Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Applied Numerical Mathematics 56 (2006) 1491–1518
APPLIED NUMERICAL MATHEMATICS
www.elsevier.com/locate/apnum

Space–time discontinuous Galerkin method for advection–diffusion problems on time-dependent domains

J.J. Sudirham *, J.J.W. van der Vegt, R.M.J. van Damme

Department of Applied Mathematics, University of Twente, PO Box 217, 7500 AE, Enschede, The Netherlands

Available online 10 January 2006

Abstract

This article presents a space–time discontinuous Galerkin (DG) finite element discretization of the advection–diffusion equation on time-dependent domains. In the space–time DG discretization no distinction is made between the space and time variables and discontinuous basis functions are used both in space and time. This approach results in an efficient numerical technique for physical applications which require moving and deforming elements, is suitable for *hp*-adaptation and results in a fully conservative discretization. A complete derivation of the space–time DG method for the advection–diffusion equation is given, together with the relation of the space–time discretization with the arbitrary Lagrangian Eulerian (ALE) approach. Detailed proofs of stability and error estimates are also provided. The space–time DG method is demonstrated with numerical experiments that agree well with the error analysis.

Keywords: Discontinuous Galerkin method; Space–time discretization; Advection–diffusion equation; A priori error analysis; Arbitrary Lagrangian Eulerian methods

1. Introduction

Many engineering applications require the solution of partial differential equations on time-dependent domains. Examples are fluid-structure interaction problems, Stefan problems, and non-linear free surface water waves. These problems require moving and deforming elements to accommodate for the changing of the domain boundary position and it is generally important to maintain exact conservation of certain physical quantities. In this paper we present a space–time discontinuous Galerkin (DG) finite element method which is well suited for problems on complicated time-dependent domains. This DG method is an extension to parabolic problems of the space–time DG method for non-linear hyperbolic problems presented in [16]. The aim of this paper is to provide a detailed derivation and analysis of the space–time DG method for the advection–diffusion equation on time-dependent domains and verify these results with numerical experiments.

The DG method has recently received significant interest since this technique is well suited to *hp*-adaptation and parallel computing due to its high degree of locality. Also, the DG method results in an element wise conservative

* Corresponding author.
E-mail addresses: j.j.sudirham@math.utwente.nl (J.J. Sudirham), j.j.w.vandervegt@math.utwente.nl (J.J.W. van der Vegt), r.m.j.vandamme@math.utwente.nl (R.M.J. van Damme).

0168-9274/$30.00
doi:10.1016/j.apnum.2005.11.003

discretization, which is crucial when dealing with conservation laws. DG methods for the spatial discretization of different types of partial differential equations have been investigated by Cockburn and co-workers and detailed surveys can be found in e.g. [2,8]. The main feature of the DG method is the use of basis functions which are discontinuous across element faces. For problems on time-dependent domains which require moving and deforming elements there is, however, a need of using basis functions which are also discontinuous in time as it provides greater flexibility to deal with the remeshing and deformation of the computational mesh. The use of discontinuous basis functions in time has been explored extensively by Hughes [11], Tezduyar [14] and their co-workers and demonstrated for a wide range of problems which require moving and deforming meshes. In the so-called space–time DG method we combine both DG techniques and obtain a versatile algorithm for the adaptive solution of partial differential equations on time-dependent domains.

The organization of this paper is as follows. In Section 2 we introduce the advection–diffusion equation under consideration. First, the equation is presented in the usual form, then after a description of the space–time domain, the advection–diffusion equation is reformulated in the space–time framework. This section is completed with a description of the boundary conditions imposed on different parts of the domain boundary. Section 3 starts with a description of the construction of the space–time domain and elements. In this section we also introduce the finite element spaces and several operators necessary to define the weak formulation. In Section 4 we present the derivation of the DG weak formulation for the advection–diffusion equation. This section is completed with the transformation of the weak formulation into an arbitrary Lagrangian Eulerian (ALE) formulation as this formulation is useful for the actual implementation of the algorithm. In Section 5, we show that the space–time DG formulation is consistent, coercive, stable and gives a unique solution. In Section 6 we provide error estimates in the DG norm and show the *hp*-convergence of the method. This section is completed with error estimates at a specific time level in the L^2-norm. In Section 7 we show results of numerical experiments on a time-dependent computational domain to verify the theoretical results and the accuracy of the space–time DG discretization. Finally, concluding remarks are made in Section 8.

2. Advection–diffusion equation

In this section we consider the advection–diffusion equation in the usual form and in the space–time framework. Let Ω_t be an open, bounded domain in $\mathbb{R}^d$, with d the number of spatial dimensions. The closure of Ω_t is $\overline{\Omega_t}$ and the boundary of Ω_t is denoted by $\partial\Omega_t$. The subscript t denotes the domain at time t as we consider the geometry of the spatial domain to be time-dependent. The outward normal vector to $\partial\Omega_t$ is denoted by $\bar{n} = (n_1, \ldots, n_d)$. Denoting $\bar{x} = (x_1, \ldots, x_d)$ as the spatial variables, we consider a time-dependent advection–diffusion equation:

$$\frac{\partial c}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(u_i(t,\bar{x})c\big) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(D_{ij}(t,\bar{x})\frac{\partial c}{\partial x_i}\right) = 0, \quad \text{in } \Omega_t, \tag{1}$$

where $u = (u_1, \ldots, u_d)$ is a vector field whose entries are continuous real-valued functions on $\overline{\Omega_t}$. Furthermore, $D \in \mathbb{R}^{d\times d}$ is a symmetric matrix of diffusion coefficients on $\overline{\Omega_t}$ whose entries are continuous real-valued functions. This matrix is positive definite in Ω_t and positive semi-definite on $\partial\Omega_t$. Then there exists a symmetric matrix $D^\star \in \mathbb{R}^{d\times d}$, the matrix square root $D^\star = D^{1/2}$, such that

$$D = D^\star D^\star. \tag{2}$$

In the space–time discretization we directly consider a domain in $\mathbb{R}^{d+1}$. A point $x \in \mathbb{R}^{d+1}$ has coordinates $(x_0, \bar{x})$, with $x_0 = t$ representing time. We then define the space–time domain $\mathcal{E} \subset \mathbb{R}^{d+1}$. The boundary of the space–time domain $\partial\mathcal{E}$ consists of the hypersurfaces $\Omega_0 := \{x \in \partial\mathcal{E} \mid x_0 = 0\}$, $\Omega_T := \{x \in \partial\mathcal{E} \mid x_0 = T\}$, and $\mathcal{Q} := \{x \in \partial\mathcal{E} \mid 0 < x_0 < T\}$. We reformulate the advection–diffusion equation now in the space–time framework. First, we introduce the vector function $B \in \mathbb{R}^{d+1}$ and the symmetric matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$ as:

$$B = (1, u), \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Then the advection–diffusion equation (1) can be transformed into a space–time formulation as:

$$-\nabla \cdot (-Bc + A\nabla c) = 0 \quad \text{in } \mathcal{E}, \tag{3}$$

where $\nabla = (\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$ denotes the gradient operator in $\mathbb{R}^{d+1}$. Later we will also use the notation $\overline{\nabla}$ to denote the spatial gradient operator in $\mathbb{R}^d$, defined as $\overline{\nabla} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$. The unit outward normal vector at $\partial\mathcal{E}$ is denoted with n.

As different boundary conditions are imposed on $\partial\mathcal{E}$, we discuss in more detail the subdivision of $\partial\mathcal{E}$ into different parts. The boundary $\partial\mathcal{E}$ is divided into disjoint boundary subsets Γ_S, Γ_-, and Γ_+, where each subset is defined as follows:

$$\Gamma_S := \{x \in \partial\mathcal{E}\colon \bar{n}^{\mathrm{T}} D \bar{n} > 0\}, \qquad \Gamma_- := \{x \in \partial\mathcal{E} \setminus \Gamma_S\colon B \cdot n < 0\}, \qquad \Gamma_+ := \{x \in \partial\mathcal{E} \setminus \Gamma_S\colon B \cdot n \geqslant 0\}.$$

The subscript S denotes the part of $\partial\mathcal{E}$ where matrix D is symmetric positive definite, while the subscripts $-$ and $+$ denote the inflow and outflow boundaries, respectively. We assume that Γ_S has a non-zero surface measure. Note that $\partial\mathcal{E} = \Gamma_S \cup \Gamma_- \cup \Gamma_+$. We subdivide Γ_S further into two sets: $\Gamma_S = \Gamma_{DS} \cup \Gamma_M$, with Γ_{DS} the part of Γ_S with a Dirichlet boundary condition and Γ_M the part of Γ_S with a mixed boundary condition. We also subdivide Γ_- into two parts: $\Gamma_- = \Gamma_{DB} \cup \Omega_0$, with Γ_{DB} the part of Γ_- with a Dirichlet boundary condition and Ω_0 the part of Γ_- with the initial condition. Note that $\Gamma_D = \Gamma_{DS} \cup \Gamma_{DB} \subset \partial\mathcal{E}$ is the part of the space–time domain boundary with a Dirichlet boundary condition. The boundary conditions on different parts of $\partial\mathcal{E}$ are written as

$$\begin{aligned} &c = c_0 && \text{on } \Omega_0,\\ &c = g_D && \text{on } \Gamma_D,\\ &\alpha c + n \cdot (A \nabla c) = g_M && \text{on } \Gamma_M, \end{aligned} \tag{4}$$

with $\alpha \geqslant 0$ and c_0, g_D, g_M given functions defined on the boundary. There is no boundary condition imposed on Γ_+.

3. Space–time description, finite element spaces and trace operators

3.1. Definition of space–time slabs, elements and faces

In this section we give a description of the space–time slabs, elements and faces used in the DG discretization. First, consider the time interval $\mathcal{I} = [0, T]$, partitioned by an ordered series of time levels $t_0 = 0 < t_1 < \cdots < t_{N_t} = T$. Denoting the nth time interval as $I_n = (t_n, t_{n+1})$, we have $\mathcal{I} = \bigcup_n \bar{I}_n$. The length of I_n is defined as $\Delta_n t = t_{n+1} - t_n$. Let Ω_{t_n} be an approximation to the spatial domain Ω at t_n for each $n = 0, \ldots, N_t$. A space–time slab is defined as the domain $\mathcal{E}^n = \mathcal{E} \cap (I_n \times \mathbb{R}^d)$ with boundaries Ω_{t_n}, $\Omega_{t_{n+1}}$ and $\mathcal{Q}^n = \partial\mathcal{E}^n \setminus (\Omega_{t_n} \cup \Omega_{t_{n+1}})$.

We now describe the construction of the space–time elements $\mathcal{K}$ in the space–time slab $\mathcal{E}^n$. Let the domain Ω_{t_n} be divided into N_n non-overlapping spatial elements K^n. At t_{n+1} the spatial elements K^{n+1} are obtained by mapping the elements K^n to their new position. Each space–time element $\mathcal{K}$ is obtained by connecting elements K^n and K^{n+1} using linear interpolation in time. A sketch of the space–time slab $\mathcal{E}^n$ and element $\mathcal{K}$ for two spatial dimensions is shown in Fig. 1. We denote by $h_{\mathcal{K}}$ the radius of the smallest sphere containing each element $\mathcal{K}$. The element boundary $\partial\mathcal{K}$ is the union of open faces of $\mathcal{K}$, which contains three parts K^n, K^{n+1}, and $\mathcal{Q}^n_{\mathcal{K}} = \partial\mathcal{K} \setminus (K^n \cup K^{n+1})$. We denote by $n_{\mathcal{K}}$ the unit outward space–time normal vector on $\partial\mathcal{K}$. The definition of the space–time domain is completed with the tessellation $\mathcal{T}_h^n$, which consists of all space–time elements in $\mathcal{E}^n$, and $\mathcal{T}_h = \bigcup_n \mathcal{T}_h^n$, which consists of all space–time elements in $\mathcal{E}$.

Next, we consider several sets of faces S. The set of all faces in $\overline{\mathcal{E}}$ is denoted with $\mathcal{F}$, the set of all interior faces in $\mathcal{E}$ with $\mathcal{F}_{\mathrm{int}}$, and the set of all boundary faces on $\partial\mathcal{E}$ with $\mathcal{F}_{\mathrm{bnd}}$. In the space–time slab $\mathcal{E}^n$ we denote the set of all faces with $\mathcal{F}^n$ and the set of all interior faces with $\mathcal{S}_I^n$. The faces separating two space–time slabs are denoted as $\mathcal{S}_S^n$. Several sets of boundary faces are defined as follows. The set of faces on Γ_{DS} and Γ_{DB} are denoted with $\mathcal{S}_{DS}^n$ and $\mathcal{S}_{DB}^n$, respectively. These sets are grouped into $\mathcal{S}_D^n$. The set of faces with a mixed boundary condition is denoted with $\mathcal{S}_M^n$. The set of faces with either a Dirichlet or a mixed boundary condition is denoted as $\mathcal{S}_{DM}^n$. The sets $\mathcal{S}_I^n$ and $\mathcal{S}_D^n$ are grouped into $\mathcal{S}_{ID}^n$.

Depending on whether the advective flux on $\mathcal{S}_{DS}^n$ is inflow or outflow, we subdivide $\mathcal{S}_{DS}^n$ further into $\mathcal{S}_{DSm}^n$ and $\mathcal{S}_{DSp}^n$, where $B \cdot n < 0$ on $\mathcal{S}_{DSm}^n$ and $B \cdot n \geqslant 0$ on $\mathcal{S}_{DSp}^n$. The sets $\mathcal{S}_{DB}^n$ and $\mathcal{S}_{DSm}^n$ are grouped into $\mathcal{S}_{DBSm}^n$ while the sets $\mathcal{S}_M^n$ and $\mathcal{S}_{DSp}^n$ are grouped into $\mathcal{S}_{MDSp}^n$. These sets are important when we discuss the advective flux in Section 4.2.

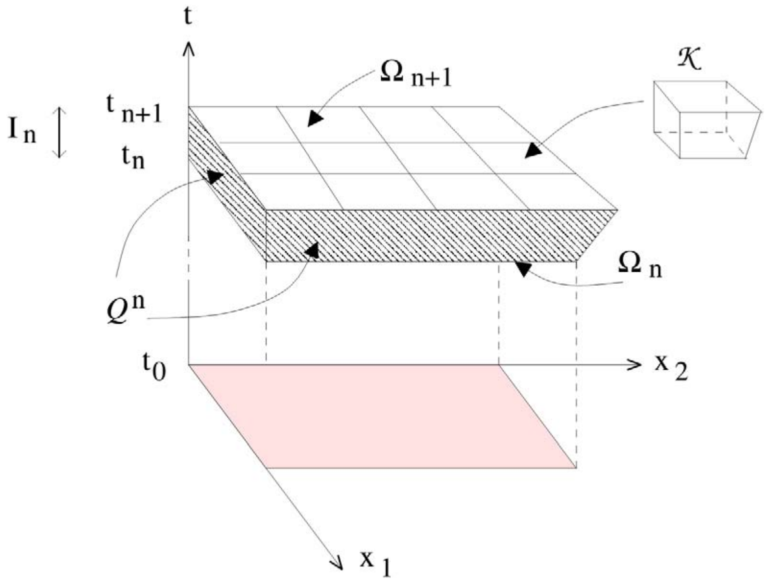

Fig. 1. Space–time slab $\mathcal{E}^n$ with space–time element $\mathcal{K}$.

