Theory of Programming Languages
程序设计语言理论

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第3章 一种简单的语言$\mathcal{L}${num, str}

3.1 概述-1

- ❖ $\mathcal{L}${num, str}
  - ➢ 支持在自然数上的基本算术运算以及字符串上的简单计算
  - ➢ 包含一种能在特定作用域内将表达式值绑定到一个变量的语言构造
- ❖ 具体语法 **(Concrete Syntax)** [PFPL, 7]
  - ➢ 将表达式表示成字符串的一种手段 (写在纸上或用键盘输入)
  - ➢ 通常希望有好的可读性并且没有二义性.

3.1 概述-2

- ❖ 抽象语法**(Abstract Syntax)** [PFPL, 8]
  - ➢ 揭示语言的层次结构和绑定结构
    e.g. 抽象语法树(abstract syntax tree, AST), 抽象绑定树(abstact binding tree, ABT)
- ❖ 语法对象**(Syntactic Objects)** [PFPL, 5,6]
  - ➢ 字符串, 名字, AST, ……
- ❖ 分析 **(Parsing)**
  - ➢ 将具体语法翻译成抽象语法的过程

3.1 概述-3

- ❖ 静态语义**(Static Semantics)** [PFPL, 9]
  - ➢ 由一组用来约束程序形成的规则组成，称为类型系统.
- ❖ 动态语义**(Dynamic Semantics)** [PFPL, 10,12]
  - ➢ 描述程序将如何执行
  - ➢ 表示方法
    - – 结构语义(Structural semantics) [PFPL, 10]
      - ▪ 上下文语义(Contextual semantics)
    - – 求值语义(Evaluation semantics), [PFPL, 12]

3.2 语法对象

3.2.1 符号和字符串-1

- 符号(**symbols**): 字符、变量名、域名等等.
  - 断言
    x sym：x是一个符号
    $x \# y$，其中 x sym 且 y sym，x和y是不同的符号
- 字符串(**strings**): 字符、变量名、域名等等.
  - 字母表(alphabet)Σ ：一组字符的集合.
  - 断言
    c char：c是一个字符.
    $\Sigma \vdash s$ str：在Σ上定义字符串，由以下规则归纳定义

$$\frac{}{\Sigma \vdash \varepsilon \text{ str}} \qquad \frac{\Sigma \vdash c \text{ char} \quad \Sigma \vdash s \text{ str}}{\Sigma \vdash c \cdot s \text{ str}} \qquad (5.1)$$

一个字符串本质上是一个字符序列，空串是空序列。

3.2.1 符号和字符串-2

- 字符串的归纳原理
  - To show P s whenever s str, it is enough to show
    1) P ε, and
    2) if P s and c sym, then P(c · s)
- 字符串的连接
  - 断言 $s_1 \hat{}\ s_2 = s$ str：s是字符串s_1和s_2连接组成的串.
  - 归纳定义：

$$\frac{}{\varepsilon \hat{}\ s = s} \qquad \frac{s_1 \hat{}\ s_2 = s}{(c \cdot s_1) \hat{}\ s_2 = c \cdot s} \qquad (5.2)$$

该断言具有模式 $(\forall, \forall, \exists!)$

3.2.2 抽象语法树-1

- 抽象语法树**Abstract Syntax Tree(AST)** [PFPL, 5.3]
  - an ordered tree in which certain symbols (operators) label the nodes
  - Each operator is assigned an arity (number of children)
- **Operator signature, Ω**
  - 是一组形如 $\text{ar}(o) = n$的断言，其中o sym、n nat
    如果$\Omega \vdash \text{ar}(o) = n$ 和$\Omega \vdash \text{ar}(o) = n'$，则 $n = n'$ nat.
  - 归纳定义

$$\frac{\Omega \vdash \text{ar}(o) = \text{zero}}{o() \text{ ast}} \qquad \frac{\Omega \vdash \text{ar}(o) = n \quad a_1 \text{ ast} \quad \cdots \quad a_n \text{ ast}}{o(a_1, \cdots, a_n) \text{ ast}} \qquad (5.3)$$

3.2.2 抽象语法树-2

- 结构归纳原理(Principle of Structural Induction)
  - **To show $\mathcal{P}(a \text{ ast})$, it is enough to show that $\mathcal{P}$ is closed under Rules (5.3). That is,**
    if $\Omega \vdash \text{ar}(o) = n$ **, then we are to show that**
    if $\mathcal{P}(a_1 \text{ ast}), \cdots, \mathcal{P}(a_n \text{ ast})$
    then $\mathcal{P}(o(a_1, \cdots, a_n) \text{ ast})$
  - 例如，**AST**高度(断言模式为$(\forall, \exists!)$)的归纳定义

$$\frac{\text{hgt}(a_1) = h_1 \quad \cdots \quad \text{hgt}(a_n) = h_n \quad \max(h_1, \cdots, h_n) = h}{\text{hgt}(o(a_1, \cdots, a_n)) = \text{succ}(h)} \qquad (5.4)$$

3.2.2 抽象语法树-3

- 变量与代换(**Variables and Substitution**)
  - **Variables are represented by names, and are given meaning by substitution.**
  - 假设$\mathcal{X} = x_1 \text{ ast}, \cdots, x_n \text{ ast}$ 是参数集
    $\{x_1, \cdots, x_n\} | x_1 \text{ ast}, \cdots, x_n \text{ ast}$ (x_1 sym$, \cdots, x_n$ sym)是一组假设序列；$x \# \mathcal{X}$表示 $x \notin \{x_1, \cdots, x_n\}$
  - 断言 $\mathcal{X} \vdash a$ ast由以下规则归纳定义

$$\frac{}{\mathcal{X}, x \text{ ast} \vdash x \text{ ast}}$$

$$\frac{\Omega \vdash \text{ar}(o) = n \quad \mathcal{X} \vdash a_1 \text{ ast} \quad \cdots \quad \mathcal{X} \vdash a_n \text{ ast}}{\mathcal{X} \vdash o(a_1, \cdots, a_n) \text{ ast}} \qquad (5.5)$$

3.2.2 抽象语法树-4

- 变量与代换(**Variables and Substitution**)
  - 归纳原理：To show $\mathcal{P}(\mathcal{X} \vdash a \text{ ast})$, it is enough to show
  1) $\mathcal{P}(\mathcal{X}, x \text{ ast} \vdash x \text{ ast})$.
  2) If $\Omega \vdash \text{ar}(o) = n$, and if
     $\mathcal{P}(\mathcal{X} \vdash a_1 \text{ ast}), \cdots, \mathcal{P}(\mathcal{X} \vdash a_n \text{ ast})$
     then $\mathcal{P}(\mathcal{X} \vdash o(a_1, \cdots, a_n) \text{ ast})$.

$\mathcal{X}$中的参数都被当成原子对象，每个参数有自己的**AST**.

3.2.2 抽象语法树-5

变量代换

- 断言 $\mathcal{X} \vdash [a/x]b = c$：c是用a代换b中的x所得的结果
- 归纳定义

$$\frac{}{\mathcal{X}, x \text{ ast} \vdash [a/x]x = a} \quad (5.6a)$$

$$\frac{x \# y}{\mathcal{X}, x \text{ ast}, y \text{ ast} \vdash [a/x]y = y} \quad (5.6b)$$

$$\frac{\mathcal{X} \vdash [a/x]b_1 = c_1 \quad \cdots \quad \mathcal{X} \vdash [a/x]b_n = c_n}{\mathcal{X} \vdash [a/x]o(b_1, \cdots, b_n) = o(c_1, \cdots, c_n)} \quad (5.6c)$$

- Theorem 5.1.如果$\mathcal{X} \vdash a$ ast 且$\mathcal{X}, x$ ast $\vdash b$ ast($x \# \mathcal{X}$) 则存在一个唯一的c使得$\mathcal{X} \vdash [a/x]b = c$且 $\mathcal{X} \vdash c$ ast

3.2.2 抽象语法树-6

证明(Theorem 5.1)：

在上下文$\mathcal{X}, x$ ast 上对b进行结构归纳：

1. 由于$\mathcal{X}, x$ ast $\vdash x$ ast，需要证明存在唯一的c使得：
   $\mathcal{X} \vdash [a/x]x = c$
   考虑规则(5.6a)，故选择c为a是充分且必要的.
2. 如果$\mathcal{X}, x$ ast$, y$ ast ($x \# \mathcal{X}$),则由规则(5.6b)，选择c为y是充分且必要的.
3. 如果$b = o(b_1, \cdots, b_n)$，则由归纳假设，存在唯一的使得 $\mathcal{X} \vdash [a/x]b_1 = c_1, \cdots, \mathcal{X} \vdash [a/x]b_n = c_n$
   由规则(5.6c)，c只能取 $o(c_1, \cdots, c_n)$

3.2.3 抽象绑定树-1

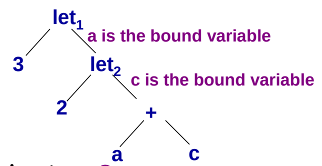
抽象语法树：反映了语法的层次结构

抽象绑定树(Abstract binding trees, ABT)：增加了绑定(binding)和作用域(scope)的概念.

- **ABT**
  - ABT: extends AST with an abstractor
  - Abstractor x.a: 将变量 x 绑定到ABT a, a 称为绑定的作用域. 受约束的变量x仅在a内是有意义的
  - arity of an operator
    - 是自然数的有限序列$(n_1, \cdots, n_k)$，k表示参数的个数，n_i 表示第i个参数中约束变量的数目(valence).
    - e.g. let x be exp1 in exp2   ar(let)=(0,1)
      //exp2 has a variable named x.

3.2.3 抽象绑定树-2

let₁ a be 3 in let₂ c be 2 in a+c   ar(let)=(0,1)



- operator signature Ω
  一组有限的形如 $\text{ar}(o) = (n_1, \cdots, n_k)$的断言
- Ω上的良形(well-formed)ABT由参数化假言断言描述
  $\{x_1, \cdots, x_k\} | x_1 \text{ abt}^0, \cdots, x_k \text{ abt}^0 \vdash a \text{ abt}^n$
  $x_1, \cdots, x_k$ 是a中的自由变量. $\mathcal{X} | \mathcal{A} \vdash a \text{ abt}^n$ $\mathcal{A} \vdash a \text{ abt}^n$

3.2.3 抽象绑定树-3

- 良形abt的归纳定义

$$\frac{}{\mathcal{X}, x | \mathcal{A}, x \text{ abt}^0 \vdash x \text{ abt}^0} \quad (6.1a)$$

$$\frac{\text{ar}(o) = (n_1, \cdots, n_k) \quad \mathcal{X} | \mathcal{A} \vdash a_1 \text{ abt}^{n_1} \quad \cdots \quad \mathcal{X} | \mathcal{A} \vdash a_k \text{ abt}^{n_k}}{\mathcal{X} | \mathcal{A} \vdash o(a_1, \cdots, a_k) \text{ abt}^0} \quad (6.1b)$$

$$\frac{\mathcal{X}, x' | \mathcal{A}, x' \text{ abt}^0 \vdash [x' \leftrightarrow x]a \text{ abt}^n \quad (x' \notin \mathcal{X})}{\mathcal{X} | \mathcal{A} \vdash x.a \text{ abt}^{n+1}} \quad (6.1c)$$

$\mathcal{X}, x' | \mathcal{A}, x' \text{ abt}^0 \vdash [x' \leftrightarrow x]a \text{ abt}^n \quad (x' \notin \mathcal{X})$ 表示用x'替换a中的约束变量x所得的体是良形的.

则 $\mathcal{X} | \mathcal{A} \vdash x.a \text{ abt}^{n+1}$，即抽象子x.a相对于 $\mathcal{A}$是良形的

3.2.3 抽象绑定树-4

- 结构归纳原理
  To show that $\mathcal{P}(\mathcal{X} | \mathcal{A} \vdash a \text{ abt}^n)$ whenever $\mathcal{X} | \mathcal{A} \vdash a \text{ abt}^n$ it suffices to show the following:

1) $\mathcal{P}(\mathcal{X}, x | \mathcal{A}, x \text{ abt}^0 \vdash x \text{ abt}^0)$.

2) For any operator, o, of arity $(m_1, \ldots, m_k)$, if
   $\mathcal{P}(\mathcal{X} | \mathcal{A} \vdash a_1 \text{ abt}^{m_1}), \cdots, \mathcal{P}(\mathcal{X} | \mathcal{A} \vdash a_k \text{ abt}^{m_k})$
   then $\mathcal{P}(\mathcal{X} | \mathcal{A} \vdash o(a_1, \cdots, a_k) \text{ abt}^0)$

3) If $\mathcal{P}(\mathcal{X}, x' | \mathcal{A}, x' \text{ abt}^0 \vdash [x' \leftrightarrow x]a \text{ abt}^n)$ for some/any $x' \notin \mathcal{X}$, then $\mathcal{P}(\mathcal{X} | \mathcal{A} \vdash x.a \text{ abt}^{n+1})$.

