Coq Introduction

张昱

Department of Computer Science and Technology
University of Science and Technology of China

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Coq Introduction

- A proof assistant for a logical framework known as the ***C***alculus of ***I***nductive ***C***onstructions.
- Allows:
  - to define functions or predicates,
  - to state mathematical theorems and software specifications,
  - to develop interactively formal proofs of these theorems,
  - to check these proofs by a relatively small certification "kernel".
- Version
  - The current stable version of Coq is the 8.1. It is available for Unix and Windows 95/98/NT/XP systems.

Useful Links

- Coq Home page
  http://coq.inria.fr/
  - The Coq Proof Assistant – A Tutorial v8.1
  - The Coq Proof Assistant – Reference Manual v8.1
- Coq Art Home page
  http://www.labri.fr/perso/casteran/CoqArt/index.html
- Coq Tutorial in POPL08
  http://www.cis.upenn.edu/~plclub/popl08-tutorial/

Coq Tools-1

- **Coq IDE**

  http://coq.inria.fr/distrib-eng.html

- **Proof General (PG)**

  http://proofgeneral.inf.ed.ac.uk/

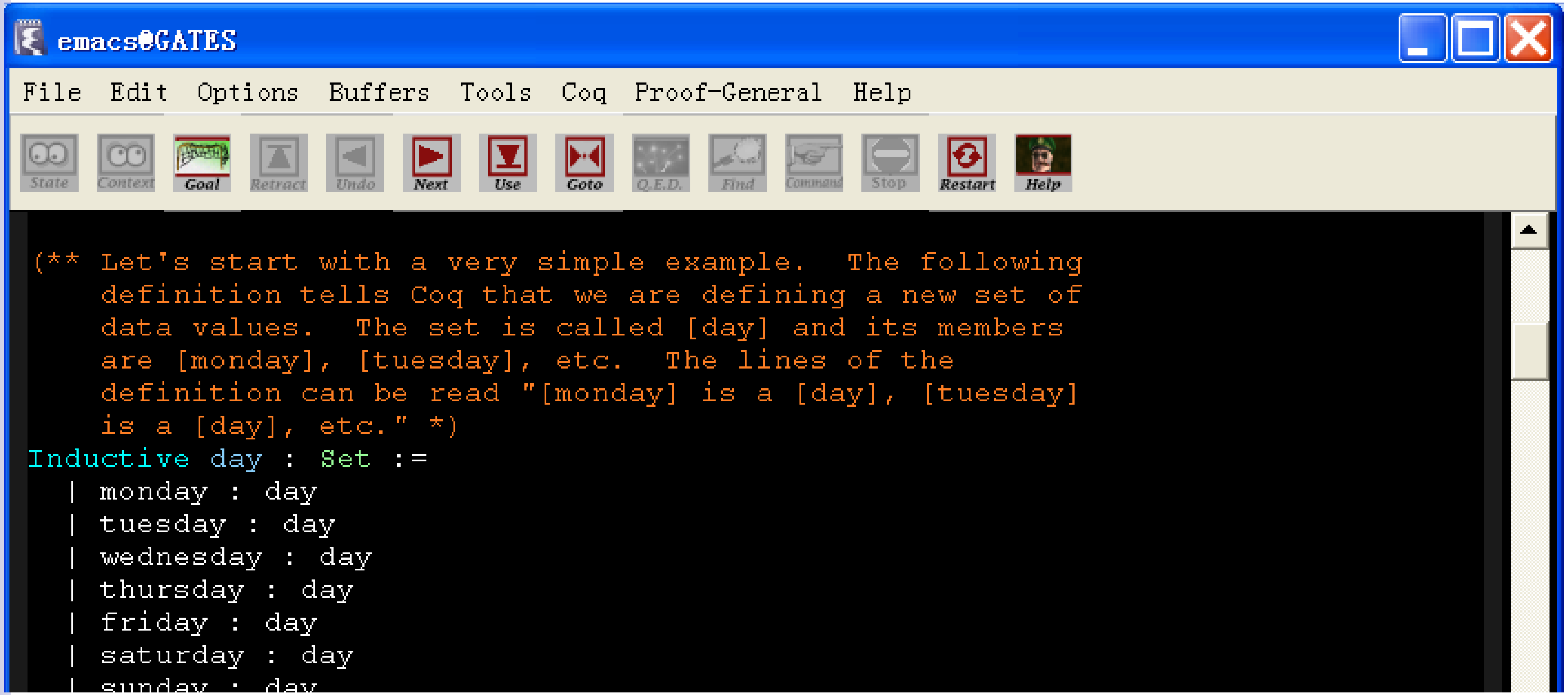
  - a generic front-end for *proof assistants* (also known as *interactive theorem provers*), based on the customizable text editor Emacs

