Asymptotic Preserving Methods

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We here consider

- Model: a multiscale linear kinetic transport model
- Numerical methods: asymptotic preserving (AP) methods based on
  - discontinuous Galerkin (DG) spatial discretizations
  - implicit-explicit Runge-Kutta time discretizations

For AP methods solving other models or based on other discretizations, see review papers, *Pareschi-Russo 2011, Jin 2012, Degond 2013*

Outline

- Introduction: *problem setup, AP methods, DG methods*
- DG based AP methods:
  - Formulation
  - Theoretical results
  - Numerical examples
  - Other developments
- Concluding remarks

Problem Setup

Consider the dynamics of some particles (*e.g. neutron or photon*) in one dimension.

- Free-streaming, with $v_0 > 0$

$$\partial_t \alpha + v_0 \partial_x \alpha = 0$$

$$\partial_t \beta - v_0 \partial_x \beta = 0$$

- Also, the interaction with the medium ($\sigma > 0$: scattering constant)

$$\partial_t \alpha + v_0 \partial_x \alpha = \frac{\sigma}{2}(\beta - \alpha) \tag{1a}$$

$$\partial_t \beta - v_0 \partial_x \beta = \underbrace{\frac{\sigma}{2}(\alpha - \beta)}_{=\sigma\left(\frac{\alpha+\beta}{2} - \beta\right)} \tag{1b}$$

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$$\partial_t \beta - v_0 \partial_x \beta = \underbrace{\frac{\sigma}{2}(\alpha - \beta)}_{=\sigma\left(\frac{\alpha+\beta}{2} - \beta\right)} \tag{1b}$$

Introduce a new function $f(x, v, t)$, with $v \in \{v_0, -v_0\}$, satisfying

$$f(x, v = v_0, t) := \alpha(x, t), \quad f(x, v = -v_0, t) := \beta(x, t),$$

and define $\langle f \rangle = \langle f \rangle(x, t)$ as

$$\begin{aligned} \langle f \rangle &= \frac{1}{2}\Big(\alpha(x, t) + \beta(x, t)\Big) \\ &= \frac{1}{2}\Big(f(x, v = v_0, t) + f(x, v = -v_0, t)\Big). \end{aligned}$$

Then the system for the particle dynamics[1] can be rewritten as

$$\underbrace{\partial_t f + v\partial_x f}_{transport} = \underbrace{\sigma\Big(\langle f \rangle - f\Big).}_{scattering} \tag{2}$$

[1] *Stochastically, the scattering process can be described by a 2-state continuous time Markov chain.*

Dimensionless Form

$$\partial_t f + v\partial_x f = \sigma\big(\langle f\rangle - f\big)$$

- Non-dimensional variables: with
  - $1/\sigma$: mean free time
  - v_0: the typical velocity
  - $1/\sigma \cdot v_0$: mean free path

we set

$$\tilde{t} = \frac{t}{1/\sigma}, \quad \tilde{v} = \frac{v}{v_0}, \quad \tilde{x} = \frac{x}{1/\sigma \cdot v_0}$$

Diffusive Scaling

- Define the dimensionless Knudsen number

$$\varepsilon = \frac{\text{mean-free path}}{\text{characteristic length of the problem}} = \frac{1/\sigma \cdot v_0}{L},$$

where the spatial domain is $[0, L]$.

When $\varepsilon \ll 1$, the problem is diffusive over long time. To see this, we apply a diffusive scaling:

$$\hat{x} = \frac{\tilde{x}}{1/\varepsilon}, \quad \hat{t} = \frac{\tilde{t}}{1/\varepsilon^2},$$

and this leads to

$$\varepsilon \partial_t f + v \partial_x f = \frac{1}{\varepsilon} \left(\langle f \rangle - f \right), \quad x \in [0, 1], \; v \in \{-1, 1\}. \qquad (3)$$

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Diffusive Behavior as $\varepsilon \ll 1$