3.2.3 抽象绑定树-5

➢ 例，abt的size s用断言 $|a \text{ abt}^n| = s$ 定义

一般地，参数化假言断言

$|x_1 \text{ abt}^0| = 1, \cdots, |x_k \text{ abt}^0| = 1 \vdash |a \text{ abt}^n| = s$

由以下规则归纳定义

$$\frac{}{\mathcal{S}, |x \text{ abt}^0| = 1 \vdash |x \text{ abt}^0| = 1} \quad (6.2a)$$

$$\frac{\mathcal{S} \vdash |a_1 \text{ abt}^{n_1}| = s_1 \quad \cdots \quad \mathcal{S} \vdash |a_m \text{ abt}^{n_m}| = s_m \quad s = s_1 + \cdots + s_m + 1}{\mathcal{S} \vdash |o(a_1, \cdots, a_m) \text{ abt}^0| = s} \quad (6.2b)$$

$$\frac{\mathcal{S}, |x' \text{ abt}^0| = 1 \vdash |[x' \leftrightarrow x]a \text{ abt}^n| = s}{\mathcal{S} \vdash |x.a \text{ abt}^{n+1}| = s + 1} \quad (6.2c)$$

Theorem 6.1.每个良形的abt有唯一的size.

3.2.3 抽象绑定树-6

❖ Apartness judgement: $\mathcal{A} \vdash x \# a \text{ abt}^n$ ($\mathcal{A} \vdash a \text{ abt}^n$)

the abt a does not involve the variable x except possibly as a bound variable.

Rules

$$\frac{x \# y}{\mathcal{A} \vdash x \# y \text{ abt}^0} \quad (6.3a)$$

$$\frac{\mathcal{A} \vdash x \# a_1 \text{ abt}^{n_1} \quad \cdots \quad \mathcal{A} \vdash x \# a_k \text{ abt}^{n_k}}{\mathcal{A} \vdash x \# o(a_1, \cdots, a_k) \text{ abt}^0} \quad (6.3b)$$

$$\frac{\mathcal{A}, y \text{ abt}^0 \vdash x \# a \text{ abt}^n}{\mathcal{A} \vdash x \# y.a \text{ abt}^{n+1}} \quad (6.3c)$$

➢ x is free in an abt,a, written $x \in a \text{ abt}$, iff it is not the case that $x \# a \text{ abt}$.

3.2.3 抽象绑定树-7

❖ Renaming of Bound Variables (α-equivalence)

两个abt是α-等价的，当且仅当它们只在约束变量的选取上有不同之处. $\mathcal{A} \vdash a =_\alpha b \text{ abt}^n$

Rules

$$\frac{}{\mathcal{A}, x \text{ abt}^0 \vdash x =_\alpha x \text{ abt}^0} \quad (6.4a)$$

$$\frac{\mathcal{A} \vdash a_1 =_\alpha b_1 \text{ abt}^{n_1} \quad \cdots \quad \mathcal{X} \vdash a_k =_\alpha b_k \text{ abt}^{n_k}}{\mathcal{X} \vdash o(a_1, \cdots, a_k) =_\alpha o(b_1, \cdots, b_k) \text{ abt}^0} \quad (6.4b)$$

$$\frac{\mathcal{A}, z \text{ abt}^0 \vdash [z \leftrightarrow x]a =_\alpha [z \leftrightarrow y]b \text{ abt}^n}{\mathcal{A} \vdash x.a =_\alpha y.b \text{ abt}^{n+1}} \quad (6.4c)$$

➢ Theorem 6.3.α-等价是自反的、对称的和传递的.

3.2.3 抽象绑定树-8

❖ Substitution:

代换是将一个abt中一个自由变量的所有出现替换为另一个abt

$\mathcal{A} \vdash [a/x]b = c \text{ abt}^n$

Rules

$$\frac{}{\mathcal{A} \vdash [a/x]x = a \text{ abt}^0} \quad (6.5a)$$

$$\frac{x \# y}{\mathcal{A} \vdash [a/x]y = y \text{ abt}^0} \quad (6.5b)$$

$$\frac{\mathcal{A} \vdash [a/x]b_1 = c_1 \text{ abt}^{n_1} \quad \cdots \quad \mathcal{X} \vdash [a/x]b_k = c_k \text{ abt}^{n_k}}{\mathcal{X} \vdash [a/x]o(b_1, \cdots, b_k) = o(c_1, \cdots, c_k) \text{ abt}^0} \quad (6.5c)$$

$$\frac{\mathcal{A}, y' \text{ abt}^0 \vdash [a/x]([y' \leftrightarrow y]b) = b' \text{ abt}^n \quad y' \# \mathcal{A} \quad y' \neq x}{\mathcal{A} \vdash [a/x]y.b = y'.b' \text{ abt}^n} \quad (6.5d)$$

➢ 由规则(6.5d), 有 $y.[y \leftrightarrow y']b' =_\alpha y'.b'$

3.2.3 抽象绑定树-9

❖ Substitution

Theorem 6.4.

1. If $\mathcal{A} \vdash a \text{ abt}^0$ and $\mathcal{A}, x \text{ abt}^0 \vdash b \text{ abt}^n$, then there exists $\mathcal{A} \vdash c \text{ abt}^n$ such that $\mathcal{A} \vdash [a/x]b = c \text{ abt}^n$.
2. If $\mathcal{A} \vdash a \text{ abt}^0$, $\mathcal{A} \vdash [a/x]b = c \text{ abt}^n$ and $\mathcal{A} \vdash [a/x]b = c' \text{ abt}^n$, then $\mathcal{A} \vdash c =_\alpha c' \text{ abt}^n$.

证明: 1. 对 $\mathcal{A}, x \text{ abt}^0 \vdash b \text{ abt}^n$ 归纳证明

2. 对 $\mathcal{A} \vdash [a/x]b = c \text{ abt}^n$和$\mathcal{A} \vdash [a/x]b = c' \text{ abt}^n$

联立归纳证明。

3.2.3 抽象绑定树-10

❖ Substitution

Theorem 6.5.

If $\mathcal{A} \vdash a =_\alpha a' \text{ abt}^0$, $\mathcal{A}, x \text{ abt}^0 \vdash b =_\alpha b' \text{ abt}^n$,

$\mathcal{A} \vdash [a/x]b = c \text{ abt}^n$ and $\mathcal{A} \vdash [a'/x]b' = c' \text{ abt}^n$,

then $\mathcal{A} \vdash c =_\alpha c' \text{ abt}^n$.

Proof. By rule induction on $\mathcal{A}, x \text{ abt}^0 \vdash b =_\alpha b' \text{ abt}^n$

3.2.3 抽象绑定树-11

若假设所有断言关于abt α-等价，则抽象子的形成规则可以简写为(x#A):

$$\frac{\mathcal{A}, x \text{ abt}^0 \vdash a \text{ abt}^n}{\mathcal{A} \vdash x.a \text{ abt}^{n+1}} \quad (6.6)$$

3.3 具体语法(Concrete Syntax)

- a means of representing expressions as strings (written on a page or entered using a keyboard)
- usually designed to enhance readability and to eliminate ambiguity.

3.3.1 Lexical Structure

3.3.2 Context-Free Grammars

3.3.3 Grammatical Structure

3.3.4 Ambiguity

3.3.5 Informal Conventions

3.3.1 Lexical Structure-1

- **Lexical analysis (lexing)**
  - characters ➔ symbols (tokens)
    - white space (spaces, tabs, newlines, comments, ...) -- discarded by the lexical analyzer
- **Lexical structure of $\mathcal{L}\{\text{num}, \text{str}\}$**

Item	itm	::=	kwd | id | num | lit | spl
Keyword	kwd	::=	$\mathtt{l}\cdot\mathtt{e}\cdot\mathtt{t}\cdot\epsilon$ | $\mathtt{b}\cdot\mathtt{e}\cdot\epsilon$ | $\mathtt{i}\cdot\mathtt{n}\cdot\epsilon$
Identifier	id	::=	ltr (ltr | dig)*
Numeral	num	::=	dig dig*
Literal	lit	::=	qum (ltr | dig)*qum
Special	spl	::=	+ | * | ^ | (|) | |
Letter	ltr	::=	a | b | ...
Digit	dig	::=	0 | 1 | ...
Quote	qum	::=	"

3.3.1 Lexical Structure-2

- **Rules for translating lexical items into tokens**

$$\frac{s \text{ str}}{\mathtt{ID}[s] \text{ tok}} \qquad \frac{}{\mathtt{LET} \text{ tok}} \qquad \frac{}{\mathtt{ADD} \text{ tok}} \qquad \frac{}{\mathtt{VB} \text{ tok}}$$

$$\frac{n \text{ nat}}{\mathtt{NUM}[n] \text{ tok}} \qquad \frac{}{\mathtt{BE} \text{ tok}} \qquad \frac{}{\mathtt{MUL} \text{ tok}} \qquad \frac{}{\mathtt{LP} \text{ tok}}$$

$$\frac{s \text{ str}}{\mathtt{LIT}[s] \text{ tok}} \qquad \frac{}{\mathtt{IN} \text{ tok}} \qquad \frac{}{\mathtt{CAT} \text{ tok}} \qquad \frac{}{\mathtt{RP} \text{ tok}}$$

(7.1)

3.3.1 Lexical Structure-3

- **Judgements for lexical analysis**
  - s inp $\longleftrightarrow t$ tokstr   Scan input
  - s itm $\longleftrightarrow t$ tok   Scan an item
  - s kwd $\longleftrightarrow t$ tok   Scan a keyword
  - s id $\longleftrightarrow t$ tok   Scan an identifier
  - s num $\longleftrightarrow t$ tok   Scan a number
  - s spl $\longleftrightarrow t$ tok   Scan a symbol
  - s lit $\longleftrightarrow t$ tok   Scan a string literal
  - s whs   Skip white space

e.g. let a be 34 in a*12

be itm $\longleftrightarrow$ BE tok a id $\longleftrightarrow$ $\mathtt{ID}[a]$ tok

3.3.1 Lexical Structure-4

- **Rules for lexical analysis**

$$\frac{}{\epsilon \text{ inp} \longleftrightarrow \epsilon \text{ tokstr}}$$

$$\frac{s = s_1 \hat{\ } s_2 \hat{\ } s_3 \text{ str} \quad s_1 \text{ whs} \quad s_2 \text{ itm} \longleftrightarrow t \text{ tok} \quad s_3 \text{ inp} \longleftrightarrow ts \text{ tokstr}}{s \text{ inp} \longleftrightarrow t \cdot ts \text{ tokstr}}$$

$$\frac{s \text{ kwd} \longleftrightarrow t \text{ tok}}{s \text{ itm} \longleftrightarrow t \text{ tok}} \qquad \frac{s \text{ id} \longleftrightarrow t \text{ tok}}{s \text{ itm} \longleftrightarrow t \text{ tok}} \qquad \frac{s \text{ num} \longleftrightarrow t \text{ tok}}{s \text{ itm} \longleftrightarrow t \text{ tok}}$$

$$\frac{s \text{ lit} \longleftrightarrow t \text{ tok}}{s \text{ itm} \longleftrightarrow t \text{ tok}} \qquad \frac{s \text{ spl} \longleftrightarrow t \text{ tok}}{s \text{ itm} \longleftrightarrow t \text{ tok}} \qquad \frac{s = \mathtt{l}\cdot\mathtt{e}\cdot\mathtt{t}\cdot\varepsilon \text{ str}}{s \text{ kwd} \longleftrightarrow \mathtt{LET} \text{ tok}}$$

$$\frac{s = \mathtt{b}\cdot\mathtt{e}\cdot\varepsilon \text{ str}}{s \text{ kwd} \longleftrightarrow \mathtt{BE} \text{ tok}} \qquad \frac{s = \mathtt{i}\cdot\mathtt{n}\cdot\varepsilon \text{ str}}{s \text{ kwd} \longleftrightarrow \mathtt{IN} \text{ tok}}$$

$$\frac{s = s_1 \hat{\ } s_2 \text{ str} \quad s_1 \text{ ltr} \quad s_2 \text{ lord}}{s \text{ id} \longleftrightarrow \mathtt{ID}[s] \text{ tok}} \quad (7.2)$$

3.3.1 Lexical Structure-5

❖ **Rules for lexical analysis(cont'd)**

$$\frac{s = s_1 \hat{}\, s_2 \ \mathsf{str} \quad s_1 \ \mathsf{dig} \quad s_2 \ \mathsf{dgs} \quad s \ \mathsf{num} \longleftrightarrow n \ \mathsf{nat}}{s \ \mathsf{num} \longleftrightarrow \mathtt{NUM}[n] \ \mathsf{tok}}$$

$$\frac{s = s_1 \hat{}\, s_2 \hat{}\, s_3 \ \mathsf{str} \quad s_1 \ \mathsf{qum} \quad s_2 \ \mathsf{lord} \quad s_3 \ \mathsf{qum}}{s \ \mathsf{lit} \longleftrightarrow \mathtt{LIT}[s_2] \ \mathsf{tok}}$$

$$\frac{s = + \cdot \varepsilon \ \mathsf{str}}{s \ \mathsf{spl} \longleftrightarrow \mathtt{ADD} \ \mathsf{tok}} \qquad \frac{s = * \cdot \varepsilon \ \mathsf{str}}{s \ \mathsf{spl} \longleftrightarrow \mathtt{MUL} \ \mathsf{tok}} \qquad \frac{s = \hat{} \cdot \varepsilon \ \mathsf{str}}{s \ \mathsf{spl} \longleftrightarrow \mathtt{CAT} \ \mathsf{tok}}$$

$$\frac{s = (\cdot \varepsilon \ \mathsf{str}}{s \ \mathsf{spl} \longleftrightarrow \mathtt{LP} \ \mathsf{tok}} \qquad \frac{s =) \cdot \varepsilon \ \mathsf{str}}{s \ \mathsf{spl} \longleftrightarrow \mathtt{RP} \ \mathsf{tok}} \qquad \frac{s = | \cdot \varepsilon \ \mathsf{str}}{s \ \mathsf{spl} \longleftrightarrow \mathtt{VB} \ \mathsf{tok}}$$