Coq Tools-2

To get started ...(e.g. using PG and for Windows)

- Install Coq
  Notice: don't select installing coqide and GDK
- Install Emacs & PG
- Modify the environment variables
  - Add
    - COQBIN = "(coq install dir)\coq\bin"
    - COQLIB = "(coq install dir)\coq\lib"
    - HOME ="(where PG is installed)"
  - Append the bin dir of coq and emacs to the value of Path
- Type coqtop.opt in command line to check …
- Run Emacs/bin/runemacs.exe

Coq Tools-3


emacs@GATES
File Edit Options Buffers Tools Coq Proof-General Help
State Context Goal Retract Undo Next Use Goto Q.E.D. Find Command Stop Restart Help
(** Let's start with a very simple example. The following
definition tells Coq that we are defining a new set of
data values. The set is called [day] and its members
are [monday], [tuesday], etc. The lines of the
definition can be read "[monday] is a [day], [tuesday]
is a [day], etc." *)
Inductive day : Set :=
| monday : day
| tuesday : day
| wednesday : day
| thursday : day
| friday : day
| saturday : day

What does Coq system provide?

- **A specification language named Gallina**
  - consists in a sequence of *declarations* and *definitions*.
  - Its terms can represent programs as well as properties of these programs and proofs of these properties.
  - Using Curry-Howard isomorphism, programs, properties and proofs are formalized in the same language called CiC, that is a λ-calculus with a rich type system.
  - All logical judgments in COQ are typing judgments.
- **Type-checker**
  - checks the correctness of proofs, in other words that checks that a program complies to its specification.
- **The proof engine**
  - provides an interactive proof assistant to build proofs using specific programs called tactics.

Gallina - Declarations

❖ Declarations

- **A declaration associates a name with a specification.**
  - Declared objects play the role of axioms or parameters in mathematics.

    $\textbf{Axiom}\ ident : term$

    $\textbf{Parameter}\ ident_1 \cdots ident_n : term$
- **Specifications**
  - logical propositions: Prop
  - mathematical collections: Set
  - abstract types: Type
- **Every valid expression e in Gallina is associated with a specification, itself a valid expression, called its type τ(E).** $e : \tau(E)$

Gallina - Inductive Definitions

❖ Inductive Definitions

- **Simple inductive types**

$$\textbf{Inductive}\ ident : sort :=$$
$$\mid\ ident_1 : type_1$$
$$\mid\ \ldots$$
$$\mid\ ident_n : type_n.$$

ident: the name of the inductively defined type

sort : the universes where it lives

$ident_1, \ldots, ident_n$: the names of its constructors

$type_1, \ldots, type_n$: the types of its constructors

Gallina - Inductive Definitions

- **Simple inductive types**

```
Inductive yesno : Set :=
  | yes : yesno
  | no : yesno.
```

defines the following four objects at once:

```
yesno is defined
yesno_rect is defined
yesno_ind is defined
yesno_rec is defined
```

```
Check yesno.
```

```
yesno
     : Set
```

A new Set is declared, with name yesno.

Gallina - Inductive Definitions

- **Simple inductive types**

```
Inductive yesno : Set :=
  | yes : yesno
  | no : yesno.
```

defines the following four objects at once:

```
yesno is defined
yesno_rect is defined
yesno_ind is defined
yesno_rec is defined
```

```
Check yesno_rect.
```

```
yesno_rect
     : forall P : yesno -> Type, P yes -> P no -> forall y : yesno, P y
```

Destructor (Elimination principle on Type): expresses structural induction/recursion principle over objects of yesno.