3.2. Finite element spaces and trace operators

First we recall standard definition of the Sobolev spaces $H^s(\mathcal{D})$ (see e.g. [4]), with s a non-negative integer, in a domain $\mathcal{D} \subset \mathbb{R}^n$, n is either d or $d+1$:

$$H^s(\mathcal{D}) := \{v \in L^2(\mathcal{D}): \partial^\gamma v \in L^2(\mathcal{D}) \text{ for } |\gamma| \leqslant s\},$$

where ∂^γ denotes the weak derivative (see [4]) and γ the multi-index symbol, $\gamma = (\gamma_1, \dots, \gamma_n)$, with γ_i non-negative integers. The length of γ is given by $|\gamma| = \sum_{i=1}^n \gamma_i$. When $s = 0$ the space is denoted as $L^2(\mathcal{D})$ which is equipped with the standard inner-product and norm:

$$(w, v)_\mathcal{D} := \int_\mathcal{D} wv \, \mathrm{d}\mathcal{K}, \qquad \|v\|_{L^2(\mathcal{D})} := (v, v)_\mathcal{D}^{1/2},$$

and for $s \geqslant 1$, the Sobolev norm and semi-norm are defined as:

$$\|v\|_{H^s(\mathcal{D})} := \Big(\sum_{|\gamma| \leqslant s} \|\partial^\gamma v\|_{L^2(\mathcal{D})}^2\Big)^{1/2}, \qquad |v|_{H^s(\mathcal{D})} := \Big(\sum_{|\gamma| = s} \|\partial^\gamma v\|_{L^2(\mathcal{D})}^2\Big)^{1/2}.$$

We now introduce anisotropic Sobolev spaces on the domain $\mathcal{D} \subset \mathbb{R}^{d+1}$ such as in [9]. Here we restrict the definition of anisotropy to the case where the Sobolev index can be different for the temporal and spatial variables. All spatial variables have, however, the same index. Let (s_t, s_s) be a pair of non-negative integers, with s_t, s_s corresponding to the temporal and spatial Sobolev index, respectively. For $\gamma_t, \gamma_s \geqslant 0$, the anisotropic Sobolev space of order (s_t, s_s) on $\mathcal{D}$ is defined by

$$H^{(s_t, s_s)}(\mathcal{D}) := \{v \in L^2(\mathcal{D}): \partial^{\gamma_t}\partial^{\gamma_s} v \in L^2(\mathcal{D}) \text{ for } \gamma_t \leqslant s_t,\ |\gamma_s| \leqslant s_s\},$$

with associated norm and semi-norm:

$$\|v\|_{H^{(s_t,s_s)}(\mathcal{D})} := \Big(\sum_{\substack{\gamma_t \leqslant s_t \\ |\gamma_s| \leqslant s_s}} \|\partial^{\gamma_t}\partial^{\gamma_s} v\|_{L^2(\mathcal{D})}^2\Big)^{1/2}, \qquad |v|_{H^{(s_t,s_s)}(\mathcal{D})} := \Big(\sum_{\substack{\gamma_t = s_t \\ |\gamma_s| = s_s}} \|\partial^{\gamma_t}\partial^{\gamma_s} v\|_{L^2(\mathcal{D})}^2\Big)^{1/2}.$$

Next, we introduce mappings of the space–time elements. Following the discussion in [9], we assume that each element $\mathcal{K} \in \mathcal{T}_h$ is an image of a fixed master element $\widehat{\mathcal{K}}$, with $\widehat{\mathcal{K}}$ an open unit hypercube in $\mathbb{R}^{d+1}$, constructed via two

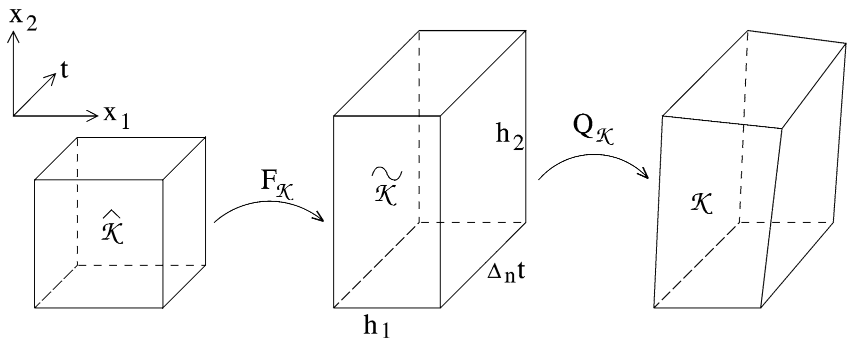

Fig. 2. Construction of elements $\mathcal{K}$ via composition of affine maps and diffeomorphisms (for $d = 2$).

mappings $Q_{\mathcal{K}} \circ F_{\mathcal{K}}$, where $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ is an affine mapping and $Q_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$ is a (regular enough) diffeomorphism (see Fig. 2). The definition of the Sobolev space $H^{(s_t,s_s)}(\widetilde{\mathcal{K}})$ on $\widetilde{\mathcal{K}}$ follows the definition of the standard Sobolev space, while the Sobolev space $H^{(s_t,s_s)}(\mathcal{K})$ is defined as follows:

$$H^{(s_t,s_s)}(\mathcal{K}) := \left\{ v \in L^2(\mathcal{K}) \colon\ v \circ Q_{\mathcal{K}} \in H^{(s_t,s_s)}(\widetilde{\mathcal{K}}) \right\}.$$

Since the DG method is a non-conforming method, it is necessary to introduce the concept of a broken anisotropic Sobolev space. To each element $\mathcal{K}$ we assign a pair of nonnegative integers $(s_{t,\mathcal{K}}, s_{s,\mathcal{K}})$ and collect them in the vectors $\mathbf{s}_t = \{s_{t,\mathcal{K}} \colon\ \mathcal{K} \in \mathcal{T}_h\}$ and $\mathbf{s}_s = \{s_{s,\mathcal{K}} \colon\ \mathcal{K} \in \mathcal{T}_h\}$. Then we assign to $\mathcal{T}_h$ the broken Sobolev space $H^{(\mathbf{s}_t,\mathbf{s}_s)}(\mathcal{E}, \mathcal{T}_h) := \{v \in L^2(\mathcal{E}) \colon\ v|_{\mathcal{K}} \in H^{(s_{t,\mathcal{K}},s_{s,\mathcal{K}})}(\mathcal{K}),\ \forall \mathcal{K} \in \mathcal{T}_h\}$, equipped with the broken Sobolev norm and corresponding semi-norm, respectively,

$$\|v\|_{H^{(\mathbf{s}_t,\mathbf{s}_s)}(\mathcal{E},\mathcal{T}_h)} := \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{H^{(s_{t,\mathcal{K}},s_{s,\mathcal{K}})}(\mathcal{K})} \right)^{1/2}, \qquad |v|_{H^{(\mathbf{s}_t,\mathbf{s}_s)}(\mathcal{E},\mathcal{T}_h)} := \left(\sum_{\mathcal{K}\in\mathcal{T}_h} |v|^2_{H^{(s_{t,\mathcal{K}},s_{s,\mathcal{K}})}(\mathcal{K})} \right)^{1/2}.$$

For $v \in H^{(1,1)}(\mathcal{E}, \mathcal{T}_h)$, we define the broken gradient $\nabla_h v$ of v by $(\nabla_h v)|_{\mathcal{K}} := \nabla(v|_{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h$.

Now we introduce the finite element spaces associated with the tessellation $\mathcal{T}_h$ that will be used in this paper. To each element $\mathcal{K}$ we assign a pair of nonnegative integers $p_{\mathcal{K}} = (p_{t,\mathcal{K}}, p_{s,\mathcal{K}})$ as local polynomial degrees, where the subscripts t and s denote time and space, and collect them into vectors $\mathbf{p}_t = \{p_{t,\mathcal{K}} \colon\ \mathcal{K} \in \mathcal{T}_h\}$ and $\mathbf{p}_s = \{p_{s,\mathcal{K}} \colon\ \mathcal{K} \in \mathcal{T}_h\}$. Defining $\mathcal{Q}_{p_{t,\mathcal{K}},p_{s,\mathcal{K}}}(\widehat{\mathcal{K}})$ as the set of all tensor-product polynomials on $\widehat{\mathcal{K}}$ of degree $p_{t,\mathcal{K}}$ in the time direction and degree $p_{s,\mathcal{K}}$ in each spatial coordinate direction, we then introduce the finite element space of discontinuous piecewise polynomial functions as

$$V_h^{(p_t,p_s)} := \left\{ v \in L^2(\mathcal{E}) \colon\ v|_{\mathcal{K}} \circ Q_{\mathcal{K}} \circ F_{\mathcal{K}} \in \mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\widehat{\mathcal{K}}),\ \forall \mathcal{K} \in \mathcal{T}_h \right\}.$$

In the derivation and analysis of the numerical discretization we also make use of the auxiliary space $\Sigma_h^{(p_t,p_s)}$:

$$\Sigma_h^{(p_t,p_s)} = \left\{ \tau \in L^2(\mathcal{E})^{d+1} \colon\ \tau|_{\mathcal{K}} \circ Q_{\mathcal{K}} \circ F_{\mathcal{K}} \in \left[\mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\widehat{\mathcal{K}}) \right]^{d+1},\ \forall \mathcal{K} \in \mathcal{T}_h \right\}.$$

The so-called traces of $v \in V_h^{(p_t,p_s)}$ on $\partial\mathcal{K}$ are defined as: $v_{\mathcal{K}}^{\pm} = \lim_{\epsilon\downarrow 0} v(x \pm \epsilon n_{\mathcal{K}})$. The traces of $\tau \in \Sigma_h^{(p_t,p_s)}$ are defined similarly.

Next, we define the *average* $\{\{\cdot\}\}$ and *jump* $[\![\cdot]\!]$ operators as trace operators for the sets $\mathcal{F}_{\text{int}}$ and $\mathcal{F}_{\text{bnd}}$. Note that functions $v \in V_h^{(p_t,p_s)}$ and $\tau \in \Sigma_h^{(p_t,p_s)}$ are in general multivalued on a face $S \in \mathcal{F}_{\text{int}}$. Introducing the functions $v_i := v|_{\mathcal{K}_i}$, $\tau_i := \tau|_{\mathcal{K}_i}$, $n_i := n|_{\partial\mathcal{K}_i}$, we define the average operator on $S \in \mathcal{F}_{\text{int}}$ as:

$$\{\{v\}\} = \frac{1}{2}(v_i^- + v_j^-), \quad \{\{\tau\}\} = \frac{1}{2}(\tau_i^- + \tau_j^-), \quad \text{on } S \in \mathcal{F}_{\text{int}},$$

while the jump operator is defined as:

$$[\![v]\!] = v_i^- n_i + v_j^- n_j, \quad [\![\tau]\!] = \tau_i^- \cdot n_i + \tau_j^- \cdot n_j, \quad \text{on } S \in \mathcal{F}_{\text{int}},$$

with i and j the indices of the elements $\mathcal{K}_i$ and $\mathcal{K}_j$ which connect to the face $S \in \mathcal{F}_{\text{int}}$. On a face $S \in \mathcal{F}_{\text{bnd}}$, the average and jump operators are defined as:

$$\{\{v\}\} = v^-, \quad \{\{\tau\}\} = \tau^-, \quad [\![v]\!] = v^- n, \quad [\![\tau]\!] = \tau^- \cdot n, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}.$$

Note that the jump $[\![v]\!]$ is a vector parallel to the normal vector n and the jump $[\![\tau]\!]$ is a scalar quantity. We also need the spatial jump operator $\langle\langle\cdot\rangle\rangle$ for functions $v \in V_h^{(p_t,p_s)}$, which is defined as:

$$\langle\langle v\rangle\rangle = v_i^- \bar{n}_i + v_j^- \bar{n}_j, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad \langle\langle v\rangle\rangle = v^- \bar{n}, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}.$$

3.3. Lifting operators

In this section we introduce several lifting operators. The lifting operators discussed in this section are similar to the ones introduced in [2,6]. These operators are required for the derivation of the space–time DG formulation in Section 4 and also for the analysis in Sections 5 and 6.

First, we introduce the local lifting operator $r_S : (L^2(S))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$\int_{\mathcal{E}} r_S(\phi) \cdot q \, \mathrm{d}\mathcal{E} = -\int_S \phi \cdot \{\{q\}\} \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}, \ \forall S \in \bigcup_n \mathcal{S}_{ID}^n. \tag{5}$$

The support of the operator r_S is limited to the element(s) that share the face S. Then we introduce the global lifting operator $R : (L^2(\bigcup_n \mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$\int_{\mathcal{E}} R(\phi) \cdot q \, \mathrm{d}\mathcal{E} = \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_{\mathcal{E}} r_S(\phi) \cdot q \, \mathrm{d}\mathcal{E}, \quad \forall q \in \Sigma_h^{(p_t,p_s)}. \tag{6}$$

We specify the above lifting operators for the Dirichlet boundary condition. Let $\mathcal{P}$ be the L^2 projection on $\Sigma_h^{(p_t,p_s)}$, and replace ϕ by $\mathcal{P} g_D n$ in (5). Then on faces $S \in \bigcup_n \mathcal{S}_D^n$ we have

$$\int_{\mathcal{E}} r_S(\mathcal{P} g_D n) \cdot q \, \mathrm{d}\mathcal{E} = -\int_S g_D n \cdot q \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}, \ \forall S \in \bigcup_n \mathcal{S}_D^n. \tag{7}$$

For the global lifting operators, we proceed in a similar way. Using the projection operator $\mathcal{P}$, we replace ϕ by $\mathcal{P} g_D n$ in (6) and (5) to have:

$$\int_{\mathcal{E}} R(\mathcal{P} g_D n) \cdot q \, \mathrm{d}\mathcal{E} = -\sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S g_D n \cdot q \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}. \tag{8}$$

Using (6) and (8), we then introduce $R_{ID} : (L^2(\bigcup_n \mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$R_{ID}(\phi) = R(\phi) - R(\mathcal{P} g_D n). \tag{9}$$

The spatial part of the lifting operators R and r_S, denoted by $\overline{R}$ and $\bar{r}_S$, are obtained by eliminating the first component of R and r_S, respectively.

4. Space–time DG discretization for the advection–diffusion equation

In this section, we describe the derivation of the space–time DG weak formulation for the advection–diffusion equation. As shown in e.g. [2,6], it is beneficial for a DG discretization to rewrite the second order partial differential equation (3) into a system of first order equations. Following the same approach, we introduce an auxiliary variable $\sigma = A\nabla c$ to obtain the following system of first order equations:

$$\sigma = A\nabla c, \tag{10}$$

$$-\nabla \cdot (-Bc + \sigma) = 0. \tag{11}$$

In the next two sections we discuss the derivation of the weak formulation for (10) and (11).

4.1. Weak formulation for the auxiliary variable

First, we consider the auxiliary equation (10). By multiplying this equation with an arbitrary test function $\tau \in \Sigma_h^{(p_t,p_s)}$ and integrating over an element $\mathcal{K} \in \mathcal{T}_h$, we obtain:

$$\int_{\mathcal{K}} \sigma \cdot \tau \, \mathrm{d}\mathcal{K} = \int_{\mathcal{K}} A\nabla c \cdot \tau \, \mathrm{d}\mathcal{K}, \quad \forall \tau \in \Sigma_h^{(p_t,p_s)}.$$

Next, we substitute σ and c with their numerical approximations $\sigma_h \in \Sigma_h^{(p_t,p_s)}$ and $c_h \in V_h^{(p_t,p_s)}$. After integration by parts twice and summation over all elements, we have for all $\tau \in \Sigma_h^{(p_t,p_s)}$ the following formulation:

$$\int_{\mathcal{E}} \sigma_h \cdot \tau \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}} A\nabla_h c_h \cdot \tau \, \mathrm{d}\mathcal{E} + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{c}_h - c_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}. \tag{12}$$

The variable $\hat{c}_h$ is the *numerical flux* that must be introduced to account for the multivalued trace on $\partial\mathcal{K}$.

We recall the following relation (see [2, relation (3.3)]), which holds for vectors τ and scalars ϕ, piecewise smooth on $\mathcal{T}_h$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} (\tau \cdot n)\phi \, \mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}} \int_S \{\{\tau\}\} \cdot [\![\phi]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S [\![\tau]\!] \{\{\phi\}\} \, \mathrm{d}S. \tag{13}$$

When applied to the last contribution in (12) and using the symmetry of the matrix A, this results in

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{c}_h - c_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}} \int_S [\![\hat{c}_h - c_h]\!] \cdot \{\{A\tau\}\} \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S \{\{\hat{c}_h - c_h\}\} [\![A\tau]\!] \, \mathrm{d}S. \tag{14}$$

We consider now the choice for the numerical flux $\hat{c}_h$. There are several options listed in [2]. After a thorough study concerning the consistency, conservation properties, and matrix sparsity of each option, we choose the following numerical flux, which is similar to the choices in [3,5,6]:

$$\hat{c}_h = \{\{c_h\}\} \quad \text{on } S \in \mathcal{F}_{\mathrm{int}}, \qquad \hat{c}_h = g_D \quad \text{on } S \in \bigcup_n \mathcal{S}_D^n, \qquad \hat{c}_h = c_h^- \quad \text{elsewhere}. \tag{15}$$

Note that on faces $S \in \mathcal{S}_S^n$, which are the element boundaries K^n and K^{n+1}, the normal vector n has values $n = (\pm 1, \underbrace{0, \ldots, 0}_{d\times})$ and thus $An = (\underbrace{0, \ldots, 0}_{(d+1)\times})$. Hence there is no coupling between the space–time slabs. Substituting the choices for the numerical flux (15) into (14) and using the fact that entries of the matrix A are continuous functions, we obtain for each space–time slab $\mathcal{E}^n$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\partial\mathcal{K}} A(\hat{c}_h - c_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K} = -\sum_{S\in\mathcal{S}_{ID}^n} \int_S [\![c_h]\!] \cdot A\{\{\tau\}\} \, \mathrm{d}S + \sum_{S\in\mathcal{S}_D^n} \int_S g_D n \cdot A\tau \, \mathrm{d}S. \tag{16}$$

After summation over all space–time slabs, and using the symmetry of matrix A we can introduce the lifting operator (9) into (16) to obtain

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{c}_h - c_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K} = \int_{\mathcal{E}} AR_{ID}\big([\![c_h]\!]\big) \cdot \tau \, \mathrm{d}\mathcal{E}. \tag{17}$$

Introducing (17) into (12), we obtain for all $\tau \in \Sigma_h^{(p_t,p_s)}$:

$$\int_{\mathcal{E}} \sigma_h \cdot \tau \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}} A\nabla_h c_h \cdot \tau \, \mathrm{d}\mathcal{E} + \int_{\mathcal{E}} AR_{ID}\big([\![c_h]\!]\big) \cdot \tau \, \mathrm{d}\mathcal{E},$$

which implies that we can express $\sigma_h \in \Sigma_h^{(p_t,p_s)}$ as:

$$\sigma_h = A\nabla_h c_h + AR_{ID}\big([\![c_h]\!]\big) \quad \text{a.e. } \forall x \in \mathcal{E}. \tag{18}$$

4.2. Weak formulation of space–time DG method

The weak formulation for the advection–diffusion equation is obtained if we multiply (11) with arbitrary test functions $v \in V_h^{(p_t,p_s)}$, integrate by parts over element $\mathcal{K}$, and then substitute c, σ with their numerical approximations $c_h \in V_h^{(p_t,p_s)}$, $\sigma_h \in \Sigma_h^{(p_t,p_s)}$:

$$\int_{\mathcal{E}} (-Bc_h + \sigma_h) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} (-B\hat{c}_h + \hat{\sigma}_h) \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = 0. \tag{19}$$

Here we replaced c_h, σ_h on $\partial\mathcal{K}$ with the numerical fluxes $\hat{c}_h$, $\hat{\sigma}_h$, to account for the multivalued traces on $\partial\mathcal{K}$.