(7.2)

3.3.1 Lexical Structure-6

❖ **e.g. a*12**

$$\dfrac{\dfrac{\dfrac{a\ \mathrm{str} \quad a\ \mathrm{ltr}}{a\ \mathrm{id} \longleftrightarrow \mathtt{ID}[a]\ \mathrm{tok}}}{a\ \mathrm{itm} \longleftrightarrow \mathtt{ID}[a]\ \mathrm{tok}} \quad \dfrac{\dfrac{\dfrac{*\ \mathrm{str}}{*\ \mathrm{spl} \longleftrightarrow \mathtt{MUL}\ \mathrm{tok}}}{*\ \mathrm{itm} \longleftrightarrow \mathtt{MUL}\ \mathrm{tok}} \quad \dfrac{\dfrac{\dfrac{1\ \mathrm{dig} \quad 2\ \mathrm{dgs} \quad 12\ \mathrm{num} \longleftrightarrow 12\ \mathrm{nat}}{12\ \mathrm{num} \longleftrightarrow \mathtt{NUM}[12]\ \mathrm{tok}}}{12\ \mathrm{itm} \longleftrightarrow \mathtt{NUM}[12]\ \mathrm{tok}}}{12\ \mathrm{inp} \longleftrightarrow \mathtt{NUM}[12]\ \mathrm{tokstr}}}{*12\ \mathrm{inp} \longleftrightarrow \mathtt{MUL} \cdot \mathtt{NUM}[12]\ \mathrm{tokstr}}}{a * 12\ \mathrm{inp} \longleftrightarrow \mathtt{ID}[a] \cdot \mathtt{MUL} \cdot \mathtt{NUM}[12]\ \mathrm{tokstr}}$$

3.3.2 Context-Free Grammars-1

❖ **Components of a Grammar**

- **tokens, or terminals,**
- **syntactic classes, or non-terminals,**
- **rules, or productions,**
  - A ::= α ,
    A: non-terminal,
    α: a string of terminals and non-terminals
  - A ::= α_1 | · · · | α_n, (compound production)

3.3.2 Context-Free Grammars-2

❖ **Context-free Grammar**

- It determines a simultaneous inductive definition of its syntactic classes
- Regard each non-terminal, A, as a judgement, s A, over strings of terminals.
- To each production, $A ::= s_1\ A_1\ s_2 \cdots s_n\ A_n\ s_{n+1}$ (7.3)

  we associate a rule:

$$\frac{s'_1\ A_1 \quad \cdots \quad s'_n\ A_n}{s_1\ s'_1\ s_2 \cdots s_n\ s'_n\ s_{n+1}\ A} \qquad (7.4)$$

and it can be rewritten as follows:

$$\frac{s'_1\ A_1 \quad \cdots \quad s'_n\ A_n \quad s = s_1 \hat{}\, s'_1 \hat{}\, s_2 \hat{} \cdots s_n \hat{}\, s'_n \hat{}\, s_{n+1}}{s\ A} \qquad (7.5)$$

3.3.3 Grammatical Structure-1

❖ **Grammatical Structure of $\mathcal{L}\{\mathrm{num}, \mathrm{str}\}$**

Expression	exp	::=	num | lit | id | LP exp RP | exp ADD exp | exp MUL exp | exp CAT exp | VB exp VB | LET id BE exp IN exp
Number	num	::=	NUM[n] (n nat)
String	lit	::=	LIT[s] (s str)
Identifier	id	::=	ID[s] (s str)

- **String**: let a be 3 in a*12
- **tokstr**: LET ID[a] BE NUM[3] IN ID[a] MUL NUM[12]

3.3.3 Grammatical Structure-2

❖ **Rules for interpreting a grammar**

$$\frac{s\ \mathrm{num}}{s\ \mathrm{exp}} \qquad \frac{s_1\ \mathrm{exp} \quad s_2\ \mathrm{exp}}{s_1\ \mathtt{ADD}\ s_2\ \mathrm{exp}} \qquad \frac{s\ \mathrm{exp}}{\mathtt{VB}\ s\ \mathtt{VB}\ \mathrm{exp}}$$

$$\frac{s\ \mathrm{lit}}{s\ \mathrm{exp}} \qquad \frac{s_1\ \mathrm{exp} \quad s_2\ \mathrm{exp}}{s_1\ \mathtt{MUL}\ s_2\ \mathrm{exp}} \qquad \frac{s\ \mathrm{exp}}{\mathtt{LP}\ s\ \mathtt{RP}\ \mathrm{exp}}$$

$$\frac{s\ \mathrm{id}}{s\ \mathrm{exp}} \qquad \frac{s_1\ \mathrm{exp} \quad s_2\ \mathrm{exp}}{s_1\ \mathtt{CAT}\ s_2\ \mathrm{exp}} \qquad \frac{s\ \mathrm{str}}{\mathtt{LIT}[s]\ \mathrm{lit}} \qquad \frac{s\ \mathrm{str}}{\mathtt{ID}[s]\ \mathrm{id}}$$

$$\frac{s_1\ \mathrm{id} \quad s_2\ \mathrm{exp} \quad s_3\ \mathrm{exp}}{\mathtt{LET}\ s_1\ \mathtt{BE}\ s_2\ \mathtt{IN}\ \mathrm{exp}} \qquad \frac{n\ \mathrm{nat}}{\mathtt{NUM}[n]\ \mathrm{num}} \qquad (7.6)$$

$$\frac{s = s_1\ \mathtt{MUL}\ s_2\ \mathrm{str} \quad s_1\ \mathrm{exp} \quad s_2\ \mathrm{exp}}{s\ \mathrm{exp}} \qquad (7.7)$$

3.3.4 Ambiguity

- **Principal goal of concrete syntax design**: readability, eliminate ambiguity
- **Example**: 1 + 2*3
  NUM[1] ADD NUM[2] MUL NUM[3]
- Ambiguity is a purely syntactic property of grammars.
- **Grammatical structure of $\mathcal{L}$\{num str\}** (eliminate ambiguity)

Factor	fct	::=	num | lit | id | LP prg RP
Term	trm	::=	fct | fct MUL trm | VB fct VB
Expression	exp	::=	trm | trm ADD exp | trm CAT exp
Program	prg	::=	exp | LET id BE trm IN prg

3.3.5 Informal Conventions

- **The concrete syntax of $\mathcal{L}$\{num str\}**

Expr

$e ::= n \mid "s" \mid s \mid e_1 + e_2 \mid e_1 * e_2 \mid e_1 \mathbin{\hat{}} e_2 \mid |e| \mid \texttt{let}\ x\ \texttt{be}\ e_1\ \texttt{in}\ e_2$

3.4 Abstract Syntax

- expose the hierarchical and binding structure of the language
- eliminate ambiguity

3.4.1 Abstract Syntax Trees

3.4.2 Parsing Into Abstract Syntax Trees

3.4.3 Parsing Into Abstract Binding Trees

3.4.4 Informal Conventions

3.4.1 Abstract Syntax Trees-1

Concrete Syntax	Token String	Abstract Syntax
n	NUM[n]	num[n]
"s"	LIT[s]	str[s]
s	ID[s]	id[s]
e_1+e_2	e_1 ADD e_2	plus(e_1; e_2)
e_1*e_2	e_1 MUL e_2	times(e_1; e_2)
e_1^e_2	e_1 CAT e_2	cat(e_1; e_2)
| e |	VB e VB	len(e)
let s be e_1 in e_2	LET ID[s] BE e_1 IN e_2	let[s](e_1; e_2)
(e)	LP e RP	e

3.4.1 Abstract Syntax Trees-2

- **Arities to operators**(AST of $\mathcal{L}$\{num, str\}):

$$\begin{aligned} \mathrm{ar}(\texttt{num}[n]) &= 0 \quad (n\ \mathrm{nat}) & \mathrm{ar}(\texttt{plus}) &= 2 \\ \mathrm{ar}(\texttt{str}[s]) &= 0 \quad (s\ \mathrm{str}) & \mathrm{ar}(\texttt{times}) &= 2 \\ \mathrm{ar}(\texttt{id}[s]) &= 0 \quad (s\ \mathrm{str}) & \mathrm{ar}(\texttt{cat}) &= 2 \\ \mathrm{ar}(\texttt{len}) &= 1 & \mathrm{ar}(\texttt{let}[s]) &= 2 \end{aligned}$$

- **Inductive definition of the abstract syntax**

$$\frac{n\ \mathrm{nat}}{\texttt{num}[n]\ \mathrm{ast}} \qquad \frac{a_1\ \mathrm{ast} \quad a_2\ \mathrm{ast}}{\texttt{plus}(a_1;a_2)\ \mathrm{ast}} \qquad \frac{a\ \mathrm{ast}}{\texttt{len}(a)\ \mathrm{ast}}$$

$$\frac{s\ \mathrm{str}}{\texttt{str}[s]\ \mathrm{ast}} \qquad \frac{a_1\ \mathrm{ast} \quad a_2\ \mathrm{ast}}{\texttt{times}(a_1;a_2)\ \mathrm{ast}} \qquad \frac{s\ \mathrm{str} \quad a_1\ \mathrm{ast} \quad a_2\ \mathrm{ast}}{\texttt{let}[s](a_1;a_2)\ \mathrm{ast}}$$

$$\frac{s\ \mathrm{str}}{\texttt{id}[s]\ \mathrm{ast}} \qquad \frac{a_1\ \mathrm{ast} \quad a_2\ \mathrm{ast}}{\texttt{cat}(a_1;a_2)\ \mathrm{ast}} \qquad (8.1)$$

3.4.2 Parsing Into ASTs-1

- **Parsing**: translation from concrete to abstract syntax
- **Parsing judgements** for $\mathcal{L}$\{num, str\}

s prg $\longleftrightarrow a$ ast	Parse as a program
s exp $\longleftrightarrow a$ ast	Parse as an expression
s trm $\longleftrightarrow a$ ast	Parse as a term
s fct $\longleftrightarrow a$ ast	Parse as a factor
s num $\longleftrightarrow a$ ast	Parse as a number
s lit $\longleftrightarrow a$ ast	Parse as a literal
s id $\longleftrightarrow a$ ast	Parse as a identifier

3.4.2 Parsing Into ASTs-2

❖ **Inductive definition**

Factor fct ::= num | lit | id | LP prg RP

$$\frac{n\ \text{nat}}{\texttt{NUM}[n]\ \text{num} \longleftrightarrow \texttt{num}[n]\ \text{ast}} \qquad \frac{s\ \text{str}}{\texttt{LIT}[s]\ \text{lit} \longleftrightarrow \texttt{str}[s]\ \text{ast}}$$

$$\frac{s\ \text{str}}{\texttt{ID}[s]\ \text{id} \longleftrightarrow \texttt{id}[s]\ \text{ast}} \qquad \frac{s\ \text{num} \longleftrightarrow a\ \text{ast}}{s\ \text{fct} \longleftrightarrow a\ \text{ast}}$$

$$\frac{s\ \text{lit} \longleftrightarrow a\ \text{ast}}{s\ \text{fct} \longleftrightarrow a\ \text{ast}} \qquad \frac{s\ \text{id} \longleftrightarrow a\ \text{ast}}{s\ \text{fct} \longleftrightarrow a\ \text{ast}} \qquad \frac{s\ \text{prg} \longleftrightarrow a\ \text{ast}}{\texttt{LP}\ s\ \texttt{RP}\ \text{fct} \longleftrightarrow a\ \text{ast}}$$

Term trm ::= fct | fct MUL trm | VB fct VB

$$\frac{s\ \text{fct} \longleftrightarrow a\ \text{ast}}{s\ \text{trm} \longleftrightarrow a\ \text{ast}} \qquad \frac{s_1\ \text{fct} \longleftrightarrow a_1\ \text{ast} \quad s_2\ \text{trm} \longleftrightarrow a_2\ \text{ast}}{s_1\ \texttt{MUL} s_2\ \text{trm} \longleftrightarrow \texttt{times}(a_1; a_2)\ \text{ast}}$$

$$\frac{s\ \text{fct} \longleftrightarrow a\ \text{ast}}{\texttt{VB}\ s\ \texttt{VB}\ \text{trm} \longleftrightarrow \texttt{len}(a)\ \text{ast}}$$

(8.2)

3.4.2 Parsing Into ASTs-3

❖ **Inductive definition (cont'd)**

Expression exp ::= trm | trm ADD exp | trm CAT exp

$$\frac{s\ \text{trm} \longleftrightarrow a\ \text{ast}}{s\ \text{exp} \longleftrightarrow a\ \text{ast}} \qquad \frac{s_1\ \text{trm} \longleftrightarrow a_1\ \text{ast} \quad s_2\ \text{exp} \longleftrightarrow a_2\ \text{ast}}{s_1\ \texttt{ADD}\ s_2\ \text{exp} \longleftrightarrow \texttt{plus}(a_1; a_2)\ \text{ast}}$$

$$\frac{s_1\ \text{trm} \longleftrightarrow a_1\ \text{ast} \quad s_2\ \text{exp} \longleftrightarrow a_2\ \text{ast}}{s_1\ \texttt{CAT}\ s_2\ \text{exp} \longleftrightarrow \texttt{cat}(a_1; a_2)\ \text{ast}}$$

Program prg ::= exp | LET id BE exp IN prg

$$\frac{s\ \text{exp} \longleftrightarrow a\ \text{ast}}{s\ \text{prg} \longleftrightarrow a\ \text{ast}}$$

$$\frac{s_1\ \text{fct} \longleftrightarrow \texttt{id}[s]\ \text{ast} \quad s_2\ \text{exp} \longleftrightarrow a_2\ \text{ast} \quad s_3\ \text{prg} \longleftrightarrow a_3\ \text{ast}}{\texttt{LET}\ s_1\ \texttt{BE}\ s_2\ \texttt{IN}\ s_3\ \text{prg} \longleftrightarrow \texttt{let}[s](a_2; a_3)\ \text{ast}}$$

(8.2)

3.4.2 Parsing Into ASTs-4

❖ **Concrete Syntax**

let a be 3 in a * 12

tokstr

LET ID[*a*] BE NUM[3] IN ID[*a*] MUL NUM[12]

prg

id — exp — prg,exp,trm

trm — fct,id — trm,fct,num

fct

num

❖ **AST**

let[*a*] (num[3]; times (id[*a*]; num[12])) ast

2

2

3.4.2 Parsing Into ASTs-5

❖ **Theorem 8.1**

If s prg $\longleftrightarrow$ *a* ast, *then s* prg *and a* ast,

……(其他分析断言Parsing judgements有类似的性质)

证明：对规则(8.2)归纳证明.