Gallina - Inductive Definitions

- **Simple inductive types**

```
Inductive yesno : Set :=
  | yes : yesno
  | no : yesno.
```

defines the following four objects at once:

```
yesno is defined
yesno_rect is defined
yesno_ind is defined
yesno_rec is defined
```

```
Check yesno_ind.
```

```
yesno_ind
     : forall P : yesno -> Prop, P yes -> P no -> forall y : yesno, P y
```

Destructor (Elimination principle on Prop): expresses structural induction/recursion principle over objects of yesno.

Gallina - Inductive Definitions

- **Simple inductive types**

```
Inductive yesno : Set :=
  | yes : yesno
  | no : yesno.
```

defines the following four objects at once:

```
yesno is defined
yesno_rect is defined
yesno_ind is defined
yesno_rec is defined
```

```
Check yesno_rec.
```

```
yesno_rec
     : forall P : yesno -> Set, P yes -> P no -> forall y : yesno, P y
```

Destructor (Elimination principle on Set): expresses structural induction/recursion principle over objects of yesno.

Gallina - Inductive Definitions

- **Simple inductive types**

```
Inductive nat : Set :=
  | O : nat
  | S : nat -> nat.
```

defines the following four objects at once:

```
nat is defined
nat_rect is defined
nat_ind is defined
nat_rec is defined
```

```
Check nat_ind.
```

```
nat_ind
     : forall P : nat -> Prop,
       P O -> (forall n : nat, P n -> P (S n)) -> forall n : nat, P n
```

Destructor (Elimination principle on Prop): expresses structural induction/recursion principle over objects of nat.

Gallina - Inductive Definitions

- Simple annotated inductive types

```
Inductive evenP : nat -> Prop :=
  | even_O : evenP O
  | even_SS : forall n:nat, evenP n -> evenP (S (S n)).
```

defines the following two objects at once:

```
evenP is defined
evenP_ind is defined
```

```
Check evenP_ind.
```

```
evenP_ind
     : forall P : nat -> Prop,
       P O ->
       (forall n : nat, evenP n -> P n -> P (S (S n))) ->
       forall n : nat, evenP n -> P n
```

Gallina - Inductive Definitions

- Simple annotated inductive types

```
Inductive evenT : nat -> Type :=
  | evenT_O : evenT O
  | evenT_SS : forall n:nat, evenT n -> evenT (S (S n)).
```

defines the following four objects at once:

```
evenT is defined
evenT_rect is defined
evenT_ind is defined
evenT_rec is defined
```

```
Check evenT_ind.
```

```
evenT_ind
     : forall P : forall n : nat, evenT n -> Prop,
       P O evenT_O ->
       (forall (n : nat) (e : evenT n), P n e -> P (S (S n)) (evenT_SS n e)) ->
       forall (n : nat) (e : evenT n), P n e
```

Gallina - Inductive Definitions

- **Mutually defined inductive types**

```
Inductive evenM : nat -> Prop :=
  | evenM_O : evenM O
  | evenM_S : forall n, oddM n -> evenM (S n)
with oddM : nat -> Prop :=
  | oddM S: forall n, evenM n -> oddM (S n).
```

defines the following four objects at once:

```
evenM, oddM are defined
evenM_ind is defined
oddM_ind is defined
```

```
Check evenM ind.
```

```
evenM_ind
     : forall P : nat -> Prop,
       P O ->
       (forall n : nat, oddM n -> P (S n)) -> forall n : nat, evenM
 n -> P n
```

Gallina - Inductive Definitions

➢ Mutually defined inductive types

```
Inductive evenM : nat -> Prop :=
  | evenM_O : evenM O
  | evenM_S : forall n, oddM n -> evenM (S n)
with oddM : nat -> Prop :=
  | oddM_S: forall n, evenM n -> oddM (S n).
```

defines the following four objects at once:

```
evenM, oddM are defined
evenM_ind is defined
oddM_ind is defined
```

```
Check oddM_ind.
```

```
oddM_ind
     : forall P : nat -> Prop,
       (forall n : nat, evenM n -> P (S n)) -> forall n : nat, oddM
 n -> P n
```