The next step is to find appropriate choices for the numerical fluxes. We separate the numerical fluxes into an *advective flux* $B\hat{c}_h$ and a *diffusive flux* $\hat{\sigma}_h$. For the advective flux, the obvious choice is an upwind flux, as described in [16]. However, for simplicity of proving the stability of the discretization, the upwind flux is written as the sum of an average plus a jump penalty, as suggested in [7]. Thus, we write the numerical flux $B\hat{c}_h$ as:

$$B\hat{c}_h = \{\{Bc_h\}\} + C_S[\![c_h]\!]. \tag{20}$$

The parameter C_S is chosen as:

$$C_S = \frac{1}{2}|B \cdot n| \quad \text{on } S \in \mathcal{F}_{\text{int}}. \tag{21}$$

For conciseness of the proofs discussed later in Sections 5 and 6 we extend the definition of C_S to the boundary of the space–time domain as:

$$C_S = \begin{cases} -B \cdot n/2, & \text{on } S \in (\bigcup_n \mathcal{S}^n_{DBSm} \cup \Omega_0), \\ +B \cdot n/2, & \text{on } S \in (\bigcup_n \mathcal{S}^n_{MDSp} \cup \Gamma_+). \end{cases} \tag{22}$$

If we substitute τ and ϕ in relation (13) with $\{\{Bc_h\}\} + C_S[\![c_h]\!]$ and v, respectively, the summation over the boundaries $\partial\mathcal{K}$ can be written as a sum over all faces as follows:

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} \big(\{\{Bc_h\}\} + C_S[\![c_h]\!]\big) \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = \sum_{S \in \mathcal{F}_{\text{int}}} \int_S \big(\{\{Bc_h\}\} + C_S[\![c_h]\!]\big) \cdot [\![v]\!] \, \mathrm{d}S + \sum_{S \in \mathcal{F}_{\text{bnd}}} \int_S Bc_h \cdot n v \, \mathrm{d}S. \tag{23}$$

Now we consider the numerical flux $\hat{\sigma}_h$. From [2], we have several options for this numerical flux. For similar reasons as in Section 4.1, we choose $\hat{\sigma}_h = \{\{\sigma_h\}\}$, which is the same as in [5,6]. By replacing $\hat{\sigma}_h$ with $\{\{\sigma_h\}\}$, then using (13) the contribution with $\hat{\sigma}_h$ in (19) can also be written as a sum over all faces $S \in \mathcal{F}$:

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} \{\{\hat{\sigma}_h\}\} \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = \sum_{S \in \mathcal{F}} \int_S \{\{\sigma_h\}\} \cdot [\![v]\!] \, \mathrm{d}S. \tag{24}$$

Using (23)–(24) and (18) (to eliminate σ_h), the primal formulation for c_h is obtained:

$$\int_{\mathcal{E}} \big(-Bc_h + A\nabla_h c_h + AR_{ID}\big([\![c_h]\!]\big)\big) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} + \sum_{S \in \mathcal{F}_{\text{int}}} \int_S \big(\{\{Bc_h\}\} + C_S[\![c_h]\!]\big) \cdot [\![v]\!] \, \mathrm{d}S + \sum_{S \in \mathcal{F}_{\text{bnd}}} \int_S Bc_h \cdot n v \, \mathrm{d}S$$
$$- \sum_{S \in \mathcal{F}} \int_S \big(A\{\{\nabla_h c_h\}\} + A\big\{\big\{R_{ID}\big([\![c_h]\!]\big)\big\}\big\}\big) \cdot [\![v]\!] \, \mathrm{d}S = 0. \tag{25}$$

This relation can be simplified using the following steps. Due to the symmetry of the matrix A and using the lifting operator R_{ID} (9) we have the relation

$$\int_{\mathcal{E}} AR_{ID}\big([\![c_h]\!]\big) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} = -\sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \int_S A[\![c_h]\!] \cdot \{\{\nabla_h v\}\} \, \mathrm{d}S + \sum_{S \in \bigcup_n \mathcal{S}^n_D} \int_S Ag_D n \cdot \nabla_h v \, \mathrm{d}S. \tag{26}$$

Further, the lifting operator R_{ID} has nonzero values only on faces $S \in \mathcal{S}^n_{ID}$. Using R, R_{ID} (see (6) and (9)) we obtain the following relation

$$-\sum_{S \in \mathcal{F}} \int_S A\big\{\big\{R_{ID}\big([\![c_h]\!]\big)\big\}\big\} \cdot [\![v]\!] \, \mathrm{d}S = \int_{\mathcal{E}} AR\big([\![c_h]\!]\big) \cdot R\big([\![v]\!]\big) \, \mathrm{d}\mathcal{E} - \int_{\mathcal{E}} AR(\mathcal{P}g_D n) \cdot R\big([\![v]\!]\big) \, \mathrm{d}\mathcal{E}. \tag{27}$$

Following a similar approach as in [6], we replace each term in (27) with the local lifting operator r_S, defined in Section 3.3, and make the following simplifications:

$$\int_{\mathcal{E}} AR(\llbracket c_h \rrbracket) \cdot R(\llbracket v \rrbracket)\,\mathrm{d}\mathcal{E} \cong \sum_{S\in\bigcup_n \mathcal{S}^n_{ID}} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} Ar_S(\llbracket c_h \rrbracket) \cdot r_S(\llbracket v \rrbracket)\,\mathrm{d}\mathcal{K}, \tag{28}$$

$$\int_{\mathcal{E}} AR(\mathcal{P}g_D n) \cdot R(\llbracket v \rrbracket)\,\mathrm{d}\mathcal{E} \cong \sum_{S\in\bigcup_n \mathcal{S}^n_{D}} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} Ar_S(\mathcal{P}g_D n) \cdot r_S(\llbracket v \rrbracket)\,\mathrm{d}\mathcal{K}. \tag{29}$$

In Section 5 we will derive a sufficient condition for the constant $\eta_{\mathcal{K}} > 0$ to guarantee a stable and unique solution. The advantage of this replacement is that the stiffness matrix in the weak formulation using the local lifting operators is considerably sparser than the stiffness matrix resulting from the weak formulation with global lifting operators. We refer to [2,6] for a further explanation.

Substituting relations (26)–(27) into (25), using relations (28)–(29), and considering the structure of matrix A, we then obtain:

$$\begin{aligned}
&-\int_{\mathcal{E}} Bc_h \cdot \nabla_h v\,\mathrm{d}\mathcal{E} + \int_{\mathcal{E}} D\overline{\nabla}_h c_h \cdot \overline{\nabla}_h v\,\mathrm{d}\mathcal{E} - \sum_{S\in\bigcup_n \mathcal{S}^n_{ID}} \int_S D\langle\langle c_h \rangle\rangle \cdot \{\{\overline{\nabla}_h v\}\}\,\mathrm{d}S \\
&\quad + \sum_{S\in\bigcup_n \mathcal{S}^n_{D}} \int_S g_D D\bar{n} \cdot \overline{\nabla}_h v\,\mathrm{d}S + \sum_{S\in\mathcal{F}_{\text{int}}} \int_S \big(\{\{Bc_h\}\} + C_S\llbracket c_h \rrbracket\big) \cdot \llbracket v \rrbracket\,\mathrm{d}S \\
&\quad + \sum_{S\in\mathcal{F}_{\text{bnd}}} \int_S Bc_h \cdot nv\,\mathrm{d}S - \sum_{S\in\bigcup_n \mathcal{S}^n_{ID}} \int_S D\{\{\overline{\nabla}_h c_h\}\} \cdot \langle\langle v \rangle\rangle\,\mathrm{d}S \\
&\quad - \sum_{S\in\mathcal{F}_{\text{bnd}}\setminus\bigcup_n \mathcal{S}^n_{D}} \int_S D\overline{\nabla}_h c_h \cdot \bar{n} v\,\mathrm{d}S + \sum_{S\in\bigcup_n \mathcal{S}^n_{ID}} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\llbracket c_h \rrbracket) \cdot \bar{r}_S(\llbracket v \rrbracket)\,\mathrm{d}\mathcal{K} \\
&\quad - \sum_{S\in\bigcup_n \mathcal{S}^n_{D}} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\mathcal{P}g_D n) \cdot \bar{r}_S(\llbracket v \rrbracket)\,\mathrm{d}\mathcal{K} = 0.
\end{aligned} \tag{30}$$

Here we used the spatial gradient operator $\overline{\nabla}$, the spatial jump operator $\langle\langle\cdot\rangle\rangle$ (see Section 3.2) and the spatial lifting operator $\bar{r}_S$ (see Section 3.3). Next, we introduce the following boundary and initial conditions:

$$\begin{aligned}
D\overline{\nabla}_h c_h \cdot \bar{n} &= g_M - \alpha c_h && \text{on } S \in \bigcup_n \mathcal{S}^n_M, \\
c_h &= g_D && \text{on } S \in \bigcup_n \mathcal{S}^n_{DBSm}, \\
c_h &= c_0 && \text{on } \Omega_0,
\end{aligned}$$

into (30). We introduce now the bilinear form $a: V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$:

$$a(c_h, v) = a_a(c_h, v) + a_d(c_h, v), \tag{31}$$

with $a_a: V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$, $a_d: V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$ defined as:

$$\begin{aligned}
a_a(c_h, v) = &-\int_{\mathcal{E}} Bc_h \cdot \nabla_h v\,\mathrm{d}\mathcal{E} + \sum_{S\in\mathcal{F}_{\text{int}}} \int_S \big(\{\{Bc_h\}\} + C_S\llbracket c_h \rrbracket\big) \cdot \llbracket v \rrbracket\,\mathrm{d}S \\
&+ \sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)} \int_S B \cdot nc_h v\,\mathrm{d}S,
\end{aligned} \tag{32}$$

$$a_d(c_h, v) = \int_{\mathcal{E}} D\overline{\nabla}_h c_h \cdot \overline{\nabla}_h v \,\mathrm{d}\mathcal{E} - \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \int_S \big(D\langle\langle c_h\rangle\rangle \cdot \{\{\overline{\nabla}_h v\}\} + D\{\{\overline{\nabla}_h c_h\}\} \cdot \langle\langle v\rangle\rangle\big)\,\mathrm{d}S$$
$$+ \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S\big([\![c_h]\!]\big) \cdot \bar{r}_S\big([\![v]\!]\big)\,\mathrm{d}\mathcal{K} + \sum_{S\in\bigcup_n \mathcal{S}_M^n} \int_S \alpha c_h v \,\mathrm{d}S, \tag{33}$$

and the linear form $\ell : V_h^{(p_t,p_s)} \to \mathbb{R}$ defined as:

$$\ell(v) = -\sum_{S\in\bigcup_n \mathcal{S}_D^n} \int_S g_D D\bar{n} \cdot \overline{\nabla}_h v \,\mathrm{d}S + \sum_{S\in\bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\mathcal{P}g_D n) \cdot \bar{r}_S\big([\![v]\!]\big)\,\mathrm{d}\mathcal{K} + \sum_{S\in\bigcup_n \mathcal{S}_M^n} \int_S g_M v \,\mathrm{d}S$$
$$- \sum_{S\in\bigcup_n \mathcal{S}_{DBSm}^n} \int_S B g_D \cdot n v \,\mathrm{d}S + \int_{\Omega_0} c_0 v \,\mathrm{d}\Omega. \tag{34}$$

Note that the term $\sum_{S\in\mathcal{F}_{\text{bnd}}\setminus\bigcup_n \mathcal{S}_{DM}^n} \int_S D\overline{\nabla}_h c_h \cdot \bar{n} v \,\mathrm{d}S$ is dropped from the bilinear form $a_d(\cdot,\cdot)$ since on $S \in \mathcal{F}_{\text{bnd}} \setminus \bigcup_n \mathcal{S}_{DM}^n$ the matrix D is zero.

The space–time DG discretization for (1) can now be formulated as follows.

Find a $c_h \in V_h^{(p_t,p_s)}$ such that:

$$a(c_h, v) = \ell(v), \quad \forall v \in V_h^{(p_t,p_s)}. \tag{35}$$

This formulation is the most straightforward for the analysis discussed in Sections 5 and 6, but for practical implementations, an arbitrary Lagrangian Eulerian (ALE) formulation is preferable. Therefore, in this paper, we also present the ALE form of the space–time weak formulation (35). The relation between the space–time and ALE formulation discussed here follows the derivation in [16].

Using a result from [16], the space–time normal vector n can be split into two parts: $n = (n_t, \bar{n})$, with n_t the temporal part and $\bar{n}$ the spatial part of the space–time normal vector n. Next, we consider the normal vector n on the faces $S \in \mathcal{F}_{\text{int}}$, which consist of two sets: $\mathcal{F}_{\text{int}} = \bigcup_n (\mathcal{S}_I^n \cup \mathcal{S}_S^n)$. On $S \in \mathcal{S}_S^n$, the space–time normal vector is $n = (\pm 1, \underbrace{0, \ldots, 0}_{d\times})$ and is not affected by the mesh velocity. On the faces $S \in \mathcal{S}_I^n$ the space–time normal vector depends on the mesh velocity u_g:

$$n = (-u_g \cdot \bar{n}, \bar{n}), \tag{36}$$

which also holds on the boundary faces $S \in \mathcal{F}_{\text{bnd}} \setminus (\Omega_0 \cup \Omega_T)$.

If we recall the bilinear and linear forms in (32)–(34), then only $a_a(\cdot,\cdot)$ and $\ell(\cdot)$ are needed to be rewritten into the ALE formulation by splitting the normal vector n into a temporal and spatial part. The bilinear form a_d in (33) remains valid for the ALE formulation since it does not depend on n_t. We now consider the contribution $\{\{Bc_h\}\} \cdot [\![v]\!]$ in (32). On $S \in \bigcup_n \mathcal{S}_I^n$, this contribution can be written in the ALE formulation using (36) as:

$$\{\{Bc_h\}\} \cdot [\![v]\!] = \{\{c_h\}\}(u - u_g) \cdot \langle\langle v\rangle\rangle,$$

while on $S \in \mathcal{S}_S^n$ this term does not change. Next, consider the term $[\![c_h]\!] \cdot [\![v]\!]$. Since the normal vector n has length one, we immediately obtain

$$[\![c_h]\!] \cdot [\![v]\!] = \big(c_h^+ - c_h^-\big)\big(v^+ - v^-\big),$$

and thus this contribution also does not depend on the mesh velocity u_g.

The bilinear form $a_a(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ in the ALE formulation are now equal to:

$$a_a(c_h, v) = -\int_{\mathcal{E}} Bc_h \cdot \nabla_h v \,\mathrm{d}\mathcal{E} + \sum_{S\in\bigcup_n \mathcal{S}_I^n} \int_S \big(\{\{c_h\}\}(u - u_g) \cdot \langle\langle v\rangle\rangle + C_S[\![c_h]\!] \cdot [\![v]\!]\big)\,\mathrm{d}S$$
$$+ \sum_{S\in\bigcup_n \mathcal{S}_S^n} \int_S \big(\{\{Bc_h\}\} + C_S[\![c_h]\!]\big) \cdot [\![v]\!]\,\mathrm{d}S + \sum_{S\in(\bigcup_n \mathcal{S}_{MDSp}^n \cup \Gamma_+)} \int_S (u - u_g) \cdot \bar{n} c_h v \,\mathrm{d}S, \tag{37}$$

$$\ell(v) = -\sum_{S\in\bigcup_n \mathcal{S}_D^n}\int_S g_D D\bar{n}\cdot\overline{\nabla}_h v\,\mathrm{d}S + \sum_{S\in\bigcup_n \mathcal{S}_D^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_{\mathcal{K}}\int_{\mathcal{K}} D\bar{r}_S(\mathcal{P}g_D n)\cdot\bar{r}_S\big([\![v]\!]\big)\,\mathrm{d}\mathcal{K} + \sum_{S\in\bigcup_n \mathcal{S}_M^n}\int_S g_M v\,\mathrm{d}S$$
$$-\sum_{S\in\bigcup_n \mathcal{S}_{DBSm}^n}\int_S g_D(u-u_g)\cdot\bar{n}v\,\mathrm{d}S + \int_{\Omega_0} c_0 v\,\mathrm{d}\Omega, \tag{38}$$

while $a_d(\cdot,\cdot)$ is given by (33).

5. Consistency, coercivity and stability of the space–time DG discretization

In this section we present an analysis of the consistency, coercivity and stability of the space time discontinuous Galerkin formulation (31)–(35). This section is divided into two subsections, Section 5.1 concerns with the main results while detailed proofs can be found in Section 5.2.

5.1. Main results

The analysis of the space–time discontinuous Galerkin formulation is considerably simplified by the introduction of a so-called DG norm, which is closely related to the bilinear form (31).