❖ **Theorem 8.2**

If s prg, *then there is a unique a such that s* prg $\longleftrightarrow$ *a* ast

……(其他分析断言Parsing judgements有类似的性质)

分析断言具有模式 (∀, ∃!)

Why introduce ABT ?

manage the binding and scope of variables in a let expression

3.4.3 Parsing Into ABTs-1

Concrete Syntax	Token String	Abstract Syntax
n	NUM[*n*]	num[*n*]
"*s*"	LIT[*s*]	str[*s*]
s	ID[*s*]	id[*s*]
e_1+e_2	e_1 ADD e_2	plus(e_1; e_2)
e_1*e_2	e_1 MUL e_2	times(e_1; e_2)
e_1^e_2	e_1 CAT e_2	cat(e_1; e_2)
| *e* |	VB *e* VB	len(*e*)
let *s* be e_1 in e_2	LET ID[*s*] BE e_1 IN e_2	let(e_1; *x*.e_2)
(*e*)	LP *e* RP	*e*

3.4.3 Parsing Into ABTs-2

- **Goal**: manage the binding and scope of variables in a let expression
- **Arities to operators** (ABT of $\mathcal{L}$**{num, str}**):

$$\begin{aligned} \mathrm{ar}(\mathtt{num}[n]) &= () & \mathrm{ar}(\mathtt{plus}) &= (0,0)\\ \mathrm{ar}(\mathtt{str}[s]) &= () & \mathrm{ar}(\mathtt{times}) &= (0,0)\\ \mathrm{ar}(\mathtt{cat}) &= (0,0) & \mathrm{ar}(\mathtt{len}) &= (0)\\ \mathrm{ar}(\mathtt{let}) &= (0,1) \end{aligned}$$

  - **Identifiers**: not as operators, but as variables.
- **Revised parsing judgements** for $\mathcal{L}$**{num, str}**

$$s \text{ prg} \longleftrightarrow a \text{ abt}$$
$$\cdots$$

3.4.3 Parsing Into ABTs-3

修订后的分析断言可以用与规则(8.1)类似的一套规则来定义，这些规则采用参数化归纳定义，其中规则的前提和结论都是具有如下形式的假言断言：

$$\mathtt{ID}[s_1] \text{ id} \longleftrightarrow x_1 \text{ abt}, \cdots, \mathtt{ID}[s_n] \text{ id} \longleftrightarrow x_n \text{ abt} \vdash s \text{ prg} \longleftrightarrow a \text{ abt}$$

其中所有的x_i是互不相同的变量名。

断言的假设部分说明标识符如何分析为变量，它遵循假言断言的自反性质：

$$\Gamma, \mathtt{ID}[s] \text{ id} \longleftrightarrow x \text{ abt} \vdash \mathtt{ID}[s] \text{ id} \longleftrightarrow x \text{ abt}$$

3.4.3 Parsing Into ABTs-4

在分析let表达式时，为维护标识符与变量之间的关联关系，会更新假设部分，以记录绑定标识符和其对应的变量之间的关联关系：

去掉?

$$\frac{\Gamma \vdash s_1 \text{ id} \longleftrightarrow x \text{ abt} \quad \Gamma \vdash s_2 \text{ exp} \longleftrightarrow a_2 \text{ abt} \quad \Gamma, s_1 \text{ id} \longleftrightarrow x \text{ abt} \vdash s_3 \text{ prg} \longleftrightarrow a_3 \text{ abt}}{\Gamma \vdash \mathtt{LET}\ s_1\ \mathtt{BE}\ s_2\ \mathtt{IN}\ s_3 \text{ prg} \longleftrightarrow \mathtt{let}(a_2;\ x.a_3) \text{ abt}} \quad \textbf{(8.3a)}$$

问题：如果内层let表达式的绑定标识符与外层let表达式的绑定标识符一样，同一id对应不同的变量，如x1和x2. 由于假言断言的交换性，导致会任意选择x1或x2应用到s3中出现的id.

利用假设不能解决标识符同名问题。

3.4.3 Parsing Into ABTs-5

解决办法：显式地维护符号表来记录标识符与其对应的变量，从而实现同名标识符的shadowing策略

分析断言的主要变化是：由假言断言

$$\Gamma \vdash s \text{ prg} \longleftrightarrow a \text{ abt}$$

改成直言断言

$$s \text{ prg} \longleftrightarrow a \text{ abt } [\sigma]$$

σ是符号表，

符号表是断言的参数，而不是在假设下执行推理的隐式机制

关于符号表的断言：

σ symtab	well-formed symbol table
$\sigma' = \sigma[\mathtt{ID}[s] \mapsto x]$	add new association
$\sigma(\mathtt{ID}[s]) = x$	lookup identifier

3.4.3 Parsing Into ABTs-6

用于分析let表达式的规则：

去掉?

$$\frac{s_1 \text{ id} \longleftrightarrow x\ [\sigma] \quad s_2 \text{ exp} \longleftrightarrow a_2 \text{ abt } [\sigma] \quad \sigma' = \sigma[s_1 \mapsto x] \quad s_3 \text{ prg} \longleftrightarrow a_3 \text{ abt } [\sigma']}{\mathtt{LET}\ s_1\ \mathtt{BE}\ s_2\ \mathtt{IN}\ s_3 \text{ prg} \longleftrightarrow \mathtt{let}(a_2;\ x.a_3) \text{ abt } [\sigma]} \quad \textbf{(8.4)}$$

该规则与(8.3a)的区别在于：必须显式管理符号表

另外必须增加一条分析标识符的规则，而不是依靠假言断言的自反性：

$$\frac{\sigma(\mathtt{ID}[s]) = x}{\mathtt{ID}[s] \text{ id} \longleftrightarrow x\ [\sigma]} \quad \textbf{(8.5)}$$

σ maps the identifier ID[s] to the variable x.

3.4.3 Parsing Into ABTs-7

- **Concrete Syntax**

  let a be 3 in a * (1 + 2)

  tokstr

  LET ID[a] BE NUM[3] IN ID[a] MUL LP NUM[1] ADD NUM[2] RP

- **AST**

  let [a] (num[3]; times (id[a]; plus (num[1]; num[2]))) ast

- **ABT**

  let(num[3]; x.times (id[a]; plus (num[1]; num[2]))) abt

3.4.3 Syntactic Conventions

❖ **The abstract syntax of $\mathcal{L}$\{num str\}**

Type	τ	::=	num | str
Expr	e	::=	x | num[n] | str[s] | plus(e_1; e_2) | times(e_1; e_2) | cat(e_1; e_2) | len(e) | let(e_1; $x.e_2$)

τ type: **τ is a well-formed type** Ω_{type}

e exp : **e is a well-formed expression** Ω_{exp}

3.5 Static Semantics

- Consist of a collection of rules for imposing constraints on the formation of programs, called a type system.
- the type of a phrase predicts the form of its value
- well-typed: A phrase is constructed consistently with these predictions.
  ill-typed
  $x : \text{nat}$ $x + 3$ well-typed $x +'' 123''$ill-typed

3.5.1 Static Semantics of L{num,str}

3.5.2 Structural Properties

3.5.1 Static Semantics of $\mathcal{L}$\{num, str\} -1

❖ **Judgement**

e : τ, where e exp and τ type.

Parametric hypothetical judgements $\mathcal{X}|\Gamma \vdash e : \tau$

$\mathcal{X}$: a finite set of variables, 通常被省去

Γ : a typing context ($x : \tau, x \in \mathcal{X}$)

❖ **Typing Rules**

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \quad (9.1a)$$

$$\frac{}{\Gamma \vdash \texttt{str}[s] : \texttt{str}} \quad (9.1b)$$

$$\frac{}{\Gamma \vdash \texttt{num}[n] : \texttt{num}} \quad (9.1c)$$

3.5.1 Static Semantics of $\mathcal{L}$\{num, str\} -2

❖ **Typing Rules**

$$\frac{\Gamma \vdash e_1 : \texttt{num} \quad \Gamma \vdash e_2 : \texttt{num}}{\Gamma \vdash \texttt{plus}(e_1; e_2) : \texttt{num}} \quad (9.1d)$$

$$\frac{\Gamma \vdash e_1 : \texttt{num} \quad \Gamma \vdash e_2 : \texttt{num}}{\Gamma \vdash \texttt{times}(e_1; e_2) : \texttt{num}} \quad (9.1e)$$

$$\frac{\Gamma \vdash e_1 : \texttt{str} \quad \Gamma \vdash e_2 : \texttt{str}}{\Gamma \vdash \texttt{cat}(e_1, e_2) : \texttt{str}} \quad (9.1f)$$

$$\frac{\Gamma \vdash e : \texttt{str}}{\Gamma \vdash \texttt{len}(e) : \texttt{num}} \quad (9.1g)$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{let}(e_1, x.e_2) : \tau_2} \quad (9.1h)$$

x不在Γ中，若e_1中有x，则通过α-等价将e_1中的x换名

3.5.1 Static Semantics of $\mathcal{L}$\{num, str\} -3

❖ **Lemma 9.1 (Unicity for Typing).**

For every typing context Γ and expression e, there exists at most one τ such that $\Gamma \vdash e : \tau$.

❖ **Lemma 9.2 (Inversion for Typing).**

Suppose that $\Gamma \vdash e : \tau$.

1. if e = x, then $\Gamma \vdash x : \tau$.
2. if e = num[n], then τ = num.
3. if e = str[s], then τ = str.
4. if e = plus(e_1; e_2) or e = times(e_1; e_2), then τ = num, $\Gamma \vdash e_1 : \texttt{num}$ and $\Gamma \vdash e_2 : \texttt{num}$.

……

3.5.1 Static Semantics of $\mathcal{L}$\{num, str\} -4

❖ **e.g.Lemma**

let (str[1], x.cat(str[123], x)) :str

$$\frac{\Gamma \vdash \texttt{str}[1] : \texttt{str} \quad \dfrac{\dfrac{}{\Gamma \vdash \texttt{str}[123] : \texttt{str}} \quad \dfrac{}{\Gamma, x : \texttt{str} \vdash x : \texttt{str}}}{\Gamma, a : \texttt{str} \vdash \texttt{cat}(\texttt{str}[123]; a) : \texttt{str}}}{\Gamma \vdash \texttt{let}(\texttt{str}[1]; x.\texttt{cat}(\texttt{str}[123]; x) : \texttt{str}}$$

3.5.2 Structural Properties-1

静态语义具有假言断言和参数化断言的结构性质。

❖ **Lemma 9.3** (proliferation)

If $\Gamma \vdash e' : \tau'$, then for any $x \# \Gamma$ and any τ type, $\Gamma, x : \tau \vdash e' : \tau'$.

❖ **Lemma 9.4** (substitution)

If $\Gamma, x : \tau \vdash e' : \tau'$ and $\Gamma \vdash e : \tau$, then $\Gamma \vdash [e/x]e' : \tau'$.

3.5.2 Structural Properties-2

❖ **Lemma 9.3** (proliferation)

If $\Gamma \vdash e' : \tau'$, then for any $x \# \Gamma$ and any τ type, $\Gamma, x : \tau \vdash e' : \tau'$.

Proof

By induction on the derivation of $\Gamma, x : \tau \vdash e' : \tau'$

Suppose $e' = \mathtt{let}(e_1, z.e_2)$, where $z \# \Gamma$ and $z \# x$

By induction we have

(A) $\Gamma, x : \tau \vdash e_1 : \tau_1$,

(B) $\Gamma, x : \tau, z : \tau_1 \vdash e_2 : \tau'$,

from which the result follows by Rule (9.1h).

Other cases….

3.5.2 Structural Properties-3

❖ **Lemma 9.4** (substitution)

If $\Gamma, x : \tau \vdash e' : \tau'$ and $\Gamma \vdash e : \tau$, then $\Gamma \vdash [e/x]e' : \tau'$.

