A space–time discontinuous Galerkin method for the time-dependent Oseen equations

J.J.W. van der Vegt [*], J.J. Sudirham

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands

Abstract

A space–time discontinuous Galerkin finite element method for the Oseen equations on time-dependent flow domains is presented. The algorithm results in a higher order accurate conservative discretization on moving and deforming meshes and is well suited for *hp*-adaptation. A detailed analysis of the stability of the numerical discretization is given which shows that the algorithm is unconditionally stable, also when equal order polynomial basis functions for the pressure and velocity are used. The accuracy of the space–time discretization is investigated using a detailed *hp*-error analysis and computations on a model problem.
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1. Introduction

Many problems in fluid dynamics require moving and deforming meshes. Important examples are fluid–structure interaction, water waves and multi-fluid flows with clearly defined interfaces. In all of these problems the computational mesh has to follow the boundary motion and interior points need to move in order to maintain a consistent mesh without grid folding. Also, when the boundary deformation becomes too large it is no longer possible to simply move the interior mesh points and a completely new mesh has to be generated. An important problem one has to face on these dynamic meshes is how to maintain a conservative numerical discretization, which preserves accuracy and efficiency, despite the mesh deformation.

An excellent technique to achieve this is the space–time discontinuous Galerkin finite element method. In this method basis functions are used which are discontinuous at element faces, both in space and time. Also, by defining the problem in four dimensional space, automatically a conservative finite element discretization is obtained on moving and deforming meshes. Since in a space–time DG method the basis functions in an element are only weakly coupled to the basis functions in neighboring elements, the algorithm is well suited for local mesh refinement (*h-adaptation*) or adjustment of the polynomial order (*p-refinement*). In addition, if a new mesh has to be generated one only has to transfer time fluxes at the interface between the old and the new mesh and it is not necessary to interpolate data from the old to the new mesh.

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[*] Corresponding author.

E-mail addresses: j.j.w.vandervegt@math.utwente.nl (J.J.W. van der Vegt), j.j.sudirham@math.utwente.nl (J.J. Sudirham).

In a series of articles, we have developed a solution adaptive space–time DG method for the compressible Euler and Navier–Stokes equations [16,20,21] and applied this technique to a number of aerodynamical applications. In addition, in [19] we have given a detailed error and stability analysis for a space–time DG method for the advection–diffusion equation. The objective of this article is to extend this work to a space–time DG discretization of the time-dependent Oseen equations, which are the linearized Navier–Stokes equations. The Oseen equations are the main component in an iterative scheme to solve the incompressible Navier–Stokes equations, for instance with a Picard iteration. A thorough understanding of the space–time DG algorithm for the Oseen equations is therefore a prerequisite before applying it to the incompressible Navier–Stokes equations.

The discontinuous Galerkin discretization for the steady Stokes, Oseen and incompressible Navier–Stokes has been considered by Cockburn, Kanschat, and Schötzau in a series of articles [8–10]. Also, the analysis in [17,18] provides important information on the construction of DG algorithms for incompressible flows, but space–time DG algorithms for these equations have not been developed so far. The main focus of this article is therefore the extension of the DG method for incompressible flows to the space–time framework and to analyze the stability and accuracy of the resulting algorithm. An important aspect in space–time DG algorithms is also the efficient solution of the algebraic equations resulting from the space–time discretization. An excellent technique for this is provided by multigrid methods, which have been extensively analyzed for discontinuous Galerkin discretizations by Hemker, Hoffmann and van Raalte in [13–15], but the application of these techniques to the Oseen equations is presently still a topic of ongoing research.

There are several important points that have to be considered in the development of a DG discretization for the Oseen equations, which are also relevant for a space–time DG method.

The first issue is the DG discretization of the viscous terms. The obvious choice is to discretize the viscous terms in a similar way as done for elliptic equations (for a unified analysis, see [3]), but now extended to vector fields. There are basically two approaches, the techniques proposed by Bassi and Rebay [4] and Brezzi et al. [6,7], and the local discontinuous Galerkin method proposed by Cockburn and Shu in [8–10]. Based on our experience with the advection–diffusion equation [19] and the compressible Navier–Stokes equations [16], we will use the Bassi and Rebay and Brezzi approach for the DG discretization of the viscous contribution since this method results in a very compact stencil. The local discontinuous Galerkin method provides, however, also a very useful discretization technique and many of the results in this article also apply to this method.

The second issue is the pressure stabilization. The analysis presented in [17,18] shows the importance of the pressure stabilization operator for the choice of the polynomial degrees in the approximation of the velocity and pressure in a DG discretization. Without a stabilization term, the DG method can only be proven stable when the polynomial degree used in the approximation of the pressure is one less than the polynomial degree for the approximation of the velocity. By adding a stabilization term similar to the one used for elliptic equations, stability is proven when equal polynomial degrees are used for the velocity and the pressure.

The organization of this article is as follows. First, the Oseen equations are written in the space–time formulation in Section 2. After the introduction of the finite element spaces and trace operators in Section 3, we give a complete derivation of the space–time DG formulation for the Oseen equations on time-dependent domains in Section 4. The analysis of the stability and accuracy of the space–time DG discretization for the Oseen equations is presented in Section 5. Numerical experiments on a model problem are given in Section 6. Finally, concluding remarks are drawn in Section 7.

2. The Oseen equations

In this section we introduce the Oseen equations and set some notations. Let Ω_t be an open, bounded, time-dependent domain in $\mathbb{R}^d$ at time t, where d is the number of spatial dimensions. The closure of Ω_t is $\bar{\Omega}_t$ and the boundary of Ω_t is denoted by $\partial\Omega_t$. Denoting $\bar{x} = (x_1, \ldots, x_d)$ as the spatial variables, we consider in Ω_t the time-dependent Oseen equations for the velocity field $u \in \mathbb{R}^d$ and the kinematic pressure $p := p/\rho \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} + \bar{\nabla} \cdot (u \otimes \bar{w}) - \nu \bar{\nabla} \cdot \bar{\nabla} u + \bar{\nabla} p = f, \quad \text{in } \Omega_t, \tag{1a}$$

$$\bar{\nabla} \cdot u = 0, \quad \text{in } \Omega_t, \tag{1b}$$

with ρ the fluid density, $\nu \in \mathbb{R}^+$ the kinematic viscosity, $\bar{w} \in \mathbb{R}^d$ a given convective divergence free velocity field, and $f \in \mathbb{R}^d$ the force vector. We introduce the product between two vectors $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ as $a \otimes b \in \mathbb{R}^{m \times n}$ with elements

$(a \otimes b)_{ij} = a_i b_j$. The notation $\bar{\nabla}$ is used for the spatial gradient operator in $\mathbb{R}^d$, and defined as $\bar{\nabla} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$. We also define the divergence of a tensor $A \in \mathbb{R}^{m \times n}$ as: $\bar{\nabla} \cdot A = \sum_j \frac{\partial}{\partial x_j} A_{ij}$.

The first step to obtain a space–time discretization is to consider the Oseen equations directly in a domain in $\mathbb{R}^{d+1}$. A point $x \in \mathbb{R}^{d+1}$ has coordinates $(x_0, \bar{x})$, with $x_0 = t$ representing time. We introduce the space–time domain $\mathcal{E} \subset \mathbb{R}^{d+1}$. The boundary of the space–time domain $\partial \mathcal{E}$ consists of the hypersurfaces $\Omega_0 := \{x \in \partial \mathcal{E} \mid x_0 = 0\}$, $\Omega_T := \{x \in \partial \mathcal{E} \mid x_0 = T\}$, and $\mathcal{Q} := \{x \in \partial \mathcal{E} \mid 0 < x_0 < T\}$. Introducing the gradient operator in $\mathbb{R}^{d+1}$ as $\nabla = (\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})$ and the vector $w = (1, \bar{w})$, the Oseen equations (1) can be transformed into a space–time formulation as:

$$\nabla \cdot (u \otimes w) - \nu \bar{\nabla} \cdot \bar{\nabla} u + \bar{\nabla} \cdot (\mathbb{I}_d p) = f, \quad \text{in } \mathcal{E},$$
$$\bar{\nabla} \cdot u = 0, \quad \text{in } \mathcal{E},$$

with $\mathbb{I}_d$ the $d \times d$ identity matrix.

Since different boundary conditions are imposed on $\partial \mathcal{E}$, we first discuss the subdivision of $\partial \mathcal{E}$ into different parts. The boundary $\partial \mathcal{E}$ is divided into disjoint subsets Γ_m and Γ_p, with:

$$\Gamma_m := \{x \in \partial \mathcal{E} : w \cdot n < 0\}, \qquad \Gamma_p := \{x \in \partial \mathcal{E} : w \cdot n \geqslant 0\}.$$

The subscripts m and p denote the inflow and outflow boundaries, respectively, and n refers to the space–time normal vector at $\partial \mathcal{E}$. We subdivide Γ_m further into two sets: Γ_{Dm} and Ω_0, with Γ_{Dm} the part of Γ_m with a Dirichlet boundary condition and Ω_0 the part of Γ_m with the initial condition. The part Γ_p is divided into three sets: Ω_T, Γ_{Dp} and Γ_N, with Ω_T the part of $\partial \mathcal{E}$ at the final time T, Γ_{Dp} the part of Γ_p with a Dirichlet boundary condition and Γ_N the part of Γ_p with a Neumann boundary condition. Note that $\Gamma_D = \Gamma_{Dm} \cup \Gamma_{Dp}$ is the part of the space–time boundary with a Dirichlet boundary condition. The boundary conditions on different parts of $\partial \mathcal{E}$ are written as

$$u = u_0 \quad \text{on } \Omega_0, \tag{3a}$$
$$u = g_D \quad \text{on } \Gamma_D, \tag{3b}$$
$$\bar{n} \cdot \bar{\nabla} u = g_N \quad \text{on } \Gamma_N, \tag{3c}$$
$$p = p_N \quad \text{on } \Gamma_N, \tag{3d}$$

with u_0 a given initial velocity field and g_D, g_N, p_N given data defined on (part of) the boundary. There is no boundary condition imposed on Ω_T.

3. Space–time elements, finite element spaces and trace operators

3.1. Definition of space–time slabs, elements and faces

As already introduced in the previous section, we consider the Oseen equations in a space–time domain $\mathcal{E}$. This requires the definition of space–time slabs. For this purpose the time interval $[0, T]$ is partitioned into several parts. The nth time interval is denoted by I_n, with its length defined as $\Delta_n t = t_{n+1} - t_n$. A space–time slab is now defined as the domain $\mathcal{E}^n = \mathcal{E} \cap (I_n \times \mathbb{R}^d)$ with boundaries Ω_{t_n}, $\Omega_{t_{n+1}}$ and $\mathcal{Q}^n = \partial \mathcal{E}^n \setminus (\Omega_{t_n} \cup \Omega_{t_{n+1}})$.

Next, we construct the space–time elements $\mathcal{K}$ in the space–time slab $\mathcal{E}^n$. Let the domain Ω_{t_n} be divided into N_n non-overlapping spatial elements K^n. At t_{n+1} the spatial elements K^{n+1} are obtained by mapping the elements K^n to their new position. Each space–time element $\mathcal{K}$ is then obtained by connecting elements K^n and K^{n+1} using linear interpolation in time. In case of curved domain boundaries, a higher order accurate interpolation is used for elements connected to the domain boundary. We then denote by $h_{\mathcal{K}}$ the radius of the smallest sphere containing each element $\mathcal{K}$. The element boundary $\partial \mathcal{K}$ is the union of open faces of $\mathcal{K}$, which contains three parts K^n, K^{n+1}, and $\mathcal{Q}^n_{\mathcal{K}} = \partial \mathcal{K} \setminus (K^n \cup K^{n+1})$. We denote by $n_{\mathcal{K}}$ the unit outward space–time normal vector on $\partial \mathcal{K}$. The definition of the space–time domain is completed with the tessellation $\mathcal{T}^n_h$ in each space–time slab and $\mathcal{T}_h = \bigcup_n \mathcal{T}^n_h$ is the tessellation in the space–time domain.

We consider several sets of faces in the tessellation. The set of all faces in $\mathcal{E}$ is denoted with $\mathcal{F}$, the set of all interior faces in $\mathcal{E}$ with $\mathcal{F}_{\text{int}}$, and the set of all boundary faces on $\partial \mathcal{E}$ with $\mathcal{F}_{\text{bnd}}$. In the space–time slab $\mathcal{E}^n$ we denote the set of all faces with $\mathcal{F}^n$ and the set of all interior faces with $\mathcal{S}^n_I$. The faces separating two space–time slabs are denoted as $\mathcal{S}^n_S$.
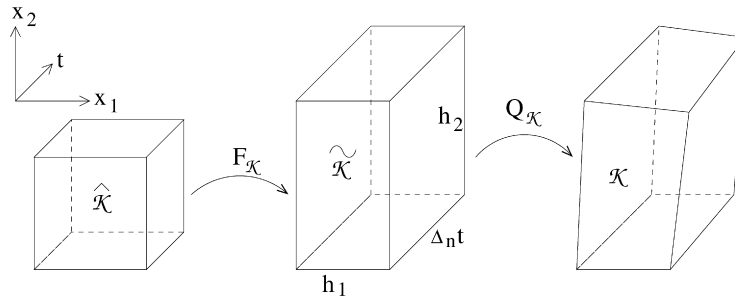
Fig. 1. Construction of elements $\mathcal{K}$ via composition of affine maps and diffeomorphisms (for $d = 2$).

Several sets of boundary faces are defined as follows. The set of faces with a Dirichlet boundary condition is denoted with $\mathcal{S}_D^n$. This set can be divided further into the sets $\mathcal{S}_{Dm}^n$ and $\mathcal{S}_{Dp}^n$, which correspond to the faces with a Dirichlet boundary condition on Γ_m and Γ_p, respectively. The set of faces with a Neumann boundary condition is denoted with $\mathcal{S}_N^n$. Further, the sets $\mathcal{S}_I^n$ and $\mathcal{S}_D^n$ are grouped into $\mathcal{S}_{ID}^n$ and the sets $\mathcal{S}_I^n, \mathcal{S}_D^n$ and $\mathcal{S}_N^n$ are grouped into $\mathcal{S}_{IDN}^n$.

3.2. Finite element spaces and trace operators

The space–time DG discretization requires the use of anisotropic Sobolev spaces on the domain $\mathcal{D} \subset \mathbb{R}^{d+1}$, see for instance [11]. The definition of standard Sobolev spaces follows [5]. Here we restrict the anisotropy to the case where the Sobolev index can be different for the temporal and spatial variables. All spatial variables have, however, the same index. Let (s_t, s_s) be a pair of non-negative integers, with s_t, s_s corresponding to the temporal and spatial Sobolev index, respectively. For $\gamma_t, \gamma_s \geq 0$, the anisotropic Sobolev space of order (s_t, s_s) on $\mathcal{D}$ is defined by

$$H^{(s_t, s_s)}(\mathcal{D}) := \left\{ v \in L^2(\mathcal{D}) : \partial^{\gamma_t} \partial^{\gamma_s} v \in L^2(\mathcal{D}) \text{ for } \gamma_t \leq s_t, |\gamma_s| \leq s_s \right\},$$

with associated norm and semi-norm:

$$\|v\|_{s_t, s_s, \mathcal{D}} := \left(\sum_{\substack{\gamma_t \leq s_t \\ |\gamma_s| \leq s_s}} \|\partial^{\gamma_t} \partial^{\gamma_s} v\|_{0, \mathcal{D}}^2 \right)^{\frac{1}{2}}, \qquad |v|_{s_t, s_s, \mathcal{D}} := \left(\sum_{\substack{\gamma_t = s_t \\ |\gamma_s| = s_s}} \|\partial^{\gamma_t} \partial^{\gamma_s} v\|_{0, \mathcal{D}}^2 \right)^{\frac{1}{2}}.$$

Since the DG method is a non-conforming method, it is necessary to introduce the concept of a broken anisotropic Sobolev space. To each element $\mathcal{K}$ we assign a pair of non-negative integers $(s_{t,\mathcal{K}}, s_{s,\mathcal{K}})$ and collect them in the vectors $\mathbf{s}_t = \{s_{t,\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$ and $\mathbf{s}_s = \{s_{s,\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$. Then we assign to $\mathcal{T}_h$ the broken Sobolev space $H^{(\mathbf{s}_t, \mathbf{s}_s)}(\mathcal{E}, \mathcal{T}_h) := \{v \in L^2(\mathcal{E}) : v|_\mathcal{K} \in H^{(s_{t,\mathcal{K}}, s_{s,\mathcal{K}})}(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_h\}$, equipped with the broken Sobolev norm and corresponding semi-norm, respectively,

$$\|v\|_{\mathbf{s}_t, \mathbf{s}_s, \mathcal{T}_h} := \left(\sum_{\mathcal{K} \in \mathcal{T}_h} \|v\|_{s_{t,\mathcal{K}}, s_{s,\mathcal{K}}, \mathcal{K}}^2 \right)^{\frac{1}{2}}, \qquad |v|_{\mathbf{s}_t, \mathbf{s}_s, \mathcal{T}_h} := \left(\sum_{\mathcal{K} \in \mathcal{T}_h} |v|_{s_{t,\mathcal{K}}, s_{s,\mathcal{K}}, \mathcal{K}}^2 \right)^{\frac{1}{2}}.$$

For $v \in H^{(1,1)}(\mathcal{E}, \mathcal{T}_h)$, we define the broken gradient $\nabla_h v$ of v by $(\nabla_h v)|_\mathcal{K} := \nabla(v|_\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_h$ and similarly for $v \in H^{(0,1)}(\mathcal{E}, \mathcal{T}_h)$ the broken spatial gradient $\bar{\nabla}_h v$ with $(\bar{\nabla}_h v)|_\mathcal{K} := \bar{\nabla}(v|_\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_h$.