Definition 1. The DG norm $|||\cdot|||_{\mathrm{DG}}$ corresponding to the bilinear form (31) can be defined on $H^{(0,1)}(\mathcal{E}) + V_h^{(p_t,p_s)}$, with $H^{(0,1)}(\mathcal{E})$ the anisotropic Sobolev space defined in Section 3.2, $\alpha \geqslant 0$ and $D^\star$ a symmetric positive semi-definite matrix, as:

$$|||v|||_{\mathrm{DG}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h}\|v\|_{L^2(\mathcal{K})}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\big\|D^\star\overline{\nabla}_h v\big\|_{L^2(\mathcal{K})}^2 + \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\big\|D^\star\bar{r}_S\big([\![v]\!]\big)\big\|_{L^2(\mathcal{K})}^2$$
$$+\sum_{S\in\bigcup_n \mathcal{S}_M^n}\|\sqrt{\alpha}v\|_{L^2(S)}^2 + \sum_{S\in\mathcal{F}}\big\|C_S^{1/2}\big|[\![v]\!]\big|\big\|_{L^2(S)}^2.$$

First, we discuss the consistency of the space–time DG method (35). This formulation is consistent when (35) is also satisfied by $c\in H^2(\mathcal{E})$, the solution of (3)–(4):

$$a(c,v)=\ell(v),\quad \forall v\in H^{(1,1)}(\mathcal{E},\mathcal{T}_h). \tag{39}$$

The proof for consistency is straightforward. We replace c_h in (31) by c. Since c solves (3)–(4), we have $\{\{Bc\}\} = Bc$ on $S\in\mathcal{F}$, $[\![c]\!]=0$ and $[\![\nabla_h c]\!]=0$ on $S\in\mathcal{F}_{\mathrm{int}}$, $[\![c]\!]=g_D n$ on $S\in\mathcal{S}_D^n$, and $\{\{\nabla_h c\}\}=\nabla c$ on $S\in\mathcal{S}_{ID}^n$. If we use these relations into (31), perform integration by parts, and use the boundary conditions (4), we obtain $\ell(v)$. Subtracting (35) from (39) yields the Galerkin orthogonality property

$$a(c-c_h,v)=0,\quad \forall v\in V_h^{(p_t,p_s)}. \tag{40}$$

The next result concerns the coercivity of the bilinear form $a(\cdot,\cdot)$. In order to prove the coercivity, we first introduce the following inequality, which is a direct extension of the one discussed in [2, p. 1763], to the space–time discretization,

$$\|v\|_{L^2(\mathcal{E})} \leqslant C_p\Bigg(\sum_{\mathcal{K}\in\mathcal{T}_h}\big\|D^\star\overline{\nabla}_h v\big\|_{L^2(\mathcal{K})}^2 + \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n}\sum_{\mathcal{K}\in\mathcal{T}_h}\big\|D^\star\bar{r}_S\big([\![v]\!]\big)\big\|_{L^2(\mathcal{K})}^2\Bigg)^{1/2}. \tag{41}$$

The constant C_p in this inequality follows from the discrete Poincaré inequality in [1, Lemma 2.1]. We then prove the coercivity in the following lemma.

Lemma 5.1. *Let $\eta_0=\min_{\mathcal{K}\in\mathcal{T}_h}\eta_{\mathcal{K}}$. Assume that $\eta_0>N_f$, with N_f the number of faces of each element $\mathcal{K}\in\mathcal{T}_h$. Then, if*

$$\frac{\beta_c}{C_p^2} + \inf_{x\in\mathcal{E}}\overline{\nabla}\cdot u(x) \geqslant b_0 > 0, \tag{42}$$

with $\beta_c = \min(1-\epsilon, \eta_0 - \frac{N_f}{\epsilon}) > 0$ *for* $\epsilon \in (\frac{N_f}{\eta_0}, 1)$*, there exists a constant* $\beta_a > 0$*, independent of the mesh size* $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$*, such that*

$$a(v,v) \geqslant \beta_a |||v|||^2_{\mathrm{DG}}, \quad \forall v \in V_h^{(p_t,p_s)}, \tag{43}$$

for $0 \leqslant p_t \leqslant 1$ *and* $p_s \geqslant 0$*, with* $\beta_a = \min(\frac{b_0}{2}, \frac{\beta_c}{2})$.

The proof, which is given in Section 5.2.1, is an extension to the space–time framework of the analysis given in [6,7]. The condition $\overline{\nabla} \cdot u \geqslant 0$ for $\forall x \in \mathcal{E}$ such as in [10] is relaxed using the assumption (42).

The next result shows that the solution to (35) is bounded by known data.

Lemma 5.2. *Assume that the parameters* $\eta_0, \beta_a, \beta_c, b_0$ *are such that Lemma* 5.1 *is satisfied and let* $\eta_m = \max_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}}$*. Then the solution to the weak formulation* (35) *satisfies the following upper bound*:

$$\beta_a^2 |||c_h|||^2_{\mathrm{DG}} \leqslant \sum_{\mathcal{K}\in\mathcal{T}_h} \big\| D^\star \overline{R}(\mathcal{P} g_D n)\big\|^2_{L^2(\mathcal{K})} + \eta_m^2 \sum_{S\in\bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\| D^\star \bar{r}_S(\mathcal{P} g_D n)\big\|^2_{L^2(\mathcal{K})}$$
$$+ \sum_{S\in\bigcup_n \mathcal{S}_M^n} \big\|\alpha^{-1/2} g_M\big\|^2_{L^2(S)} + 4 \sum_{S\in\bigcup_n \mathcal{S}_{DB}^n} \big\| C_S^{1/2} g_D\big\|^2_{L^2(S)} + 4\big\|C_S^{1/2} c_0\big\|^2_{L^2(\Omega_0)}.$$

The proof, given in Section 5.2.2, is an extension to space–time framework of the analysis given in [10]. It mainly consists of applying the Schwarz and arithmetic–geometric mean inequalities to linear form $\ell(\cdot)$ and making use of the result from Lemma 5.1.

The upper bound for the solution given by Lemma 5.2 is independent of $h_{\mathcal{K}}$, the radius of the smallest sphere containing each space–time element, hence also from the time step $\Delta_n t$, since $\Delta_n t \leqslant h_{\mathcal{K}}$. This result shows that the space–time DG discretization is unconditionally stable when the proper stabilization coefficient η_0 is chosen.

The next result states the existence of a unique solution of (35). The proof, which is given in Section 5.2.3, is obtained by using the coercivity given in Lemma 5.1.

Theorem 5.3. *Assume that* $\eta_0 > N_f$*, with* N_f *the number of faces of each element* $\mathcal{K} \in \mathcal{T}_h$*, and the parameters* β_a, β_c *are chosen such that Lemma* 5.1 *is satisfied. Then the space–time discontinuous Galerkin discretization given by* (35) *is unconditionally stable and has a unique solution for basis functions which are constant or linear in time.*

5.2. Detailed proofs

5.2.1. Proof of coercivity in Lemma 5.1

To prove Lemma 5.1, we first consider $a_a(c_h, v)$. Take $c_h = v$ in (32), use the relation: $vB\cdot\nabla_h v = -\frac{1}{2}(\nabla_h \cdot B)v^2 + \frac{1}{2}\nabla_h\cdot(Bv^2)$, and apply Gauss' Theorem for $a_a(v,v)$ to obtain the following relation:

$$a_a(v,v) = \frac{1}{2}\int_{\mathcal{E}} (\nabla_h\cdot B)v^2\,\mathrm{d}\mathcal{E} - \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} (B\cdot n)v^2\,\mathrm{d}\partial\mathcal{K}$$
$$+ \sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S \big(\{\{Bv\}\} + C_S[\![v]\!]\big)\cdot[\![v]\!]\,\mathrm{d}S + \sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)}\int_S B\cdot n v^2\,\mathrm{d}S.$$

Using the identity (13) and the fact that vector B is a continuous function, the last equation is written further as

$$a_a(v,v) = \frac{1}{2}\int_{\mathcal{E}} (\nabla\cdot B)v^2\,\mathrm{d}\mathcal{E} - \frac{1}{2}\sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S B\cdot[\![v^2]\!]\,\mathrm{d}S$$
$$- \frac{1}{2}\sum_{S\in(\bigcup_n \mathcal{S}^n_{DBSm}\cup\Omega_0)}\int_S B\cdot n v^2\,\mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S \{\{Bv\}\}\cdot[\![v]\!]\,\mathrm{d}S$$
$$+ \frac{1}{2}\sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)}\int_S B\cdot n v^2\,\mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S C_S[\![v]\!]\cdot[\![v]\!]\,\mathrm{d}S. \tag{44}$$

Due to the continuity of vector B, on faces $S \in \mathcal{F}_{\text{int}}$ we have:

$$\int_S \{\{Bv\}\} \cdot [\![v]\!] \, \mathrm{d}S = \frac{1}{2} \int_S B \cdot [\![v^2]\!] \, \mathrm{d}S. \tag{45}$$

As a consequence of (45) and using the definition of C_S in (21)–(22), we can write the final form of $a_a(v, v)$ as:

$$a_a(v, v) = \frac{1}{2} \int_{\mathcal{E}} (\nabla \cdot B) v^2 \, \mathrm{d}\mathcal{E} + \sum_{S \in \mathcal{F}} \left\| C_S^{1/2} \left| [\![v]\!] \right| \right\|^2_{L^2(S)}. \tag{46}$$

Next, we consider $a_d(v, v)$ in (33) with $c_h = v$. Using the global lifting operator $\overline{R}$, which is the spatial part of the lifting operator R defined in (6), and the fact that matrix $D^\star$ is symmetric, we can write $a_d(v, v)$ as:

$$\begin{aligned} a_d(v, v) = {} & \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \overline{\nabla}_h v \right\|^2_{L^2(\mathcal{K})} + 2 \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} D^\star \overline{\nabla}_h v \cdot D^\star \overline{R}\big([\![v]\!]\big) \, \mathrm{d}\mathcal{K} \\ & + \sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \left\| D^\star \bar{r}_S\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})} + \sum_{S \in \bigcup_n \mathcal{S}^n_M} \| \sqrt{\alpha} v \|^2_{L^2(S)}. \end{aligned} \tag{47}$$

Using the Schwarz and arithmetic–geometric mean inequalities we obtain

$$2 \int_{\mathcal{K}} D^\star \overline{\nabla}_h v \cdot D^\star \overline{R}\big([\![v]\!]\big) \, \mathrm{d}\mathcal{K} \geqslant -\epsilon \left\| D^\star \overline{\nabla}_h v \right\|^2_{L^2(\mathcal{K})} - \frac{1}{\epsilon} \left\| D^\star \overline{R}\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})}, \tag{48a}$$

with $\epsilon > 0$. As a consequence of (6) and the fact that the local lifting operator $\bar{r}_S$ is only non-zero in the elements connected to the face S, we also have

$$\left\| D^\star \overline{R}\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})} \leqslant N_f \sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \left\| D^\star \bar{r}_S\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})}, \tag{48b}$$

with N_f the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$. Introducing (48a)–(48b) into (47) and combining with (46), we deduce

$$\begin{aligned} a(v, v) \geqslant {} & \frac{1}{2} \int_{\mathcal{E}} (\overline{\nabla} \cdot u) v^2 \, \mathrm{d}\mathcal{E} + (1 - \epsilon) \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \overline{\nabla}_h v \right\|^2_{L^2(\mathcal{K})} + \left(\eta_0 - \frac{N_f}{\epsilon} \right) \sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \bar{r}_S\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})} \\ & + \sum_{S \in \bigcup_n \mathcal{S}^n_M} \| \sqrt{\alpha} v \|^2_{L^2(S)} + \sum_{S \in \mathcal{F}} \left\| C_S^{1/2} \left| [\![v]\!] \right| \right\|^2_{L^2(S)}, \end{aligned} \tag{49}$$

with η_0 defined as $\eta_0 = \min_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}$. If we take $\eta_0 > N_f$ and $\epsilon \in (\frac{N_f}{\eta_0}, 1)$, and choose $\beta_c = \min(1 - \epsilon, \eta_0 - \frac{N_f}{\epsilon}) > 0$, we obtain:

$$\begin{aligned} a(v, v) \geqslant {} & \frac{1}{2} \int_{\mathcal{E}} (\overline{\nabla} \cdot u) v^2 \, \mathrm{d}\mathcal{E} + \beta_c \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \overline{\nabla}_h v \right\|^2_{L^2(\mathcal{K})} + \beta_c \sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \bar{r}_S\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})} \\ & + \sum_{S \in \bigcup_n \mathcal{S}^n_M} \| \sqrt{\alpha} v \|^2_{L^2(S)} + \sum_{S \in \mathcal{F}} \left\| C_S^{1/2} \left| [\![v]\!] \right| \right\|^2_{L^2(S)}. \end{aligned} \tag{50}$$

Making use inequality (41) into (50) and assuming the existence of $b_0 > 0$ that satisfies (42), we then obtain:

$$\begin{aligned} a(v, v) \geqslant {} & \frac{b_0}{2} \| v \|^2_{L^2(\mathcal{E})} + \frac{\beta_c}{2} \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \overline{\nabla}_h v \right\|^2_{L^2(\mathcal{K})} + \frac{\beta_c}{2} \sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| D^\star \bar{r}_S\big([\![v]\!]\big) \right\|^2_{L^2(\mathcal{K})} \\ & + \sum_{S \in \bigcup_n \mathcal{S}^n_M} \| \sqrt{\alpha} v \|^2_{L^2(S)} + \sum_{S \in \mathcal{F}} \left\| C_S^{1/2} \left| [\![v]\!] \right| \right\|^2_{L^2(S)}. \end{aligned} \tag{51}$$

Since $\beta_c/2$ is always less than one, choosing $\beta_a = \min(\frac{b_0}{2}, \frac{\beta_c}{2})$ completes the proof of the coercivity. □

5.2.2. Proof of boundedness in Lemma 5.2

To prove Lemma 5.2, we take $v = c_h$ in (35), which results in the relation:

$$a(c_h, c_h) = \ell(c_h). \tag{52}$$

Using the lifting operator R in (8), the symmetry of matrix D, and the definition of C_S on $S \in \mathcal{F}_{\text{bnd}}$ in (22), the functional $\ell(c_h)$ can be written as:

$$\begin{aligned}\ell(c_h) = {} & \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} D^\star \overline{R}(\mathcal{P} g_D n) \cdot D^\star \overline{\nabla}_h c_h \,\mathrm{d}\mathcal{K} \\ & + \sum_{S\in\bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D^\star \bar{r}_S(\mathcal{P} g_D n) \cdot D^\star \bar{r}_S\big([\![c_h]\!]\big) \,\mathrm{d}\mathcal{K} + \sum_{S\in\bigcup_n \mathcal{S}_M^n} \int_S g_M c_h \,\mathrm{d}S \\ & + 2 \sum_{S\in\bigcup_n \mathcal{S}_{DBSm}^n} \int_S C_S g_D c_h \,\mathrm{d}S + 2\int_{\Omega_0} C_S c_0 c_h \,\mathrm{d}\Omega. \end{aligned} \tag{53}$$

Applying the Schwarz and arithmetic-geometric mean inequalities on each term in (53) and combining this result with (52) and Lemma 5.1 using $v = c_h$, we obtain the inequality

$$\begin{aligned} & \beta_a \|c_h\|^2_{L^2(\mathcal{E})} + \left(\beta_a - \frac{\epsilon_1}{2}\right) \sum_{\mathcal{K}\in\mathcal{T}_h} \|D^\star \overline{\nabla}_h c_h\|^2_{L^2(\mathcal{K})} + \left(\beta_a - \frac{\eta_m \epsilon_2}{2}\right) \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|D^\star \bar{r}_S\big([\![c_h]\!]\big)\big\|^2_{L^2(\mathcal{K})} \\ & \quad + \left(\beta_a - \frac{\epsilon_3}{2}\right) \sum_{S\in\bigcup_n \mathcal{S}_M^n} \|\sqrt{\alpha} c_h\|^2_{L^2(S)} + (\beta_a - \epsilon_4) \sum_{S\in\mathcal{F}} \big\|C_S^{1/2} \big|[\![c_h]\!]\big|\big\|^2_{L^2(S)} \\ & \leqslant \frac{1}{2\epsilon_1} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|D^\star \overline{R}(\mathcal{P} g_D n)\big\|^2_{L^2(\mathcal{K})} + \frac{\eta_m}{2\epsilon_2} \sum_{S\in\bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|D^\star \bar{r}_S(\mathcal{P} g_D n)\big\|^2_{L^2(\mathcal{K})} \\ & \quad + \frac{1}{2\epsilon_3} \sum_{S\in\bigcup_n \mathcal{S}_M^n} \big\|\alpha^{-1/2} g_M\big\|^2_{L^2(S)} + \frac{1}{\epsilon_4} \sum_{S\in\bigcup_n \mathcal{S}_{DBSm}^n} \big\|C_S^{1/2} g_D\big\|^2_{L^2(S)} + \frac{1}{\epsilon_4} \big\|C_S^{1/2} c_0\big\|^2_{L^2(\Omega_0)}, \end{aligned}$$

with $\epsilon_1, \ldots, \epsilon_4 > 0$ and $\eta_m = \max_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}}$. Next, we substitute the following coefficients: $\epsilon_1 = \beta_a$, $\epsilon_2 = \frac{\beta_a}{\eta_m}$, $\epsilon_3 = \beta_a$, and $\epsilon_4 = \frac{\beta_a}{2}$ and multiply the result with $2\beta_a$ to complete the proof. □

5.2.3. Proof of the uniqueness in Theorem 5.3

To prove the uniqueness of the solution it is sufficient to show that the following homogeneous equation:

Find a $c_h \in V_h^{(p_t,p_s)}$ such that:

$$a(c_h, v) = 0, \quad \forall v \in V_h^{(p_t,p_s)}, \quad \text{with } c_h(0,\bar{x}) = 0, \tag{54}$$

has only the trivial solution $c_h = 0$ for all $t > 0$.