Proof

By induction on $\Gamma, x : \tau \vdash e' : \tau'$

Suppose $e' = \mathtt{let}(e_1, z.e_2)$, where $z \# \Gamma$, $z \# x$ and $z \# e$

By induction we have

(A) $\Gamma \vdash [e/x]e_1 : \tau_1$,

(B) $\Gamma, z : \tau_1 \vdash [e/x]e_2 : \tau'$,

Since z#e, we have $[e/x]\mathtt{let}(e_1, z.e_2) = \mathtt{let}([e/x]e_1, z.[e/x]e_2)$

It follows by Rule (9.1h) that $\Gamma \vdash [e/x]\mathtt{let}(e_1, z.e_2) : \tau$

Other cases….

3.5.2 Structural Properties-4

❖ **Lemma 9.5** (descomposition)

If $\Gamma \vdash [e/x]e' : \tau'$ then there exists a unique type τ such that $\Gamma \vdash e : \tau$, $\Gamma, x : \tau \vdash e' : \tau'$.

Proof

Directly from the unicity of types (Lemma 9.1)

since τ is the unique type for e in the composite expression [e/x]e'.

3.6 Dynamic Semantics

- Specify how programs are to be executed.
- Methods for specifying dynamic semantics
  - Structural semantics: small-step OS.
    - Contextual semantics
  - Evaluation semantics: big-step OS
    - Environment semantics, cost semantics

3.6.1 Transition Systems [PFPL, 4]

3.6.2 Structural semantics [PFPL, 10]

3.6.3 Contextual semantics [PFPL, 10]

3.6.4 Evaluation semantics [PFPL, 12]

3.6.5 Environment semantics and Cost semantics

3.6.1 Transition Systems-1

❖ **Transition system**

S: a set of states that are related by a transition judgement,

An transition system is specified by the judgements

s state, s final, s initial, $s \mapsto s'$

- A state s is stuck, if there is no $s' \in S$ such that $s \mapsto s'$.

All final states are stuck, but not all stuck states need be final!

3.6.1 Transition Systems-2

❖ **Transition Sequence:** a sequence of states $s_0, \cdots, s_n$ such that s_0 initial and $s_i \mapsto s_{i+1}, 0 \le i < n$

A transition sequence is
- maximal: iff $s_n \not\mapsto$
- complete: iff $s_n \not\mapsto$ and s_n final
- deterministic: iff for every state s there exists at most one state s' such that $s \mapsto s'$, otherwise it is non-deterministic.

3.6.1 Transition Systems-3

❖ **Iterated Transition**

$s \mapsto^* s'$: is the reflexive, transitive closure of $s \mapsto s'$

Rules

$$\frac{}{s \mapsto^* s} \quad (4.1a)$$

$$\frac{s \mapsto s' \quad s' \mapsto^* s''}{s \mapsto^* s''} \quad (4.1b)$$

Principle of rule induction

To show that P(s,s') holds whenever $s \mapsto^* s'$,
it is enough to show:
1) P(s,s)
2) if $s \mapsto s'$ and P(s',s''), then P(s, s'') .

3.6.1 Transition Systems-4

❖ **Iterated Transition**

$s \mapsto^n s'$: n-times iterated transition judgement, $n \ge 0$

$$\frac{}{s \mapsto^0 s} \quad (4.2a)$$

$$\frac{s \mapsto s' \quad s' \mapsto^n s''}{s \mapsto^{n+1} s''} \quad (4.2b)$$

Theorem 4.1

For all states s and s', $s \mapsto^* s'$ iff $s \mapsto^k s'$ for some $k \ge 0$.

$\downarrow s$: indicate there exists some s' final such that $s \mapsto^* s'$

3.6.2 Structural semantics-1

❖ **Structural semantics of $\mathcal{L}\{\text{num, str}\}$**
- a transition system whose states are closed expressions.
- Every closed expression is an initial state
- The final states are the closed values, as defined by

$$\frac{}{\mathtt{num}[n]\ \mathsf{val}} \quad \frac{}{\mathtt{str}[s]\ \mathsf{val}} \quad (10.1)$$

❖ **Transition judgement** $e \mapsto e'$

Rule

$$\frac{n_1 + n_2 = n\ \mathsf{nat}}{\mathtt{plus}(\mathtt{num}[n_1];\mathtt{num}[n_2]) \mapsto \mathtt{num}[n]} \quad (10.2a)$$

$$\frac{e_1 \mapsto e_1'}{\mathtt{plus}(e_1;e_2) \mapsto \mathtt{plus}(e_1',e_2)} \quad (10.2b)$$

3.6.2 Structural semantics-2

❖ **Rule**

instruction transitions (10.2a, d, g) Primitive steps of evaluation

search transitions Determine the evaluation order

$$\frac{e_1\ \mathsf{val} \quad e_2 \mapsto e_2'}{\mathtt{plus}(e_1;e_2) \mapsto \mathtt{plus}(e_1;e_2')} \quad (10.2c)$$

$$\frac{s_1 \hat{}\, s_2 = s\ \mathsf{str}}{\mathtt{cat}(\mathtt{str}[s_1];\mathtt{str}[s_2]) \mapsto \mathtt{str}[s]} \quad (10.2d)$$

$$\frac{e_1 \mapsto e_1'}{\mathtt{cat}(e_1;e_2) \mapsto \mathtt{cat}(e_1';e_2)} \quad (10.2e)$$

$$\frac{e_1\ \mathsf{val} \quad e_2 \mapsto e_2'}{\mathtt{cat}(e_1;e_2) \mapsto \mathtt{cat}(e_1,e_2')} \quad (10.2f)$$

$$\frac{e_1\ \mathsf{val}}{\mathtt{let}(e_1;x.e_2) \mapsto [e_1/x]e_2} \quad (10.2g)$$

$$\frac{e_1 \mapsto e_1'}{\mathtt{let}(e_1;x.e_2) \mapsto \mathtt{let}(e_1';x.e_2)} \quad (10.2h)$$

3.6.2 Structural semantics-3

❖ **Derivation sequence:**

width: the number of steps in the sequence

$$\begin{aligned}
&\mathtt{let}(\mathtt{plus}(\mathtt{num}[1];\mathtt{num}[2]),x.\mathtt{plus}(\mathtt{plus}(x;\mathtt{num}[3]);\mathtt{num}[4]))\\
\mapsto\ &\mathtt{let}(\mathtt{num}[3];x.\mathtt{plus}(\mathtt{plus}(x;\mathtt{num}[3]);\mathtt{num}[4]))\\
\mapsto\ &\mathtt{plus}(\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]);\mathtt{num}[4])\\
\mapsto\ &\mathtt{plus}(\mathtt{num}[6];\mathtt{num}[4])\\
\mapsto\ &\mathtt{num}[10]
\end{aligned}$$

depth: the derivation tree for each step

e.g. the third transition is

$$\frac{\overline{\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]) \mapsto \mathtt{num}[6]}\,(10.2a)}{\mathtt{plus}(\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]);\mathtt{num}[4]) \mapsto \mathtt{plus}(\mathtt{num}[6];\mathtt{num}[4])}\,(10.2b)$$

3.6.2 Structural semantics-4

❖ **Principle of rule induction**

To show P(e, e') holds whenever $e \mapsto e'$, it is sufficient to show that P is closed under the rules defining the transition judgement.

❖ **Lemma 10.1 (evaluation of exp.s is deterministic)**

If $e \mapsto e'$ and $e \mapsto e''$, then e' is e''.

Proof. By simultaneous induction on the two premises using Rules (10.2).

Only one rule applies for a given e.

3.6.3 Contextual semantics-1

❖ **Contextual semantics**: variant of structural semantics

- isolate instruction steps as instruction transition judgements $e_1 \rightsquigarrow e_2$

$$\frac{m + n = p \text{ nat}}{\texttt{plus}(\texttt{num}[m];\texttt{num}[n]) \rightsquigarrow \texttt{num}[p]} \quad (10.3a)$$

$$\frac{s \hat{} t = u \text{ str}}{\texttt{cat}(\texttt{str}[s];\texttt{str}[t]) \rightsquigarrow \texttt{str}[u]} \quad (10.3b)$$

$$\frac{e_1 \text{ val}}{\texttt{let}(e_1; x.e_2) \rightsquigarrow [e_1/x]e_2} \quad (10.3c)$$

redax: LHS of each intruction; contractum: RHS

- formalize the process of locating the next instruction using an evaluation context.

3.6.3 Contextual semantics-2

Evaluation Context Judgement

$\mathcal{E}$ ectxt : determines the location of the next instruction to execute in a larger expression.

$\circ$:"hole", the position of the next instruction step

e.g. $\circ$ can be one of the following forms

plus(num[n1]; num[n2])

cat(str[s1]; str[s2])

let(e1; x.e2), where e1 val

……

3.6.3 Contextual semantics-3

Evaluation Context Judgement

Rules $\frac{}{\circ \text{ ectxt}}$

It means the next instruction may occur "here", i.e.

$$\frac{\mathcal{E}_1 \text{ ectxt}}{\texttt{plus}(\mathcal{E}_1; e_2) \text{ ectxt}} \qquad \frac{e_1 \text{ val} \quad \mathcal{E}_2 \text{ ectxt}}{\texttt{plus}(e_1; \mathcal{E}_2) \text{ ectxt}}$$

$$\frac{\mathcal{E}_1 \text{ ectxt}}{\texttt{cat}(\mathcal{E}_1; e_2) \text{ ectxt}} \qquad \frac{e_1 \text{ val} \quad \mathcal{E}_2 \text{ ectxt}}{\texttt{cat}(\mathcal{E}_1; e_2) \text{ ectxt}}$$

$$\frac{\mathcal{E}_1 \text{ ectxt}}{\texttt{let}(\mathcal{E}_1; x.e_2) \text{ ectxt}} \quad (10.4)$$

The remaining rules correspond one-for-one to the search rules of the structural semantics.

3.6.3 Contextual semantics-4

❖ **Evaluation Context**

- a template instantiated by replacing the hole with an instruction to be executed.

Judgement $e' = \mathcal{E}\{e\}$

e' is the result of filling the hole in $\mathcal{E}$ with e.

(10.5)

$$\frac{}{e = \circ\{e\}} \qquad \frac{e_1 = \mathcal{E}_1\{e\}}{\texttt{cat}(e_1; e_2) = \texttt{cat}(\mathcal{E}_1; e_2)\{e\}}$$

$$\frac{e_1 = \mathcal{E}_1\{e\}}{\texttt{plus}(e_1; e_2) = \texttt{plus}(\mathcal{E}_1; e_2)\{e\}} \qquad \frac{e_1 \text{ val} \quad e_1 = \mathcal{E}_2\{e\}}{\texttt{cat}(e_1; e_2) = \texttt{cat}(e_1; \mathcal{E}_2)\{e\}}$$

$$\frac{e_1 \text{ val} \quad e_1 = \mathcal{E}_2\{e\}}{\texttt{plus}(e_1; e_2) = \texttt{plus}(e_1; \mathcal{E}_2)\{e\}} \qquad \frac{e_1 = \mathcal{E}_1\{e\}}{\texttt{let}(e_1; x.e_2) = \texttt{let}(\mathcal{E}_1; x.e_2)\{e\}}$$

3.6.3 Contextual semantics-4

❖ **Dynamic semantics for $\mathcal{L}$\{num str\}**

$$\frac{e = \mathcal{E}\{e_0\} \quad e_0 \rightsquigarrow e_0' \quad e' = \mathcal{E}\{e_0'\}}{e \mapsto e'} \quad (10.6)$$

A transition from e to e' consists of

1. decomposing e into an evaluation context and an instruction,
2. execution of that instruction, and
3. replacing the instruction by the result of its execution in the same spot within e to obtain e'.

3.6.3 Contextual semantics-5

The structural and contextual semantics define the same transition relation.

Theorem 10.2. $e \mapsto_s e'$ if, and only if, $e \mapsto_c e'$.

$$\frac{e = \mathcal{E}\{e_0\} \quad e_0 \rightsquigarrow e_0' \quad e' = \mathcal{E}\{e_0'\}}{e \mapsto e'} \quad (10.6)$$

$$\frac{e_0 \rightsquigarrow e_0'}{\mathcal{E}\{e_0\} \mapsto \mathcal{E}\{e_0'\}} \quad (10.7)$$

3.6.4 Evaluation semantics-1

- **Evaluation semantics(ES)**: a relation between a phrase and its value.
- **Evaluation judgement** $e \Downarrow v$

  specify the value v of a closed expression e

$$\frac{}{\mathtt{num}[n] \Downarrow \mathtt{num}[n]} \qquad \frac{e_1 \Downarrow \mathtt{num}[n_1] \quad e_2 \Downarrow \mathtt{num}[n_2] \quad n_1 + n_2 = n \ \mathsf{nat}}{\mathtt{plus}(e_1; e_2) \Downarrow \mathtt{num}[n]}$$

$$\frac{}{\mathtt{str}[s] \Downarrow \mathtt{str}[s]} \qquad \frac{e_1 \Downarrow \mathtt{str}[s_1] \quad e_2 \Downarrow \mathtt{str}[s_2] \quad s_1 \hat{}\, s_2 = s \ \mathsf{str}}{\mathtt{cat}(e_1; e_2) \Downarrow \mathtt{str}[s]}$$

$$\frac{e_1 \Downarrow v_1 \quad [v_1/x]e_2 \Downarrow v_2}{\mathtt{let}(e_1; x.e_2) \Downarrow v_2} \quad (12.1)$$

3.6.4 Evaluation semantics-2

- **Principle of rule induction**

  To show P(e, v) holds, it is enough to show that P is closed under the rules defining the evaluation judgement.