We now discuss the finite element spaces associated with the tessellation $\mathcal{T}_h$ that will be used in this article. First, we introduce mappings of the space–time elements. Following the discussion in [11], we assume that each element $\mathcal{K} \in \mathcal{T}_h$ is an image of a fixed master element $\hat{\mathcal{K}}$, with $\hat{\mathcal{K}}$ an open unit hypercube in $\mathbb{R}^{d+1}$, constructed via two mappings $Q_\mathcal{K} \circ F_\mathcal{K}$, where $F_\mathcal{K} : \hat{\mathcal{K}} \to \tilde{\mathcal{K}}$ is an affine mapping and $Q_\mathcal{K} : \tilde{\mathcal{K}} \to \mathcal{K}$ is a (regular enough) diffeomorphism (see Fig. 1).

To each element $\mathcal{K}$ we assign a pair of non-negative integers $(p_{t,\mathcal{K}}, p_{s,\mathcal{K}})$ as local polynomial degrees, where the subscripts t and s denote time and space, respectively, and collect them into vectors $p_t = \{p_{t,\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$ and $p_s = \{p_{s,\mathcal{K}} : \mathcal{K} \in \mathcal{T}_h\}$. We define $\mathcal{Q}_{p_{t,\mathcal{K}}, p_{s,\mathcal{K}}}(\hat{\mathcal{K}})$ as the set of all tensor-product polynomials on $\hat{\mathcal{K}}$ of degree $p_{t,\mathcal{K}}$ in the time direction and degree $p_{s,\mathcal{K}}$ in each spatial coordinate direction. The finite element spaces of discontinuous polynomial functions are defined as follows

$$V_h^{(p_t,p_s)} := \big\{ v \in L^2(\mathcal{E})^d \colon v|_{\mathcal{K}} \circ Q_{\mathcal{K}} \circ F_{\mathcal{K}} \in [\mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}})]^d, \forall \mathcal{K} \in \mathcal{T}_h \big\},$$

$$\mathcal{Q}_h^{(p_t,p_s)} := \big\{ q \in L^2(\mathcal{E}) \colon q|_{\mathcal{K}} \circ Q_{\mathcal{K}} \circ F_{\mathcal{K}} \in \mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h \big\}.$$

In the derivation and analysis of the numerical discretization we also make use of auxiliary spaces $\Sigma_h^{(p_t,p_s)}$ and $\bar{\Sigma}_h^{(p_t,p_s)}$:

$$\Sigma_h^{(p_t,p_s)} := \big\{ \tau \in L^2(\mathcal{E})^{d\times(d+1)} \colon \tau|_{\mathcal{K}} \circ Q_{\mathcal{K}} \circ F_{\mathcal{K}} \in [\mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}})]^{d\times(d+1)}, \forall \mathcal{K} \in \mathcal{T}_h \big\},$$

$$\bar{\Sigma}_h^{(p_t,p_s)} := \big\{ \bar{\tau} \in L^2(\mathcal{E})^{d\times d} \colon \bar{\tau}|_{\mathcal{K}} \circ Q_{\mathcal{K}} \circ F_{\mathcal{K}} \in [\mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}})]^{d\times d}, \forall \mathcal{K} \in \mathcal{T}_h \big\}.$$

The so called traces of $v \in V_h^{(p_t,p_s)}$ on $\partial\mathcal{K}$ are defined as: $v_{\mathcal{K}}^{\pm} = \lim_{\epsilon\downarrow0} v(x \pm \epsilon n_{\mathcal{K}})$. The traces of $q \in \mathcal{Q}_h^{(p_t,p_s)}$, $\tau \in \Sigma_h^{(p_t,p_s)}$, and $\bar{\tau} \in \bar{\Sigma}_h^{(p_t,p_s)}$ are defined similarly.

Next, we define several trace operators for the sets $\mathcal{F}_{\text{int}}$ and $\mathcal{F}_{\text{bnd}}$. Note that functions $v \in V_h^{(p_t,p_s)}$, $q \in \mathcal{Q}_h^{(p_t,p_s)}$, $\tau \in \Sigma_h^{(p_t,p_s)}$ and $\bar{\tau} \in \bar{\Sigma}_h^{(p_t,p_s)}$ are in general multivalued on a face $S \in \mathcal{F}_{\text{int}}$. Introducing the functions $v_i := v|_{\mathcal{K}_i}$, $q_i := q|_{\mathcal{K}_i}$, $\tau_i := \tau|_{\mathcal{K}_i}$, $\bar{\tau}_i := \bar{\tau}|_{\mathcal{K}_i}$, we define the average operator $\{\!\{\cdot\}\!\}$ on $S \in \mathcal{F}_{\text{int}}$ as:

$$\{\!\{v\}\!\} = (v_i + v_j)/2, \quad \{\!\{q\}\!\} = (q_i + q_j)/2, \quad \{\!\{\tau\}\!\} = (\tau_i + \tau_j)/2, \quad \{\!\{\bar{\tau}\}\!\} = (\bar{\tau}_i + \bar{\tau}_j)/2,$$

while on $S \in \mathcal{F}_{\text{bnd}}$, we set accordingly

$$\{\!\{v\}\!\} = v, \quad \{\!\{q\}\!\} = q, \quad \{\!\{\tau\}\!\} = \tau, \quad \{\!\{\bar{\tau}\}\!\} = \bar{\tau}.$$

We also introduce the jump operators $[\![\cdot]\!]$ and $\langle\!\langle\cdot\rangle\!\rangle$. For functions $q \in \mathcal{Q}_h^{(p_t,p_s)}$, $\tau \in \Sigma_h^{(p_t,p_s)}$ and $\bar{\tau} \in \bar{\Sigma}_h^{(p_t,p_s)}$, the jump operators are defined on $S \in \mathcal{F}_{\text{int}}$ as:

$$[\![q]\!] = q_i n_i + q_j n_j, \quad [\![\tau]\!] = \tau_i \cdot n_i + \tau_j \cdot n_j,$$

$$\langle\!\langle q \rangle\!\rangle = q_i \bar{n}_i + q_j \bar{n}_j, \quad \langle\!\langle \bar{\tau} \rangle\!\rangle = \bar{\tau}_i \cdot \bar{n}_i + \bar{\tau}_j \cdot \bar{n}_j,$$

with $n_i, \bar{n}_i$ the outward normal vector on $\partial\mathcal{K}_i$ and its spatial part, respectively. For functions $v \in V_h^{(p_t,p_s)}$, we define the jump operators on $S \in \mathcal{F}_{\text{int}}$ as follows:

$$[\![\![v]\!]\!] = v_i \otimes n_i + v_j \otimes n_j, \quad \langle\!\langle\!\langle v \rangle\!\rangle\!\rangle = v_i \otimes \bar{n}_i + v_j \otimes \bar{n}_j, \quad \langle\!\langle v \rangle\!\rangle = v_i \cdot \bar{n}_i + v_j \cdot \bar{n}_j.$$

By taking all functions from the neighboring element equal to zero, the definitions of jump operators are also valid on boundary faces $S \in \mathcal{F}_{\text{bnd}}$. Note that $\langle\!\langle v \rangle\!\rangle$ is scalar, $[\![q]\!] \in \mathbb{R}^{d+1}$, $\langle\!\langle q \rangle\!\rangle \in \mathbb{R}^d$ are vectors, and $[\![\![v]\!]\!] \in \mathbb{R}^{d\times(d+1)}$, $\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle \in \mathbb{R}^{d\times d}$ are matrices. The jumps $[\![\tau]\!]$, $\langle\!\langle\bar{\tau}\rangle\!\rangle \in \mathbb{R}^d$ are vectors.

3.3. Lifting operators

The derivation of the space–time DG formulation requires several lifting operators. The main purpose of the lifting operator is to eliminate the auxiliary variables σ, introduced in Section 4.1, by extending data defined at the element faces into the whole space–time domain.

First, we introduce the local lifting operator $\mathcal{L}_S \colon (L^2(S))^{d\times(d+1)} \to \Sigma_h^{(p_t,p_s)}$:

$$\int_{\mathcal{E}} \mathcal{L}_S(\vartheta) : \tau \, \mathrm{d}\mathcal{E} = \int_S \vartheta : \{\!\{\tau\}\!\} \, \mathrm{d}S, \quad \forall \tau \in \Sigma_h^{(p_t,p_s)}, \ \forall S \in \bigcup_n \mathcal{S}_{ID}^n, \tag{4}$$

where the dyadic product between two matrices $A, B \in \mathbb{R}^{m\times n}$ is defined as: $A : B = \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$. Note, the local lifting operator $\mathcal{L}_S$ is only non-zero in the two elements connected to the face S. The global lifting operator $\mathcal{L} \colon (L^2(\bigcup_n \mathcal{S}_{ID}^n))^{d\times(d+1)} \to \Sigma_h^{(p_t,p_s)}$ is now introduced as

$$\int_{\mathcal{E}} \mathcal{L}(\vartheta) : \tau \, \mathrm{d}\mathcal{E} = \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_{\mathcal{E}} \mathcal{L}_S(\vartheta) : \tau \, \mathrm{d}\mathcal{E}, \quad \forall \tau \in \Sigma_h^{(p_t,p_s)}. \tag{5}$$

We also specify the above lifting operators for the Dirichlet boundary condition. On faces $S \in \bigcup_n \mathcal{S}_D^n$ we have

$$\int_{\mathcal{E}} \mathcal{L}_S(\mathcal{P}g_D \otimes n) : \tau \, d\mathcal{E} = \int_S g_D \otimes n : \tau \, dS, \quad \forall \tau \in \Sigma_h^{(p_t, p_s)}, \ \forall S \in \bigcup_n \mathcal{S}_D^n,$$

with $\mathcal{P}$ the L^2 projection on $\Sigma_h^{(p_t, p_s)}$. For the global lifting operators, we proceed as follows. We replace ϑ by $\mathcal{P}g_D \otimes n$ in (4) and (5) to obtain the following global lifting operator for the Dirichlet boundary:

$$\int_{\mathcal{E}} \mathcal{L}(\mathcal{P}g_D \otimes n) : \tau \, d\mathcal{E} = \sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S g_D \otimes n : \tau \, dS, \quad \forall \tau \in \Sigma_h^{(p_t, p_s)}. \tag{6}$$

Using (5) and (6), we then introduce $\mathcal{L}_{ID} : (L^2(\bigcup_n \mathcal{S}_{ID}^n))^{d \times (d+1)} \to \Sigma_h^{(p_t, p_s)}$ as:

$$\mathcal{L}_{ID}(\vartheta) = -\mathcal{L}(\vartheta) + \mathcal{L}(\mathcal{P}g_D \otimes n). \tag{7}$$

Later in this article, we will also use the spatial part of the lifting operators, denoted by $\bar{\mathcal{L}}, \bar{\mathcal{L}}_S$, which are obtained by eliminating the first component of $\mathcal{L}, \mathcal{L}_S$, respectively.

4. Space–time DG discretization for the Oseen equations

In this section we give a derivation of the space–time DG formulation for the Oseen equations (2). Since the DG basis functions are discontinuous at the element faces we cannot directly obtain a weak formulation for the Oseen equations because this requires the trace of first order derivatives at the element faces, which are not uniquely defined for discontinuous basis functions. For this purpose we transform the Oseen equations into a first order system by the introduction of an auxiliary variable $\sigma = \bar{\nabla}u$:

$$\sigma = \bar{\nabla}u, \quad \text{in } \mathcal{E}, \tag{8a}$$

$$\nabla \cdot (u \otimes w) - \nu \bar{\nabla} \cdot \sigma + \bar{\nabla} \cdot (\mathbb{I}_d p) = f, \quad \text{in } \mathcal{E}, \tag{8b}$$

$$\bar{\nabla} \cdot u = 0, \quad \text{in } \mathcal{E}. \tag{8c}$$

These equations are completed with the boundary conditions (3a)–(3d). In the next three sections we give the derivation of the space–time DG formulation for (8).

In the remainder of this article, we will assume that $w \in H(\text{div }0, \mathcal{E}) := \{v \in (H^{(1,1)}(\mathcal{E}))^{d+1} : \nabla \cdot v = 0\}$, $f \in (L^2(\mathcal{E}))^d$, $g_D \in (L^2(\bigcup_n S_D^n))^d$, $g_N, p_N \in L^2(\bigcup_n S_N^n)$, $u_0 \in (L^2(\Omega_0))^d$.

4.1. Space–time DG formulation for the auxiliary variable

The space–time DG formulation for the auxiliary equation (8a) is obtained by multiplying (8a) with an arbitrary test function $\bar{\tau} \in \bar{\Sigma}_h^{(p_t, p_s)}$, substituting σ, u with the approximations $\sigma_h \in \bar{\Sigma}_h^{(p_t, p_s)}, u_h \in V_h^{(p_t, p_s)}$, and integration over a space–time element $\mathcal{K} \in \mathcal{T}_h^n$. Next, we perform integration by parts on the right hand side with respect to $x_1, \ldots, x_d$ twice and, after summation over all elements $\mathcal{K} \in \mathcal{T}_h^n$, we obtain for all $\bar{\tau} \in \bar{\Sigma}_h^{(p_t, p_s)}$:

$$\int_{\mathcal{E}^n} \sigma_h : \bar{\tau} \, d\mathcal{E} = \int_{\mathcal{E}^n} \bar{\nabla}_h u_h : \bar{\tau} \, d\mathcal{E} + \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}_{\mathcal{K}}^n} (\hat{u}_h^\sigma - u_h) \otimes \bar{n} : \bar{\tau}^- \, d\partial\mathcal{K}. \tag{9}$$

The variable $\hat{u}_h^\sigma$ is the *numerical flux* that must be defined to account for the multivalued trace at $\mathcal{Q}_{\mathcal{K}}^n$. Note that since we perform the integration by parts on the spatial variables, we only have to consider the weak formulation in the space–time slab $\mathcal{E}^n$ since there are no fluxes between the different space–time slabs.

We recall the following relation, which is an extension of the identity introduced in [3] to tensors $\bar{\tau}$ and vectors v, piecewise smooth on $\mathcal{T}_h$:

$$\sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}_{\mathcal{K}}^n} v \otimes \bar{n} : \bar{\tau}^- \, d\partial\mathcal{K} = \sum_{S \in \mathcal{S}_{IDN}^n} \int_S \langle\!\langle\!\langle v \rangle\!\rangle\!\rangle : \{\!\{\bar{\tau}\}\!\} \, dS + \sum_{S \in \mathcal{S}_I^n} \int_S \{\!\{v\}\!\} \cdot \langle\!\langle \bar{\tau} \rangle\!\rangle \, dS, \tag{10}$$

and can be proved by a straightforward calculation. When applied to the last term in (9), this results in

$$\sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}_{\mathcal{K}}^n} (\hat{u}_h^\sigma - u_h) \otimes \bar{n} : \bar{\tau}^- \, \mathrm{d}\partial\mathcal{K} = \sum_{S \in \mathcal{S}_{IDN}^n} \int_S \langle\!\langle\!\langle \hat{u}_h^\sigma - u_h \rangle\!\rangle\!\rangle : \{\!\{ \bar{\tau} \}\!\} \, \mathrm{d}S + \sum_{S \in \mathcal{S}_I^n} \int_S \{\!\{ \hat{u}_h^\sigma - u_h \}\!\} \cdot \langle\!\langle \bar{\tau} \rangle\!\rangle \, \mathrm{d}S. \tag{11}$$

For the numerical flux $\hat{u}_h^\sigma$, we make a similar choice as in [6,19], now applied to vector functions:

$$\hat{u}_h^\sigma = \{\!\{ u_h \}\!\} \quad \text{on } \mathcal{S}_I^n, \qquad \hat{u}_h^\sigma = g_D \quad \text{on } \mathcal{S}_D^n, \qquad \hat{u}_h^\sigma = u_h \quad \text{on } \mathcal{S}_N^n. \tag{12}$$

Replacing $\hat{u}_h^\sigma$ in (11) with the numerical flux (12), the weak formulation (9) is now equal to:

$$\int_{\mathcal{E}^n} \sigma_h : \bar{\tau} \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}^n} \bar{\nabla}_h u_h : \bar{\tau} \, \mathrm{d}\mathcal{E} - \sum_{S \in \mathcal{S}_{ID}^n} \int_S \langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle : \{\!\{ \bar{\tau} \}\!\} \, \mathrm{d}S + \sum_{S \in \mathcal{S}_D^n} \int_S g_D \otimes \bar{n} : \bar{\tau} \, \mathrm{d}S. \tag{13}$$