We proceed as follows. Assume that c_h is a solution of (54) and take $v = c_h$ in (31). Then we rewrite (43) as:

$$\begin{aligned} a(c_h, c_h) \geqslant \beta_a \sum_{n=0}^{N_t-1} \Bigg(& \sum_{\mathcal{K}\in\mathcal{T}_h^n} \|c_h\|^2_{L^2(\mathcal{K})} + \sum_{\mathcal{K}\in\mathcal{T}_h^n} \|D^\star \overline{\nabla}_h c_h\|^2_{L^2(\mathcal{K})} + \sum_{S\in\mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h^n} \big\|D^\star \bar{r}_S\big([\![c_h]\!]\big)\big\|^2_{L^2(\mathcal{K})} \\ & + \sum_{S\in\mathcal{S}_M^n} \|\sqrt{\alpha} c_h\|^2_{L^2(S)} + \sum_{S\in\mathcal{F}^n} \big\|C_S^{1/2} \big|[\![c_h]\!]\big|\big\|^2_{L^2(S)} \Bigg). \end{aligned}$$

Consider now the space–time slab for $n = 0$. The coercivity condition, in combination with the initial condition $c_h^+ = 0$ at $t = 0$ and (54), imply that $c_h = 0$ in the first space–time slab when constant or linear polynomials in time are used. We can continue this argument to the other space–time slabs and obtain that $c_h = 0$ is the only solution possible for the homogeneous equation. Hence the DG algorithm has a unique solution c_h for constant or linear basis functions in time. The unconditional stability of the DG algorithm is a direct consequence of Lemma 5.2. □

6. Error estimates and *hp*-convergence

First, let us define the projection $\mathcal{P}: L^2(\mathcal{E}) \to V_h^{(p_t,p_s)}$ as:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} (\mathcal{P}w, v)_{\mathcal{K}} = \sum_{\mathcal{K}\in\mathcal{T}_h} (w, v)_{\mathcal{K}}, \quad \forall v \in V_h^{(p_t,p_s)}, \tag{55}$$

which can be used to decompose the global error $c - c_h$ as:

$$c - c_h = (c - \mathcal{P}c) + (\mathcal{P}c - c_h) \equiv \rho + \theta, \tag{56}$$

with ρ the interpolation error and θ the discretization error. In the next section we discuss upper bounds for the interpolation error ρ.

6.1. Bounds for the interpolation error

In this section we present upper bounds for the interpolation error $\rho = c - \mathcal{P}c$. These estimates are an extension of the bounds for the interpolation error derived in [9] to general dimensions. We restrict the derivations for a separate polynomial degree $p_{t,\mathcal{K}}$ in time and a polynomial degree $p_{s,\mathcal{K}}$ in each spatial variable.

Lemma 6.1. *Assume that $\mathcal{K}$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_{\mathcal{K}}$, $F_{\mathcal{K}}$, with $F_{\mathcal{K}}: \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $Q_{\mathcal{K}}: \widetilde{\mathcal{K}} \to \mathcal{K}$. Assume also that $h_{i,\mathcal{K}}$, $i = 1, \ldots, d$ is the edge length of $\widetilde{\mathcal{K}}$ in the x_i direction, and $\Delta_n t$ the edge length in the x_0 direction (see illustration in Fig. 2 for $d = 2$). Let $c|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}+1,k_{s,\mathcal{K}}+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}, k_{s,\mathcal{K}} \geqslant 0$. Let $\mathcal{P}$ denote the L^2 projection of c onto the finite element space $V_h^{(p_t,p_s)}$, then the projection error $\rho = c - \mathcal{P}c$ in $\mathcal{K}$ and its trace at the boundary $\partial\mathcal{K}$ obey the error bounds*:

$$\|\rho\|^2_{L^2(\mathcal{K})} \leqslant C Z_{\mathcal{K}}, \tag{57}$$

$$\|\overline{\nabla}_h \rho\|^2_{L^2(\mathcal{K})} \leqslant C N_{\mathcal{K}}, \tag{58}$$

$$\|\rho\|^2_{L^2(\partial\mathcal{K})} \leqslant C(A_{\mathcal{K}} + B_{\mathcal{K}}), \tag{59}$$

where

$$Z_{\mathcal{K}} = \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2s_{\mathcal{K}}} \left\|\partial_i^{s_{\mathcal{K}}} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} + \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}}\right)^{2s_{0,\mathcal{K}}} \left\|\partial_0^{s_{0,\mathcal{K}}} c\right\|^2_{L^2(\widetilde{\mathcal{K}})},$$

$$N_{\mathcal{K}} = \sum_{i=1}^{d} \frac{h_{i,\mathcal{K}}^{2t_{\mathcal{K}}}}{p_{s,\mathcal{K}}^{2t_{\mathcal{K}}-1}} \left\|\partial_i^{t_{\mathcal{K}}+1} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} + \sum_{i=1}^{d}\sum_{j\neq i} \frac{h_{j,\mathcal{K}}^{2t_{\mathcal{K}}+2}}{p_{s,\mathcal{K}}^{2t_{\mathcal{K}}}} \left\|\partial_j^{t_{\mathcal{K}}+1}\partial_i c\right\|^2_{L^2(\widetilde{\mathcal{K}})} + \sum_{i=1}^{d} \frac{(\Delta_n t)^{2t_{0,\mathcal{K}}+2}}{p_{t,\mathcal{K}}^{2t_{0,\mathcal{K}}}} \left\|\partial_0^{t_{0,\mathcal{K}}+1}\partial_i c\right\|^2_{L^2(\widetilde{\mathcal{K}})},$$

$$\begin{aligned}
A_{\mathcal{K}} = {} & \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2t_{\mathcal{K}}+1} \left\|\partial_i^{t_{\mathcal{K}}+1} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} + \sum_{i=1}^{d}\sum_{j\neq i} \frac{1}{h_{i,\mathcal{K}}} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2s_{\mathcal{K}}} \left\|\partial_j^{s_{\mathcal{K}}} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} \\
& + \sum_{i=1}^{d}\sum_{j\neq i} \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2q_{\mathcal{K}}} \left\|\partial_j^{q_{\mathcal{K}}}\partial_i c\right\|^2_{L^2(\widetilde{\mathcal{K}})},
\end{aligned}$$

$$\begin{aligned}
B_{\mathcal{K}} = {} & \sum_{i=1}^{d} \frac{1}{h_{i,\mathcal{K}}} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}}\right)^{2s_{0,\mathcal{K}}} \left\|\partial_0^{s_{0,\mathcal{K}}} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} + \sum_{i=1}^{d} \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}}\right)^{2q_{0,\mathcal{K}}} \left\|\partial_0^{q_{0,\mathcal{K}}}\partial_i c\right\|^2_{L^2(\widetilde{\mathcal{K}})} \\
& + \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}}\right)^{2t_{0,\mathcal{K}}+1} \left\|\partial_0^{t_{0,\mathcal{K}}+1} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} + \frac{1}{\Delta_n t} \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2s_{\mathcal{K}}} \left\|\partial_i^{s_{\mathcal{K}}} c\right\|^2_{L^2(\widetilde{\mathcal{K}})} \\
& + \frac{\Delta_n t}{p_{t,\mathcal{K}}} \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2q_{\mathcal{K}}} \left\|\partial_i^{q_{\mathcal{K}}}\partial_0 c\right\|^2_{L^2(\widetilde{\mathcal{K}})},
\end{aligned}$$

with $p_{t,\mathcal{K}}$ and $p_{s,\mathcal{K}}$ the local polynomial degree in time and space, respectively, on element $\mathcal{K}$, $0 < s_{0,\mathcal{K}} \leqslant \min(p_{t,\mathcal{K}}+1, k_{t,\mathcal{K}}+1)$, $0 < s_{\mathcal{K}} \leqslant \min(p_{s,\mathcal{K}}+1, k_{s,\mathcal{K}}+1)$, $0 < q_{0,\mathcal{K}} \leqslant \min(p_{t,\mathcal{K}}+1, k_{t,\mathcal{K}})$, $0 < q_{\mathcal{K}} \leqslant \min(p_{s,\mathcal{K}}+1, k_{s,\mathcal{K}})$, $0 < t_{0,\mathcal{K}} \leqslant \min(p_{t,\mathcal{K}}, k_{t,\mathcal{K}})$, and $0 < t_{\mathcal{K}} \leqslant \min(p_{s,\mathcal{K}}, k_{s,\mathcal{K}})$. The constant C has a positive value that depends only on the spatial dimension d and the mapping $Q_{\mathcal{K}}$.

Remark 6.2. In particular, when c is sufficiently smooth and the spatial shape of element $\mathcal{K}$ is regular: $h_{\mathcal{K}} = h_{i,\mathcal{K}}$, $i = 1, \dots, d$, we obtain the following leading terms for each estimate given in Lemma 6.1:

$$\|\rho\|^2_{L^2(\mathcal{K})} \leqslant C\left(\frac{h_{\mathcal{K}}^{2p_{s,\mathcal{K}}+2}}{p_{s,\mathcal{K}}^{2p_{s,\mathcal{K}}+2}} + \frac{\Delta_n t^{2p_{t,\mathcal{K}}+2}}{p_{t,\mathcal{K}}^{2p_{t,\mathcal{K}}+2}}\right)|c|^2_{H^{(p_{t,\mathcal{K}}+1,p_{s,\mathcal{K}}+1)}(\mathcal{K})},$$

$$\|\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} \leqslant C\left(\frac{h_{\mathcal{K}}^{2p_{s,\mathcal{K}}}}{p_{s,\mathcal{K}}^{2p_{s,\mathcal{K}}-1}} + \frac{\Delta_n t^{2p_{t,\mathcal{K}}+2}}{p_{t,\mathcal{K}}^{2p_{t,\mathcal{K}}}}\right)|c|^2_{H^{(p_{t,\mathcal{K}}+1,p_{s,\mathcal{K}}+1)}(\mathcal{K})},$$

$$\|\rho\|^2_{L^2(\partial\mathcal{K})} \leqslant C\left(\frac{h_{\mathcal{K}}^{2p_{s,\mathcal{K}}+1}}{p_{s,\mathcal{K}}^{2p_{s,\mathcal{K}}+1}} + \frac{\Delta_n t^{2p_{t,\mathcal{K}}+1}}{p_{t,\mathcal{K}}^{2p_{t,\mathcal{K}}+1}}\right)|c|^2_{H^{(p_{t,\mathcal{K}}+1,p_{s,\mathcal{K}}+1)}(\mathcal{K})}.$$

The proof for Lemma 6.1 is a straightforward extension of Lemmas 3.13 and 3.17 in [9] to general dimensions and therefore only the main steps are summarized. For details we refer to [9]. The first bound (57) follows directly from Lemma 3.13 in [9]. The second bound (58) is obtained as follows. First, the bound for the partial derivative in each spatial variable in Lemma 3.13 [9] is extended to general dimensions. The upper bound for the gradient is then obtained by adding all the bounds for partial derivatives in the spatial variables. The third bound (59) is obtained in a similar way. First, the bound of the interpolation error at each face of $\mathcal{K}$ is derived, which is an extension of Lemma 3.17 in [9] to general dimensions. Then the upper bounds for the boundary faces of $\partial\mathcal{K}$ are added up.

We also need an upper bound for the following term:

$$\sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \|D^\star \bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{E})}. \tag{60}$$

The upper bound for this term is obtained through the following technique. First, we use a similar derivation as in [12, Lemma 7.2] to express an upper bound of (60) in terms of the interpolation error ρ at the boundary:

$$\sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \|D^\star \bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{E})} \leqslant C\overline{D}\sum_{\mathcal{K}}\sum_{i=1}^{d} h^{-1}_{i,\mathcal{K}} p^2_{s,\mathcal{K}} \|\rho\|^2_{L^2(\partial\mathcal{K}_i)}, \tag{61}$$

with $\overline{D} = \max_{\mathcal{K}\in\mathcal{T}_h} \|D\|_{L^\infty(\mathcal{K})}$, $\partial\mathcal{K}_i$ the boundary of $\mathcal{K}$ in the x_i direction, $i = 1, \dots, d$, and the constant C depends on the mapping $Q_{\mathcal{K}}$. After that the upper bound for ρ on each $\partial\mathcal{K}_i$ (an extension of Lemma 3.17 in [9] to general dimensions) is used. The result is shown in the following lemma.

Lemma 6.3. *Assume that $\mathcal{K}$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_{\mathcal{K}}$, $F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $Q_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$. Assume also that $h_{i,\mathcal{K}}$, $i = 1, \dots, d$ is the edge length of $\widetilde{\mathcal{K}}$ in the x_i direction, and $\Delta_n t$ the edge length in the x_0 direction. Let $c|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}+1,k_{s,\mathcal{K}}+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}, k_{s,\mathcal{K}} \geqslant 0$. Let $\mathcal{P}$ denote the L^2-projection of c onto the finite element space $V_h^{(p_t,p_s)}$, then the following estimate holds*:

$$\sum_{S \in \bigcup_n \mathcal{S}^n_{ID}} \|D^\star \bar{r}_S([\![\rho]\!])\|^2_{L^2(\mathcal{E})} \leqslant C\overline{D}\sum_{\mathcal{K}\in\mathcal{T}_h}(R_{\mathcal{K}} + T_{\mathcal{K}}),$$

with $\overline{D} = \max_{\mathcal{K}\in\mathcal{T}_h} \|D\|_{L^\infty(\mathcal{K})}$ and

$$R_{\mathcal{K}} = \sum_{i=1}^{d} \frac{p_{s,\mathcal{K}}^2}{h_{i,\mathcal{K}}} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2t_{\mathcal{K}}+1} \left\|\partial_i^{t_{\mathcal{K}}+1} c\right\|_{L^2(\widetilde{\mathcal{K}})}^2 + \sum_{i=1}^{d}\sum_{j\neq i} \left(\frac{p_{s,\mathcal{K}}}{h_{i,\mathcal{K}}}\right)^2 \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2s_{\mathcal{K}}} \left\|\partial_j^{s_{\mathcal{K}}} c\right\|_{L^2(\widetilde{\mathcal{K}})}^2$$

$$+ \sum_{i=1}^{d}\sum_{j\neq i} p_{s,\mathcal{K}} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}}\right)^{2q_{\mathcal{K}}} \left\|\partial_j^{q_{\mathcal{K}}} \partial_i c\right\|_{L^2(\widetilde{\mathcal{K}})}^2,$$

$$T_{\mathcal{K}} = \sum_{i=1}^{d} \frac{p_{s,\mathcal{K}}^2}{h_{i,\mathcal{K}}\Delta_n t} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}}\right)^{2s_{0,\mathcal{K}}} \left\|\partial_0^{s_{0,\mathcal{K}}} c\right\|_{L^2(\widetilde{\mathcal{K}})}^2 + \sum_{i=1}^{d} p_{s,\mathcal{K}} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}}\right)^{2q_{0,\mathcal{K}}} \left\|\partial_0^{q_{0,\mathcal{K}}} \partial_i c\right\|_{L^2(\widetilde{\mathcal{K}})}^2,$$

with $p_{t,\mathcal{K}}$ and $p_{s,\mathcal{K}}$ the local polynomial degree in time and space, respectively, on element $\mathcal{K}$, $0 < s_{0,\mathcal{K}} \leqslant \min(p_{t,\mathcal{K}}+1, k_{t,\mathcal{K}}+1)$, $0 < s_{\mathcal{K}} \leqslant \min(p_{s,\mathcal{K}}+1, k_{s,\mathcal{K}}+1)$, $0 < q_{0,\mathcal{K}} \leqslant \min(p_{t,\mathcal{K}}+1, k_{t,\mathcal{K}})$, $0 < q_{\mathcal{K}} \leqslant \min(p_{s,\mathcal{K}}+1, k_{s,\mathcal{K}})$, $0 < t_{0,\mathcal{K}} \leqslant \min(p_{t,\mathcal{K}}, k_{t,\mathcal{K}})$, and $0 < t_{\mathcal{K}} \leqslant \min(p_{s,\mathcal{K}}, k_{s,\mathcal{K}})$. The constant C has a positive value that depends only on the spatial dimension d and the mapping $Q_{\mathcal{K}}$.

Remark 6.4. In particular, when c is sufficiently smooth and the spatial shape of element $\mathcal{K}$ is regular: $h_{\mathcal{K}} = h_{i,\mathcal{K}}$, $i = 1, \dots, d$, we obtain the following leading terms for the estimate given in Lemma 6.3:

$$\sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \left\|D^\star \bar{r}_S(\llbracket\rho\rrbracket)\right\|_{L^2(\mathcal{E})}^2 \leqslant C\overline{D} \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{h_{\mathcal{K}}^{2p_{s,\mathcal{K}}}}{p_{s,\mathcal{K}}^{2p_{s,\mathcal{K}}-1}} + \frac{p_{s,\mathcal{K}}^2}{h_{\mathcal{K}}} \frac{\Delta_n t^{2p_{t,\mathcal{K}}+1}}{p_{t,\mathcal{K}}^{2p_{t,\mathcal{K}}+2}}\right) |c|_{H^{(p_{t,\mathcal{K}}+1, p_{s,\mathcal{K}}+1)}(\mathcal{K})}^2.$$

6.2. Global estimates

As a first step in obtaining global estimates, we need an estimate for θ in terms of ρ, which is given by the following lemma.