  1. Show that P(num[n], num[n]) .
  2. Show that P(str[s], str[s]) .
  3. Show that P(plus(e_1; e_2), num[n]), assuming $n_1 + n_2 = n$ nat, P(e_1, num[n_1]) and P(e_2, num[n_2]).
  4. Show that P(cat(e_1; e_2), str[s]), assuming s_1^s_2 = s str, P(e_1, str[s_1]) and P(e_2, str[s_2]).
  5. Show that P(let(e_1; x.e_2), v_2) , assuming P(e_1, v_1) and P([v_1/x]e_2, v_2).

  Lemma 12.1 If $e \Downarrow v$, then v val.

3.6.4 Evaluation semantics-3

Theorem 12.2 For all closed expressions e and values v, $e \mapsto^* v$ iff $e \Downarrow v$.

Lemma 12.3 If $e \Downarrow v$, then $e \mapsto^* v$.

Proof. By induction on the definition of the evaluation judgement.

Suppose: $\mathtt{plus}(e_1; e_2) \Downarrow \mathtt{num}[n]$ by the rule (12.1).

By induction, $e_1 \mapsto^* \mathtt{num}[n_1]$ and $e_2 \mapsto^* \mathtt{num}[n_2]$

$$\begin{aligned} \mathtt{plus}(e_1; e_2) &\mapsto^* \mathtt{plus}(\mathtt{num}[n_1]; e_2) \\ &\mapsto^* \mathtt{plus}(\mathtt{num}[n_1]; \mathtt{num}[n_2]) \\ &\mapsto \mathtt{num}[n_1 + n_2] \end{aligned}$$

3.6.4 Evaluation semantics-4

Lemma 12.4 If $e \mapsto e'$ and $e' \Downarrow v$, then $e \Downarrow v$

Proof. By induction on the definition of the transition judgement.

Suppose: $\mathtt{plus}(e_1; e_2) \mapsto \mathtt{plus}(e_1'; e_2)$ where $e_1 \mapsto e_1'$ by the rule (10.2).

Suppose further: $\mathtt{plus}(e_1'; e_2) \Downarrow \mathtt{num}[n]$ so that $e_1' \Downarrow \mathtt{num}[n_1]$ and $e_2 \Downarrow \mathtt{num}[n_2]$ and $n_1 + n_2 = n \ \mathsf{nat}$

By induction, $e_1 \Downarrow \mathtt{num}[n_1]$ and hence

$$\mathtt{plus}(e_1; e_2) \Downarrow \mathtt{num}[n]$$

3.6.5 Environment S. and Cost S.-1

- **Environment semantics**
  - Substitution: replace let-bound variables by their bindings during evaluation.
    Maintain the invariant that only closed expressions are ever considered

  In practice, we do not perform substitution

  - record the bindings of variables in some sort of data structure
  - **environment** $\mathcal{E}$: set of hypotheses of the form $x \Downarrow v$, x is a variable, v is a value

3.6.5 Environment S. and Cost S.-2

❖ **Environment semantics**

Judgement $\mathcal{E} \vdash e \Downarrow v$

$\mathcal{E}$:an env. governing some finite set of variables

Rules

$$\frac{}{\mathcal{E}, x \Downarrow v \vdash x \Downarrow v}$$

$$\frac{\mathcal{E} \vdash e_1 \Downarrow \texttt{num}[n_1] \quad \mathcal{E} \vdash e_2 \Downarrow \texttt{num}[n_2]}{\mathcal{E} \vdash \texttt{plus}(e_1; e_2) \Downarrow \texttt{num}[n_1 + n_2]}$$

$$\frac{\mathcal{E} \vdash e_1 \Downarrow \texttt{str}[s_1] \quad \mathcal{E} \vdash e_2 \Downarrow \texttt{num}[s_2]}{\mathcal{E} \vdash \texttt{cat}(e_1; e_2) \Downarrow \texttt{str}[s_1 \hat{}\, s_2]} \qquad \textbf{(12.2)}$$

$$\frac{\mathcal{E} \vdash e_1 \Downarrow v_1 \quad \mathcal{E}, x \Downarrow v_1 \vdash e_2 \Downarrow v_2}{\mathcal{E} \vdash \texttt{let}(e_1; x.e_2) \Downarrow v_2}$$

3.6.5 Environment S. and Cost S.-3

❖ **Cost semantics**

➢ SS provides time complexity for program,but ES does not provide such a direct notion of complexity

Judgement : $e \Downarrow^n v$, e evaluates to v in n steps

Rules

$$\frac{}{\texttt{num[n]} \Downarrow^0 \texttt{num[n]}} \qquad \frac{}{\texttt{str[s]} \Downarrow^0 \texttt{str[s]}}$$

$$\frac{e_1 \Downarrow^{k_1} \texttt{num}[n_1] \quad e_2 \Downarrow^{k_2} \texttt{num}[n_2]}{\texttt{plus}(e_1; e_2) \Downarrow^{k_1+k_2+1} \texttt{num}[n_1 + n_2]}$$

$$\frac{e_1 \Downarrow^{k_1} \texttt{str}[s_1] \quad e_2 \Downarrow^{k_2} \texttt{str}[s_2]}{\texttt{cat}(e_1; e_2) \Downarrow^{k_1+k_2+1} \texttt{str}[s_1 + s_2]} \qquad \textbf{(12.3)}$$

$$\frac{e_1 \Downarrow^{k_1} \quad [v_1/x]e_2 \Downarrow^{k_2} v_2}{\texttt{let}(e_1; x.e_2) \Downarrow^{k_1+k_2+1} v_2}$$

一个例子

❖ 程序

let a be 3+3 in let b be 4 in a+b

❖ 记号串

LET ID[a] BE NUM[3] ADD NUM[3] IN LET ID[b] BE NUM[4] IN ID[a] ADD ID[b]

❖ 分析生成抽象绑定树(**ABT**)

Factor	fct	::=	num | lit | id | LP prg RP
Term	trm	::=	fct | fct MUL trm | VB fct VB
Expression	exp	::=	trm | trm ADD exp | trm CAT exp
Program	prg	::=	exp | LET id BE trm IN prg

一个例子-分析生成ABT

LET ID[a] BE NUM[3] ADD NUM[3] IN LET ID[b] BE NUM[4] IN ID[a] ADD ID[b]

Factor	fct	::=	num | lit | id | LP prg RP
Term	trm	::=	fct | fct MUL trm | VB fct VB
Expression	exp	::=	trm | trm ADD exp | trm CAT exp
Program	prg	::=	exp | LET id BE trm IN prg

$$\Gamma, \texttt{ID}[s]\ \text{id} \longleftrightarrow x\ \text{abt} \vdash \texttt{ID}[s]\ \text{id} \longleftrightarrow x\ \text{abt}$$

$$\frac{\Gamma \vdash s_2\ \text{exp} \longleftrightarrow a_2\ \text{abt} \quad \Gamma, s_1\ \text{id} \longleftrightarrow x\ \text{abt} \vdash s_3\ \text{prg} \longleftrightarrow a_3\ \text{abt}}{\Gamma \vdash \texttt{LET}\ s_1\ \texttt{BE}\ s_2\ \texttt{IN}\ s_3\ \text{prg} \longleftrightarrow \texttt{let}(a_2;\ x.a_3)\ \text{abt}}$$

$$\frac{}{\mathcal{A}, x\ abt^0 \vdash x\ abt^0}$$

$$\frac{\text{ar}\ (o) = (n_1, \cdots, n_k) \quad \mathcal{A} \vdash a_1\ \text{abt}^{n_1} \quad \cdots \quad \mathcal{A} \vdash a_k\ \text{abt}^{n_k}}{\mathcal{A} \vdash o(a_1, \cdots, a_k)\ \text{abt}^0}$$

$$\frac{x \# \mathcal{A} \quad \mathcal{A}, x\ \text{abt}^0 \vdash a\ \text{abt}^n}{\mathcal{A} \vdash x.a\ \text{abt}^{n+1}}$$

自底向上构造（ Γ 被省去）

➢ 对**ID[a]**和**ID[b]**分析，得到对应的抽象绑定树

$$\frac{a\ \text{str}}{\texttt{ID}[a]\ \text{id} \longleftrightarrow x_1\ \text{abt}^0 \vdash \texttt{ID}[a]\ \text{id} \longleftrightarrow x_1\ \text{abt}^0}$$

$$\frac{b\ \text{str}}{\texttt{ID}[b]\ \text{id} \longleftrightarrow x_2\ \text{abt}^0 \vdash \texttt{ID}[b]\ \text{id} \longleftrightarrow x_2\ \text{abt}^0}$$

一个例子-分析生成ABT

LET ID[a] BE NUM[3] ADD NUM[3] IN LET ID[b] BE NUM[4] IN ID[a] ADD ID[b]

Factor	fct	::=	num | lit | id | LP prg RP
Term	trm	::=	fct | fct MUL trm | VB fct VB
Expression	exp	::=	trm | trm ADD exp | trm CAT exp
Program	prg	::=	exp | LET id BE trm IN prg

$$\Gamma, \texttt{ID}[s]\ \text{id} \longleftrightarrow x\ \text{abt} \vdash \texttt{ID}[s]\ \text{id} \longleftrightarrow x\ \text{abt}$$

$$\frac{\Gamma \vdash s_2\ \text{exp} \longleftrightarrow a_2\ \text{abt} \quad \Gamma, s_1\ \text{id} \longleftrightarrow x\ \text{abt} \vdash s_3\ \text{prg} \longleftrightarrow a_3\ \text{abt}}{\Gamma \vdash \texttt{LET}\ s_1\ \texttt{BE}\ s_2\ \texttt{IN}\ s_3\ \text{prg} \longleftrightarrow \texttt{let}(a_2;\ x.a_3)\ \text{abt}}$$

$$\frac{}{\mathcal{A}, x\ abt^0 \vdash x\ abt^0}$$

$$\frac{\text{ar}\ (o) = (n_1, \cdots, n_k) \quad \mathcal{A} \vdash a_1\ \text{abt}^{n_1} \quad \cdots \quad \mathcal{A} \vdash a_k\ \text{abt}^{n_k}}{\mathcal{A} \vdash o(a_1, \cdots, a_k)\ \text{abt}^0}$$

$$\frac{x \# \mathcal{A} \quad \mathcal{A}, x\ \text{abt}^0 \vdash a\ \text{abt}^n}{\mathcal{A} \vdash x.a\ \text{abt}^{n+1}}$$

➢ 对**NUM[3]**和**NUM[4]**分析，得到对应的抽象绑定树

$$\cfrac{\cfrac{\cfrac{\cfrac{3\ \text{nat}}{\vdash \texttt{NUM}[3]\ \text{num} \longleftrightarrow \texttt{num}[3]\ \text{abt}^0}}{\vdash \texttt{NUM}[3]\ \text{fct} \longleftrightarrow \texttt{num}[3]\ \text{abt}^0}}{\vdash \texttt{NUM}[3]\ \text{trm} \longleftrightarrow \texttt{num}[3]\ \text{abt}^0}}{\vdash \texttt{NUM}[3]\ \text{exp} \longleftrightarrow \texttt{num}[3]\ \text{abt}^0} \qquad \cfrac{\cfrac{\cfrac{\cfrac{4\ \text{nat}}{\vdash \texttt{NUM}[4]\ \text{num} \longleftrightarrow \texttt{num}[4]\ \text{abt}^0}}{\vdash \texttt{NUM}[4]\ \text{fct} \longleftrightarrow \texttt{num}[4]\ \text{abt}^0}}{\vdash \texttt{NUM}[4]\ \text{trm} \longleftrightarrow \texttt{num}[4]\ \text{abt}^0}}{\vdash \texttt{NUM}[4]\ \text{exp} \longleftrightarrow \texttt{num}[4]\ \text{abt}^0}$$

一个例子-分析生成ABT

LET ID[a] BE NUM[3] ADD NUM[3] IN LET ID[b] BE NUM[4] IN ID[a] ADD ID[b]

Factor	fct	::=	num | lit | id | LP prg RP
Term	trm	::=	fct | fct MUL trm | VB fct VB
Expression	exp	::=	trm | trm ADD exp | trm CAT exp
Program	prg	::=	exp | LET id BE trm IN prg

$$\Gamma, \texttt{ID}[s]\ \text{id} \longleftrightarrow x\ \text{abt} \vdash \texttt{ID}[s]\ \text{id} \longleftrightarrow x\ \text{abt}$$

$$\frac{\Gamma \vdash s_2\ \text{exp} \longleftrightarrow a_2\ \text{abt} \quad \Gamma, s_1\ \text{id} \longleftrightarrow x\ \text{abt} \vdash s_3\ \text{prg} \longleftrightarrow a_3\ \text{abt}}{\Gamma \vdash \texttt{LET}\ s_1\ \texttt{BE}\ s_2\ \texttt{IN}\ s_3\ \text{prg} \longleftrightarrow \texttt{let}(a_2;\ x.a_3)\ \text{abt}}$$

$$\frac{}{\mathcal{A}, x\ abt^0 \vdash x\ abt^0}$$

$$\frac{\text{ar}\ (o) = (n_1, \cdots, n_k) \quad \mathcal{A} \vdash a_1\ \text{abt}^{n_1} \quad \cdots \quad \mathcal{A} \vdash a_k\ \text{abt}^{n_k}}{\mathcal{A} \vdash o(a_1, \cdots, a_k)\ \text{abt}^0}$$

$$\frac{x \# \mathcal{A} \quad \mathcal{A}, x\ \text{abt}^0 \vdash a\ \text{abt}^n}{\mathcal{A} \vdash x.a\ \text{abt}^{n+1}}$$