We sum (13) over all space–time slabs and the next step is to derive an equation for σ_h which can be used in the DG formulation for (8b). The main benefit of the elimination of the auxiliary variables at the discrete level is a significant reduction in the number of equations and variables in the DG discretization. For this purpose, the last two terms on the right-hand side of (13) are replaced with the spatial part of the lifting operator $\mathcal{L}_{ID}$, defined in (7), to obtain:

$$\int_{\mathcal{E}} \bar{\mathcal{L}}_{ID}\big(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle\big) : \bar{\tau} \, \mathrm{d}\mathcal{E} = - \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle : \{\!\{ \bar{\tau} \}\!\} \, \mathrm{d}S + \sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S g_D \otimes \bar{n} : \bar{\tau} \, \mathrm{d}S. \tag{14}$$

Introducing (14) into (13) and using the fact that this relation must be valid for arbitrary test functions $\bar{\tau}$, we can express $\sigma_h \in \bar{\Sigma}_h^{(p_t, p_s)}$ as

$$\sigma_h = \bar{\nabla}_h u_h + \bar{\mathcal{L}}_{ID}\big(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle\big), \quad \text{a.e. } \forall x \in \mathcal{E}. \tag{15}$$

4.2. Space–time DG formulation for the momentum equations

The space–time DG formulation for the momentum equations in the Oseen equations is obtained by multiplying (8b) with arbitrary test functions $v \in V_h^{(p_t, p_s)}$ and integrating over an element $\mathcal{K} \in \mathcal{T}_h^n$, such that for all $v \in V_h^{(p_t, p_s)}$ the following relation is satisfied:

$$\int_{\mathcal{K}} \big(\nabla \cdot (u \otimes w)\big) \cdot v \, \mathrm{d}\mathcal{K} - \int_{\mathcal{K}} (v \bar{\nabla} \cdot \sigma) \cdot v \, \mathrm{d}\mathcal{K} + \int_{\mathcal{K}} (\bar{\nabla} \cdot \mathbb{I}_d p) \cdot v \, \mathrm{d}\mathcal{K} = \int_{\mathcal{K}} f \cdot v \, \mathrm{d}\mathcal{K}. \tag{16}$$

The functions u, σ, p are then replaced by their approximations $u_h \in V_h^{(p_t, p_s)}$, $\sigma_h \in \bar{\Sigma}_h^{(p_t, p_s)}$, $p_h \in \mathcal{Q}_h^{(p_t, p_s)}$, respectively. Next, we integrate by parts each term on the left hand side of (16). For the first term, integration by parts is with respect to $x_0, \ldots, x_d$, while for the second and third term the integration by parts is with respect to $x_1, \ldots, x_d$. After summation over all elements $\mathcal{K} \in \mathcal{T}_h$, we have for all $v \in V_h^{(p_t, p_s)}$:

$$T_c + T_d + T_p = \int_{\mathcal{E}} f \cdot v \, \mathrm{d}\mathcal{E}, \tag{17}$$

with

$$T_c := - \int_{\mathcal{E}} u_h \otimes w : \nabla_h v \, \mathrm{d}\mathcal{E} + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} \hat{u}_h^c \otimes w : v \otimes n \, \mathrm{d}\partial\mathcal{K}, \tag{18a}$$

$$T_d := \int_{\mathcal{E}} v\sigma_h : \bar{\nabla}_h v \, \mathrm{d}\mathcal{E} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{Q}_{\mathcal{K}}^n} v\hat{\sigma}_h : v \otimes \bar{n} \, \mathrm{d}\partial\mathcal{K}, \tag{18b}$$

$$T_p := - \int_{\mathcal{E}} \mathbb{I}_d p_h : \bar{\nabla}_h v \, \mathrm{d}\mathcal{E} + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{Q}_{\mathcal{K}}^n} \mathbb{I}_d \hat{p}_h : v \otimes \bar{n} \, \mathrm{d}\partial\mathcal{K}, \tag{18c}$$

related, respectively, to the convective, diffusive, and pressure terms. Here we replaced u_h, σ_h, p_h at $\partial\mathcal{K}$ with the numerical fluxes $\hat{u}_h^c, \hat{\sigma}_h, \hat{p}_h$, to account for the multivalued traces at $\partial\mathcal{K}$. The next step is to find appropriate choices for the numerical fluxes and we discuss the derivation for each term separately.

First, we consider the convective term T_c (18a), which includes the *convective flux* $\hat{u}_h^c$. The obvious choice is an upwind flux, as in [9]. However, for simplicity of proving stability of the discretization, the upwind flux is written as the sum of an average plus a jump penalty, as in [7,19]. Thus, we write the numerical flux $\hat{u}_h^c \otimes w$ as:

$$\hat{u}_h^c \otimes w = \{\!\{u_h\}\!\} \otimes w + C_S [\![u_h]\!]. \tag{19}$$

The parameter C_S is chosen as:

$$C_S = \frac{1}{2}|w \cdot n| \quad \text{on } S \in \mathcal{F}_{\text{int}}. \tag{20}$$

For conciseness of the proof discussed later in Section 5, we extend the definition of C_S to the boundary of the space–time domain as:

$$C_S = \begin{cases} -w \cdot n/2, & \text{on } S \subset \varGamma_m, \\ +w \cdot n/2, & \text{on } S \subset \varGamma_p. \end{cases} \tag{21}$$

We recall identity (10), this time for tensors τ and vectors v, piecewise smooth on $\mathcal{T}_h$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\partial\mathcal{K}} v\otimes n:\tau\,\mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}}\int_S [\![v]\!]:\{\!\{\tau\}\!\}\,\mathrm{d}S + \sum_{S\in\mathcal{F}_{\text{int}}}\int_S \{\!\{v\}\!\}\cdot[\![\tau]\!]\,\mathrm{d}S. \tag{22}$$

If we substitute τ in relation (22) with (19) and use the boundary conditions on $\varGamma_{Dm}$ and $\varOmega_0$, we obtain the final form of T_c:

$$T_c = -\int_{\mathcal{E}} u_h\otimes w:\nabla_h v\,\mathrm{d}\mathcal{E} + \sum_{S\in\mathcal{F}_{\text{int}}}\int_S \big(\{\!\{u_h\}\!\}\otimes w + C_S[\![u_h]\!]\big):[\![v]\!]\,\mathrm{d}S$$
$$+ \sum_{S\subset\varGamma_p}\int_S u_h\otimes w:v\otimes n\,\mathrm{d}S + \sum_{S\in\bigcup_n\mathcal{S}_{Dm}^n}\int_S g_D\otimes w:v\otimes n\,\mathrm{d}S - \int_{\varOmega_0} u_0\cdot v\,\mathrm{d}S. \tag{23}$$

Next, we consider the diffusive term T_d (18b). We recall identity (10) again. When applied to the second term in (18b), we obtain:

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{Q_{\mathcal{K}}^n} \nu\hat{\sigma}_h:v\otimes\bar{n}\,\mathrm{d}\partial\mathcal{K} = \sum_{S\in\bigcup_n\mathcal{S}_{IDN}^n}\int_S \nu\{\!\{\hat{\sigma}_h\}\!\}:\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\,\mathrm{d}S + \sum_{S\in\bigcup_n\mathcal{S}_I^n}\int_S \nu\langle\!\langle\hat{\sigma}_h\rangle\!\rangle\cdot\{\!\{v\}\!\}\,\mathrm{d}S. \tag{24}$$

For the numerical flux $\hat{\sigma}_h$, we make the same choice as in [6,19], now applied to tensor functions:

$$\hat{\sigma}_h = \{\!\{\sigma_h\}\!\} \quad \text{on } \mathcal{S}_I^n, \qquad \hat{\sigma}_h = \sigma_h \quad \text{on } \mathcal{S}_D^n\cup\mathcal{S}_N^n, \tag{25}$$

and substitute (25) into (24). Using the boundary condition (3c) on $S\in\bigcup_n\mathcal{S}_N^n$ and after replacement of σ_h with (15), the term T_d becomes:

$$T_d = \int_{\mathcal{E}} \nu\bar{\nabla}_h u_h:\bar{\nabla}_h v\,\mathrm{d}\mathcal{E} + \int_{\mathcal{E}} \nu\bar{\mathcal{L}}_{ID}\big(\langle\!\langle\!\langle u_h\rangle\!\rangle\!\rangle\big):\bar{\nabla}_h v\,\mathrm{d}\mathcal{E} - \sum_{S\in\bigcup_n\mathcal{S}_N^n}\int_S \nu g_N\cdot v\,\mathrm{d}S$$
$$- \sum_{S\in\bigcup_n\mathcal{S}_{ID}^n}\int_S \nu\{\!\{\bar{\nabla}_h u_h\}\!\}:\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\,\mathrm{d}S - \sum_{S\in\bigcup_n\mathcal{S}_{ID}^n}\int_S \nu\{\!\{\bar{\mathcal{L}}_{ID}\big(\langle\!\langle\!\langle u_h\rangle\!\rangle\!\rangle\big)\}\!\}:\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\,\mathrm{d}S. \tag{26}$$

We can further evaluate the second and fourth term in (26). Using the lifting operator $\bar{\mathcal{L}}_{ID}$ defined in (14) with $\bar{\tau} = \bar{\nabla}_h v$ we have the relation

$$\int_{\mathcal{E}} \nu\bar{\mathcal{L}}_{ID}\big(\langle\!\langle\!\langle u_h\rangle\!\rangle\!\rangle\big):\bar{\nabla}_h v\,\mathrm{d}\mathcal{E} = -\sum_{S\in\bigcup_n\mathcal{S}_{ID}^n}\int_S \nu\langle\!\langle\!\langle u_h\rangle\!\rangle\!\rangle:\{\!\{\bar{\nabla}_h v\}\!\}\,\mathrm{d}S + \sum_{S\in\bigcup_n\mathcal{S}_D^n}\int_S \nu g_D\otimes\bar{n}:\bar{\nabla}_h v\,\mathrm{d}S, \tag{27}$$

and by considering only the spatial part of the lifting operators $\mathcal{L}, \mathcal{L}_{ID}$, defined in (5) and (7), we obtain

$$\sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S v \{\!\{\bar{\mathcal{L}}_{ID}(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle)\}\!\} : \langle\!\langle\!\langle v \rangle\!\rangle\!\rangle \, \mathrm{d}S = -\int_{\mathcal{E}} v \bar{\mathcal{L}}(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle) : \bar{\mathcal{L}}(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{E}$$

$$+ \int_{\mathcal{E}} v \bar{\mathcal{L}}(\mathcal{P} g_D \otimes \bar{n}) : \bar{\mathcal{L}}(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{E}. \tag{28}$$

In order to ensure that only contributions from neighboring elements occur in the discretization, which improves both the computational efficiency and memory use, the contributions from the global lifting operator $\bar{\mathcal{L}}$ in (28) are replaced with the local lifting operator $\bar{\mathcal{L}}_S$ (defined in (4) and (5)), using the following simplifications

$$\int_{\mathcal{E}} v \bar{\mathcal{L}}(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle) : \bar{\mathcal{L}}(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{E} \cong \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} v \bar{\mathcal{L}}_S(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle) : \bar{\mathcal{L}}_S(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{K}, \tag{29a}$$

$$\int_{\mathcal{E}} v \bar{\mathcal{L}}(\mathcal{P} g_D \otimes \bar{n}) : \bar{\mathcal{L}}(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{E} \cong \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} v \bar{\mathcal{L}}_S(\mathcal{P} g_D \otimes \bar{n}) : \bar{\mathcal{L}}_S(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{K}, \tag{29b}$$

with the parameter $\eta_{\mathcal{K}}^u$ a positive constant. Later in Section 5 we discuss the minimum value for $\eta_{\mathcal{K}}^u$ in order to have a stable method. Introducing the relations (27)–(29b) into (26), the term T_d can be written in its final form

$$T_d = \int_{\mathcal{E}} v \bar{\nabla}_h u_h : \bar{\nabla}_h v \, \mathrm{d}\mathcal{E} - \sum_{S \in \bigcup_n \mathcal{S}_N^n} v g_N \cdot v \mathrm{d}S - \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S v \langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle : \{\!\{\bar{\nabla}_h v\}\!\} \, \mathrm{d}S$$

$$- \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S v \{\!\{\bar{\nabla}_h u_h\}\!\} : \langle\!\langle\!\langle v \rangle\!\rangle\!\rangle \, \mathrm{d}S + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} v \bar{\mathcal{L}}_S(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle) : \bar{\mathcal{L}}_S(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{K}$$

$$+ \sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S v g_D \otimes \bar{n} : \bar{\nabla}_h v \mathrm{d}S - \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} v \bar{\mathcal{L}}_S(\mathcal{P} g_D \otimes \bar{n}) : \bar{\mathcal{L}}_S(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{K}. \tag{30}$$

Finally, we consider the last term T_p given by (18c). We use the relation (10), this time for vectors v and scalars q, and replace $\hat{p}_h$ in (18c) with:

$$\hat{p}_h = \{\!\{p_h\}\!\} \quad \text{on } \mathcal{S}_I^n, \qquad \hat{p}_h = p_h \quad \text{on } \mathcal{S}_D^n, \qquad \hat{p}_h = p_N \quad \text{on } \mathcal{S}_N^n. \tag{31}$$

Note that on faces $S \in \bigcup_n \mathcal{S}_{ID}^n$, we follow a similar approach as described in [10,17]. The condition on $S \in \bigcup_n \mathcal{S}_N^n$ is required to have a well-posed problem for certain flow conditions. We obtain then the following expression for T_p:

$$T_p = -\int_{\mathcal{E}} p_h \bar{\nabla}_h \cdot v \, \mathrm{d}\mathcal{E} + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \{\!\{p_h\}\!\} \langle\!\langle v \rangle\!\rangle \, \mathrm{d}S + \sum_{S \in \bigcup_n \mathcal{S}_N^n} \int_S p_N v \cdot \bar{n} \, \mathrm{d}S. \tag{32}$$

Introducing all terms in (23), (30) and (32) into (17), we obtain the space–time DG discretization for the momentum equations (8b) in the Oseen equations:

Find $(u_h, p_h) \in V_h^{(p_t, p_s)} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ such that $\forall v \in V_h^{(p_t, p_s)}$:

$$\mathcal{O}_h(u_h, v; w) + \mathcal{A}_h(u_h, v) + \mathcal{B}_h(p_h, v) = N_h(v; w) + F_h(v) + G_h(v). \tag{33}$$

Here, the forms $\mathcal{O}_h : V_h^{(p_t, p_s)} \times V_h^{(p_t, p_s)} \times H(\mathrm{div}\, 0, \mathcal{E}) \to \mathbb{R}$, $\mathcal{A}_h : V_h^{(p_t, p_s)} \times V_h^{(p_t, p_s)} \to \mathbb{R}$, and $\mathcal{B}_h : \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R} \times V_h^{(p_t, p_s)} \to \mathbb{R}$ are defined as:

$$\mathcal{O}_h(u_h, v; w) := -\int_{\mathcal{E}} u_h \otimes w : \nabla_h v \, \mathrm{d}\mathcal{E} + \sum_{S \in \mathcal{F}_{\mathrm{int}}} \int_S \left(\{\!\{u_h\}\!\} \otimes w + C_S [\![u_h]\!]\right) : [\![v]\!] \, \mathrm{d}S$$

$$+ \sum_{S \subset \Gamma_p} \int_S u_h \otimes w : v \otimes n \, \mathrm{d}S, \tag{34}$$

$$\mathcal{A}_h(u_h, v) := \int_{\mathcal{E}} \nu \bar{\nabla}_h u_h : \bar{\nabla}_h v \, d\mathcal{E} - \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \nu \langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle : \{\!\{ \bar{\nabla}_h v \}\!\} \, dS - \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \nu \{\!\{ \bar{\nabla}_h u_h \}\!\} : \langle\!\langle\!\langle v \rangle\!\rangle\!\rangle \, dS$$

$$+ \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} \nu \bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle\big) : \bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle\big) \, d\mathcal{K}, \tag{35}$$

$$\mathcal{B}_h(p_h, v) := - \int_{\mathcal{E}} p_h \bar{\nabla}_h \cdot v \, d\mathcal{E} + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \{\!\{ p_h \}\!\} \langle\!\langle v \rangle\!\rangle \, dS, \tag{36}$$

and the linear forms $N_h : V_h^{(p_t, p_s)} \times H(\mathrm{div}\, 0, \mathcal{E}) \to \mathbb{R}$, $F_h : V_h^{(p_t, p_s)} \to \mathbb{R}$, $G_h : V_h^{(p_t, p_s)} \to \mathbb{R}$ as:

$$N_h(v; w) := - \sum_{S \in \bigcup_n \mathcal{S}_{Dm}^n} \int_S g_D \otimes w : v \otimes n \, dS + \int_{\Omega_0} u_0 \cdot v \, dS, \tag{37}$$

$$F_h(v) := - \sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S \nu g_D \otimes \bar{n} : \bar{\nabla}_h v \, dS + \sum_{S \in \bigcup_n \mathcal{S}_N^n} \int_S \nu g_N \cdot v \, dS$$

$$+ \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} \nu \bar{\mathcal{L}}_S(\mathcal{P} g_D \otimes \bar{n}) : \bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle\big) \, d\mathcal{K}, \tag{38}$$

$$G_h(v) := \int_{\mathcal{E}} f \cdot v \, d\mathcal{E} - \sum_{S \in \bigcup_n \mathcal{S}_N^n} \int_S p_N v \cdot \bar{n} \, dS. \tag{39}$$