Lemma 6.5. *There exists a constant $\beta_a > 0$, defined in Lemma* 5.1, *independent of the mesh size $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$, such that the function θ defined in* (56) *satisfies the inequality*

$$\frac{1}{4}\beta_a^2 |||\theta|||_{\mathrm{DG}}^2 \leqslant \sum_{\mathcal{K}\in\mathcal{T}_h} \left\|(D^\star)^{-1} u\right\|_{L^\infty(\mathcal{K})}^2 \|\rho\|_{L^2(\mathcal{K})}^2 + (N_f+1) \sum_{\mathcal{K}\in\mathcal{T}_h} \left\|D^\star \overline{\nabla}_h \rho\right\|_{L^2(\mathcal{K})}^2$$

$$+ \left(N_f + \eta_m^2\right) \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \left\|D^\star \bar{r}_S(\llbracket\rho\rrbracket)\right\|_{L^2(\mathcal{K})}^2 + \frac{1}{2} \sum_{S\in\bigcup_n \mathcal{S}_M^n} \|\sqrt{\alpha}\rho\|_{L^2(S)}^2$$

$$+ 2\sum_{S\in\mathcal{F}_{\mathrm{int}}} \left\|C_S^{1/2} \{\{\rho\}\}\right\|_{L^2(S)}^2 + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \left\|C_S^{1/2} \left|\llbracket\rho\rrbracket\right|\right\|_{L^2(S)}^2 + \sum_{S\in(\bigcup_n \mathcal{S}_{MDSp}^n \cup \Gamma_+)} 2\left\|C_S^{1/2} \left|\llbracket\rho\rrbracket\right|\right\|_{L^2(S)}^2,$$

with $\beta_a = \min(\frac{b_0}{2}, \frac{\beta_c}{2})$, $0 < \beta_c = \min(1-\epsilon, \eta_0 - \frac{N_f}{\epsilon})$, for $\epsilon \in (\frac{N_f}{\eta_0}, 1)$, and b_0 satisfies (42).

The proof for this lemma is given in Section 6.4.1.

Applying the triangle inequality to (56), we obtain the following bound on the global error $c - c_h$ in the DG norm:

$$|||c - c_h|||_{\mathrm{DG}} \leqslant |||\rho|||_{\mathrm{DG}} + |||\theta|||_{\mathrm{DG}}. \tag{62}$$

Using Lemma 6.5, the error in the DG norm can now be expressed solely in terms of the projection error ρ. Introducing the estimates for ρ given by Lemmas 6.1 and 6.3, the error bound can be formulated in the next theorem.

Theorem 6.6. *Suppose that $\mathcal{K}$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_{\mathcal{K}} \circ F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $Q_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$. Suppose also that $h_{i,\mathcal{K}}$, $i = 1, \dots, d$ is the edge length of $\widetilde{\mathcal{K}}$ in the x_i direction, and $\Delta_n t$ the edge length in the x_0 direction. Let $c|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}+1, k_{s,\mathcal{K}}+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}, k_{s,\mathcal{K}} \geqslant 0$, and $c_h \in V_h^{(p_t,p_s)}$ be the discontinuous Galerkin approximation to c defined by* (35). *Then, the following error bound holds*:

$$|||c - c_h|||_{\mathrm{DG}}^2 \leqslant \mathcal{C}\left(a_1 \sum_{\mathcal{K}} Z_{\mathcal{K}} + a_2 \sum_{\mathcal{K}} N_{\mathcal{K}} + a_3 \sum_{\mathcal{K}} (R_{\mathcal{K}} + T_{\mathcal{K}}) + a_4 \sum_{\mathcal{K}} (A_{\mathcal{K}} + B_{\mathcal{K}})\right),$$

with $Z_{\mathcal{K}}, N_{\mathcal{K}}, A_{\mathcal{K}}, B_{\mathcal{K}}$ defined in Lemma 6.1*, $R_{\mathcal{K}}, T_{\mathcal{K}}$ in Lemma* 6.3*, β_a in Lemma* 6.5*,*

$$a_1 = 1 + 4u_D^2\beta_a^2, \qquad a_2 = \left(1 + 4(N_f + 1)/\beta_a^2\right)\overline{D},$$
$$a_3 = \left(1 + 4\left(N_f + \eta_m^2\right)/\beta_a^2\right)\overline{D}, \qquad a_4 = \left(1 + 2/\beta_a^2\right)\bar{\alpha} + \left(1 + 20/\beta_a^2\right)\overline{C}_S,$$

and

$$\overline{D} = \max_{\mathcal{K}\in\mathcal{T}_h} \|D\|_{L^\infty(\mathcal{K})}, \qquad \bar{\alpha} = \max_{\mathcal{K}\in\mathcal{T}_h} \|\alpha\|_{L^\infty(\mathcal{K})},$$
$$\overline{C}_S = \max_{\mathcal{K}\in\mathcal{T}_h} \|C_S\|_{L^\infty(\mathcal{K})}, \qquad u_D = \max_{\mathcal{K}\in\mathcal{T}_h} \left\|(D^\star)^{-1}u\right\|_{L^\infty(\mathcal{K})}.$$

The constant $\mathcal{C}$ is a positive constant that depends on the spatial dimension d and the mapping $Q_{\mathcal{K}}$.

Corollary 6.7. *When c is sufficiently smooth, the spatial shapes of all elements $\mathcal{K} \in \mathcal{T}_h$ are regular: $h = h_{\mathcal{K}}$, $\forall \mathcal{K} \in \mathcal{T}_h$, and uniform polynomial degrees (p_t, p_s) are used for all elements $\mathcal{K} \in \mathcal{T}_h$, then we obtain the error bound*

$$|||c - c_h|||_{\mathrm{DG}}^2 \leqslant \mathcal{C}\left(a_1\left(\frac{h^{2p_s+2}}{p_s^{2p_s+2}} + \frac{\Delta_n t^{2p_t+2}}{p_t^{2p_t+2}}\right) + a_2\left(\frac{h^{2p_s}}{p_s^{2p_s-1}} + \frac{\Delta_n t^{2p_t+2}}{p_t^{2p_t}}\right) + a_3\left(\frac{h^{2p_s}}{p_s^{2p_s-1}} + \frac{p_s^2}{h}\frac{\Delta_n t^{2p_t+1}}{p_t^{2p_t+2}}\right) + a_4\left(\frac{h^{2p_s+1}}{p_s^{2p_s+1}} + \frac{\Delta_n t^{2p_t+1}}{p_t^{2p_t+1}}\right)\right)|c|_{H^{(p_t+1,p_s+1)}(\mathcal{E})}^2.$$

6.3. Error estimates at specific time levels

The error estimate given by Theorem 6.6 is useful to determine the dependence of the error in the complete space–time domain on the spatial mesh size, time step and the polynomial degrees. It is, however, also important to know the error at a specific time level. In this section we provide an error estimate in the L^2 norm for the domain Ω_T at time T. Following a similar procedure as in [15], we consider the following *backward problem* in time, related to (1):

$$-\frac{\partial z}{\partial t} + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(u_i(t,\bar{x})z\right) - \sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(D_{ij}(t,\bar{x})\frac{\partial z}{\partial x_i}\right) = 0, \quad \text{for } t < T, \tag{63}$$

with homogeneous boundary conditions at $\partial\mathcal{E} \setminus (\Omega_0 \cup \Omega_T)$ and the following initial condition:

$$z = \phi \quad \text{at } \Omega_T, \tag{64}$$

with $\phi \in L^2(\Omega_T)$. Replacing t by $t_{N_t} + 0 - t$, the analogue of the weak formulation (35) for (63) is as follows.

Find a $z_h \in V_h^{(p_t,p_s)}$, such that for all $w \in V_h^{(p_t,p_s)}$, the following relation is satisfied:

$$a(w, z_h) = \ell^\star(w), \tag{65}$$

with

$$\ell^\star(w) = \sum_{S\subset\Omega_T}\int_S B\phi \cdot n w\, \mathrm{d}S = (\phi, w)_{\Omega_T}, \tag{66}$$

where the bilinear form $a(\cdot,\cdot)$ is defined in (31). Note that by replacing t by $t_{N_t} + 0 - t$, the definitions of the inflow–outflow boundaries and the DG norm remain the same. In addition, the backward problem has a unique solution and other results obtained for the original problem can be translated to this case, such as the orthogonality relation. We start with an estimate for the discretization error $\theta = \mathcal{P}c - c_h$ at time T.

Lemma 6.8. *Assume that the conditions of Lemma* 5.1 *are satisfied. Let c_h be the solution of* (35)*, z_h the solution of* (65)*, and $\theta = \mathcal{P}c - c_h$, then the following inequality holds*:

$$(\phi, \theta)_{\Omega_T} \leqslant \left(C_e|||\rho|||_{\mathrm{DG}} + \left(2\sum_{S\in\mathcal{F}_{\mathrm{int}}}\left\|C_S^{1/2}\{\{\rho\}\}\right\|_{L^2(S)}^2\right)^{1/2}\right)|||z_h|||_{\mathrm{DG}}, \tag{67}$$

with $C_e = 5 + 2\sqrt{N_f} + \eta_m + u_D$, and $u_D = \max_{\mathcal{K}\in\mathcal{T}_h}\|(D^\star)^{-1}u\|_{L^\infty(\mathcal{K})}$.

The proof is given in Section 6.4.2. An estimate for the DG norm of the solution z_h of the backward problem is provided by the next lemma.

Lemma 6.9. *The solution z_h to* (65) *satisfies the following upper bound:*

$$\frac{1}{2}\beta_a^2 |||z_h|||_{\mathrm{DG}}^2 \leqslant (\phi,\phi)_{\Omega_T},$$

with $\beta_a > 0$ satisfies Lemma 5.1.

The proof is given in Section 6.4.2. Using Lemma 6.9, the estimate given by (67) can further be written as

$$(\phi,\theta)_{\Omega_T} \leqslant \frac{\sqrt{2}}{\beta_a}\left(C_e|||\rho|||_{\mathrm{DG}} + \left(2\sum_{S\in\mathcal{F}_{\mathrm{int}}} \left\| C_S^{1/2}\{\{\rho\}\}\right\|_{L^2(S)}^2\right)^{1/2}\right)\|\phi\|_{L^2(\Omega_T)}.$$

After using the relation

$$\|\theta\|_{L^2(\Omega_T)} = \sup_{0\neq\phi\in L^2(\Omega_T)} \frac{(\phi,\theta)_{\Omega_T}}{\|\phi\|_{L^2(\Omega_T)}},$$

we then have

$$\|\theta\|_{L^2(\Omega_T)} \leqslant \frac{\sqrt{2}}{\beta_a}\left(C_e|||\rho|||_{\mathrm{DG}} + \left(2\sum_{S\in\mathcal{F}_{\mathrm{int}}} \left\| C_S^{1/2}\{\{\rho\}\}\right\|_{L^2(S)}^2\right)^{1/2}\right). \tag{68}$$

Using the *hp*-estimates for ρ in Lemma 6.1, we obtain the following bound.

Theorem 6.10. *Suppose that $\mathcal{K}$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_\mathcal{K} \circ F_\mathcal{K}$, with $F_\mathcal{K}: \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $Q_\mathcal{K}: \widetilde{\mathcal{K}} \to \mathcal{K}$. Suppose also that $h_{i,\mathcal{K}}, i = 1,\ldots,d$ is the edge length of $\widetilde{\mathcal{K}}$ in the x_i direction, and $\Delta_n t$ the edge length in the x_0 direction. Let $c|_\mathcal{K} \in H^{(k_{t,\mathcal{K}}+1,k_{s,\mathcal{K}}+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}, k_{s,\mathcal{K}} \geqslant 0$ and $c_h \in V_h^{(p_t,p_s)}$ be the discontinuous Galerkin approximation to c defined by* (35)*. Then the following error bound holds*:

$$\|c - c_h\|_{L^2(\Omega_T)}^2 \leqslant \mathcal{C}\left(\sum_\mathcal{K}\big(b_1 Z_\mathcal{K} + b_2(N_\mathcal{K} + R_\mathcal{K} + T_\mathcal{K}) + (b_3+b_4)(A_\mathcal{K} + B_\mathcal{K})\big)\right),$$

with $Z_\mathcal{K}, N_\mathcal{K}, A_\mathcal{K}, B_\mathcal{K}$ defined in Lemma 6.1*, $R_\mathcal{K}, T_\mathcal{K}$ in Lemma* 6.3*,*

$$b_1 = 2C_e^2/\beta_a^2, \qquad b_2 = \big(2C_e^2/\beta_a^2\big)\overline{D},$$
$$b_3 = \big(2C_e^2/\beta_a^2\big)\bar{\alpha}, \qquad b_4 = \big(2C_e^2/\beta_a^2 + 4/\beta_a^2 + 1\big)\overline{C}_S,$$

the coefficients $\overline{D}, \bar{\alpha}, \overline{C}_S$ given in Theorem 6.6*, C_e in Lemma* 6.8*, and β_a satisfies Lemma* 5.1*. The constant $\mathcal{C}$ has a positive value that depends only on the spatial dimension d and the mapping $Q_\mathcal{K}$.*

The proof of this theorem is immediate using (68) and Lemma 6.1.

Corollary 6.11. *When c is sufficiently smooth, the spatial shapes of all elements $\mathcal{K} \in \mathcal{T}_h$ are regular*: *$h = h_\mathcal{K}$, $\forall\mathcal{K} \in \mathcal{T}_h$, and uniform polynomial degrees (p_t, p_s) are used for all elements $\mathcal{K} \in \mathcal{T}_h$, then we obtain the error bound*

$$\|c - c_h\|_{L^2(\Omega_T)}^2 \leqslant \mathcal{C}\Bigg(b_1\left(\frac{h^{2p_s+2}}{p_s^{2p_s+2}} + \frac{\Delta_n t^{2p_t+2}}{p_t^{2p_t+2}}\right) + b_2\left(2\frac{h^{2p_s}}{p_s^{2p_s-1}} + \frac{\Delta_n t^{2p_t+2}}{p_t^{2p_t}} + \frac{p_s^2}{h}\frac{\Delta_n t^{2p_t+2}}{p_t^{2p_t}}\right)$$
$$+ (b_3+b_4)\left(\frac{h^{2p_s+1}}{p_s^{2p_s+1}} + \frac{\Delta_n t^{2p_t+1}}{p_t^{2p_t+1}}\right)\Bigg)|c|_{H^{(p_t+1,p_s+1)}(\mathcal{E})}^2.$$

6.4. Proofs

6.4.1. Proof of the upper bound for the discretization error θ in Lemma 6.5

The proof of Lemma 6.5 starts with the orthogonality relation (40) and the decomposition of the error (56), which imply that

$$a(\theta+\rho, v) = 0, \quad \forall v \in V_h^{(p_t,p_s)}. \tag{69}$$

Taking $v=\theta$, we obtain $a(\theta,\theta) = -a(\rho,\theta)$. We continue with the derivation of an estimate for $|a(\rho,\theta)|$. First, we consider the bilinear form $a_a(\rho,\theta)$. Since $\theta \in V_h^{(p_t,p_s)}$, which is polynomial, we have $\frac{\partial\theta}{\partial t} \in V_h^{(p_t,p_s)}$ and we can use the L^2 orthogonality relation for the projection $\mathcal{P}$, given by (55), to obtain:

$$\begin{aligned} a_a(\rho,\theta) = &-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} (D^\star)^{-1}u\rho\cdot D^\star\overline{\nabla}_h\theta\,\mathrm{d}\mathcal{K} + \sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S \{\{B\rho\}\}\cdot[\![\theta]\!]\,\mathrm{d}S \\ &+\sum_{S\in\mathcal{F}_{\mathrm{int}}}\int_S C_S[\![\rho]\!]\cdot[\![\theta]\!]\,\mathrm{d}S + \sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)}\int_S B\cdot n\rho\theta\,\mathrm{d}S. \end{aligned} \tag{70}$$

Using the same argument as in [7], that is by using (21) and the continuity property of B, we have: $|\{\{B\rho\}\}\cdot n| = |B\cdot n||\{\{\rho\}\}| = 2C_S|\{\{\rho\}\}|$. Then, by using the Schwarz inequality together with the arithmetic-geometric mean inequality in the form $pq \leqslant \frac{p^2}{\beta} + \frac{\beta q^2}{4}$, we have the following estimate:

$$\begin{aligned} \left|a_a(\rho,\theta)\right| \leqslant &\frac{1}{\beta}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|(D^\star)^{-1}u\right\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} + \frac{2}{\beta}\sum_{S\in\mathcal{F}_{\mathrm{int}}}\left\|C_S^{1/2}\{\{\rho\}\}\right\|^2_{L^2(S)} \\ &+\frac{1}{\beta}\sum_{S\in\mathcal{F}_{\mathrm{int}}}\left\|C_S^{1/2}\big|[\![\rho]\!]\big|\right\|^2_{L^2(S)} + \frac{2}{\beta}\sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)}\left\|C_S^{1/2}\big|[\![\rho]\!]\big|\right\|^2_{L^2(S)} \\ &+\frac{1}{4}\beta\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^\star\overline{\nabla}_h\theta\|^2_{L^2(\mathcal{K})} + \frac{3}{4}\beta\sum_{S\in\mathcal{F}_{\mathrm{int}}}\left\|C_S^{1/2}\big|[\![\theta]\!]\big|\right\|^2_{L^2(S)} \\ &+\frac{1}{2}\beta\sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)}\left\|C_S^{1/2}\big|[\![\theta]\!]\big|\right\|^2_{L^2(S)}. \end{aligned} \tag{71}$$

Next, we consider the bilinear form $a_d(\rho,\theta)$. Using the lifting operator R, the bilinear form can be written as:

$$\begin{aligned} a_d(\rho,\theta) = &\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D\overline{\nabla}_h\rho\cdot\overline{\nabla}_h\theta\,\mathrm{d}\mathcal{K} + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\overline{R}\big([\![\rho]\!]\big)\cdot D\overline{\nabla}_h\theta\,\mathrm{d}\mathcal{K} \\ &+\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} D\overline{\nabla}_h\rho\cdot\overline{R}\big([\![\theta]\!]\big)\,\mathrm{d}\mathcal{K} + \sum_{S\in\bigcup_n \mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\eta_{\mathcal{K}}\int_{\mathcal{K}} D\bar{r}_S\big([\![\rho]\!]\big)\cdot\bar{r}_S\big([\![\theta]\!]\big)\,\mathrm{d}\mathcal{K} \\ &+\sum_{S\in\bigcup_n \mathcal{S}^n_M}\int_S \alpha\rho\theta\,\mathrm{d}S. \end{aligned} \tag{72}$$