To check the diffusive behavior, let

$$\rho = \frac{\alpha + \beta}{2} = \langle f \rangle, \quad J = \frac{\alpha - \beta}{2\varepsilon} = \frac{1}{\varepsilon}\langle v f \rangle, \tag{4}$$

the equation under the diffusive scaling becomes

$$\partial_t \rho + \partial_x J = 0, \tag{5a}$$

$$\varepsilon^2 \partial_t J + \partial_x \rho = -J. \tag{5b}$$

When $\varepsilon \ll 1$,

$$\begin{aligned} (5b) &\Rightarrow J = -\partial_x \rho + O(\varepsilon^2) \\ &\Rightarrow \partial_t \rho = \partial_{xx} \rho + O(\varepsilon^2) \end{aligned}$$

Model: a more general form

Linear kinetic transport equation under the diffusive scaling:

$$\varepsilon\partial_t f + v\partial_x f = \frac{\sigma(x)}{\varepsilon}\mathcal{L}(f)$$

- $f = f(x, v, t)$: the density distribution function of particles, depending on position $x \in \Omega_x$, velocity $v \in \Omega_v$, time t, with both Ω_x, Ω_v being bounded
- $\sigma(x)$: the scaled scattering function
- $\mathcal{L}(f) = \langle f\rangle - f$: the normalized scattering operator, where $\langle f\rangle = \int_{\Omega_v} f d\mu$, with μ being a measure associated with Ω_v, satisfying $\int_{\Omega_v} 1 d\mu$=1.
- The operator $\mathcal{L}$ acts only on v, and $\text{Null}(\mathcal{L}) = \{f : f = \langle f\rangle\}$=Span$\{1\}$.
- $\varepsilon > 0$: the Knudsen number, defined as the ratio of the mean free path and the characteristic length

Example 1 (Telegraph or Goldstein-Taylor equation). $\Omega_v = \{-1, 1\}$. μ is a discrete measure, with $\langle f \rangle = \frac{1}{2}\Big(f(x, v = 1, t) + f(x, v = -1, t)\Big)$, and $\sigma(x) = 1$

Example 2 (One-group transport equation in slab geometry $\Omega_v = [-1, 1]$, and $d\mu = \frac{1}{2}dv$, with dv being Lebesgue measure.

A more general form of Example 2 is[2]

$$\varepsilon \partial_t f + v \partial_x f = \frac{\sigma_S(x)}{\varepsilon} \mathcal{L}(f) - \varepsilon \sigma_a(x) f + \varepsilon S(x) \tag{6}$$

[2] *The one-dimensional problem* (6) *is a simplified model for physically more relevant kinetic transport models in neutron transport theory, radiative transfer theory etc.*

- Diffusive behavior as $\varepsilon \ll 1$: *As* $\varepsilon \to 0$, *the limiting equation*[3] *is*

$$\partial_t \rho = \kappa \partial_x (\sigma^{-1}(x) \partial_x \rho). \tag{7}$$

  Here $\rho = \langle f \rangle$, *with* $\kappa = \langle v^2 \rangle$: $\kappa = 1$ *(example 1)*, $\kappa = 1/3$ *(example 2)*

- Initial and boundary layers: *non well-prepared initial, anisotropic boundary data*

[3] *by perturbation method; Kurtz 1973, Bensoussan-Lions-Papanicolaou 1979*

Asymptotic Preserving (AP) Method

◇ **On the continuous level:**

$$\mathcal{F}^{\varepsilon} : \varepsilon \partial_t f + v \partial_x f = \frac{1}{\varepsilon}\mathcal{L}(f); \qquad \mathcal{F}^{0} : \partial_t \rho = \kappa \partial_{xx} \rho$$

◇ **On the discrete level:** let $\mathcal{F}_{\delta}^{\varepsilon}$ be a numerical method for $\mathcal{F}^{\varepsilon}$, with δ being some discretization parameters such as Δx, Δt.

$$\begin{array}{ccc} \mathcal{F}^{\varepsilon} & \xrightarrow{\varepsilon \to 0} & \mathcal{F}^{0} \\ (\delta \to 0) \uparrow & & \uparrow (\delta \to 0) \\ \mathcal{F}_{\delta}^{\varepsilon} & \xrightarrow{\varepsilon \to 0} & \mathcal{F}_{\delta}^{0} \end{array}$$

The method $\mathcal{F}_{\delta}^{\varepsilon}$ is said to be *asymptotic preserving* if it preserves the asymptotic limit at the discrete level. That is, as $\varepsilon \to 0$, the scheme $\mathcal{F}_{\delta}^{\varepsilon}$ becomes a consistent and stable numerical method $\mathcal{F}_{\delta}^{0}$ for the limiting equation $\mathcal{F}^{0}$ on under-resolved meshes, with $\delta \gg \varepsilon$.