4.3. Space–time DG formulation for the continuity equation

The space–time DG formulation for the continuity equation is obtained by multiplying (8c) with arbitrary test functions $q \in \mathcal{Q}_h^{(p_t, p_s)}$, replacing u with $u_h \in V_h^{(p_t, p_s)}$ and integrating over the element $\mathcal{K} \in \mathcal{T}_h^n$. After integration by parts twice with respect to $x_1, \ldots, x_d$ and summation over all elements $\mathcal{K} \in \mathcal{T}_h$ we obtain:

$$\int_{\mathcal{E}} (\bar{\nabla}_h \cdot u_h) q \, d\mathcal{E} + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{Q}_{\mathcal{K}}^n} (\hat{u}_h^p - u_h) \cdot \bar{n} q \, d\partial\mathcal{K} = 0, \quad \forall q \in \mathcal{Q}_h^{(p_t, p_s)}, \tag{40}$$

with $\hat{u}_h^p$ the numerical flux that has to be introduced to account for the multivalued traces on $\mathcal{Q}_{\mathcal{K}}^n$. Using (10), we can write (40) as

$$\int_{\mathcal{E}} (\bar{\nabla}_h \cdot u_h) q \, d\mathcal{E} + \sum_{S \in \bigcup_n \mathcal{S}_{IDN}^n} \int_S \{\!\{ \hat{u}_h^p - u_h \}\!\} \cdot \langle\!\langle q \rangle\!\rangle \, dS + \sum_{S \in \bigcup_n \mathcal{S}_I^n} \int_S \langle\!\langle \hat{u}_h^p - u_h \rangle\!\rangle \{\!\{ q \}\!\} \, dS = 0. \tag{41}$$

The next step is to find an appropriate numerical flux $\hat{u}_h^p$. The following approach is considered for the numerical flux $\hat{u}_h^p$ on $S \in \bigcup_n \mathcal{S}_I^n$:

$$\hat{u}_h^p = \{\!\{ u_h \}\!\} + \gamma \langle\!\langle p_h \rangle\!\rangle, \tag{42}$$

with $\gamma > 0$. This approach is first introduced for a DG discretization in [9]. The term containing the pressure is called the pressure stabilization. On boundary faces we choose:

$$\hat{u}_h^p = g_D \quad \text{on } \mathcal{S}_D^n, \qquad \hat{u}_h^p = u_h \quad \text{on } \mathcal{S}_N^n. \tag{43}$$

Introducing the numerical fluxes (42)–(43) into (41), we obtain the final form of the weak formulation for the continuity equation (8c):

Find $(u_h, p_h) \in V_h^{(p_t, p_s)} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ such that $\forall q \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$:

$$-\mathcal{B}_h(q, u_h) + \mathcal{C}_h(p_h, q) = H_h(q), \tag{44}$$

with $\mathcal{B}_h$ defined in (36), while $\mathcal{C}_h : \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R} \to \mathbb{R}$ and $H_h : \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R} \to \mathbb{R}$ are defined as:

$$\mathcal{C}_h(p_h, q) := \sum_{S \in \bigcup_n \mathcal{S}_I^n} \int_S \gamma \langle\!\langle p_h \rangle\!\rangle \cdot \langle\!\langle q \rangle\!\rangle \, \mathrm{d}S, \tag{45}$$

$$H_h(q) := - \sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S g_D \cdot \bar{n} q \, \mathrm{d}S. \tag{46}$$

The space–time DG formulation for the Oseen equations (8) is now stated as:
Find $(u_h, p_h) \in V_h^{(p_t, p_s)} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ such that $\forall (v, q) \in V_h^{(p_t, p_s)} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$:

$$\mathcal{O}_h(u_h, v; w) + \mathcal{A}_h(u_h, v) + \mathcal{B}_h(p_h, v) = N_h(v; w) + F(v) + G_h(v), \tag{47}$$

$$-\mathcal{B}_h(q, u_h) + \mathcal{C}_h(p_h, q) = H_h(q). \tag{48}$$

5. Stability and error analysis

In this section we discuss the stability and error analysis for the space–time DG formulation for the Oseen equations given by (47)–(48).

The first main result in this section is Theorem 5.9, stating that the space–time DG discretization for the Oseen equations has a unique solution and is unconditionally stable, also when equal order polynomials are used for the velocity and pressure. A central point in the stability analysis will be the derivation of an inf–sup condition and an upper bound for the solution of the Oseen equations in terms of initial and boundary data and the source term.

The second main result is an a priori *hp*-error estimate, given in Theorem 5.13 and Corollary 5.14, which shows the dependence of the error on the mesh size h and polynomial order p.

The analysis of the space–time DG formulation (47)–(48) is considerably simplified by the introduction of the following DG norm and norm for vector functions.

Definition 5.1. The DG norm $\|\!|\cdot|\!\|_{\mathrm{DG}}$, related to the bilinear form (35), is defined on $H^{(0,1)}(\mathcal{E}) + V_h^{(p_t, p_s)}$ as:

$$\|\!|v|\!\|_{\mathrm{DG}}^2 = \sum_{\mathcal{K} \in \mathcal{T}_h} \|v\|_{0,0,\mathcal{K}}^2 + \sum_{\mathcal{K} \in \mathcal{T}_h} \|\bar{\nabla}_h v\|_{0,0,\mathcal{K}}^2 + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \|\bar{\mathcal{L}}_S(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle)\|_{0,0,\mathcal{K}}^2,$$

with $H^{(0,1)}(\mathcal{E})$ the anisotropic Sobolev space defined in Section 3.2. The seminorm $|\cdot|_{\mathcal{S}}$, with $\mathcal{S}$ is a set of faces, is defined on $\mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ as:

$$|q|_{\mathcal{S}}^2 = \sum_{S \in \mathcal{S}} \int_S \left|\langle\!\langle q \rangle\!\rangle\right|^2 \mathrm{d}S. \tag{49}$$

5.1. Continuity, coercivity, and inf–sup condition

The first result in this section establishes the continuity property of the bilinear form $\mathcal{A}_h$.

Lemma 5.2. *Let* $v_m = \max_{x \in \mathcal{E}} v(x)$, $\eta_m^u = \max_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u$, *and* N_f *be the number of faces of each element* $\mathcal{K} \in \mathcal{T}_h$. *Then there exists a constant* $\alpha_{\mathcal{A}} = \eta_m^u + 2\sqrt{N_f} + 1 > 0$, *independent of the mesh size* $h = \max_{\mathcal{K} \in \mathcal{T}_h} h_{\mathcal{K}}$, *such that*

$$\left|\mathcal{A}_h(u_h, v)\right| \leqslant v_m \alpha_{\mathcal{A}} \|\!|u_h|\!\|_{\mathrm{DG}} \|\!|v|\!\|_{\mathrm{DG}}, \quad \forall u_h, v \in V_h^{(p_t, p_s)}.$$

Proof. We consider the bilinear form $\mathcal{A}_h$ in the form

$$\mathcal{A}_h(u_h, v) = \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} v \bar{\nabla}_h u_h : \bar{\nabla}_h v \, \mathrm{d}\mathcal{K} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} v \bar{\nabla}_h u_h : \bar{\mathcal{L}}(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{K} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} v \bar{\mathcal{L}}(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle) : \bar{\nabla}_h v \, \mathrm{d}\mathcal{K}$$

$$+ \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} v \bar{\mathcal{L}}_S(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle) : \bar{\mathcal{L}}_S(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle) \, \mathrm{d}\mathcal{K}. \tag{50}$$

As a consequence of (5), we have

$$\big\|\bar{\mathcal{L}}\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2 \leqslant N_f \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \big\|\bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2, \tag{51}$$

with N_f the number of faces of each element $\mathcal{K}\in\mathcal{T}_h$. Application of Schwarz' inequality and the use of inequality (51) yields:

$$\big|\mathcal{A}_h(u_h, v)\big| \leqslant \nu_m \alpha_{\mathcal{A}} \|u_h\|_{\mathrm{DG}} \|v\|_{\mathrm{DG}}, \tag{52}$$

with $\nu_m = \max_{x\in\mathcal{E}} \nu(x)$, $\eta_m^u = \max_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}}^u$, and $\alpha_{\mathcal{A}} = \eta_m^u + 2\sqrt{N_f} + 1$. $\quad\square$

Next, we establish the coercivity of $\mathcal{A}_h$.

Lemma 5.3. *Let* $\nu_0 = \min_{x\in\mathcal{E}} \nu(x)$, $\eta_0^u = \min_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}}^u$, *and* N_f *be the number of faces of each element* $\mathcal{K}\in\mathcal{T}_h$. *Then there exists a constant* $\bar{\beta}_{\mathcal{A}} > 0$, *independent of the mesh size* $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$, *such that*

$$\mathcal{A}_h(v, v) \geqslant \nu_0 \bar{\beta}_{\mathcal{A}} \|v\|_{\mathrm{DG}}^2, \quad \forall v \in V_h^{(p_t, p_s)},$$

with $\bar{\beta}_{\mathcal{A}} = \frac{\beta_{\mathcal{A}}}{2} \min(1, 1/C_p^2)$, *where* $\beta_{\mathcal{A}} = \min(1-\epsilon, \eta_0^u - \frac{N_f}{\epsilon})$ *for* $\epsilon \in (\frac{N_f}{\eta_0^u}, 1)$, *and* C_p^2 *the coefficient in the discrete Poincaré inequality ([2], Lemma 2.1).*

Proof. We start with replacing u_h in (50) with v:

$$\mathcal{A}_h(v, v) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nu \bar{\nabla}_h v : \bar{\nabla}_h v \, \mathrm{d}\mathcal{K} - 2 \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nu \bar{\nabla}_h v : \bar{\mathcal{L}}\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big) \, \mathrm{d}\mathcal{K}$$
$$+ \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \eta_{\mathcal{K}}^u \int_{\mathcal{K}} \nu \bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big) : \bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big) \, \mathrm{d}\mathcal{K}. \tag{53}$$

Using the Schwarz and arithmetic-geometric mean inequalities we have the inequality:

$$2 \int_{\mathcal{K}} \bar{\nabla}_h v : \bar{\mathcal{L}}\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big) \, \mathrm{d}\mathcal{K} \leqslant \epsilon \|\bar{\nabla}_h v\|_{0,0,\mathcal{K}}^2 + \frac{1}{\epsilon} \big\|\bar{\mathcal{L}}\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2, \tag{54}$$

with $\epsilon > 0$. Introducing inequalities (51) and (54) into (53), we deduce

$$\mathcal{A}_h(v, v) \geqslant \nu_0(1-\epsilon) \sum_{\mathcal{K}\in\mathcal{T}_h} \|\bar{\nabla}_h v\|_{0,0,\mathcal{K}}^2 + \nu_0\left(\eta_0 - \frac{N_f}{\epsilon}\right) \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|\bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2. \tag{55}$$

If we take the parameters $\eta_0^u > N_f$ and $\epsilon \in (\frac{N_f}{\eta_0^u}, 1)$, then for $0 < \beta_{\mathcal{A}} = \min(1-\epsilon, \eta_0^u - \frac{N_f}{\epsilon})$, we obtain

$$\mathcal{A}_h(v, v) \geqslant \nu_0 \beta_{\mathcal{A}} \sum_{\mathcal{K}\in\mathcal{T}_h} \|\bar{\nabla}_h v\|_{0,0,\mathcal{K}}^2 + \nu_0 \beta_{\mathcal{A}} \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|\bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2. \tag{56}$$

Next, we recall the discrete inequality which is first discussed in [3], p. 1763, but now applied to vector functions in the space–time discretization:

$$\|v\|_{0,0,\mathcal{E}} \leqslant C_P \left(\sum_{\mathcal{K}\in\mathcal{T}_h} \|\bar{\nabla}_h v\|_{0,0,\mathcal{K}}^2 + \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|\bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2 \right)^{1/2}. \tag{57}$$

Using (57) in (56), we then obtain:

$$\mathcal{A}_h(v, v) \geqslant \nu_0 \left(\frac{\beta_{\mathcal{A}}}{2C_p^2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|_{0,0,\mathcal{K}}^2 + \frac{\beta_{\mathcal{A}}}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \|\bar{\nabla}_h v\|_{0,0,\mathcal{K}}^2 + \frac{\beta_{\mathcal{A}}}{2} \sum_{S\in\bigcup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K}\in\mathcal{T}_h} \big\|\bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle v\rangle\!\rangle\!\rangle\big)\big\|_{0,0,\mathcal{K}}^2 \right). \tag{58}$$

Choosing $\bar{\beta}_{\mathcal{A}} = \frac{\beta_{\mathcal{A}}}{2} \min(1, 1/C_p^2)$ completes the proof. $\quad\square$

For the analysis of the DG algorithm we also need that the bilinear form $\mathcal{B}_h$ (36) is continuous, which is stated in the next lemma.

Lemma 5.4. *Let N_f be the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$. Then there exists a constant $\alpha_\mathcal{B} = \sqrt{N_f} + 1 > 0$, independent of the mesh size $h = \max_{\mathcal{K} \in \mathcal{T}_h} h_\mathcal{K}$, such that*

$$\left| \mathcal{B}_h(q, v) \right| \leqslant \alpha_\mathcal{B} \|q\|_{0,0,\mathcal{E}} \|v\|_{\mathrm{DG}}, \quad \forall(q, v) \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R} \times V_h^{(p_t, p_s)}.$$

Proof. First, we consider the bilinear form $\mathcal{B}_h(q, v)$ in the form

$$\mathcal{B}_h(q, v) = -\sum_{\mathcal{K} \in \mathcal{T}_h} \int_\mathcal{K} I_d q : \bar{\nabla}_h v \, \mathrm{d}\mathcal{K} + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_\mathcal{K} I_d q : \bar{\mathcal{L}}\big(\langle\!\langle\!\langle v \rangle\!\rangle\!\rangle\big) \, \mathrm{d}\mathcal{K}. \tag{59}$$

Application of Schwarz' inequality and inequality (51) yields:

$$\left| \mathcal{B}_h(q, v) \right| \leqslant \|q\|_{0,0,\mathcal{E}} \|v\|_{\mathrm{DG}} + \sqrt{N_f} \|q\|_{0,0,\mathcal{E}} \|v\|_{\mathrm{DG}}. \tag{60}$$

Choosing $\alpha_\mathcal{B} = \sqrt{N_f} + 1$ completes the proof. $\quad\square$

We also need to show that the bilinear form $\mathcal{B}_h$ satisfies the *inf–sup* condition. First, we introduce the *inf–sup* condition for the Stokes equations in the domain Ω_t as follows:

$$\inf_{0 \neq q \in L^2(\Omega_t)/\mathbb{R}} \sup_{0 \neq v \in (H_0^1(\Omega_t))^d} \frac{-\int_{\Omega_t} q \bar{\nabla}_h \cdot v \, \mathrm{d}\Omega}{|v|_{1,\Omega_t} \|q\|_{0,\Omega_t}} \geqslant C_{\Omega_t} > 0, \tag{61}$$

with the constant C_{Ω_t} depending only on Ω_t. For a proof see [12]. If we fix now $q \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$, then the *inf–sup* condition (61) guarantees that there exists a $z(t) \in (H_0^1(\Omega_t))^d$, with $t \in [0, T]$, such that:

$$-\int_{\Omega_t} q \bar{\nabla}_h \cdot z(t) \, \mathrm{d}\Omega = \|q\|_{0,\Omega_t}^2, \quad \text{with } \left\| z(t) \right\|_{1,\Omega_t} \leqslant C_{\Omega_t}^{-1} \|q\|_{0,\Omega_t}, \tag{62}$$

where we use the Poincaré inequality to change $|z(t)|_{1,\Omega_t}$ into $\|z(t)\|_{1,\Omega_t}$. Integrating in time from $t = 0$ to $t = T$, then results in the relation:

$$-\int_\mathcal{E} q \bar{\nabla}_h \cdot z \, \mathrm{d}\mathcal{E} = \|q\|_{0,0,\mathcal{E}}^2, \quad \text{with } \|z\|_{0,1,\mathcal{E}} \leqslant C_\mathcal{E}^{-1} \|q\|_{0,0,\mathcal{E}}. \tag{63}$$

In the next lemma, we establish an *inf–sup* condition for the bilinear form $\mathcal{B}_h(\cdot, \cdot)$, defined in (36).

Lemma 5.5. *The following inf–sup condition holds for $(q, v) \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R} \times V_h^{(p_t, p_s)}$:*

$$\mathcal{B}_h(q, v) \geqslant C_{\mathrm{inf}} \|q\|_{0,0,\mathcal{E}}^2 \left(1 - \frac{|q|_{\bigcup_n \mathcal{S}_l^n}}{\|q\|_{0,0,\mathcal{E}}} \right), \quad \forall q \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}, \tag{64}$$

with $C_{\mathrm{inf}} > 0$ solely depending on $C_\mathcal{E}^{-1}$ and the interpolation bounds.