Applying the Schwarz' inequality, inequality (48b) and arithmetic-geometric mean inequality yields:

$$\begin{aligned} \left|a_d(\rho,\theta)\right| \leqslant &\frac{N_f+1}{\beta}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^\star\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} + \frac{N_f+\eta_m^2}{\beta}\sum_{S\in\bigcup_n \mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|D^\star\bar{r}_S\big([\![\rho]\!]\big)\right\|^2_{L^2(\mathcal{K})} \\ &+\frac{1}{2\beta}\sum_{S\in\bigcup_n \mathcal{S}^n_M}\|\sqrt{\alpha}\rho\|^2_{L^2(S)} + \frac{\beta}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^\star\overline{\nabla}_h\theta\|^2_{L^2(\mathcal{K})} \\ &+\frac{\beta}{2}\sum_{S\in\bigcup_n \mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|D^\star\bar{r}_S\big([\![\theta]\!]\big)\right\|^2_{L^2(\mathcal{K})} + \frac{\beta}{2}\sum_{S\in\bigcup_n \mathcal{S}^n_M}\|\sqrt{\alpha}\theta\|^2_{L^2(S)}, \end{aligned} \tag{73}$$

with $\eta_m = \max_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}}$. Adding (71) and (73), combining the result with the coercivity estimate (43) for $v = \theta$, and taking $\beta = \beta_a$, with β_a defined in Lemma 5.1, we deduce:

$$\begin{aligned}\frac{\beta_a}{4}|||\theta|||^2_{\mathrm{DG}} \leqslant{} & \frac{1}{\beta_a}\sum_{\mathcal{K}\in\mathcal{T}_h}\big\|(D^\star)^{-1}u\big\|^2_{L^\infty(\mathcal{K})}\|\rho\|^2_{L^2(\mathcal{K})} + \frac{N_f+1}{\beta_a}\sum_{\mathcal{K}\in\mathcal{T}_h}\|D^\star\overline{\nabla}_h\rho\|^2_{L^2(\mathcal{K})} \\ & + \frac{N_f+\eta_m^2}{\beta_a}\sum_{S\in\bigcup_n \mathcal{S}^n_{ID}}\sum_{\mathcal{K}\in\mathcal{T}_h}\big\|D^\star\bar{r}_S([\![\rho]\!])\big\|^2_{L^2(\mathcal{K})} \\ & + \frac{1}{2\beta_a}\sum_{S\in\bigcup_n \mathcal{S}^n_M}\|\sqrt{\alpha}\rho\|^2_{L^2(S)} + \frac{2}{\beta_a}\sum_{S\in\mathcal{F}_{\mathrm{int}}}\big\|C_S^{1/2}\{\{\rho\}\}\big\|^2_{L^2(S)} \\ & + \frac{1}{\beta_a}\sum_{S\in\mathcal{F}_{\mathrm{int}}}\big\|C_S^{1/2}\big|[\![\rho]\!]\big|\big\|^2_{L^2(S)} + \frac{2}{\beta_a}\sum_{S\in(\bigcup_n \mathcal{S}^n_{MDSp}\cup\Gamma_+)}\big\|C_S^{1/2}\big|[\![\rho]\!]\big|\big\|^2_{L^2(S)}.\end{aligned} \tag{74}$$

Multiplying the last equation with β_a completes the proof of Lemma 6.5. □

6.4.2. Proof of the upper bound for ϕ in Lemma 6.8

The proof of Lemma 6.8 starts with introducing $w = \theta$ in (65) and using (69):

$$(\phi,\theta)_{\Omega_T} = a_a(\theta, z_h) + a_d(\theta, z_h) \leqslant \big|a_a(\rho, z_h)\big| + \big|a_d(\rho, z_h)\big|.$$

We estimate now each term separately. First, we derive an estimate for the bilinear form $a_a(\rho, z_h)$. Since $\frac{\partial z_h}{\partial t} \in V_h^{(p_t,p_s)}$, the contribution $\int_{\mathcal{K}} \rho \frac{\partial z_h}{\partial t}\, \mathrm{d}\mathcal{K}$ is zero due to the orthogonality relation (55) and hence the bilinear form a_a is similar to (70). Using the Schwarz' inequality, we can estimate a_a as:

$$\big|a_a(\rho, z_h)\big| \leqslant \left(C_c|||\rho|||_{\mathrm{DG}} + \left(2\sum_{S\in\mathcal{F}_{\mathrm{int}}}\big\|C_S^{1/2}\{\{\rho\}\}\big\|^2_{L^2(S)}\right)^{1/2}\right)|||z_h|||_{\mathrm{DG}},$$

with $C_c = 3 + u_D$ and $u_D = \max_{\mathcal{K}\in\mathcal{T}_h}\|(D^\star)^{-1}u\|_{L^\infty(\mathcal{K})}$. Next, we consider $a_d(\rho, z_h)$, which is of the form (72). Using inequality (48b), we obtain the upper bound for the bilinear form a_d as follows:

$$\big|a_d(\rho, z_h)\big| \leqslant C_d|||\rho|||_{\mathrm{DG}}|||z_h|||_{\mathrm{DG}},$$

with $C_d = 2 + 2\sqrt{N_f} + \eta_m$. Collecting all the terms we obtain the estimate

$$(\phi,\theta)_{\Omega_T} \leqslant \left(C_e|||\rho|||_{\mathrm{DG}} + \left(2\sum_{S\in\mathcal{F}_{\mathrm{int}}}\big\|C_S^{1/2}\{\{\rho\}\}\big\|^2_{L^2(S)}\right)^{1/2}\right)|||z_h|||_{\mathrm{DG}},$$

with $C_e = C_c + C_d$. □

6.4.3. Proof of the upper bound for z_h in Lemma 6.9

To prove Lemma 6.9 we proceed as follows. First, we take $w = z_h$ in (65). Then we use the Schwarz and arithmetic-geometric mean inequalities and the definition of C_S on $S \in \mathcal{F}_{\mathrm{bnd}}$ (22) to obtain:

$$a(z_h, z_h) \leqslant \frac{1}{2\alpha_1}(\phi,\phi)_{\Omega_T} + \alpha_1\sum_{S\in\mathcal{F}}\big\|C_S^{1/2}\big|[\![z_h]\!]\big|\big\|^2_{L^2(S)}, \tag{75}$$

with $\alpha_1 > 0$ an arbitrary constant. Since Lemma 5.1 also applies to the backward problem, we can state that

$$a(z_h, z_h) \geqslant \beta_a|||z_h|||^2_{\mathrm{DG}}, \tag{76}$$

with $\beta_a > 0$ defined in Lemma 5.1. Combining (75) and (76) and choosing $\alpha_1 = \frac{\beta_a}{2}$, we obtain:

$$\frac{1}{2}\beta_a|||z_h|||^2_{\mathrm{DG}} \leqslant \frac{1}{\beta_a}(\phi,\phi)_{\Omega_T}.$$

Multiplying the last equation with β_a completes the proof. □

7. Numerical results

In this section we present a number of numerical experiments in two spatial dimensions in order to verify the error analysis discussed in this paper. We provide results for the following time-dependent advection–diffusion equation:

$$\frac{\partial c}{\partial t} + u\sum_{i=1}^{2}\frac{\partial c}{\partial x_i} - D\sum_{i=1}^{2}\frac{\partial^2 c}{\partial x_i^2} = 0, \quad (0,1)^2, \tag{77}$$

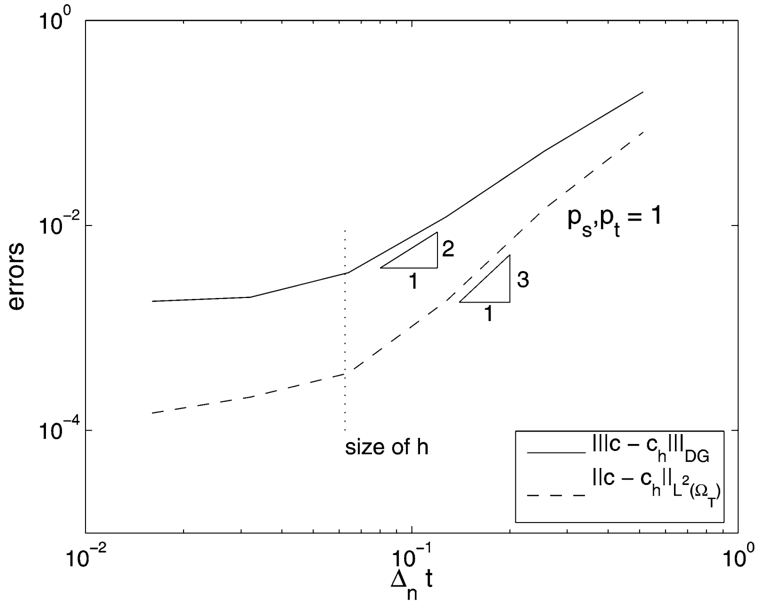

Fig. 3. Convergence of space–time DG method when $u = 1$, $D = 0$ under $\Delta_n t$-refinement.

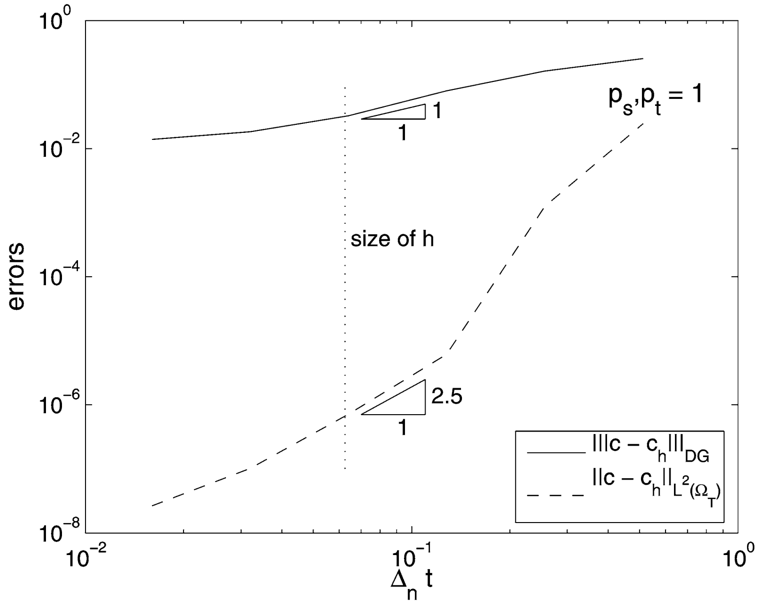

Fig. 4. Convergence of space–time DG method when $u = 1$, $D = 1$ under $\Delta_n t$-refinement.

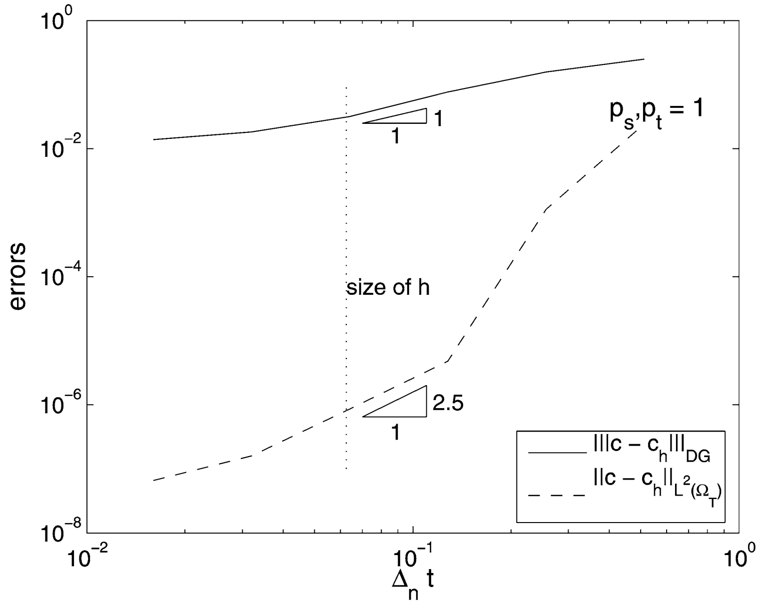

Fig. 5. Convergence of space–time DG method when $u = 0$, $D = 1$ under $\Delta_n t$-refinement.

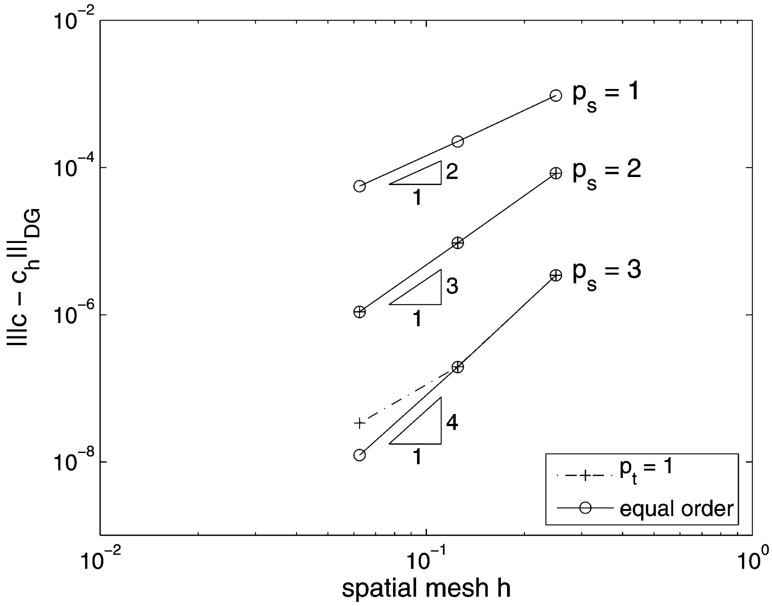

Fig. 6. Convergence of space–time DG method when $u = 1$, $D = 0$ under h-refinement.

with u and $D \geqslant 0$ constants. The initial condition is

$$c(0, x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2),$$

and the boundary conditions are chosen so that the analytical solution is given by

$$c(t, x_1, x_2) = \sin\big(\pi(x_1 - ut)\big)\sin\big(\pi(x_2 - ut)\big)\exp\big(-2D\pi^2 t\big).$$

We consider three cases: (1) advection problem ($u = 1$, $D = 0$), (2) advection–diffusion problem ($u = 1$, $D = 1$), and (3) diffusion problem ($u = 0$, $D = 1$).

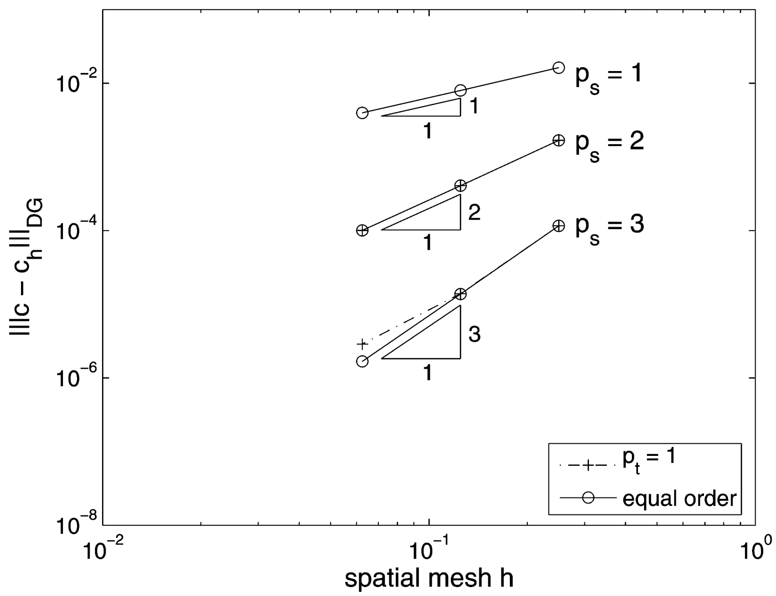

Fig. 7. Convergence of space–time DG method when $u = 1$, $D = 1$ under h-refinement.

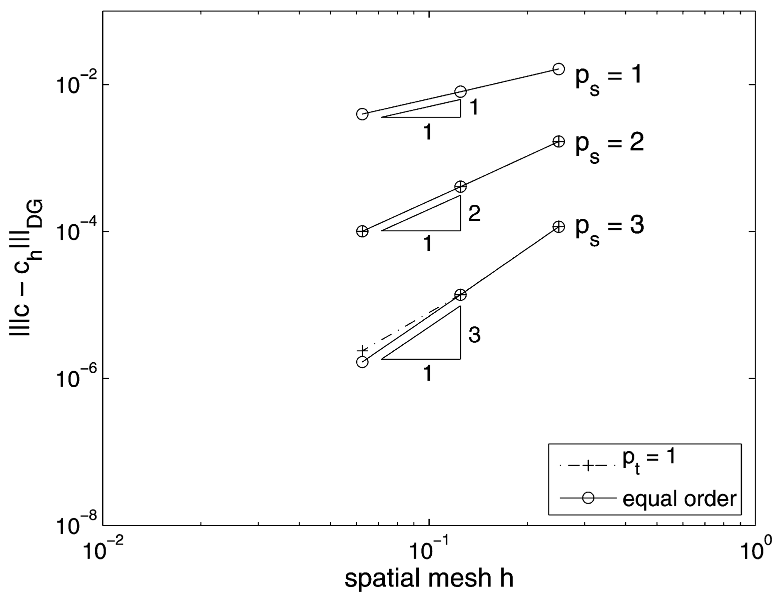

Fig. 8. Convergence of space–time DG method when $u = 0$, $D = 1$ under h-refinement.