➢ 对**ID[a] ADD ID[b]**分析，得到对应的抽象绑定树

$$\frac{ar(\texttt{plus}) = (0,0) \quad x_1\ \text{abt}^0 \vdash x_1\ \text{abt}^0 \quad x_2\ \text{abt}^0 \vdash x_2\ \text{abt}^0}{x_1\ \text{abt}^0, x_2\ \text{abt}^0 \vdash \texttt{plus}(x_1; x_2)\ \text{abt}^0}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{a\ \text{str}}{\texttt{ID}[a]\ \text{id} \longleftrightarrow x_1\ \text{abt}^0}}{\texttt{ID}[a]\ \text{fct} \longleftrightarrow x_1\ \text{abt}^0}}{\texttt{ID}[a]\ \text{trm} \longleftrightarrow x_1\ \text{abt}^0} \quad \cfrac{\cfrac{\cfrac{\cfrac{b\ \text{str}}{\texttt{ID}[b]\ \text{id} \longleftrightarrow x_2\ \text{abt}^0}}{\texttt{ID}[b]\ \text{fct} \longleftrightarrow x_2\ \text{abt}^0}}{\texttt{ID}[b]\ \text{trm} \longleftrightarrow x_2\ \text{abt}^0}}{\texttt{ID}[b]\ \text{exp} \longleftrightarrow x_2\ \text{abt}^0}}{\texttt{ID}[a]\ \texttt{ADD}\ \texttt{ID}[b]\ \text{exp} \longleftrightarrow \texttt{plus}(x_1; x_2)\ \text{abt}^0}$$

An Example-Parsing into ABT

LET ID[a] BE NUM[3] ADD NUM[3] IN LET ID[b] BE NUM[4] IN ID[a] ADD ID[b]

Factor	fct	::=	num | lit | id | LP prg RP
Term	trm	::=	fct | fct MUL trm | VB fct VB
Expression	exp	::=	trm | trm ADD exp | trm CAT exp
Program	prg	::=	exp | LET id BE trm IN prg

$$\Gamma, \mathtt{ID}[s]\ \mathsf{id} \longleftrightarrow x\ \mathsf{abt} \vdash \mathtt{ID}[s]\ \mathsf{id} \longleftrightarrow x\ \mathsf{abt}$$

$$\frac{\Gamma \vdash s_2\ \mathsf{exp} \longleftrightarrow a_2\ \mathsf{abt} \quad \Gamma, s_1\ \mathsf{id} \longleftrightarrow x\ \mathsf{abt} \vdash s_3\ \mathsf{prg} \longleftrightarrow a_3\ \mathsf{abt}}{\Gamma \vdash \mathtt{LET}\ s_1\ \mathtt{BE}\ s_2\ \mathtt{IN}\ s_3\ \mathsf{prg} \longleftrightarrow \mathtt{let}(a_2;\ x.a_3)\ \mathsf{abt}}$$

$$\frac{}{\mathcal{A}, x\ abt^0 \vdash x\ abt^0}$$

$$\frac{\mathrm{ar}\ (o) = (n_1, \cdots, n_k) \quad \mathcal{A} \vdash a_1\ \mathrm{abt}^{n_1} \quad \cdots \quad \mathcal{A} \vdash a_k\ \mathrm{abt}^{n_k}}{\mathcal{A} \vdash o(a_1, \cdots, a_k)\ \mathrm{abt}^0}$$

$$\frac{x \# \mathcal{A} \quad \mathcal{A}, x\ \mathrm{abt}^0 \vdash a\ \mathrm{abt}^n}{\mathcal{A} \vdash x.a\ \mathrm{abt}^{n+1}}$$

- 对LET ID[b] BE NUM[4] IN ID[a] ADD ID[b]分析

$$\frac{x_1\ \mathrm{abt}^0, x_2\ \mathrm{abt}^0 \vdash \mathtt{plus}(x_1;x_2)\ \mathrm{abt}^0}{x_1\ \mathrm{abt}^0 \vdash x_2.\mathtt{plus}(x_1;x_2))\ \mathrm{abt}^1}$$

$$\frac{ar(\mathtt{let}) = (0,1) \quad \vdash \mathtt{num}[4]\ \mathrm{abt}^0 \quad x_1\ \mathrm{abt}^0 \vdash x_2.\mathtt{plus}(x_1;x_2)\ \mathrm{abt}^1}{x_1\ \mathrm{abt}^0 \vdash \mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))\ \mathrm{abt}^0}$$

$$\frac{\vdash \mathtt{NUM}[4]\ \mathrm{exp} \longleftrightarrow \mathtt{num}[4]\ \mathrm{abt}^0 \quad \mathtt{ID}[a]\ \mathrm{id} \longleftrightarrow x_1\ \mathrm{abt}^0, \mathtt{ID}[b]\ \mathrm{id} \longleftrightarrow x_2\ \mathrm{abt}^0 \vdash \mathtt{ID}[a]\ \mathtt{ADD}\ \mathtt{ID}[b]\ \mathrm{prg} \longleftrightarrow \mathtt{plus}(x_1;x_2)\ \mathrm{abt}^0}{\mathtt{ID}[a]\ \mathrm{id} \longleftrightarrow x_1\ \mathrm{abt}^0 \vdash \mathtt{LET}\ \mathtt{ID}[b]\ \mathtt{BE}\ \mathtt{NUM}[4]\ \mathtt{IN}\ \mathtt{ID}[a]\ \mathtt{ADD}\ \mathtt{ID}[b]\ \mathrm{exp} \longleftrightarrow \mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))\ \mathrm{abt}^0}$$

一个例子-静态语义

- ABT let(plus(num[3];num[3]); x_1.let(num[4]; x_2.plus(x_1; x_2)))
- 类型检查 (static semantics)

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{}{\Gamma \vdash \mathtt{num}[n] : \mathtt{num}} \qquad \frac{\Gamma \vdash e_1 : \mathtt{num} \quad \Gamma \vdash e_2 : \mathtt{num}}{\Gamma \vdash \mathtt{plus}(e_1;e_2) : \mathtt{num}} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \mathtt{let}(e_1;x.e_2) : \tau_2}$$

自底向上的类型检查

$$\frac{x_1 : \mathtt{num} \vdash x_1 : \mathtt{num} \quad x_2 : \mathtt{num} \vdash x_2 : \mathtt{num}}{x_1 : \mathtt{num}, x_2 : \mathtt{num} \vdash \mathtt{plus}(x_1;x_2) : \mathtt{num}}$$

$$\frac{\vdash \mathtt{num}[4] : \mathtt{num} \quad x_1 : \mathtt{num}, x_2 : \mathtt{num} \vdash \mathtt{plus}(x_1;x_2) : \mathtt{num}}{x_1 : \mathtt{num} \vdash \mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2)) : \mathtt{num}}$$

$$\frac{\vdash \mathtt{num}[3] : \mathtt{num}}{\vdash \mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]) : \mathtt{num}}$$

$$\frac{\vdash \mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]) : \mathtt{num} \quad x_1 : \mathtt{num} \vdash \mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2)) : \mathtt{num}}{\vdash \mathtt{let}(\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]);x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))) : \mathtt{num}}$$

一个例子-结构语义

- ABT let(plus(num[3];num[3]); x_1.let(num[4]; x_2.plus(x_1; x_2)))
- 执行 (Structural Semantics)

$$\frac{}{\mathtt{num}[n]\ \mathsf{val}} \qquad \frac{}{\mathtt{str}[s]\ \mathsf{val}} \qquad \frac{n_1 + n_2 = n\ \mathsf{nat}}{\mathtt{plus}(\mathtt{num}[n_1];\mathtt{num}[n_2]) \mapsto \mathtt{num}[n]} \qquad \frac{e_1\ \mathsf{val} \quad e_2 \mapsto e_2'}{\mathtt{plus}(e_1;e_2) \mapsto \mathtt{plus}(e_1;e_2')}$$

$$\frac{e_1 \mapsto e_1'}{\mathtt{plus}(e_1;e_2) \mapsto \mathtt{plus}(e_1';e_2)} \qquad \frac{e_1\ \mathsf{val}}{\mathtt{let}(e_1;x.e_2) \mapsto [e_1/x]e_2} \qquad \frac{e_1 \mapsto e_1'}{\mathtt{let}(e_1;x.e_2) \mapsto \mathtt{let}(e_1';x.e_2)}$$

$$\begin{aligned} & \mathtt{let}(\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]);x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))) \\ \mapsto\ & \mathtt{let}(\mathtt{num}[6];x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2)) \\ \mapsto\ & \mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(\mathtt{num}[6];x_2)) \\ \mapsto\ & \mathtt{plus}(\mathtt{num}[6];\mathtt{num}[4]) \\ \mapsto\ & \mathtt{num}[10] \end{aligned}$$

一个例子-上下文语义

- ABT let(plus(num[3];num[3]); x_1.let(num[4]; x_2.plus(x_1; x_2)))
- 执行 (Contextual Semantics)

$$\frac{m + n = p\ \mathsf{nat}}{\mathtt{plus}(\mathtt{num}[m];\mathtt{num}[n]) \leadsto \mathtt{num}[p]} \qquad \frac{e_1\ \mathsf{val}}{\mathtt{let}(e_1;x.e_2) \leadsto [e_1/x]e_2} \qquad \frac{}{e = \circ\{e\}}$$

$$\frac{e_1 = \mathcal{E}_1\{e\}}{\mathtt{plus}(e_1;e_2) = \mathtt{plus}(\mathcal{E}_1;e_2)\{e\}} \qquad \frac{e_1\ \mathsf{val} \quad e_1 = \mathcal{E}_2\{e\}}{\mathtt{plus}(e_1;e_2) = \mathtt{plus}(e_1;\mathcal{E}_2)\{e\}} \qquad \frac{e_1 = \mathcal{E}_1\{e\}}{\mathtt{let}(e_1;x.e_2) = \mathtt{let}(\mathcal{E}_1;x.e_2)\{e\}}$$

$$\frac{e = \mathcal{E}\{e_0\} \quad e_0 \leadsto e_0' \quad e' = \mathcal{E}\{e_0'\}}{e \mapsto e'}$$

$$\begin{aligned} & \mathtt{let}(\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]);x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))) \\ =\ & \mathtt{let}(\circ;x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))\{\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3])\} \\ \mapsto\ & \mathtt{let}(\circ;x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))\{\mathtt{num}[6]\} \\ =\ & \circ\{\mathtt{let}(\mathtt{num}[6];x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))\} \\ \mapsto\ & \circ\{\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(\mathtt{num}[6];x_2))\} \\ \mapsto\ & \circ\{\mathtt{plus}(\mathtt{num}[6];\mathtt{num}[4])\} \\ \mapsto\ & \circ\{\mathtt{num}[10]\} \end{aligned}$$

一个例子-求值语义

- ABT let(plus(num[3];num[3]); x_1.let(num[4]; x_2.plus(x_1; x_2)))
- 执行 (Evaluation Semantics)

$$\frac{}{\mathtt{num}[n] \Downarrow \mathtt{num}[n]} \qquad \frac{e_1 \Downarrow \mathtt{num}[n_1] \quad e_2 \Downarrow \mathtt{num}[n_2] \quad n_1 + n_2 = n\ \mathsf{nat}}{\mathtt{plus}(e_1;e_2) \Downarrow \mathtt{num}[n]} \qquad \frac{e_1 \Downarrow v_1 \quad [v_1/x]e_2 \Downarrow v_2}{\mathtt{let}(e_1;x.e_2) \Downarrow v_2}$$

自底向上

$$\frac{\mathtt{num}[3] \Downarrow \mathtt{num}[3] \quad 3 + 3 = 6\ \mathsf{nat}}{\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]) \Downarrow \mathtt{num}[6]}$$

$$\frac{\mathtt{num}[4] \Downarrow \mathtt{num}[4] \quad \mathtt{plus}(\mathtt{num}[6];\mathtt{num}[4]) \Downarrow \mathtt{num}[10]}{\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(\mathtt{num}[6];x_2)) \Downarrow \mathtt{num}[10]}$$

$$\frac{\mathtt{plus}(\mathtt{num}[3];\mathtt{num}[3]) \Downarrow \mathtt{num}[6] \quad \mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(\mathtt{num}[6];x_2)) \Downarrow \mathtt{num}[10]}{\mathtt{let}(\mathtt{num}[6];x_1.\mathtt{let}(\mathtt{num}[4];x_2.\mathtt{plus}(x_1;x_2))) \Downarrow \mathtt{num}[10]}$$

3.7 类型和语言

- 类型安全

  表达静态语义和动态语义之间的一致性

  – 静态语义预测表达式值将具有某种形式，使得表达式的动态语义是良定义的.