Proof. To prove the *inf–sup* condition, we follow similar steps as in [18]. First, we fix $q \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ and define the L^2-projection $\mathcal{P}_{K(t)} : (L^2(K(t)))^d \to V_h^{(0, p_s)}|_{K(t)}$, with $K(t) = \mathcal{K} \cap \{t\}$ the space–time element at time t, as:

$$\int_{K(t)} (\mathcal{P}_{K(t)} v) \cdot \phi \, \mathrm{d}K = \int_{K(t)} v \cdot \phi \, \mathrm{d}K, \quad \forall\phi \in V_h^{(0, p_s)}|_{K(t)}. \tag{65}$$

Next, for $q \in \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$, we consider the bilinear form $\mathcal{B}_h(q, z)$, for $z \in L^2([0, T], H_0^1(\Omega_t))^d$:

$$\mathcal{B}_h(q, z) = -\int_\mathcal{E} q \bar{\nabla}_h \cdot z \, \mathrm{d}\mathcal{E} + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \{\!\{q\}\!\} \langle\!\langle z \rangle\!\rangle \, \mathrm{d}S.$$

Since $\langle\!\langle z \rangle\!\rangle = 0$ on $S \in \bigcup_n \mathcal{S}_{ID}^n$, we can use (63) to obtain

$$\mathcal{B}_h(q,z) = \|q\|_{0,0,\mathcal{E}}^2. \tag{66}$$

We consider now the bilinear form $\mathcal{B}_h(q,v)$, with $v = \mathcal{P}_{K(t)} z$:

$$
\begin{aligned}
\mathcal{B}_h(q,v) &= \mathcal{B}_h(q,z) + \mathcal{B}_h(q, \mathcal{P}_{K(t)}z - z), \\
&= \|q\|_{0,0,\mathcal{E}}^2 - \int_{\mathcal{E}} q \bar{\nabla}_h \cdot (\mathcal{P}_{K(t)}z - z)\, d\mathcal{E} + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \{\!\{q\}\!\} \langle\!\langle \mathcal{P}_{K(t)}z - z \rangle\!\rangle \, dS, \\
&= \|q\|_{0,0,\mathcal{E}}^2 + \int_{\mathcal{E}} \bar{\nabla}_h q \cdot (\mathcal{P}_{K(t)}z - z)\, d\mathcal{E} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{Q}_{\mathcal{K}}^n} q \bar{n} \cdot (\mathcal{P}_{K(t)}z - z)\, d\partial\mathcal{K} \\
&\quad + \sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \int_S \{\!\{q\}\!\} \langle\!\langle \mathcal{P}_{K(t)}z - z \rangle\!\rangle \, dS.
\end{aligned}
\tag{67}
$$

The last equation is obtained using integration by parts with respect to $x_1, \ldots, x_d$, and the fact that $\bar{n} = 0$ at Ω_0 and Ω_T. Applying identity (10) (for vectors) into (67), using the orthogonality property of the L^2-projection $\mathcal{P}_{K(t)}$ and the fact that $z = 0$ at $\partial\Omega_t$, we then obtain:

$$\mathcal{B}_h(q,v) = \|q\|_{0,0,\mathcal{E}}^2 - \sum_{S \in \bigcup_n \mathcal{S}_I^n} \int_S \langle\!\langle q \rangle\!\rangle \cdot \{\!\{\mathcal{P}_{K(t)}z - z\}\!\} \, dS, \quad \forall q \in \mathcal{Q}_h^{(p_t,p_s)}/\mathbb{R}. \tag{68}$$

We estimate the second term on the right-hand side in (68) as follows:

$$
\begin{aligned}
\left| \sum_{S \in \bigcup_n \mathcal{S}_I^n} \int_S \langle\!\langle q \rangle\!\rangle \cdot \{\!\{\mathcal{P}_{K(t)}z - z\}\!\} \, dS \right| &\leqslant \left(\sum_{S \in \bigcup_n \mathcal{S}_I^n} \int_S |\langle\!\langle q \rangle\!\rangle|^2 \, dS \right)^{1/2} \left(\sum_{S \in \bigcup_n \mathcal{S}_I^n} \int_S |\{\!\{\mathcal{P}_{K(t)}z - z\}\!\}|^2 \, dS \right)^{1/2}, \\
&\leqslant C |q|_{\bigcup_n \mathcal{S}_I^n} \left(\sum_{\mathcal{K} \in \mathcal{T}_h} \|\bar{\nabla}_h z\|_{0,0,\mathcal{K}}^2 \right)^{1/2},
\end{aligned}
\tag{69}
$$

using the seminorm defined in (49) for $\gamma \approx h$ and a standard interpolation estimate. Using (69), we obtain the following inequality for $\mathcal{B}_h(q,v)$:

$$\mathcal{B}_h(q,v) \geqslant \|q\|_{0,0,\mathcal{E}}^2 - C|q|_{\bigcup_n \mathcal{S}_I^n} \left(\sum_{\mathcal{K} \in \mathcal{T}_h} \|\bar{\nabla}_h z\|_{0,0,\mathcal{K}}^2 \right)^{1/2} = C\|q\|_{0,0,\mathcal{E}}^2 \left(1 - \frac{|q|_{\bigcup_n \mathcal{S}_I^n} \|z\|_{0,1,\mathcal{E}}}{\|q\|_{0,0,\mathcal{E}}^2} \right). \tag{70}$$

Together with the fact that $\|z\|_{0,1,\mathcal{E}} \leqslant C_{\mathcal{E}}^{-1} \|q\|_{0,0,\mathcal{E}}$ this completes the proof. $\quad\square$

We also show that for a divergence free convective velocity field w, the trilinear form $\mathcal{O}_h$ (34) satisfies a stability relation.

Lemma 5.6. *For $w \in H(\mathrm{div}\,0, \mathcal{E})$, the trilinear form $\mathcal{O}_h$ satisfies the following stability relation*

$$\mathcal{O}_h(v,v;w) = \sum_{S \in \mathcal{F}} \|C_S^{1/2}|[\![v]\!]|\|_{0,0,S}^2, \quad \forall v \in V_h^{(p_t,p_s)}.$$

Proof. First we replace u_h in (34) with v:

$$
\begin{aligned}
\mathcal{O}_h(v,v;w) &= -\int_{\mathcal{E}} v \otimes w : \nabla_h v \, d\mathcal{E} + \sum_{S \in \mathcal{F}_{\mathrm{int}}} \int_S \left(\{\!\{v\}\!\} \otimes w + C_S [\![v]\!] \right) : [\![v]\!] \, dS \\
&\quad + \sum_{S \in \Gamma_p} \int_S v \otimes w : v \otimes n \, dS.
\end{aligned}
\tag{71}
$$

Using the following relation: $v \otimes w : \nabla_h v = \frac{1}{2}(\nabla_h \cdot w)(v \cdot v) - \frac{1}{2}\nabla_h \cdot ((v \otimes w) \cdot v)$ and applying the Gauss theorem, we can write (71) as

$$\mathcal{O}_h(v, v; w) = \frac{1}{2}\int_{\mathcal{E}} (\nabla_h \cdot w)(v \cdot v)\,d\mathcal{E} - \frac{1}{2}\sum_{\mathcal{K} \in \mathcal{T}_h}\int_{\partial \mathcal{K}} v \otimes w : v \otimes n\,d\partial\mathcal{K} + \sum_{S \in \mathcal{F}_{\text{int}}}\int_S \{\!\{v\}\!\} \otimes w : [\![v]\!]\,dS$$
$$+ \sum_{S \in \mathcal{F}_{\text{int}}}\int_S C_S[\![v]\!] : [\![v]\!]\,dS + \sum_{S \subset \Gamma_p}\int_S v \otimes w : v \otimes n\,dS. \tag{72}$$

Using (22) and the fact that w is a continuous function, we then obtain:

$$\mathcal{O}_h(v, v; w) = \frac{1}{2}\int_{\mathcal{E}} (\nabla_h \cdot w)(v \cdot v)\,d\mathcal{E} - \frac{1}{2}\sum_{S \in \mathcal{F}_{\text{int}}}\int_S w \cdot [\![v \cdot v]\!]\,dS$$
$$+ \sum_{S \in \mathcal{F}_{\text{int}}}\int_S \{\!\{v\}\!\} \otimes w : [\![v]\!]\,dS + \sum_{S \in \mathcal{F}_{\text{int}}}\int_S C_S[\![v]\!] : [\![v]\!]\,dS$$
$$- \frac{1}{2}\sum_{S \subset \Gamma_m}\int_S v \otimes w : v \otimes n\,dS + \frac{1}{2}\sum_{S \subset \Gamma_p}\int_S v \otimes w : v \otimes n\,dS. \tag{73}$$

Due to the continuity of w, on faces $S \in \mathcal{F}_{\text{int}}$, we have:

$$\int_S \{\!\{v\}\!\} \otimes w : [\![v]\!]\,dS = \frac{1}{2}\int_S w \cdot [\![v \cdot v]\!]\,dS. \tag{74}$$

Finally, as a consequence of (74), the fact that $\nabla_h \cdot w = 0$, and using the definition of C_S in (20)–(21), we obtain that the trilinear form $\mathcal{O}_h$ satisfies the stability relation in Lemma 5.6. $\quad\square$

5.2. Global stability

In this section we discuss a global stability result. First, we define the product space $\mathcal{Z}_h^{(p_t,p_s)} = V_h^{(p_t,p_s)} \times \mathcal{Q}_h^{(p_t,p_s)}/\mathbb{R}$, endowed with the norm:

$$\left\|(v, q)\right\|_{\text{DG}}^2 = \|v\|_{\text{DG}}^2 + |q|_{\bigcup_n \mathcal{S}_I^n}^2 + \sum_{S \in \mathcal{F}}\left\|C_S^{1/2}|[\![v]\!]\right\|_{0,0,S}^2. \tag{75}$$

If we define the forms $\tilde{\mathcal{A}} : \mathcal{Z}_h^{(p_t,p_s)} \times \mathcal{Z}_h^{(p_t,p_s)} \to \mathbb{R}$ and $\tilde{\mathcal{L}} : \mathcal{Z}_h^{(p_t,p_s)} \to \mathbb{R}$ as:

$$\tilde{\mathcal{A}}(u_h, p_h; v, q) := \mathcal{A}_h(u_h, v) + \mathcal{B}_h(p_h, v) - \mathcal{B}_h(q, u_h) + \mathcal{C}_h(p_h, q) + \mathcal{O}_h(u_h, v; w), \tag{76}$$
$$\tilde{\mathcal{L}}(v, q) := N_h(v; w) + F_h(v) + G_h(v) + H_h(q), \tag{77}$$

then (47)–(48) are equivalent with:

Find $(u_h, p_h) \in \mathcal{Z}_h^{(p_t,p_s)}$, such that:

$$\tilde{\mathcal{A}}(u_h, p_h; v, q) = \tilde{\mathcal{L}}(v, q), \quad \forall(v, q) \in \mathcal{Z}_h^{(p_t,p_s)}. \tag{78}$$

First, we discuss the consistency of the space–time DG method (47)–(48). The formulation is consistent when (47)–(48) is also satisfied by $(u, p) \in H^{(1,2)}(\mathcal{E})^d \times H^{(0,1)}(\mathcal{E})/\mathbb{R}$, the solution of (2)–(3):

$$\tilde{\mathcal{A}}(u, p; v, q) = \tilde{\mathcal{L}}(v, q), \quad \forall(v, q) \in H^{(1,1)}(\mathcal{E})^d \times L^2(\mathcal{E})/\mathbb{R}. \tag{79}$$

The proof for consistency is straightforward. We replace (u_h, p_h) in (76) by (u, p). Since (u, p) solves (2)–(3), we have $[\![u]\!] = 0$, $\langle\!\langle\!\langle u \rangle\!\rangle\!\rangle = 0$, $\langle\!\langle p \rangle\!\rangle = 0$, $\{\!\{u\}\!\} = u$, $\{\!\{p\}\!\} = p$ on $\mathcal{F}_{\text{int}}$. If we introduce these relations into (76), perform integration by parts, and use the boundary conditions (3), we obtain $\tilde{\mathcal{L}}(v, q)$. Subtracting (79) from (78) yields the Galerkin orthogonality property:

$$\tilde{\mathcal{A}}(u - u_h, p - p_h; v, q) = 0, \quad \forall(v, q) \in \mathcal{Z}_h^{(p_t,p_s)}. \tag{80}$$

Next, we show the coercivity of $\tilde{\mathcal{A}}$.

Lemma 5.7. *Let* $v_0 = \min_{x \in \mathcal{E}} v(x)$ *and assume that the constant* $\bar{\beta}_{\mathcal{A}}$ *satisfies the conditions given in Lemma* 5.3. *Then there exists a constant* $C_{\tilde{\mathcal{A}}} > 0$ *such that*

$$\tilde{\mathcal{A}}(v, q; v, q) \geqslant C_{\tilde{\mathcal{A}}} \big\|\hspace{-0.6mm}\big\|(v, q)\big\|\hspace{-0.6mm}\big\|_{\mathrm{DG}}^2, \quad \forall (v, q) \in \mathcal{Z}_h^{(p_t, p_s)}, \tag{81}$$

with $C_{\tilde{\mathcal{A}}} = \min(v_0 \bar{\beta}_{\mathcal{A}}, \gamma, 1)$.

Proof. First we replace (u_h, p_h) in (76) with (v, q) to obtain:

$$\tilde{\mathcal{A}}(v, q; v, q) = \mathcal{A}_h(v, v) + \mathcal{C}_h(q, q) + \mathcal{O}_h(v, v; w).$$

Using Lemmas 5.3 and 5.6 and the norm $|\cdot|_{\bigcup_n \mathcal{S}_I^n}$, we obtain:

$$\tilde{\mathcal{A}}(v, q; v, q) \geqslant v_0 \bar{\beta}_{\mathcal{A}} \|v\|_{\mathrm{DG}}^2 + \gamma |q|_{\bigcup_n \mathcal{S}_I^n}^2 + \sum_{S \in \mathcal{F}} \big\| C_S^{1/2} |[\![v]\!]| \big\|_{0,0,S}^2.$$

Taking $C_{\tilde{\mathcal{A}}} = \min(v_0 \bar{\beta}_{\mathcal{A}}, \gamma, 1)$ completes the proof. $\square$

In the next lemma we show that the solution to (47)–(48) is bounded by known data. For this purpose we define the L^2-projection $\mathcal{P}_S : (L^2(S))^d \to \gamma_S(V_h^{(p_t, p_s)})$, with γ_S the trace at $S \in \bigcup_n \mathcal{S}_D^n$, as:

$$\int_S \mathcal{P}_S v \cdot \psi \, \mathrm{d}S = \int_S v \cdot \psi \, \mathrm{d}S, \quad \forall \psi \in \gamma_S(V_h^{(p_t, p_s)}),$$

and we introduce the lifting operator $\hat{\mathcal{L}}_S(\gamma_S(V_h^{(p_t, p_s)})) \to V_h^{(p_t, p_s)}$, which satisfies:

$$\hat{\mathcal{L}}_S(\mathcal{P}_S g_D) = \mathcal{P}_S g_D \quad \text{at } S \in \bigcup_n \mathcal{S}_D^n.$$

In addition, we will need the face bubble functions Ψ_S, with $S \subset \partial \mathcal{K}$, which are defined as $\Psi_S = \hat{\Psi}_i \circ F_{\mathcal{K}}^{-1} \circ Q_{\mathcal{K}}^{-1}$, with i the index of the face $\hat{S}_i \subset \partial \hat{\mathcal{K}}$ which is connected to the face $S \subset \partial \mathcal{K}$ through the mapping $Q_{\mathcal{K}} \circ F_{\mathcal{K}}$. The face bubble functions $\hat{\Psi}_i$ satisfy the conditions

$$\hat{\Psi}_i(\hat{x}) > 0, \quad \forall \hat{x} \in \hat{\mathcal{K}}, \tag{82}$$
$$\hat{\Psi}_i = 0, \quad \text{at } \overline{\hat{S}}_j, \forall j \in \{1, \ldots, N_f\} \text{ and } j \neq i,$$

hence $\hat{\Psi}_i = 0$ at $\partial \hat{S}_i$. For any $q_h \in \mathcal{Q}_h^{(p_t, p_s)}$, we then have the inequality

$$|q_h|_S^2 \leqslant C_i |\Psi_S q_h|_S^2, \quad \forall S \in \mathcal{F}, \tag{83}$$

with $C_i > 0$ a constant independent of the mesh size. The proof of this inequality follows directly from the fact that all norms are equivalent in a finite dimensional space and condition (82). See for instance [1].

