $x{:}\tau$ **in** e_2 相当于$e_1[\tau_2](\Lambda(t)\lambda(x{:}\tau)\ e_2)$

将client代码 e_2 打包成一个多态函数 $\forall(t.\tau \to u)$，

将存在类型的结果类型（即$\forall(u.\forall(t.\tau \to u) \to u)$ 中的 u ）实例化为τ_2，

再将$e_1[\tau_2]$ 应用到多态client程序 e_2 上

e_1最终为一个pack值，即为一个多态函数

$$\Lambda(u)\lambda(x{:}\forall(t.\tau \to u))x[\rho](e)$$

Type Quantification is not sufficient

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- Quantification over types
  - $\forall(t.\tau)$ models generics
  - $\exists(t.\tau)$ models abstraction

⇒ Not sufficient to model many programming situations of practical interest.

- Examples (not just type quantification)
  - Abstract families of types
    - e.g. τ list An infinite collection types sharing a common collection of operations on them
  - Interrelated abstract types
    - e.g. a type of trees whose nodes have a forest of child and a type of forests whose elements are trees

*Constructors and Kinds

- **Quantification over kinds, than just types, e.g. over**
  - **type constructors: functions mapping types to types**
  - **type structures: tuples of types**
- **Kinds: classifying constructors**
  - **Static layer: use kinds to classify constructors**
  - **Dynamic layer: use types to classify expressions(terms)**

The two-layer architecture models *phase distinction.*

- Constructors are the static data of the language.
- Expressions (terms) are the dynamic data of the language.

F_ω $\lambda^{\to,\forall_k,\exists_k}$ Grammar

Kind	κ	::=	Type	T	types
			Unit	1	nullary product
			Prod(κ_1; κ_2)	$\kappa_1 \times \kappa_2$	binary product
			Arr(κ_1; κ_2)	$\kappa_1 \to \kappa_2$	function

Con	c	::=	u	u	variable
			arr	$\to$	function constructor
			all{κ}	$\forall_\kappa$	universal quantifier
			some{κ}	$\exists_\kappa$	existential quantifier
			proj[l](c)	$c \cdot \mathtt{l}$	first projection
			proj[r](c)	$c \cdot \mathtt{r}$	second projection
			app(c_1; c_2)	$c_1[c_2]$	application
			unit	$\langle\rangle$	null tuple
			pair(c_1; c_2)	$\langle c_1, c_2 \rangle$	pair
			lam($u.c$)	$\lambda(u)\, c$	abstraction

*Constructors and Kinds

- More details are not discussed in this course.
- But if you are interest, you need further learn to understand:
  - More judgements specifying static semantics of constructors and kinds
    - Constructor / type /expression formation
  - Rules for constructor formation
  - Substitution lemma
  - …

*Modularity

- References: [PFPL Chapters 42-44]
- Syntax is divided into more levels
  - Expressions classified by types
  - Constructors classified by kinds
  - Modules classified by signatures

Summary: Generic Abstractions

- Parameterize modules by types
- Create general implementations
  - Can be instantiated in many ways
- Language examples
  - Ada generic packages
  - C++ templates, e.g. C++ Standard Template Library(STL)
  - ML functors
  - …

THANKS