First, we investigate the behaviour of the space–time DG discretization on a sequence of successively finer time intervals with a fixed number of elements in space and linear polynomial degrees: $p_{t,\mathcal{K}}$, $p_{s,\mathcal{K}} = 1$. We perform computations from $t = 0$ until the final time $T = 0.5$. The results are given in Figs. 3–5. When there is no diffusion process ($D = 0$), Fig. 3 shows that the error in the DG-norm as a function of the time step converges at the rate $O(\Delta_n t^2)$ when $\Delta_n t \geqslant h$, with h the spatial mesh size. This rate of convergence is better than the theoretical estimates presented in Theorem 6.6. This means that the errors in the DG-norm are dominated by the L^2-norm contribution (the first term in Theorem 6.6), while the contributions due to the jumps at the element boundaries are negligible. When there is also diffusion process present ($D = 1$), the errors in the DG-norm are dominated by the L^2-norm of the derivatives (the second term in Theorem 6.6), see Figs. 4 and 5. The errors in the DG-norm converge then at the rate $O(\Delta_n t)$, verifying

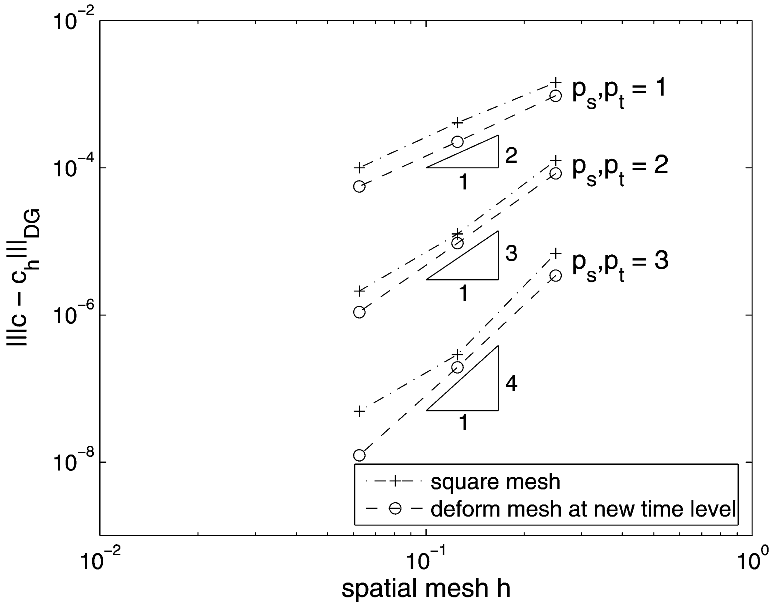

Fig. 9. Convergence of space–time DG method when $u = 1$, $D = 0$ under h-refinement for square and deformed mesh.

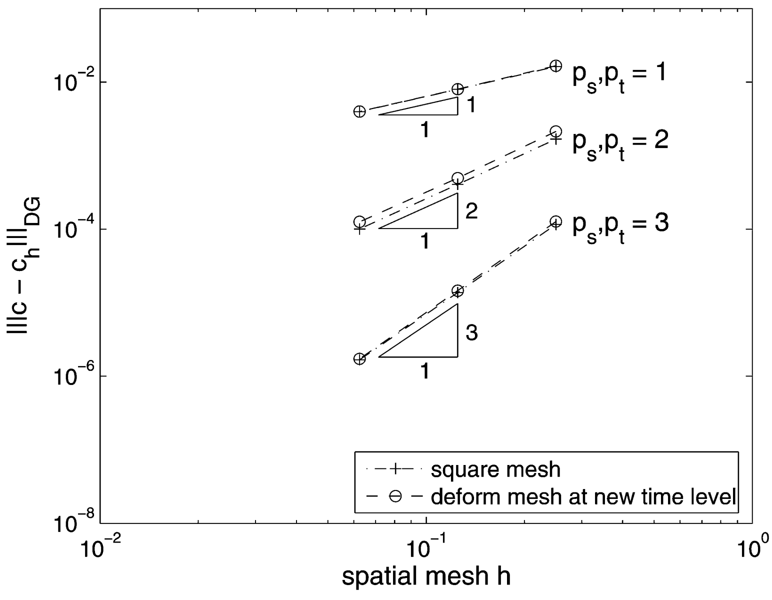

Fig. 10. Convergence of space–time DG method when $u = 1$, $D = 1$ under h-refinement for square and deformed mesh.

the theoretical estimates in Theorem 6.6. At the final time $T = 0.5$, the rates of the convergence of the space–time DG discretization are better than the theoretical estimates given in Theorem 6.10.

Next, we study the rates of convergence on meshes with a different spatial mesh size and increasing polynomial degrees. We compare the error for equal polynomial degrees: $p_{t,\mathcal{K}} = p_{s,\mathcal{K}}$ and also for linear polynomials in time: $p_{t,\mathcal{K}} = 1$. The results are shown in Figs. 6–8. When there is no diffusion ($D = 0$) and equal polynomial degrees in time and space are used, Fig. 6 shows that the error in the DG-norm converges at the rate h^{p_s+1}. This rate is better than is obtained in the theoretical estimates Theorem 6.6. This indicates that the errors in the DG-norm are dominated by the L^2-norm contribution and we can neglect contribution from the L^2-norm on the boundary $\partial\mathcal{K}$. However, when

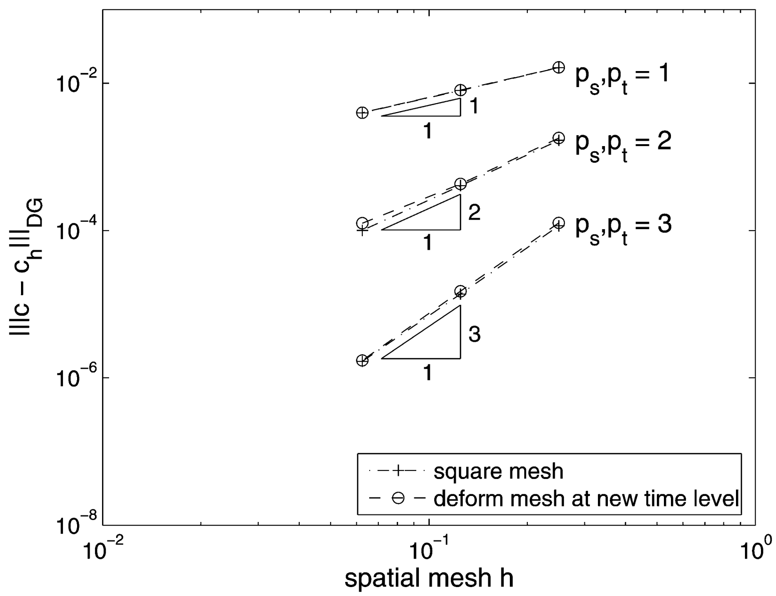

Fig. 11. Convergence of space–time DG method when $u = 0$, $D = 1$ under h-refinement for square and deformed mesh.

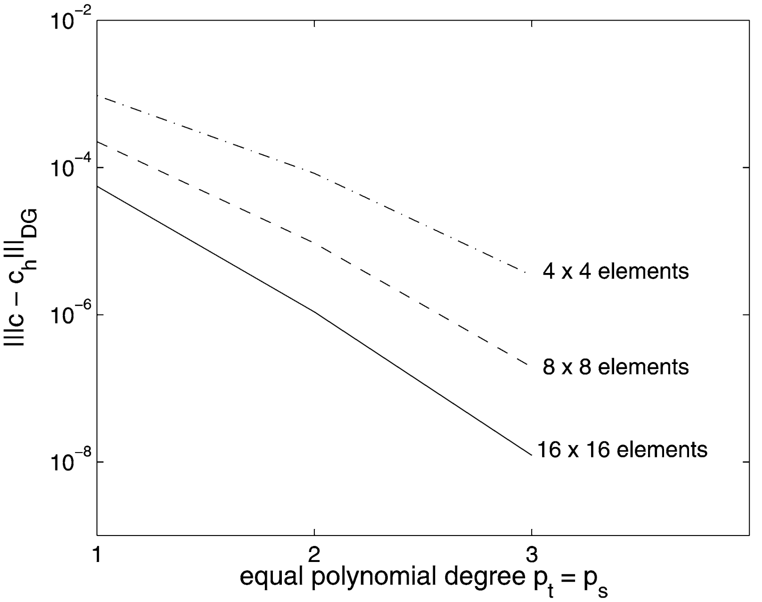

Fig. 12. Convergence of space–time DG method when $u = 1$, $D = 0$ under p-refinement.

diffusion is also present ($D = 1$), from Figs. 7 and 8 we can conclude that the errors in the DG-norm are also influenced by L^2-norm of the derivatives and hence the errors converge at the rate h^{p_s} as we expect from Theorem 6.6.

Using linear polynomials in time, we observe that as the mesh becomes finer, then the error is dominated by the error in time, but this only occurs at relatively small error levels. The tests with linear polynomials in time were performed since the analysis presented in Section 5 could only prove a unique solution for polynomials linear in time and we want to investigate the effect of restricting the polynomial degree in time on the accuracy.

We also investigate the effect of the mesh movement on the accuracy. We construct the mesh movement as follows. At t_n we have a uniform square mesh. At t_{n+1}, the uniform mesh is deformed by randomly perturbing the interior nodes. Thus the meshes at t_n and t_{n+1} are not identical, and the mesh velocity (discussed in Section 4.2) is present.

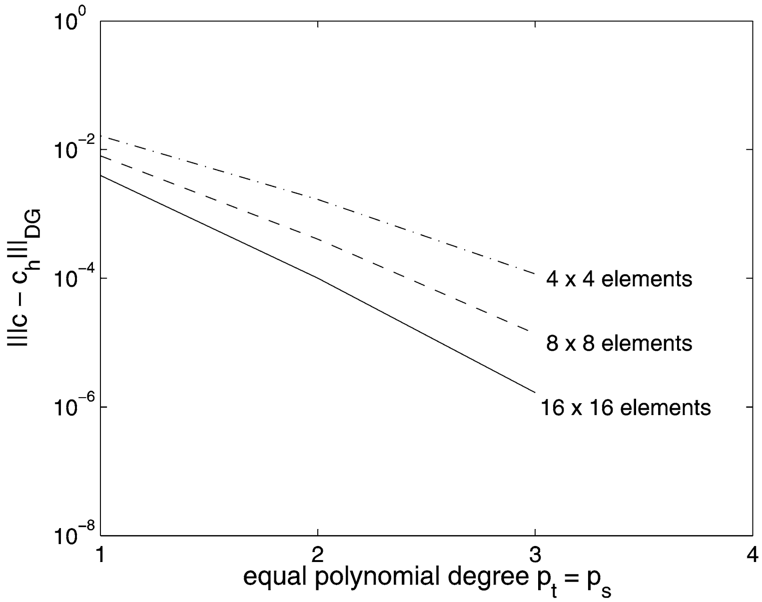

Fig. 13. Convergence of space–time DG method when $u = 1$, $D = 1$ under p-refinement.

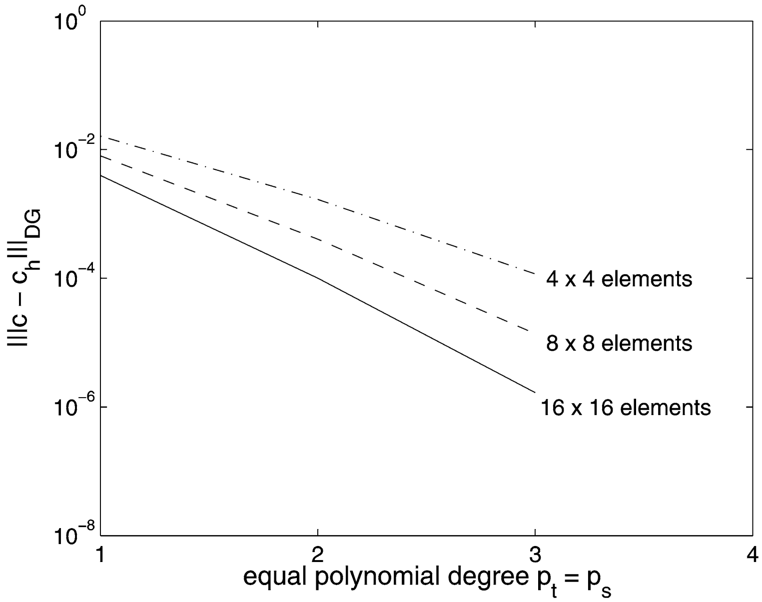

Fig. 14. Convergence of space–time DG method when $u = 0$, $D = 1$ under p-refinement.

The plots of the errors in the DG-norm on time deforming meshes are shown in Figs. 9–11. The figures show that the errors in the DG-norm on a square mesh and on a time deforming mesh converge at the same rate.

Finally, we investigate the convergence of the space–time DG method with p-refinement and the results are shown in Figs. 12–14. Here we only study the p-refinement for equal polynomial degrees in time and space: $p_{t,\mathcal{K}} = p_{s,\mathcal{K}}$ on a square mesh. We observe that on a linear-log scale, the errors in the DG-norm for all three cases become straight lines which indicate exponential convergence in p.

8. Concluding remarks

In this paper we present a new space–time DG method for the advection–diffusion equation in time-dependent domains. We study and prove the consistency, coercivity, stability and the existence of a unique solution of the method. We also present an error estimate in the DG-norm on the space–time domain and in the L^2-norm at a specific time level.

The numerical results show that for pure advection problem, the space–time DG discretization with h-refinement converges in the DG-norm faster than the theoretical estimate in Theorem 6.6. For the case when diffusion is present the convergence of the space–time DG discretization with h-refinement is numerically observed to be optimal in the DG norm, thus verifies the theoretical estimates. The use of a time deforming mesh does not influence the rates of convergence. The rates of convergence with p-refinement is numerically observed to be optimal in the DG-norm for all three cases. Further, although the space–time DG discretization was only proven to be stable for the linear polynomials in time, in the numerical simulations the algorithm performs also well for higher polynomial degrees in time.

Presently, the space–time discontinuous Galerkin method is being extended to the incompressible Navier–Stokes equations. Also, the analysis of a posteriori error estimates for the space–time DG discretization is being conducted. This will be used to control the mesh adaptation which can be done straightforwardly with a space–time DG method. The space–time DG method presented in this paper has been successfully applied to wet-chemical etching processes, for more details see [13].

Acknowledgements

This research has been conducted in the STW project TWI.5453: *Analysis and Control of Transport Phenomena in Wet-Chemical Etching Processes*. The financial support from STW is gratefully acknowledged. We also would like to thank the referees for giving valuable improvements to the paper and to Prof. P. Houston for pointing out the interpolation estimates in [9].

References

[1] D.N. Arnold, An interior penalty finite element method with discontinuous elements, SIAM J. Numer. Anal. 19 (4) (1982) 742–760.
[2] D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 39 (5) (2002) 1749–1779.
[3] F. Bassi, S. Rebay, Numerical evaluation of two discontinuous Galerkin methods for the compressible Navier–Stokes equations, Int. J. Numer. Meth. Fluids. 40 (2002) 197–207.
[4] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, second ed., Springer, Berlin, 2002.
[5] F. Brezzi, G. Manzini, D. Marini, P. Pietra, A. Russo, Discontinuous finite elements for diffusion problems, Atti del Convegno in Memoria di F. Brioschi, Milano, Instituto Lombardo di Scienze e Lettere, 1997.
[6] F. Brezzi, G. Manzini, D. Marini, P. Pietra, A. Russo, Discontinuous Galerkin approximations for elliptic problems, Numer. Methods Partial Differential Equations 16 (2000) 365–378.
[7] F. Brezzi, L.D. Marini, E. Süli, Discontinuous Galerkin methods for first-order hyperbolic problems, Math. Models Methods Appl. Sci. 14 (12) (2004) 1893–1903.
[8] B. Cockburn, C.W. Shu, Runge–Kutta discontinuous Galerkin method for convection dominated problems, J. Sci. Comput. 6 (3) (2001) 173–261.
[9] E.H. Georgoulis, Discontinuous Galerkin Methods on Shaped-Regular and Anisotropic Meshes, PhD thesis, Christ Church, University of Oxford, 2003.
[10] P. Houston, Ch. Schwab, E. Süli, Discontinuous *hp*-finite element methods for advection–diffusion–reaction problems, SIAM J. Numer. Anal. 39 (6) (2002) 2133–2163.
[11] A. Masud, T.J.R. Hughes, A space–time Galerkin/least squares finite element formulation of the Navier–Stokes equations for moving domain problems, Comput. Methods Appl. Mech. Eng. 146 (1997) 91–126.
[12] D. Schotzau, Ch. Schwab, A. Toselli, Mixed *hp*-DGFEM for incompressible flows, SIAM J. Numer. Anal. 40 (6) (2003) 2171–2194.
[13] J.J. Sudirham, R.M.J. van Damme, J.J.W. van der Vegt, Space–time discontinuous Galerkin method for wet-chemical etching of microstructures, in: Proceedings of European Congress in Applied Sciences and Engineering, ECCOMAS, Jyvaskyla, Finland, 2004.
[14] T.E. Tezduyar, M. Behr, S. Mittal, A.A. Johnson, Computation of unsteady incompressible flows with the stabilized finite element methods: Space–time formulations, iterative strategies and massively parallel implementation, in: New Methods in Transient Analysis, AMD, vol. 143, ASME, 1992.
[15] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer, Berlin, 1997.
[16] J.J.W. van der Vegt, H. van der Ven, Space–time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flows, Part I. General formulation, J. Comput. Phys. 182 (2002) 546–585.