Lemma 5.8. *Let* $C_{\tilde{\mathcal{A}}}$ *satisfy the conditions given in Lemma* 5.7, $\tilde{C}_{\tilde{\mathcal{A}}} = \frac{C_{\tilde{\mathcal{A}}}}{\max(1, C_i)}$, $v_m = \max_{x \in \mathcal{E}} v(x)$, *and* $\eta_m^u = \max_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u$. *Then the solution to* (47)–(48) *satisfies the following upper bound:*

$$\begin{aligned}
\tilde{C}_{\tilde{\mathcal{A}}}^2 \big\|\hspace{-0.6mm}\big\|(u_h, p_h)\big\|\hspace{-0.6mm}\big\|_{\mathrm{DG}}^2 \leqslant\; & 4 \sum_{\mathcal{K} \in \mathcal{T}_h} \|f\|_{0,0,\mathcal{K}}^2 + 2 v_m^2 \big(N_f + (\eta_m^u)^2\big) \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \big\| \bar{\mathcal{L}}_S\big(\langle\!\langle\!\langle \mathcal{P} g_D \rangle\!\rangle\!\rangle\big) \big\|_{0,0,\mathcal{K}}^2 \\
& + 24 \sum_{S \in \bigcup_n \mathcal{S}_D^n} \|C_S^{1/2} g_D\|_{0,0,S}^2 + 12 \|C_S^{-1/2} u_0\|_{0,0,\Omega_0}^2 + 4 v_m^2 \sum_{S \in \bigcup_n \mathcal{S}_N^n} \|C_S^{-1/2} g_N\|_{0,0,S}^2 \\
& + 4 \sum_{S \in \bigcup_n \mathcal{S}_N^n} \|C_S^{-1/2} p_N\|_{0,0,S}^2 + \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big\|_{0,1,\mathcal{E}}^2 \\
& + \frac{2 \alpha_{\mathcal{B}} \tilde{C}_{\tilde{\mathcal{A}}}}{C_{\mathrm{inf}}} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\|\hspace{-0.6mm}\big\| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big\|\hspace{-0.6mm}\big\|_{\mathrm{DG}} \sum_{S' \in \bigcup_n \mathcal{S}_D^n} \big\| \Psi_{S'} \hat{\mathcal{L}}_{S'}(\mathcal{P}_{S'} g_D) \big\|_{0,1,\mathcal{E}}
\end{aligned}$$

$$+ 4 \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \left\| \nabla_h \big(\Psi_S \hat{\mathcal{L}}_S (\mathcal{P}_S g_D) \big) \cdot w \right\|_{0,0,\mathcal{K}}^2$$

$$+ 4 \sum_{S \in \bigcup_n \mathcal{S}_D^n} \left\| C_S^{1/2} [\![\Psi_S \hat{\mathcal{L}}_S (\mathcal{P}_S g_D)]\!] \right\|_{0,0,S}^2 + 2 v_m^2 \alpha_{\mathcal{A}}^2 \sum_{S \in \bigcup_n \mathcal{S}_D^n} \left\|\!\left| \Psi_S \hat{\mathcal{L}}_S (\mathcal{P}_S g_D) \right|\!\right\|_{\mathrm{DG}}$$

$$+ 2 \tilde{C}_{\tilde{\mathcal{A}}} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big(\big| N_h \big(\Psi_S \hat{\mathcal{L}}_S (\mathcal{P}_S g_D); w \big) \big| + \big| F_h \big(\Psi_S \hat{\mathcal{L}}_S (\mathcal{P}_S g_D) \big) \big| + \big| G_h \big(\Psi_S \hat{\mathcal{L}}_S (\mathcal{P}_S g_D) \big) \big| \big).$$

Proof. We start with (83) with $q_h = p_h$, then the DG-norm (75) satisfies the inequality

$$\big\| (u_h, p_h) \big\|_{\mathrm{DG}}^2 \leqslant \max(1, C_i) \big\| (u_h, \Psi p_h) \big\|_{\mathrm{DG}}^2, \tag{84}$$

with $\Psi p_h = \sum_{S \in \mathcal{F}} \Psi_S p_h$. Next, using (84), the coercivity estimate given by Lemma 5.7, with $(v, q) = (u_h, \Psi p_h)$, and (78) we obtain

$$C_{\tilde{\mathcal{A}}} \big\| (u_h, p_h) \big\|_{\mathrm{DG}}^2 \leqslant \max(1, C_i) \tilde{\mathcal{A}}(u_h, \Psi p_h; u_h, \Psi p_h) = \max(1, C_i) \tilde{\mathcal{L}}(u_h, \Psi p_h),$$

where we used the fact that $\tilde{\mathcal{A}}(u_h, \Psi p_h; u_h, \Psi p_h)$ does not depend on Ψp_h. The final estimate is now obtained by determining an upper bound for each term in $\tilde{\mathcal{L}}$ separately.

First, we consider N_h given by (37):

$$N_h(u_h; w) = - \sum_{S \in \bigcup_n \mathcal{S}_{Dm}^n} \int_S g_D \otimes w : u_h \otimes n \, \mathrm{d}S + \int_{\Omega_0} u_0 \cdot u_h \, \mathrm{d}S,$$

$$= 2 \sum_{S \in \bigcup_n \mathcal{S}_{Dm}^n} \int_S C_S [\![g_D]\!] : [\![u_h]\!] \, \mathrm{d}S + \int_{\Omega_0} [\![u_0]\!] : [\![u_h]\!] \, \mathrm{d}S,$$

$$\leqslant \frac{3}{2} \epsilon_1 \sum_{S \in \mathcal{F}} \big\| C_S^{1/2} |[\![u_h]\!]| \big\|_{0,0,S}^2 + \frac{1}{\epsilon_1} \sum_{S \in \bigcup_n \mathcal{S}_{Dm}^n} \big\| C_S^{1/2} g_D \big\|_{0,0,S}^2 + \frac{1}{2\epsilon_1} \big\| C_S^{-1/2} u_0 \big\|_{0,0,\Omega_0}^2. \tag{85}$$

Next, we find an upper bound for F_h given by (38). Using the lifting operators defined in Section 3.3 and inequality (51), we can write F_h as:

$$F_h(u_h) = - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} v \bar{\mathcal{L}} \big(\langle\!\langle\!\langle \mathcal{P} g_D \rangle\!\rangle\!\rangle \big) : \bar{\nabla}_h u_h \, \mathrm{d}\mathcal{E}$$

$$+ \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^{u_h} \int_{\mathcal{K}} v \bar{\mathcal{L}}_S \big(\langle\!\langle\!\langle \mathcal{P} g_D \rangle\!\rangle\!\rangle \big) : \bar{\mathcal{L}}_S \big(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle \big) \, \mathrm{d}\mathcal{K} + \sum_{S \in \bigcup_n \mathcal{S}_N^n} \int_S v g_N \cdot u_h \, \mathrm{d}S,$$

$$\leqslant v_m \frac{\epsilon_2}{2} \sum_{\mathcal{K} \in \mathcal{T}_h} \big\| \bar{\nabla}_h u_h \big\|_{0,0,\mathcal{K}}^2 + v_m \left(\frac{N_f}{2\epsilon_2} + \frac{\eta_m^u}{2\epsilon_3} \right) \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \big\| \bar{\mathcal{L}}_S \big(\langle\!\langle\!\langle \mathcal{P} g_D \rangle\!\rangle\!\rangle \big) \big\|_{0,0,\mathcal{K}}^2$$

$$+ v_m \eta_m^u \frac{\epsilon_3}{2} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \big\| \bar{\mathcal{L}}_S \big(\langle\!\langle\!\langle u_h \rangle\!\rangle\!\rangle \big) \big\|_{0,0,\mathcal{K}}^2$$

$$+ v_m \frac{1}{2\epsilon_4} \sum_{S \in \bigcup_n \mathcal{S}_N^n} \big\| C_S^{-1/2} g_N \big\|_{0,0,S}^2 + v_m \frac{\epsilon_4}{2} \sum_{S \in \mathcal{F}} \big\| C_S^{1/2} |[\![u_h]\!]| \big\|_{0,0,S}^2. \tag{86}$$

An upper bound for G_h, given by (39), is obtained as follows:

$$G_h(u_h) = \int_{\mathcal{E}} f \cdot u_h \, \mathrm{d}\mathcal{E} - \sum_{S \in \bigcup_n \mathcal{S}_N^n} \int_S p_N u_h \cdot \bar{n} \, \mathrm{d}S,$$

$$\leqslant \frac{\epsilon_5}{2} \sum_{\mathcal{K} \in \mathcal{T}_h} \| u_h \|_{0,0,\mathcal{K}}^2 + \frac{1}{2\epsilon_5} \sum_{\mathcal{K} \in \mathcal{T}_h} \| f \|_{0,0,\mathcal{K}}^2$$

$$+ \frac{\epsilon_6}{2} \sum_{S \in \bigcup_n \mathcal{S}_N^n} \big\| C_S^{1/2} |[\![u_h]\!]| \big\|_{0,0,S}^2 + \frac{1}{2\epsilon_6} \sum_{S \in \bigcup_n \mathcal{S}_N^n} \| C_S^{-1/2} p_N \|_{0,0,S}^2. \tag{87}$$

Finally, for the estimation of H_h, given by (46), we first use the relation:

$$H_h(\Psi p_h) = - \sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S \Psi_S \mathcal{P}_S g_D \cdot \bar{n}\, p_h \, \mathrm{d}S. \tag{88}$$

An estimate for $H_h(\Psi p_h)$ can now be obtained by choosing the test function v in the weak formulation (33) as $v = \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D)$ in the element $\mathcal{K}$ connected to the boundary face $S \in \bigcup_n \mathcal{S}_D^n$ and zero elsewhere:

$$\sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_S \Psi_S \mathcal{P}_S g_D \cdot \bar{n}\, p_h \, \mathrm{d}S = \sum_{S \in \bigcup_n \mathcal{S}_D^n} \bigg(\int_{\mathcal{E}} p_h \bar{\nabla}_h \cdot \big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) \, \mathrm{d}\mathcal{E} + \int_{\mathcal{E}} u_h \otimes w : \nabla_h \big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) \, \mathrm{d}\mathcal{E}$$

$$- \sum_{S' \subset \Gamma_p} \int_{S'} u_h \otimes w : \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \otimes n \, \mathrm{d}S - \mathcal{A}_h\big(u_h, \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big)$$

$$+ N_h\big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D); w \big) + F_h\big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) + G_h\big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) \bigg),$$

where we used the fact that $\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) = 0$ at $\partial \mathcal{K}/S$. We estimate now the individual terms on the right-hand side. The first contribution has an upper bound:

$$\sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_{\mathcal{E}} p_h \bar{\nabla}_h \cdot \big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) \, \mathrm{d}\mathcal{E} \leqslant \| p_h \|_{0,0,\mathcal{E}} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big\|_{0,1,\mathcal{E}}$$

$$\leqslant \bigg(|p_h|_{\bigcup_n \mathcal{S}_I^n} + \frac{\alpha_{\mathcal{B}}}{C_{\mathrm{inf}}} \sum_{S' \in \bigcup_n \mathcal{S}_D^n} \big\| |\Psi_{S'} \hat{\mathcal{L}}_{S'}(\mathcal{P}_{S'} g_D)| \big\|_{\mathrm{DG}} \bigg) \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big\|_{0,1,\mathcal{E}}$$

$$\leqslant \frac{1}{2}\epsilon_7 |p_h|_{\bigcup_n \mathcal{S}_I^n}^2 + \frac{1}{2\epsilon_7} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big\|_{0,1,\mathcal{E}}^2$$

$$+ \frac{\alpha_{\mathcal{B}}}{C_{\mathrm{inf}}} \sum_{S' \in \bigcup_n \mathcal{S}_D^n} \big\| |\Psi_{S'} \hat{\mathcal{L}}_{S'}(\mathcal{P}_{S'} g_D)| \big\|_{\mathrm{DG}} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\| \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big\|_{0,1,\mathcal{E}},$$

where we used in the second inequality the *inf–sup* condition, given by Lemma 5.5, and the boundedness of B_h, given by Lemma 5.4. The second contribution has as upper bound:

$$\sum_{S \in \bigcup_n \mathcal{S}_D^n} \int_{\mathcal{E}} u_h \otimes w : \nabla_h \big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) \, \mathrm{d}\mathcal{E} \leqslant \frac{1}{2}\epsilon_8 \| u_h \|_{0,0,\mathcal{E}}^2 + \frac{1}{2\epsilon_8} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \big\| \nabla_h \big(\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \big) \cdot w \big\|_{0,0,\mathcal{K}}^2,$$

and the third contribution is bounded by:

$$\sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{S' \subset \Gamma_p} \int_{S'} u_h \otimes w : \Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D) \otimes n \, \mathrm{d}S$$

$$= 2 \sum_{S \in \bigcup_n \mathcal{S}_D^n} \sum_{S' \subset \Gamma_p} \int_{S'} C_{S'} [\![u_h]\!] : [\![\![\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D)]\!]\!] \, \mathrm{d}S,$$

$$\leqslant \frac{1}{2}\epsilon_9 \sum_{S \in \mathcal{F}} \big\| C_S^{1/2} |[\![u_h]\!]| \big\|_{0,0,S}^2 + \frac{1}{2\epsilon_9} \sum_{S \in \bigcup_n \mathcal{S}_D^n} \big\| C_S^{1/2} [\![\![\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D)]\!]\!] \big\|_{0,0,S}^2.$$

An upper bound for the remaining terms can be obtained directly using the boundedness for A_h, given by Lemma 5.2, and using the estimates given by (85)–(87) with u_h replaced with $\Psi_S \hat{\mathcal{L}}_S(\mathcal{P}_S g_D)$.

Collecting all terms, introducing the following coefficients: $\epsilon_1 = \frac{\tilde{C}_{\tilde{A}}}{12}, \epsilon_2 = \frac{\tilde{C}_{\tilde{A}}}{2v_m}, \epsilon_3 = \frac{\tilde{C}_{\tilde{A}}}{2v_m \eta_m^u}, \epsilon_4 = \frac{\tilde{C}_{\tilde{A}}}{4v_m}, \epsilon_5 = \frac{\tilde{C}_{\tilde{A}}}{4}, \epsilon_6 = \frac{\tilde{C}_{\tilde{A}}}{4}, \epsilon_7 = \tilde{C}_{\tilde{A}}, \epsilon_8 = \frac{\tilde{C}_{\tilde{A}}}{4}, \epsilon_9 = \frac{\tilde{C}_{\tilde{A}}}{4}, \epsilon_{10} = \frac{\tilde{C}_{\tilde{A}}}{2v_m \alpha_{\mathcal{A}}}$, and finally multiplying the result with $2\tilde{C}_{\tilde{A}}$ completes the proof. □

The result of Lemmas 5.7 and 5.8 immediately imply the existence of a unique solution of (47)–(48) in $\mathcal{Z}_h^{(p_t, p_s)}$.

Theorem 5.9. *Let $v_0 = \min_{x \in \mathcal{E}} v(x)$, $\eta_0^u = \min_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}}^u > N_f$, with N_f the number of faces of each element $\mathcal{K} \in \mathcal{T}_h$, and the parameter $C_{\tilde{A}}$ satisfies the conditions given in Lemma 5.7. Then the space–time discontinuous Galerkin discretization given by (47)–(48) is unconditionally stable and has a unique solution in $V_h^{(p_t, p_s)} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ with $p_t \geqslant 0$ and $p_s \geqslant 1$.*

Corollary 5.10. *The space–time DG discretization of the Oseen equations, given by (47)–(48), is unconditionally stable.*

Proof. This result is a direct consequence of Lemma 5.8, which is valid for any value of the spatial mesh size and time step, and Theorem 5.9. □

5.3. hp-Error analysis

The results obtained in Sections 5.1 and 5.2 now make it possible to analyze the *hp*-error of the space–time discretization. Since the spatial mesh size h and time step Δt are in general quite different we make the dependence of the error on these parameters explicit. For the error analysis we define the projection $\mathcal{P}_u : L^2(\mathcal{E}) \to V_h^{(p_t, p_s)}$ as:

$$\sum_{\mathcal{K} \in \mathcal{T}_h} (\mathcal{P}_u z, v)_{\mathcal{K}} = \sum_{\mathcal{K} \in \mathcal{T}_h} (z, v)_{\mathcal{K}}, \quad \forall v \in V_h^{(p_t, p_s)}, \tag{89}$$

and the projection $\mathcal{P}_p : L^2(\mathcal{E}) \to \mathcal{Q}_h^{(p_t, p_s)}$ as:

$$\sum_{\mathcal{K} \in \mathcal{T}_h} (\mathcal{P}_p q)_{\mathcal{K}} = \sum_{\mathcal{K} \in \mathcal{T}_h} (p, q)_{\mathcal{K}}, \quad \forall q \in \mathcal{Q}_h^{(p_t, p_s)}, \tag{90}$$

which can be used to decompose the global error $(u - u_h, p - p_h)$ as:

$$(u - u_h, p - p_h) = (u - \mathcal{P}_u u, p - \mathcal{P}_p p) + (\mathcal{P}_u u - u_h, \mathcal{P}_p p - p_h)$$
$$\equiv (\rho_u, \rho_p) + (\theta_u, \theta_p), \tag{91}$$

with (ρ_u, ρ_p) the interpolation error and (θ_u, θ_p) the discretization error.

In the next lemma we give an estimate for (θ_u, θ_p) in terms of (ρ_u, ρ_p).