Gallina - Definitions

❖ Definitions

- A definition gives a name to a term (definition).

$$\textbf{Definition}\ ident[(ident_1 : term_1) \cdots (ident_n : term_n)] : term_0 := term$$

```
Definition swap_yesno (b:yesno) : yesno :=
  match b with
  | yes => no
  | no => yes
  end.

Check swap_yesno.
```

```
swap_yesno
     : yesno -> yesno
```

Gallina - Definitions

❖ Definitions

```
Definition both_yes (b1:yesno) (b2:yesno) : yesno :=
  match b1 with
  | yes => b2
  | no => no
  end.

Check both_yes.
```

```
both_yes
     : yesno -> yesno -> yesno
```

Gallina - Definition of recursive functions

❖ Definition of recursive functions

$\textbf{Fixpoint}\ \textit{ident params}\{\textbf{struct}\ \textit{ident}_0\} : \textit{type}_0 := \textit{term}_0$

```
Fixpoint plus (m : nat) (n : nat) {struct m} : nat :=
  match m with
    | O => n
    | S m' => S (plus m' n)
  end.
```

```
plus is recursively defined
```

```
Check plus.
```

```
plus
     : nat -> nat -> nat
```

Gallina - Definition of recursive functions

❖ Definition of recursive functions

$$\textbf{Fixpoint}\ \mathit{ident}\ \mathit{params}\{\textbf{struct}\ \mathit{ident}_0\} : \mathit{type}_0 := \mathit{term}_0$$

```
Fixpoint even (n:nat) {struct n} : yesno :=
  match n with
  | O          => yes
  | S O        => no
  | S (S n') => even n'
  end.

Check even.
```

```
even
     : nat -> yesno
```

See bnat_1.v and make exercises.

Gallina – Statement and proofs

Statement

- A statement claims a goal of which the proof is then interactively done using tactics.

  Theorem *ident* : *type*.
  Lemma *ident* : *type*.
  Proposition *ident* : *type*.
  Fact *ident* : *type*.
  Definition *ident* : *type*.

- After a statement, Coq needs a proof.

  Proof. ⋯ **Qed**.

  A proof starts by the keyword Proof. Then Coq enters the proof editing mode until the proof is completed.

  Proof. ⋯ **Admitted**.

  Turns the current conjecture into an axiom and exits editing of current proof.

Proof engine – Common Commands

- **Displaying [Coq RM 6]**
  - **Print** *qualid*.   e.g. **Print even**.
  - **Print All**.
- **Requests to the environment [Coq RM 6]**
  - **Check** *term*.   displays the type of term.
- **Compiled files [Coq RM 6]**
  - **Require** *dirpath*. , **Require Export** *qualid*.
  - ……

Proof engine – Atomic Tactics - 1

See Chapter 8 in Coq Reference Manual for detail.

- **Explicit proof as a term [Coq RM 8.2]**
  - **exact** *term*. gives directly the exact proof term of the goal.
- **Basics [Coq RM 8.3]**
  - **intro** [*ident*]. introduce the premise of the goal.
    intros [$ident_1 \cdots ident_n$]. introduce all premises of the goal.
  - **apply** [*term*]. tries to match the current goal against the conclusion of the type of term.
    - E.g. goal: Q --(apply P->Q.)---> goal: P

    apply *term* **in** $term_p$.
    - E.g. goal: Q x: P --(apply P->Q in x.)---> goal: Q x: Q
  - **assumption.** It looks in the local context for an hypothesis which type is equal to the goal.

Proof engine – Atomic Tactics - 2

- **Basics [Coq RM 8.3]**
  - split. splits conjunction A/\ B into A and B.
  - left. right. applies upon the left/right of disjunction A \/ B , and then they are respectively equivalent to A and B.

  See prop.v for detail and make excises.

- **Conversion tactics [Coq RM 8.5]**
  - simpl. applies $\beta\eta$-reduction rule.