$\Rightarrow$ Uniform convergence in ε in the diffusive regime as $\delta \to 0$ *(Golse-Jin-Levermore 1999)*

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Why AP?

Marvin L. Adams, *Discontinuous Finite Element Transport Solutions in Thick Diffusive Problems*, Nucl. Sci. Eng., 137 (2001)

"Regions that are extremely optically thick and diffusive often appear in radiative-transfer problems of practical interest. Computational limitations force the use of spatial grids whose cells are often quite thick compared to a mean-free-path, especially in low-energy groups. ...

An obvious question is why the community does not simply solve the diffusion equation in thick diffusive regions. There are several reasons. First, a typical radiative transfer problem has a wide range of material properties. In a given energy group, some spatial regions may be quite thick and others quite thin. Further, in a given spatial region, the material may be quite thick and diffusive to photons in some energy groups but thin to photons in other groups."

"Second, if part of the problem is to be solved with diffusion theory and part with transport, then the problem must be so divided. If the division is made such that diffusion theory is applied to regions that are not very thick and diffusive, then errors will result. If, on the other hand, the division leaves some relatively thick and diffusive regions for the transport equation, then the transport solution cannot be trusted unless it is known to behave well in the thick diffusive limit."

Objective: Design and mathematically understand AP methods, particularly within the discontinuous Galerkin (DG) framework

- AP: *As* $\varepsilon \to 0$, *the scheme in the limit is consistent and stable for the limiting equation on fixed meshes.*
- Uniformly stable with respect to ε ranging from $O(1)$ to 0
- High order in space and time: $\varepsilon = O(1)$, $\varepsilon \ll 1$
- Easy to solve numerically

Review of Discontinuous Galerkin (DG) Method[4]

Model equation:

$$\partial_t u + \partial_x (cu) = 0, \quad x \in (0,1), \quad t > 0$$

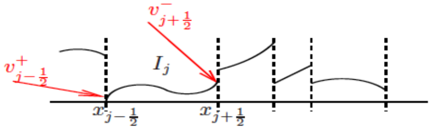
Weak form:

$$\sum_j \Big(\int_{I_j} \big(\partial_t uw - cu\partial_x w\big)dx + cu(x_{j+\frac{1}{2}})w(x_{j+\frac{1}{2}}) - cu(x_{j-\frac{1}{2}})w(x_{j-\frac{1}{2}})\Big) = 0 \quad (8)$$

Semi-discrete DG method: look for $u_h \in U_h$, such that $\forall w \in U_h$

$$\sum_j \Big(\int_{I_j} \big(\partial_t u_h w - cu_h\partial_x w\big)dx + \widehat{(cu_h)}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \widehat{(cu_h)}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}}\Big) = 0 \quad (9)$$

- Mesh: $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $j = 1, 2, \cdots$; Define $h = \max_j |I_j|$
- Discrete space: $U_h = U_h^k = \{w : w \in P^k(I_j), \ \forall j\}$, with $P^k(I_j)$ consisting of polynomials of degree up to k;
- Numerical flux: $\widehat{cu_h}$



[4] *Reed-Hill 1973, Cockburn, Shu et al 1989-1998*

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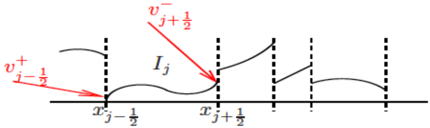

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Numerical flux $\widehat{cu_h}$: *Its choice may affect numerical stability, accuracy, and some implementation aspects of the method.*

- Upwind flux

$$\widehat{cu_h} := \begin{cases} cu_h^-, & c > 0 \\ cu_h^+, & c < 0 \end{cases}$$

Stability: $\int_0^1 |u_h(x,t))|^2 dx + |c| \sum_j (u^+_{h,j+\frac{1}{2}} - u^-_{h,j+\frac{1}{2}})^2 = \int_0^1 |u_h(x,0))|^2 dx$