Lemma 5.11. *Let $C_{\tilde{A}}$ satisfy the conditions given in Lemma 5.7, $\alpha_{\mathcal{A}}, \alpha_{\mathcal{B}}$ be defined in Lemmas 5.2 and 5.4, respectively, $\bar{w}_m = \max_{\mathcal{K} \in \mathcal{T}_h} \|\bar{w}\|_{L^\infty(\mathcal{K})}$, $v_m = \max_{x \in \mathcal{E}} v(x)$, C_S given by (20) and the stabilization coefficient $\gamma > 0$. Then the functions (θ_u, θ_p) satisfy the inequality:*

$$\frac{1}{4} C_{\tilde{A}}^2 \|(\theta_u, \theta_p)\|_{\mathrm{DG}}^2 \leqslant \left(v_m^2 \alpha_{\mathcal{A}}^2 + \bar{w}_m^2 + (1 + C_{\tilde{A}}/C_{\inf}) \alpha_{\mathcal{B}}^2 \right) \|\rho_u\|_{\mathrm{DG}}^2 + \alpha_{\mathcal{B}}^2 \|\rho_p\|_{0,0,\mathcal{E}}^2 + \gamma^2 |\rho_p|_{\bigcup_n \mathcal{S}_I^n}^2$$
$$+ \sum_{S \in \mathcal{F}_{\mathrm{int}}} \|C_S^{1/2} |\{\!\!\{\rho_u\}\!\!\}|\|_{0,0,S}^2 + \sum_{S \in \mathcal{F}} \|C_S^{1/2} |[\![\rho_u]\!]|\|_{0,0,S}^2.$$

Proof. To prove this lemma, we use the orthogonality relation (80) to obtain:

$$\tilde{\mathcal{A}}(\rho_u + \theta_u, \rho_p + \theta_p; v, q) = 0, \quad \forall (v, q) \in \mathcal{Z}_h^{(p_t, p_s)}. \tag{92}$$

Taking $(v, q) = (\theta_u, \theta_p)$, we then obtain: $\tilde{\mathcal{A}}(\theta_u, \theta_p; \theta_u, \theta_p) = -\tilde{\mathcal{A}}(\rho_u, \rho_p; \theta_u, \theta_p)$.

We continue with an estimate for $|\tilde{\mathcal{A}}(\rho_u, \rho_p; \theta_u, \theta_p)|$:

$$\left|\tilde{\mathcal{A}}(\rho_u, \rho_p; \theta_u, \theta_p)\right| \leqslant \left|\mathcal{A}_h(\rho_u, \theta_u)\right| + \left|\mathcal{B}_h(\rho_p, \theta_u)\right| + \left|\mathcal{B}_h(\theta_p, \rho_u)\right| + \left|\mathcal{C}_h(\rho_p, \theta_p)\right| + \left|\mathcal{O}_h(\rho_u, \theta_u; w)\right|. \tag{93}$$

The first term in (93) can be estimated using Lemma 5.2, while the second and the third term can be estimated using Lemma 5.4. The fourth term can be estimated easily as follows:

$$\left|\mathcal{C}_h(\rho_p, \theta_p)\right| \leqslant \gamma \, |\rho_p|_{\bigcup_n \mathcal{S}_I^n} |\theta_p|_{\bigcup_n \mathcal{S}_I^n}. \tag{94}$$

Next, we derive an estimate for the fifth term in (93):

$$\left|\mathcal{O}_h(\rho_u, \theta_u; w)\right| \leqslant \left|\int_{\mathcal{E}} \rho_u \cdot \frac{\partial \theta_u}{\partial t} \, d\mathcal{E}\right| + \left|\int_{\mathcal{E}} \rho_u \otimes \bar{w} : \bar{\nabla}_h \theta_u \, d\mathcal{E}\right| + \left|\sum_{S \in \mathcal{F}_{\text{int}}} \int_S \{\!\{\rho_u\}\!\} \otimes w : [\![\theta_u]\!] \, dS\right|$$

$$+ \left|\sum_{S \in \mathcal{F}_{\text{int}}} \int_S C_S [\![\rho_u]\!] : [\![\theta_u]\!] \, dS\right| + \left|\sum_{S \subset \Gamma_p} \int_S C_S \rho_u \otimes w : \theta_u \otimes n \, dS\right|. \tag{95}$$

Since $\theta_u \in V_h^{(p_t, p_s)}$, which is polynomial, hence $\frac{\partial \theta_u}{\partial t} \in V_h^{(p_t, p_s)}$, we can use the orthogonality relation (89) to eliminate the first term in (95). The remaining terms in (95) are estimated as follows:

$$\left|\mathcal{O}_h(\rho_u, \theta_u; w)\right| \leqslant \sum_{\mathcal{K} \in \mathcal{T}_h} \|\bar{w}\|_{L^\infty(\mathcal{K})} \|\rho_u\|_{0,0,\mathcal{K}} \|\bar{\nabla}_h \theta_u\|_{0,0,\mathcal{K}} + \sum_{S \in \mathcal{F}_{\text{int}}} \left\|C_S^{1/2} |\{\!\{\rho_u\}\!\}|\right\|_{0,0,S} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S}$$

$$+ \sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\rho_u]\!]|\right\|_{0,0,S} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S},$$

$$\leqslant \bar{w}_m \|\!|\rho_u|\!\|_{\text{DG}} \|\!|\theta_u|\!\|_{\text{DG}} + \left(\sum_{S \in \mathcal{F}_{\text{int}}} \left\|C_S^{1/2} |\{\!\{\rho_u\}\!\}|\right\|_{0,0,S}^2\right)^{1/2} \left(\sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S}^2\right)^{1/2}$$

$$+ \left(\sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\rho_u]\!]|\right\|_{0,0,S}^2\right)^{1/2} \left(\sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S}^2\right)^{1/2}, \tag{96}$$

with $\bar{w}_m = \max_{\mathcal{K} \in \mathcal{T}_h} \|\bar{w}\|_{L^\infty(\mathcal{K})}$. We also need an estimate for $\|\theta_p\|_{0,0,\mathcal{E}}$, which can be obtained directly using Lemmas 5.4 and 5.5:

$$\|\theta_p\|_{0,0,\mathcal{E}} \leqslant |\theta_p|_{\bigcup_n \mathcal{S}_I^n} + (\alpha_\mathcal{B}/C_{\text{inf}}) \|\!|\rho_u|\!\|_{\text{DG}}. \tag{97}$$

Collecting all terms we obtain the estimate:

$$\left|\tilde{\mathcal{A}}(\rho_u, \rho_p; \theta_u, \theta_p)\right| \leqslant \nu_m \alpha_\mathcal{A} \|\!|\rho_u|\!\|_{\text{DG}} \|\!|\theta_u|\!\|_{\text{DG}} + \alpha_\mathcal{B} \|\rho_p\|_{0,0,\mathcal{E}} \|\!|\theta_u|\!\|_{\text{DG}}$$

$$+ \frac{\alpha_\mathcal{B}^2}{C_{\text{inf}}} \|\!|\rho_u|\!\|_{\text{DG}}^2 + \alpha_\mathcal{B} |\theta_p|_{\bigcup_n \mathcal{S}_I^n} \|\!|\rho_u|\!\|_{\text{DG}} + \gamma \, |\rho_p|_{\bigcup_n \mathcal{S}_I^n} |\theta_p|_{\bigcup_n \mathcal{S}_I^n} + \bar{w}_m \|\!|\rho_u|\!\|_{\text{DG}} \|\!|\theta_u|\!\|_{\text{DG}}$$

$$+ \left(\sum_{S \in \mathcal{F}_{\text{int}}} \left\|C_S^{1/2} |\{\!\{\rho_u\}\!\}|\right\|_{0,0,S}^2\right)^{1/2} \left(\sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S}^2\right)^{1/2}$$

$$+ \left(\sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\rho_u]\!]|\right\|_{0,0,S}^2\right)^{1/2} \left(\sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S}^2\right)^{1/2}. \tag{98}$$

Applying the Schwarz' inequality and the arithmetic-geometric mean inequality finally yields:

$$\left|\tilde{\mathcal{A}}(\rho_u, \rho_p; \theta_u, \theta_p)\right| \leqslant \frac{3}{4} \beta \|\!|\theta_u|\!\|_{\text{DG}}^2 + \frac{1}{\beta} (\nu_m^2 \alpha_\mathcal{A}^2 + \bar{w}_m^2) \|\!|\rho_u|\!\|_{\text{DG}}^2 + \frac{1}{\beta} \alpha_\mathcal{B}^2 \|\rho_p\|_{0,0,\mathcal{E}}^2 + \frac{\alpha_\mathcal{B}^2}{C_{\text{inf}}} \|\!|\rho_u|\!\|_{\text{DG}}^2$$

$$+ \frac{1}{\beta} \alpha_\mathcal{B}^2 \|\!|\rho_u|\!\|_{\text{DG}}^2 + \frac{2}{4} \beta |\theta_p|_{\bigcup_n \mathcal{S}_I^n}^2 + \frac{1}{\beta} \gamma^2 |\rho_p|_{\bigcup_n \mathcal{S}_I^n}^2 + \frac{2}{4} \beta \sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\theta_u]\!]|\right\|_{0,0,S}^2$$

$$+ \frac{1}{\beta} \sum_{S \in \mathcal{F}_{\text{int}}} \left\|C_S^{1/2} |\{\!\{\rho_u\}\!\}|\right\|_{0,0,S}^2 + \frac{1}{\beta} \sum_{S \in \mathcal{F}} \left\|C_S^{1/2} |[\![\rho_u]\!]|\right\|_{0,0,S}^2. \tag{99}$$

Combining (99) with Lemma 5.7 for $(v, q) = (\theta_u, \theta_p)$, taking $\beta = C_{\tilde{\mathcal{A}}}$, and multiplying the result with $C_{\tilde{\mathcal{A}}}$ completes the proof. $\quad\square$

We now present upper bounds for the interpolation error (ρ_u, ρ_p), which are taken from [19].

Lemma 5.12. *Assume that $\mathcal{K}$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_{\mathcal{K}}, F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \hat{\mathcal{K}} \to \tilde{\mathcal{K}}$ and $Q_{\mathcal{K}} : \tilde{\mathcal{K}} \to \mathcal{K}$. Assume also that $h_{i,\mathcal{K}}, i = 1, \ldots, d$ is the edge length of $\tilde{\mathcal{K}}$ in the x_i direction, and $\Delta_n t$ the edge length in the x_0 direction (see illustration in Fig. 1 for $d = 2$). Let $u|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}^u+1, k_{s,\mathcal{K}}^u+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}^u, k_{s,\mathcal{K}}^u \geqslant 0$, and $p|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}^p+1, k_{s,\mathcal{K}}^p+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}^p, k_{s,\mathcal{K}}^p \geqslant 0$. Let $\mathcal{P}_u$ denote the L^2 projection of u onto the finite element space $V_h^{(p_t, p_s)}$, and $\mathcal{P}_p$ the L^2 projection of p onto the finite element space $\mathcal{Q}_h^{(p_t, p_s)}$. Then the projection error ρ_u obeys the error bounds:*

$$\|\rho_u\|_{0,0,\mathcal{K}}^2 \leqslant C Z_{\mathcal{K}}^u, \qquad \|\bar{\nabla}_h \rho_u\|_{0,0,\mathcal{K}}^2 \leqslant C N_{\mathcal{K}}^u, \qquad \|\rho_u\|_{0,0,\partial\mathcal{K}}^2 \leqslant C(A_{\mathcal{K}}^u + B_{\mathcal{K}}^u),$$

where

$$Z_{\mathcal{K}}^u = \sum_{i=1}^d \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2s_{\mathcal{K}}^u} \|\partial_i^{s_{\mathcal{K}}^u} u\|_{0,0,\tilde{\mathcal{K}}}^2 + \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^u}\right)^{2s_{0,\mathcal{K}}^u} \|\partial_0^{s_{0,\mathcal{K}}^u} u\|_{0,0,\tilde{\mathcal{K}}}^2,$$

$$N_{\mathcal{K}}^u = \sum_{i=1}^d \frac{h_{i,\mathcal{K}}^{2t_{\mathcal{K}}^u}}{(p_{s,\mathcal{K}}^u)^{2t_{\mathcal{K}}^u - 1}} \|\partial_i^{t_{\mathcal{K}}^u+1} u\|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^d \sum_{j \neq i} \frac{h_{j,\mathcal{K}}^{2t_{\mathcal{K}}^u+2}}{(p_{s,\mathcal{K}}^u)^{2t_{\mathcal{K}}^u}} \|\partial_j^{t_{\mathcal{K}}^u+1} \partial_i u\|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \sum_{i=1}^d \frac{(\Delta_n t)^{2t_{0,\mathcal{K}}^u+2}}{(p_{t,\mathcal{K}}^u)^{2t_{0,\mathcal{K}}^u}} \|\partial_0^{t_{0,\mathcal{K}}^u+1} \partial_i u\|_{0,0,\tilde{\mathcal{K}}}^2,$$

$$A_{\mathcal{K}}^u = \sum_{i=1}^d \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2t_{\mathcal{K}}^u+1} \|\partial_i^{t_{\mathcal{K}}^u+1} u\|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^d \sum_{j \neq i} \frac{1}{h_{i,\mathcal{K}}} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2s_{\mathcal{K}}^u} \|\partial_j^{s_{\mathcal{K}}^u} u\|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \sum_{i=1}^d \sum_{j \neq i} \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2q_{\mathcal{K}}^u} \|\partial_j^{q_{\mathcal{K}}^u} \partial_i u\|_{0,0,\tilde{\mathcal{K}}}^2,$$

$$B_{\mathcal{K}}^u = \sum_{i=1}^d \frac{1}{h_{i,\mathcal{K}}} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^u}\right)^{2s_{0,\mathcal{K}}^u} \|\partial_0^{s_{0,\mathcal{K}}^u} u\|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^d \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^u}\right)^{2q_{0,\mathcal{K}}^u} \|\partial_0^{q_{0,\mathcal{K}}^u} \partial_i u\|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^u}\right)^{2t_{0,\mathcal{K}}^u+1} \|\partial_0^{t_{0,\mathcal{K}}^u+1} u\|_{0,0,\tilde{\mathcal{K}}}^2 + \frac{1}{\Delta_n t} \sum_{i=1}^d \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2s_{\mathcal{K}}^u} \|\partial_i^{s_{\mathcal{K}}^u} u\|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \frac{\Delta_n t}{p_{t,\mathcal{K}}^u} \sum_{i=1}^d \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2q_{\mathcal{K}}^u} \|\partial_i^{q_{\mathcal{K}}^u} \partial_0 u\|_{0,0,\tilde{\mathcal{K}}}^2,$$

with $p_{t,\mathcal{K}}^u$ and $p_{s,\mathcal{K}}^u$ the local polynomial degree in time and space for u, respectively, on element $\mathcal{K}$, $0 < s_{0,\mathcal{K}}^u \leqslant \min(p_{t,\mathcal{K}}^u + 1, k_{t,\mathcal{K}}^u + 1)$, $0 < s_{\mathcal{K}}^u \leqslant \min(p_{s,\mathcal{K}}^u + 1, k_{s,\mathcal{K}}^u + 1)$, $0 < q_{0,\mathcal{K}}^u \leqslant \min(p_{t,\mathcal{K}}^u + 1, k_{t,\mathcal{K}}^u)$, $0 < q_{\mathcal{K}}^u \leqslant \min(p_{s,\mathcal{K}}^u + 1, k_{s,\mathcal{K}}^u)$, $0 < t_{0,\mathcal{K}}^u \leqslant \min(p_{t,\mathcal{K}}^u, k_{t,\mathcal{K}}^u)$, and $0 < t_{\mathcal{K}}^u \leqslant \min(p_{s,\mathcal{K}}^u, k_{s,\mathcal{K}}^u)$. Further we have:

$$\sum_{S \in \bigcup_n \mathcal{S}_{ID}^n} \|\bar{\mathcal{L}}_S(\llbracket \rho_u \rrbracket)\|_{0,0,\mathcal{E}}^2 \leqslant C \sum_{\mathcal{K} \in \mathcal{T}_h} (R_{\mathcal{K}}^u + T_{\mathcal{K}}^u),$$

where:

$$R_{\mathcal{K}}^u = \sum_{i=1}^d \frac{(p_{s,\mathcal{K}}^u)^2}{h_{i,\mathcal{K}}} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2t_{\mathcal{K}}^u+1} \|\partial_i^{t_{\mathcal{K}}^u+1} u\|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^d \sum_{j \neq i} \left(\frac{p_{s,\mathcal{K}}^u}{h_{i,\mathcal{K}}}\right)^2 \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}^u}\right)^{2s_{\mathcal{K}}^u} \|\partial_j^{s_{\mathcal{K}}^u} u\|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \sum_{i=1}^{d} \sum_{j \neq i} p_{s,\mathcal{K}}^u \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}^u} \right)^{2q_{\mathcal{K}}^u} \| \partial_j^{q_{\mathcal{K}}^u} \partial_i u \|_{0,0,\tilde{\mathcal{K}}}^2,$$

$$T_{\mathcal{K}}^u = \sum_{i=1}^{d} \frac{(p_{s,\mathcal{K}}^u)^2}{h_{i,\mathcal{K}} \Delta_n t} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^u} \right)^{2s_{0,\mathcal{K}}^u} \| \partial_0^{s_{0,\mathcal{K}}^u} u \|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^{d} p_{s,\mathcal{K}}^u \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^u} \right)^{2q_{0,\mathcal{K}}^u} \| \partial_0^{q_{0,\mathcal{K}}^u} \partial_i u \|_{0,0,\tilde{\mathcal{K}}}^2.$$