RULE 5 (BETA REDUCTION)
The following transformation is called β-reduction:

$$((\lambda x.M)N) \underset{\beta}{\rightarrow} M[x \rightarrow N]$$

RULE 8 (ETA REDUCTION)
The following transformation of a lambda expression is called η-reduction.

$$(\lambda x.Mx) \underset{\eta}{\rightarrow} M \ ,$$

where x may not be a free variable in M.

Proof engine – Atomic Tactics - 3

❖ Conversion tactics [Coq RM 8.5]

- **unfold** *qualid*. qualid must denote a defined transparent constant or local definition. 展开目标中出现的qualid，在替换时执行$\beta\eta$ 归约
- **red**. applies to a goal which has the form
  forall (x:T1)...(xk:Tk), c t1 ... tn
  where c is a constant. If c is transparent then it replaces c with its definition (say t) and then reduces (t t1 ... tn) according to $\beta\eta$ -reduction rules.
- **fold** *term*. term is reduced using the red tactic.

Proof engine – Atomic Tactics - 4

❖ Introductions [Coq RM 8.6]

- **constructor** *num*. applies to a goal such that the head of its conclusion is an inductive constant (say I). The argument num must be less or equal to the numbers of constructor(s) of I.

  Let ci be the i-th constructor of I, then

  constructor i

  is equivalent to

  intros; apply ci.

Proof engine – Atomic Tactics - 5

❖ Equality [Coq RM 8.8]

- **reflexivity.** applies to a goal which has the form t=u. It checks that t and u are convertible and then solve the goal.
- **symmetry.** applies to a goal which has the form t=u and changes it into u=t.
- **rewrite** *term.* is equivalent to **rewrite** $->$ *term.* the type of term must have the form term1 = term2. the tactic replaces every term1 by term2 in the goal. **rewrite** $<-$ *term.* uses term1=term2 from right to left.

Proof engine – Atomic Tactics - 6

❖ Elimination [Coq RM 8.7]

- **induction** *term.* the type of term must be an inductive constant. Then, the tactic generates subgoals, one for each possible form of term .
- **destruct** *term.* its behavior is similar to induction except that no induction hypothesis is generated. 不产生归纳项
- **decompose** $[qualid_1 \ldots qualid_n]$ *term.* recursively decompose a complex proposition in order to obtain atomic ones. e.g. decompose [and or] H.

Proof engine – Atomic Tactics - 7

❖ Elimination [Coq RM 8.7]

- elim *term*. more basic induction tactic. 根据目标的类型选择合适的destructor并应用之；它不影响目标的假设，也不会引入归纳假设.

Proof engine – Atomic Tactics - 8

❖ Inversion [Coq RM 8.10]

- **inversion** *ident*. let the type ident in the local context be (I t), where I is a inductive predicate. The tactic derives for each possible constructor ci of (I t), all the necessary conditions that should hold for the instance (I t) to be proved by ci.

❖ Contradictory

- **intros contra.** introduce contradictory assumptions introduce true = false

Proof engine –Tactic macros

❖ Ltac

- Ltac Solve := simpl; intros; auto.
- **(util.v)**

```
Ltac move_to_top x' :=
  match reverse goal with
  | H : _ |- _ => try move x' after H
  end.
Ltac Case s' := let c' := fresh "case" in set (c' := s');
   move_to_top c'.
```

User extensions – Syntax extensions

❖ Notations[Coq RM 11.1]

- ➢ Notation "A /\ B" := (and A B).
- ➢ Notation "A /\ B" := (and A B) (at level 80, right associativity).
- ➢ Notation "(x , y)" := (@pair _ _ x y) (at level 0).
- ➢ Notation "A /\ B" := (and A B) : type_scope.

Tools – Coq commands

- **Coq commands [Coq RM 12]**
  - Interactive use: coqtop.opt

    coqtop.opt –help   show the help of usage

    If we have an util.v to be compiled, we can execute

    coqtop.opt –compile util

    then util.vo is generated.
  - Batch compilation: coqc.opt

    coqc.opt util.v

Thanks!