Accuracy: $O(h^{k+1})$

- Central flux:

$$\widehat{cu_h} := \frac{cu_h^- + cu_h^+}{2}$$

Stability: $\int_0^1 |u_h(x,t))|^2 dx = \int_0^1 |u_h(x,0))|^2 dx$

Accuracy: $O(h^{k+1})$ (for even k), $O(h^k)$ (for odd k)

A viewpoint of DG discretization: discrete derivative

◇ Continuous problem:

$$\sum_j \int_{I_j} \Big(\partial_t u + \partial_x(cu)\Big)\, w dx = 0 \tag{10}$$

◇ Semi-discrete DG method: (with the upwind flux $\widehat{cu} = cu^-, \quad c > 0$)

$$\sum_j \Big(\int_{I_j} \partial_t u_h w dx - \int_{I_j} c u_h \partial_x w dx + \widehat{(cu_h)}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - \widehat{(cu_h)}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} \Big) = 0$$

$$\Leftrightarrow \sum_j \int_{I_j} (\partial_t u_h + \partial_x(cu_h)) w dx + \underbrace{\sum_j c\Big((u_h^+ - u_h^-)w^+\Big)_{j-\frac{1}{2}}}_{=\sum_j \int_{I_j} \mathcal{D}_h^B(u_h;c) w dx \ \text{(lifting operator)}} = 0$$

$$\Leftrightarrow \sum_j \int_{I_j} \Big(\partial_t u_h + \mathcal{D}_h(u_h;c)\Big) w dx = 0 \tag{11}$$

$\mathcal{D}_h(u_h;c) = \partial_x(cu_h) + \mathcal{D}_h^B(u_h;c) \in U_h$: discrete spatial derivative of cu, depending on the discrete space U_h and the numerical flux $\widehat{cu}$

AP Property of Upwind DG Method: an example

- Model: telegraph equation, with $\Omega_v = \{-1, 1\}$ and $\Omega_x = [-\pi, \pi]$

$$\varepsilon \partial_t f + v \partial_x f = \frac{1}{\varepsilon}(\langle f \rangle - f)$$

- Exact solution: $r = \frac{-2}{1+\sqrt{1-4\varepsilon^2}}$

$$f(x, v, t) = \frac{1}{r} e^{rt} \sin x + \varepsilon v e^{rt} \cos x$$

  with the periodic boundary condition.

- Method: P^0 upwind DG in space + backward Euler in time[5], with N uniform elements of size Δx in the x direction, and $\Delta t = \Delta x$

[5] *If forward Euler method is applied in time, numerical stability requires* $\Delta t = O(\varepsilon h)$.

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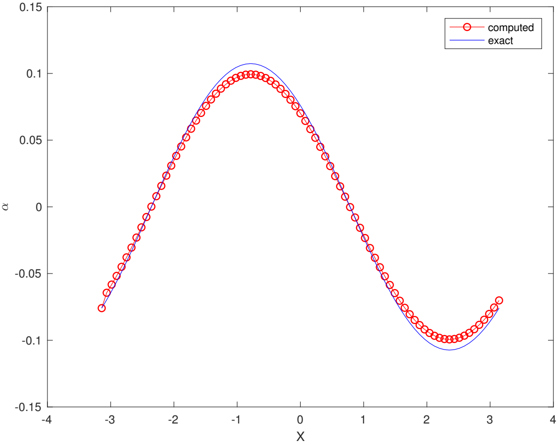
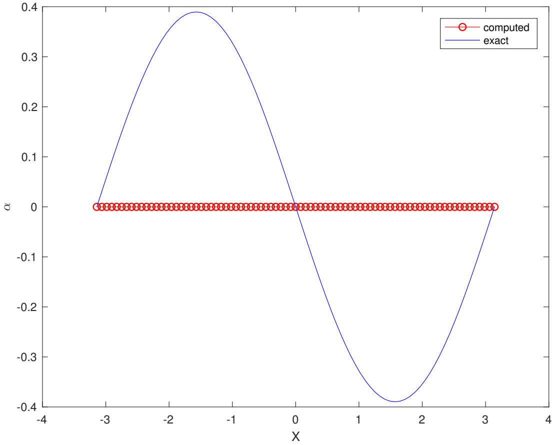
$f(x, v = 1, t)$ at the time $T = 1$ with $N = 80$, $\varepsilon = 0.5$