3.7.1 类型安全(Type Safety)[PFPL, 11]

3.7.2 运行时错误[PFPL, 11]

3.7.3 阶段上的区别[PFPL, 13]

3.7.4 引入和消去[PFPL, 13]

3.7.5 组合性[PFPL, 13]

3.7.6 变量和值[PFPL, 13]

3.7.1 类型安全-1

- **Types** $\tau ::= \text{num} \mid \text{str}$
- **Values** $v ::= \texttt{num}[n] \mid \texttt{str}[s]$, n nat, s str
- **Expr** $e ::= x \mid \texttt{num}[n] \mid \texttt{str}[s] \mid \texttt{plus}(e_1;e_2) \mid \texttt{times}(e_1;e_2) \mid \texttt{cat}(e_1;e_2) \mid \texttt{len}(e) \mid \texttt{let}(e_1;x.e_2)$
- 定型规则**Typing rules**

$$\frac{\Gamma \vdash e_1 : \text{num} \quad \Gamma \vdash e_2 : \text{num}}{\Gamma \vdash \texttt{plus}(e_1;e_2) : \text{num}}$$

- 定型的逆转**Inversion for Typing**

如果 $\Gamma \vdash e : \tau$, e=plus(e_1;e_2), 那么 τ = num, $\Gamma \vdash e_1 : \text{num}$ 且 $\Gamma \vdash e_2 : \text{num}$.

3.7.1 类型安全-2

- **Theorem 11.1 (Type safety for $\mathcal{L}$\{num str\})**

1.(preservation) 如果 $e : \tau$ 且 $e \mapsto e'$,则 $e' : \tau$.

2.(progress) 如果 $e : \tau$, 那么或者 e val, 或者存在 e' 使得 $e \mapsto e'$.

- 保持性(Preservation): 计算的每一步都保持定型.
- 进展性(Progress): 确保良类型的表达式或者是值, 或者可以被进一步计算.

e是受阻的(stuck)当且仅当它不是一个值, 而且也不存在e'使得 $e \mapsto e'$.

一个受阻的状态必然是不良类型的(ill-typed).

进展性: 良类型的程序不会到达受阻状态。

3.7.1 类型安全-保持性-1

- 保持性 如果 $e : \tau$ 并且 $e \mapsto e'$, 则 $e' : \tau$.

证明: 对转换(transition)断言的推导规则进行规则归纳.

情况 1
$$\frac{e_1 \mapsto e_1'}{\texttt{plus}(e_1;e_2) \mapsto \texttt{plus}(e_1';e_2)}$$

假设: $\texttt{plus}(e_1;e_2) : \tau$

由定型的逆转引理, 有 $\tau = \text{num}$, $e_1 : \text{num}$, $e_2 : \text{num}$

再由归纳原理, 有 $e_1' : \text{num}$

从而有 $\texttt{plus}(e_1';e_2) : \text{num}$

故得证。

3.7.1 类型安全-保持性-2

- 保持性 如果 $e : \tau$ 并且 $e \mapsto e'$, 则 $e' : \tau$.

证明: 根据转换断言规则进行规则归纳证明.

情况 2
$$\frac{e_1 \text{ val}}{\texttt{let}(e_1;x.e_2) \mapsto [e_1/x]e_2}$$

假设: $\texttt{let}(e_1;x.e_2) : \tau_2$

由定型的逆转引理9.2, 对于某些 τ_1 有 $e_1 : \tau_1$

使得 $x : \tau_1 \vdash e_1 : \tau_2$

由置换引理9.4, 可得 $[e_1/x]e_2 : \tau_2$

故得证。

…… // 其他情况

3.7.1 类型安全-保持性-3

- 保持性
  - 对保持性的证明不能按表达式e的结构进行归纳, 因为在绝大多数情况下, 会有不止一个转换规则适用于一个表达式。

  例如: 对于plus(e_1;e_2),可以有以下转换规则

$$\frac{n_1 + n_2 = n \text{ nat}}{\texttt{plus}(\texttt{num}[n_1];\texttt{num}[n_2]) \mapsto \texttt{num}[n]}$$

$$\frac{e_1 \mapsto e_1'}{\texttt{plus}(e_1;e_2) \mapsto \texttt{plus}(e_1';e_2)}$$

$$\frac{e_1 \text{ val} \quad e_2 \mapsto e_2'}{\texttt{plus}(e_1;e_2) \mapsto \texttt{plus}(e_1;e_2')}$$

3.7.1 类型安全-进展性-1

进展性 如果$e : \tau$, 则或者e val ,或者存在 e'使得 $e \mapsto e'$.

引理11.3(范式Canonical Forms):

如果e val且e:τ, 那么:

1.如果τ = num, 则e = num[n](n为某一数值)是范式。

2.如果τ = str, 则e = str[s](s为某一串)是范式。

证明 按定型规则(9.1)和值规则(10.1)进行归纳。

没有求值规则可以作用在范式e上。

表达式e求值终止, 是指存在某个范式e', 使得 $e \mapsto^* e'$

3.7.1 类型安全-进展性-2

进展性　如果 $e:\tau$，则或者 e val，或者存在 e'使得 $e \mapsto e'$.

证明：对定型推导进行归纳.

情况 1　$\dfrac{e_1:\texttt{num}\quad e_2:\texttt{num}}{\texttt{plus}(e_1;e_2):\texttt{num}}$　上下文为空表示只考虑闭项

由归纳法，有：

1) e_1 val ：由归纳法有

a) e_2 val：则由范式引理11.3，有 $e_1=\texttt{num}[n_1]$，$e_2=\texttt{num}[n_2]$

从而 $\texttt{plus}(\texttt{num}[n_1];\texttt{num}[n_2]) \mapsto \texttt{num}[n_1+n_2]$

b) 存在 e_2' 使得 $e_2 \mapsto e_2'$：由转换规则，有

$\texttt{plus}(e_1;e_2) \mapsto \texttt{plus}(e_1;e_2')$

2) 存在 e_1'，使得 $e_1 \mapsto e_1'$

$\texttt{plus}(e_1;e_2) \mapsto \texttt{plus}(e_1';e_2)$

3.7.1 类型安全-进展性-3

❖ 进展性

➢ L{num,str}的定型规则是语法制导的，所以进展性就等于按表达式e的结构进行归纳。

➢ 当定型规则不是语法制导的时候，即可能存在不止一个规则适用于一个给定的表达式。

例如：while b do c

b为假，－

b为真，c; while b do c

3.7.2 运行时错误-1

❖ 对L{num,str}进行扩展，增加除法div(e_1;e_2)运算

假设对除法运算没有指定除0的语义

$\dfrac{e_1:\texttt{num}\quad e_2:\texttt{num}}{\texttt{div}(e_1;e_2):\texttt{num}}$

这时，div(num[2];num[0])是良类型的，但求值时会受阻

➢ 解决办法1：增强类型系统，使得良类型的程序不会执行除0操作

这要求类型检查器要能证明分母非0

➔对多数程序来说，很难确定！

➢ 解决办法2：增加动态检查，使得除0会导致求值结果为错误

– 无需检查的错误(unchecked error)：由类型系统来排除

– 要检查的错误(checked error)：需要定义检查这种错误的动态语义

3.7.2 运行时错误-2

❖ 对动态检查错误的形式化-方法1

➢ 增加断言e err 以及对断言的归纳定义：

引起错误　$\dfrac{e_1\ \text{val}}{\texttt{div}(e_1;\texttt{num}[0])\ \text{err}}$

传播错误　$\dfrac{e_1\ \text{err}}{\texttt{plus}(e_1;e_2)\ \text{err}}$　$\dfrac{e_1\ \text{val}\quad e_2\ \text{err}}{\texttt{plus}(e_1;e_2)\ \text{err}}$　(11.1)

➢ 保持性定理不受影响

➢ 带错误检查的进展性：要考虑出的错误

Theorem11.5. 如果e:τ，则或者e err，或者e val，或者存在e'，使得 $e \mapsto e'$。

证明：对定型规则归纳证明，与前面的证明类似，只是现在要考虑三种情况。

3.7.2 运行时错误-3

❖ 对动态检查错误的形式化

方法1：需要一组特殊的求值规则来检查错误

方法2：通过增加error表达式，将求值与错误检查合二为一。

➢ 定型规则：增加如下规则　$\overline{\texttt{error}:\tau}$　(11.2)

➢ 动态语义：

增加引起错误的规则　$\dfrac{e_1\ \text{val}}{\texttt{div}(e_1;\texttt{num}[0]) \mapsto \texttt{error}}$　(11.3)

增加一些规则以传播错误，如

$\overline{\texttt{plus}(\texttt{error};e_2) \mapsto \texttt{error}}$　$\dfrac{e_1\ \text{val}}{\texttt{plus}(e_1;\texttt{error}) \mapsto \texttt{error}}$

3.7.3 阶段上的区别(Phase Distinction)

❖ 静态语义 vs. 动态语义

➢ 静态语义(定型规则)对程序中的结构进行约束，以保证动态语义(求值规则)是良行为的(well-behaved)

❖ 静态阶段 vs. 动态阶段

➢ 静态阶段发生在动态阶段之前，二者相互独立

➢ 静态阶段预测表达式在动态阶段所求的值的形式

❖ 类型安全定理(进展性和保持性)

➢ 静态语义所预测的是动态语义中的真集，否则动态语义会到达受阻状态

➢ 如何处理安全性的反例：或者增强静态语义保证反例被禁止；或者增强动态语义确保在运行时能对错误条件进行检查

3.7.4 引入和消去-1

❖ 和类型相关的基本操作

➢ 引入形式(Introduction)：构造属于这种类型的值

例：nat类型的引入形式是数值

str类型的引入形式是字符串

➢ 消去形式(Elimination)：这类值所能进行的运算

例：nat类型的消去形式是加、乘运算

str类型的消去形式是连接、求长度运算

就λ演算而言，引入形式为λ抽象，消去形式为λ应用。

❖ 动态语义以逆转原理(inversion principle)为基础

[逆转原理]例:假设e:τ,如果e=num[n]，那么τ=num

3.7.4 引入和消去-2

➢ 消去形式是引入形式的逆

引入是构造这种类型的值，而消去则是确定这种类型的值所能进行的运算

➢ 消去形式所得到的是由传给它的参数来确定

例： plus(e_1;e_2)的结果是一个数值，它由参数e_1和e_2的值来得到，两个参数都是数值。

➢ 可以将类型安全定理看成是对语言逆转原理的验证

例：对于plus(e_1;e_2)，

类型保持定理确保plus的参数e_1和e_2的类型必须是num，从而由范式引理，可得e_1和e_2的值必须是数值。这保证plus的进展性，它产生一个数值，其类型为num 。

3.7.4 引入和消去-3

❖ 逆转原理可以指导对动态语义的推导，但在多数情况下不能完全决定动态语义

假设对L{num,str}增加ifz(e;e_1;e_2)条件表达式，

考虑消去形式ifz(e;e_1;e_2)：e是num类型，如果e计算为0，则结果为e_1，否则为e_2。

e的求值结果将决定是继续计算e_1,还是e_2,而不是对e_1和e_2都计算。

➢ 主要参数 vs. 次要的参数

– 对于ifz(e;e_1;e_2)来说， e是主要参数，e_1和e_2是次要参数

对一个消去形式来说，其主要参数必须被计算，而次要参数则不必被计算。

3.7.4 引入和消去-4

❖ 引入形式的参数计算

假设将L{num,str}中的数值换成z和s(e)(分别表示0和后继)

z和s(e)均为num类型的引入形式。

s(e)是值的前提是要求e本身是值？还是不管e是否是值？

➢ 激进eager(严格)的运算 要求引入形式的参数是值在动态语义中就必须有关联的search规则

➢ 惰性lazy(不严格)的运算 不要求引入形式的参数是值

e.g. s(plus(z;s(z))) 是值

如果语言中所有引入形式的参数是激进的，则该语言是激进的。

如果语言中所有引入形式的参数是惰性的，则该语言是惰性的。

3.7.5 组合性

❖ 组合性(compositionality)

由定型的置换性质和传递性(见3.5.2节)可以得到类型系统的一个基本性质，称为组合性或模块性(modularity)

$$\frac{\Gamma \vdash e : \tau \quad \Gamma, x : \tau \vdash e' : \tau'}{\Gamma \vdash [e/x]e' : \tau'}$$

上述规则体现连接(linking)的本质

➢ e'中含有τ类型的自由变量x

➢ linker的任务是通过置换x,将e和e'组合起来，得到一个完整的编译单元

➢ 客户端e'可以单独进行类型检查，而与共享组件e的实现无关

类型提供了模块性的基础。

3.7.6 变量和值

如果一个变量在执行时永远只绑定到一个值上，则称该变量为值变量，否则为计算变量。

➢ 计算变量 $x_1 : \tau_1, \ldots, x_n : \tau_n \vdash e : \tau.$

$$\frac{\Gamma \vdash e : \tau \quad \Gamma, x : \tau \vdash e' : \tau'}{\Gamma \vdash [e/x]e' : \tau'}$$

x可以代换为任何类型为τ的表达式

➢ 值变量 $x_1 \text{ val}, \ldots, x_n \text{ val}\ x_1 : \tau_1, \ldots, x_n : \tau_n \vdash e : \tau,$

$$\frac{\Phi\,\Gamma \vdash e : \tau \quad \Phi \vdash e \text{ val} \quad \Phi, x \text{ val}\ \Gamma, x : \tau \vdash e' : \tau'}{\Phi\,\Gamma \vdash [e/x]e' : \tau'}$$

Φ表示一组形如 x_i val的假设

注意：e是开放的值(含有自由变量)，即

$$x_1 \text{ val}, \ldots, x_n \text{ val} \vdash e \text{ val}$$

如:x val $\vdash$ s(s(x)) val 与后继运算是eager/lazy无关

Thanks!