The projection error ρ_p obeys the error bounds:

$$\| \rho_p \|_{0,0,\mathcal{K}}^2 \leqslant C Z_{\mathcal{K}}^p, \qquad \| \rho_p \|_{0,0,\partial\mathcal{K}}^2 \leqslant C(A_{\mathcal{K}}^p + B_{\mathcal{K}}^p),$$

where

$$Z_{\mathcal{K}}^p = \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \right)^{2s_{\mathcal{K}}^p} \| \partial_i^{s_{\mathcal{K}}^p} p \|_{0,0,\tilde{\mathcal{K}}}^2 + \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^p} \right)^{2s_{0,\mathcal{K}}^p} \| \partial_0^{s_{0,\mathcal{K}}^p} p \|_{0,0,\tilde{\mathcal{K}}}^2,$$

$$A_{\mathcal{K}}^p = \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \right)^{2t_{\mathcal{K}}^p+1} \| \partial_i^{t_{\mathcal{K}}^p+1} p \|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^{d} \sum_{j \neq i} \frac{1}{h_{i,\mathcal{K}}} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \right)^{2s_{\mathcal{K}}^p} \| \partial_j^{s_{\mathcal{K}}^p} p \|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \sum_{i=1}^{d} \sum_{j \neq i} \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \left(\frac{h_{j,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \right)^{2q_{\mathcal{K}}^p} \| \partial_j^{q_{\mathcal{K}}^p} \partial_i p \|_{0,0,\tilde{\mathcal{K}}}^2,$$

$$B_{\mathcal{K}}^p = \sum_{i=1}^{d} \frac{1}{h_{i,\mathcal{K}}} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^p} \right)^{2s_{0,\mathcal{K}}^p} \| \partial_0^{s_{0,\mathcal{K}}^p} p \|_{0,0,\tilde{\mathcal{K}}}^2 + \sum_{i=1}^{d} \frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^p} \right)^{2q_{0,\mathcal{K}}^p} \| \partial_0^{q_{0,\mathcal{K}}^p} \partial_i p \|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \left(\frac{\Delta_n t}{p_{t,\mathcal{K}}^p} \right)^{2t_{0,\mathcal{K}}^p+1} \| \partial_0^{t_{0,\mathcal{K}}^p+1} p \|_{0,0,\tilde{\mathcal{K}}}^2 + \frac{1}{\Delta_n t} \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \right)^{2s_{\mathcal{K}}^p} \| \partial_i^{s_{\mathcal{K}}^p} p \|_{0,0,\tilde{\mathcal{K}}}^2$$

$$+ \frac{\Delta_n t}{p_{t,\mathcal{K}}^p} \sum_{i=1}^{d} \left(\frac{h_{i,\mathcal{K}}}{p_{s,\mathcal{K}}^p} \right)^{2q_{\mathcal{K}}^p} \| \partial_i^{q_{\mathcal{K}}^p} \partial_0 p \|_{0,0,\tilde{\mathcal{K}}}^2,$$

with $p_{t,\mathcal{K}}^p$ and $p_{s,\mathcal{K}}^p$ the local polynomial degree in time and space for p, respectively, on element $\mathcal{K}$, $0 < s_{0,\mathcal{K}}^p \leqslant \min(p_{t,\mathcal{K}}^p + 1, k_{t,\mathcal{K}}^p + 1)$, $0 < s_{\mathcal{K}}^p \leqslant \min(p_{s,\mathcal{K}}^p + 1, k_{s,\mathcal{K}}^p + 1)$, $0 < q_{0,\mathcal{K}}^p \leqslant \min(p_{t,\mathcal{K}}^p + 1, k_{t,\mathcal{K}}^p)$, $0 < q_{\mathcal{K}}^p \leqslant \min(p_{s,\mathcal{K}}^p + 1, k_{s,\mathcal{K}}^p)$, $0 < t_{0,\mathcal{K}}^p \leqslant \min(p_{t,\mathcal{K}}^p, k_{t,\mathcal{K}}^p)$, and $0 < t_{\mathcal{K}}^p \leqslant \min(p_{s,\mathcal{K}}^p, k_{s,\mathcal{K}}^p)$. The constant C has a positive value that depends only on the spatial dimension d and the mapping $Q_{\mathcal{K}}$.

Using Lemmas 5.11 and 5.12, we immediately obtain the following error estimates:

Theorem 5.13. *Suppose $\mathcal{K}$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $Q_{\mathcal{K}}, F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \hat{\mathcal{K}} \to \tilde{\mathcal{K}}$ and $Q_{\mathcal{K}} : \tilde{\mathcal{K}} \to \mathcal{K}$. Suppose also that $h_{i,\mathcal{K}}, i = 1, \dots, d$ is the edge length of $\tilde{\mathcal{K}}$ in the x_i direction, and $\Delta_n t$ the edge length in the x_0 direction. Let $u|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}^u+1, k_{s,\mathcal{K}}^u+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}^u, k_{s,\mathcal{K}}^u \geqslant 0$, and $p|_{\mathcal{K}} \in H^{(k_{t,\mathcal{K}}^p+1, k_{s,\mathcal{K}}^p+1)}(\mathcal{K})$, with $k_{t,\mathcal{K}}^p, k_{s,\mathcal{K}}^p \geqslant 0$. Let $(u_h, p_h) \in V_h^{(p_t, p_s)} \times \mathcal{Q}_h^{(p_t, p_s)}/\mathbb{R}$ be the discontinuous Galerkin approximation to (u, p) in (47)–(48). Then the following error bound holds*:

$$\left\| | (u - u_h, p - p_h) | \right\|_{\mathrm{DG}}^2 \leqslant 2C(1 + a_1) \sum_{\mathcal{K} \in \mathcal{T}_h} (Z_{\mathcal{K}}^u + N_{\mathcal{K}}^u + R_{\mathcal{K}}^u + T_{\mathcal{K}}^u) + 2C\bar{C}_S(1 + a_4) \sum_{\mathcal{K} \in \mathcal{T}_h} (A_{\mathcal{K}}^u + B_{\mathcal{K}}^u)$$

$$+ 2Ca_2 \sum_{\mathcal{K} \in \mathcal{T}_h} Z_{\mathcal{K}}^p + 2C(1 + a_3) \sum_{\mathcal{K} \in \mathcal{T}_h} (A_{\mathcal{K}}^p + B_{\mathcal{K}}^p),$$

with $\bar{C}_S = \max_{\mathcal{K} \in \mathcal{T}_h} \| C_S \|_{L^\infty(\mathcal{K})}$ and

$$a_1 = \frac{4}{C_{\tilde{\mathcal{A}}}^2} \left(v_m^2 \alpha_{\mathcal{A}}^2 + \bar{w}_m^2 + (1 + C_{\tilde{\mathcal{A}}}/C_{\inf})\alpha_{\mathcal{B}}^2 \right), \quad a_2 = \frac{4\alpha_{\mathcal{B}}^2}{C_{\tilde{\mathcal{A}}}^2}, \quad a_3 = \frac{4\gamma^2}{C_{\tilde{\mathcal{A}}}^2}, \quad a_4 = \frac{8}{C_{\tilde{\mathcal{A}}}^2}.$$

Table 1
L^2-norm of the error for u and p for different values of stabilization parameter γ

(p_t, p_s)		$\gamma = 10$	$\gamma = 100$	$\gamma = 1000$	$\gamma = 10000$
$(1, 2)$	u	$8.7806\mathrm{E}-05$	$8.7833\mathrm{E}-05$	$8.7836\mathrm{E}-05$	$8.7836\mathrm{E}-05$
	p	$5.2186\mathrm{E}-03$	$5.2162\mathrm{E}-03$	$5.2159\mathrm{E}-03$	$5.2159\mathrm{E}-03$
$(2, 2)$	u	$8.8268\mathrm{E}-05$	$8.8297\mathrm{E}-05$	$8.8300\mathrm{E}-05$	$8.8300\mathrm{E}-05$
	p	$5.5157\mathrm{E}-03$	$5.5136\mathrm{E}-03$	$5.5133\mathrm{E}-03$	$5.5133\mathrm{E}-03$
$(1, 3)$	u	$1.6869\mathrm{E}-06$	$1.6869\mathrm{E}-06$	$1.6869\mathrm{E}-06$	$1.6869\mathrm{E}-06$
	p	$1.6472\mathrm{E}-04$	$1.6471\mathrm{E}-04$	$1.6471\mathrm{E}-04$	$1.6471\mathrm{E}-04$
$(3, 3)$	u	$1.1899\mathrm{E}-06$	$1.1899\mathrm{E}-06$	$1.1899\mathrm{E}-06$	$1.1899\mathrm{E}-06$
	p	$5.4106\mathrm{E}-05$	$5.4096\mathrm{E}-05$	$5.4095\mathrm{E}-05$	$5.4099\mathrm{E}-05$

Proof. The estimate follows directly from taking the DG norm of both sides of (91) and using the triangle inequality to express the error in terms of the interpolation error (ρ_u, ρ_p) and the discretization error (θ_u, θ_p). The final error estimate is obtained using Lemma 5.11 and the interpolation estimates given by Lemma 5.12.   □

Corollary 5.14. *When u and p are sufficiently smooth, the spatial shapes of all elements $\mathcal{K} \in \mathcal{T}_h$ are regular: $h = \max_{\mathcal{K}} h_{\mathcal{K}}, \forall \mathcal{K} \in \mathcal{T}_h$, and uniform polynomial degrees (p_t^u, p_s^u) and (p_t^p, p_s^p) are used for all elements $\mathcal{K} \in \mathcal{T}_h$, then we obtain the error bound*:

$$
\begin{aligned}
\left\| \left\| (u - u_h, p - p_h) \right\| \right\|_{\mathrm{DG}}^2 \leqslant\ & 2C\Bigg((1 + a_1)\left(\frac{h^{2p_s^u}}{(p_s^u)^{2p_s^u - 1}} + \frac{(p_s^u)^2}{h} \frac{\Delta_n t^{2p_t^u + 1}}{(p_t^u)^{2p_t^u + 2}} \right) \\
& + \bar{C}_S(1 + a_4)\left(\frac{h^{2p_s^u + 1}}{(p_s^u)^{2p_s^u + 1}} + \frac{\Delta_n t^{2p_t^u + 1}}{(p_t^u)^{2p_t^u + 1}} \right) \Bigg) |u|_{p_t^u + 1, p_s^u + 1, \mathcal{E}}^2 \\
& + 2C\Bigg(a_2\left(\frac{h^{2p_s^p + 2}}{(p_s^p)^{2p_s^p + 2}} + \frac{\Delta_n t^{2p_t^p + 2}}{(p_t^p)^{2p_t^p + 2}} \right) \\
& + (1 + a_3)\left(\frac{h^{2p_s^p + 1}}{(p_s^p)^{2p_s^p + 1}} + \frac{\Delta_n t^{2p_t^p + 1}}{(p_t^p)^{2p_t^p + 1}} \right) \Bigg) |p|_{p_t^p + 1, p_s^p + 1, \mathcal{E}}^2.
\end{aligned}
$$

6. Numerical results

In this section we provide several numerical experiments in two spatial dimensions to investigate the order of accuracy of the space–time DG discretization given by (47)–(48). As exact solution in the tests, we choose the following functions:

$$u_1(t, x_1, x_2) = -\exp(x_1)\big(x_2 \cos(x_2) + \sin(x_2)\big) \exp(-t),$$
$$u_2(t, x_1, x_2) = \exp(x_1) x_2 \sin(x_2) \exp(-t),$$
$$p(t, x_1, x_2) = 2\exp(x_1) \sin(x_2) \exp(-t),$$

which solve the Oseen equations (1) with $\bar{w} = u$, a suitably chosen source vector f and initial condition u_0. The computational domain is taken to be $(-1, 1)^2$ and Dirichlet boundary conditions are imposed on the boundary.

We first study the influence of the choice of the stabilization parameter γ in the bilinear form $\mathcal{C}_h$, defined in (45), on the accuracy of the DG solution. We conduct therefore simulations for different values of γ on a mesh with 8×8 elements and different polynomial degrees, both in space and time. The results are shown in Table 1. For each simulation, the polynomial degrees for p are taken the same as for u. The results show that the choice of the stabilization parameter γ does not influence the accuracy of u and p. The parameter γ does have, however, a significant influence on the conditioning of the discretization matrix. Larger values of γ gives a better conditioning of the matrix.

Next, we study the order of accuracy of the velocity field u and the pressure p on meshes with different mesh sizes and increasing polynomial degrees. Here we use the stabilization term $\mathcal{C}_h$ with $\gamma = 10\,000$. We first study the error
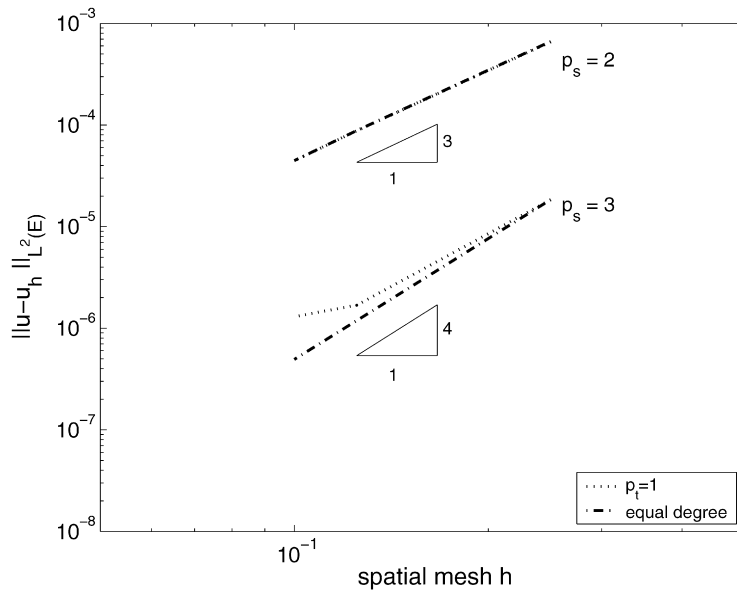
Fig. 2. $L^2(\mathcal{E})$ error for the velocity u in the space–time DG discretization of the Oseen equations under h-refinement and for different polynomial degrees.
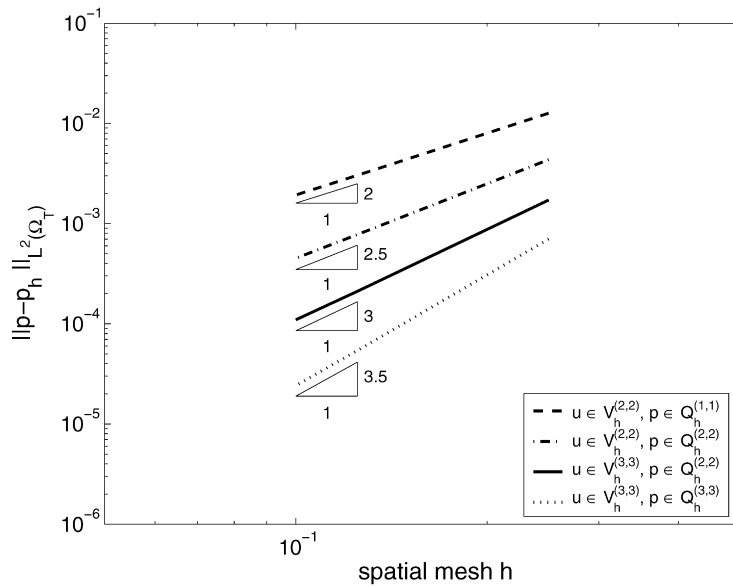

Fig. 3. $L^2(\Omega_T)$ error for the pressure p in the space–time DG discretization of the Oseen equations under h-refinement and for different polynomial degrees.

in the L^2-norm in the whole space–time domain $\mathcal{E}$ for the velocity field u. The results are shown in Fig. 2. The plots show that the rate of convergence of the space–time DG method for the velocity field is optimal in the L^2-norm. Using linear polynomials in time and higher polynomial degrees in space we observe that, as the mesh becomes finer, the error is dominated by the error in time, but this only happens, however, when the spatial error is already very small.

We also consider the L^2-norm of error for the pressure p in Ω_T, the domain at the final time T of the simulation, both when equal polynomial degrees for u and p are used and also for different polynomial degrees. The results are shown in Fig. 3. We observe that when equal polynomial degrees are used for u and p then the L^2-norm of the error in

the pressure converges at the rate $h^{p_s+0.5}$, with p_s the polynomial degree of the pressure, while when the polynomial degrees for p are one less than the polynomial degrees for u, then the pressure converges at the rate h^{p_s+1}.

7. Concluding remarks

In this article we present a space–time DG discretization for the Oseen equations in time-dependent flow domains. We prove the continuity, coercivity and stability of the method, which is shown to be unconditionally stable, and investigate the effect of the pressure stabilization operator on the stability. In addition, a full *hp*-error analysis is presented where the dependence of the error on mesh size, time step and polynomial order is made explicit.

The numerical experiments show that the convergence rate of the space–time DG solution for the velocity field is optimal in the L^2-norm, while the pressure converges at the rate $h^{p_s+0.5}$, with p_s the polynomial degree of the pressure, when equal polynomial degrees of the velocity and pressure are used and h^{p_s+1} when the polynomial degree of the velocity is larger than for the pressure. The simulations show that the algorithm also performs well for higher polynomial degrees in time.

The extension of the space–time DG discretization discussed in this article to the incompressible Navier–Stokes equations can be accomplished using a Picard iteration, in which the prescribed velocity field w in (1a) is replaced by the computed velocity field u from the previous iteration. The only main addition to algorithm then is the use of a projection operator which ensures that the discrete velocity field is divergence free. Also, the application of the presented algorithm to free boundary problems is under investigation.

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