$f(x, v = 1, t)$ at the time $T = 1$ with $N = 80$, $\varepsilon = 10^{-6}$



To understand the observation

- The scheme, rewritten in $\alpha(x,t) = f(x, v=1, t)$ and $\beta(x,t) = f(x, v=-1, t)$, is

$$\frac{\alpha_j^{n+1} - \alpha_j^n}{\Delta t} + \frac{\alpha_j^{n+1} - \alpha_{j-1}^{n+1}}{\Delta x} = \frac{\beta_j^{n+1} - \alpha_j^{n+1}}{2\varepsilon} \quad (12)$$

$$\frac{\beta_j^{n+1} - \beta_j^n}{\Delta t} - \frac{\beta_{j+1}^{n+1} - \beta_j^{n+1}}{\Delta x} = \frac{\alpha_j^{n+1} - \beta_j^{n+1}}{2\varepsilon} \quad (13)$$

  Here $\alpha_j^n \approx \alpha(x_j, t_n)$, $\beta_j^n \approx \beta(x_j, t_n)$

- Recall the model in $\rho = \langle f \rangle = \frac{\alpha+\beta}{2}$, $J = \frac{\langle vf \rangle}{\varepsilon} = \frac{\alpha-\beta}{2\varepsilon}$ is

$$\partial_t \rho + \partial_x J = 0, \quad \varepsilon^2 \partial_t J + \partial_x \rho = -J$$

  When $\varepsilon \ll 1$, it becomes $\partial_t \rho = \partial_{xx} \rho + O(\varepsilon^2)$.

- The scheme, rewritten in ρ and J, is

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + \frac{J_{j+1}^{n+1} - J_{j-1}^{n+1}}{2\Delta x} - \frac{\Delta x}{2\varepsilon} \frac{\rho_{j+1}^{n+1} - 2\rho_j^{n+1} + \rho_{j-1}^{n+1}}{\Delta x^2} = 0 \quad (14a)$$

$$\varepsilon^2 \frac{J_j^{n+1} - J_j^n}{\Delta t} + \frac{\rho_{j+1}^{n+1} - \rho_{j-1}^{n+1}}{2\Delta x} - \frac{\varepsilon \Delta x}{2} \frac{J_{j+1}^{n+1} - 2J_j^{n+1} + J_{j-1}^{n+1}}{\Delta x^2} = -J_j^{n+1} \quad (14b)$$

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(cont'd)

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$$\varepsilon^2\frac{J_j^{n+1}-J_j^n}{\Delta t}+\frac{\rho_{j+1}^{n+1}-\rho_{j-1}^{n+1}}{2\Delta x}-\frac{\varepsilon\Delta x}{2}\frac{J_{j+1}^{n+1}-2J_j^{n+1}+J_{j-1}^{n+1}}{\Delta x^2}=-J_j^{n+1}$$

- Formally when $\varepsilon \ll 1$,

$$\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}=\frac{\rho_{j+2}^{n+1}-2\rho_j^{n+1}+\rho_{j-2}^{n+1}}{(2\Delta x)^2}+\frac{\Delta x}{2\varepsilon}\frac{\rho_{j+1}^{n+1}-2\rho_j^{n+1}+\rho_{j-1}^{n+1}}{\Delta x^2}+O(\varepsilon) \tag{15}$$

On a fixed mesh, and let $\varepsilon \to 0$, the limiting scheme is inconsistent to the limiting heat equation. $\Rightarrow$ The scheme is not AP.

Remedy

- either using P^k upwind DG methods in space with $k \geq 1$,
- or applying a scale-dependent numerical flux instead of the *pure* upwind flux

$$\text{upwind flux}: \widehat{vf} = \begin{cases} vf^-, \text{if } v > 0 \\ vf^+, \text{if } v < 0 \end{cases} = \frac{v}{2}(f^+ + f^-) - \frac{|v|}{2}(f^+ - f^-) \quad (16)$$

$$\text{scale-dependent flux}: \widehat{vf} = \frac{v}{2}(f^+ + f^-) - \frac{|v|\lambda(\varepsilon)}{2}(f^+ - f^-) \quad (17)$$

where $\lambda(\varepsilon) \geq 0$, satisfying

- $\lambda(\varepsilon) = O(1)$ for $\varepsilon = O(1)$
- $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0$

Revisit the example

- use the scale-dependent numerical flux with $\lambda(\varepsilon) = \varepsilon$
- $T = 1$ and $\Delta t = \Delta x = 2\pi/N$

Table: L^1 errors and orders in $f(x, v = 1, t)$

	$\varepsilon = 0.5$		$\varepsilon = 10^{-6}$	
N	error	order	error	order
40	3.41E-02	-	2.10E-01	-
80	1.83E-02	0.90	1.13E-01	0.89
160	7.90E-03	1.21	5.67E-02	1.00
320	4.03E-03	0.97	2.89E-02	0.97
640	1.94E-03	1.05	1.44E-02	1.00

Upwind DG methods and their AP property have been extensively studied for *stationary* neutron transport or radiative transfer equations. [6]

- Reed-Hill 1973: *neutron transport equation*
- Larsen 1983: *With the upwind flux, the P^1 DG method possesses the 'thick' diffusion limit, yet the P^0 DG method doesn't.* (1d)

  Larsen-Morel 1989: *asymptotic analysis of the P^1 upwind DG method in the presence of boundary layer* (1d)
- Adams 2001: *linear P^1 or bilinear Q^1 DG methods with the upwind flux on general meshes in multi-dimensions, with the methods on certain meshes not possessing the diffusion limit*

  Guermond-Kanschat 2010: *mathematics analysis for upwind DG methods with general discrete spaces*
- Ragusa-Guermond-Kanschat 2012: *DG methods with AP property using reduced upwind stabilization*

[6] *Solvers: Adams-Larsen 2002 (transport sweep + (accelerated) source iteration)*

DG based AP Methods

following a different framework

Main ingredients

- Reformulation(s): *micro-macro decomposition*
- Temporal discretization: *globally stiffly accurate implicit-explicit (IMEX) Runge-Kutta (RK) methods; implicit-explicit strategies*
- Spatial discretization: *DG methods*

1. Reformulation

- Consider $L^2(\Omega_v; d\mu)$ with the inner product $\langle \cdot, \cdot \rangle$: $\langle f_1, f_2 \rangle = \int_{\Omega_v} f_1 f_2 d\mu$.
- Let Π be an orthogonal projection onto $\mathrm{Null}(\mathcal{L}) = \{f : f = \langle f \rangle\}$

$$\Pi f = \langle f \rangle \tag{18}$$

  And let $\mathbf{I}$ be the identify operator.
- Then we have $f = \rho + \varepsilon g$, where $\rho = \Pi f$ and $g = \frac{1}{\varepsilon}(\mathbf{I} - \Pi) f$. The model equation

$$\varepsilon \partial_t f + v \partial_x f = \frac{1}{\varepsilon}(\langle f \rangle - f)$$

  is reformulated into

Micro-macro decomposition (*Liu-Yu 2004*)

$$\partial_t \rho + \partial_x \langle v g \rangle = 0 \tag{19a}$$

$$\partial_t g + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x (v g) + \frac{1}{\varepsilon^2} v \partial_x \rho = -\frac{1}{\varepsilon^2} g \tag{19b}$$

- As $\varepsilon \to 0$,

$$\Rightarrow \quad g = -v \partial_x \rho \quad \text{(local equilibrium)} \tag{20a}$$

$$\Rightarrow \quad \partial_t \rho = \langle v^2 \rangle \partial_{xx} \rho \tag{20b}$$

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Assumptions:

- The initial data is *well-prepared*: that is, at $t = 0$

$$g + v\partial_x \rho \to 0, \quad \text{as } \varepsilon \to 0.$$

  In this case, the problem is free of initial layers in leading terms.
- Boundary conditions in x are periodic[7].

[7] *General isotropic boundary conditions are considered numerically.*

2. Temporal Discretization: first order implicit-explicit (IMEX) method ($\rho^n \approx \rho(\cdot, t_n)$, $g^n \approx g(\cdot, \cdot, t_n)$)

Implicit-explicit strategy

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \langle v g^n \rangle = 0 \tag{21a}$$

$$\frac{g^{n+1} - g^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x (v g^n) + \frac{1}{\varepsilon^2} v \partial_x \rho^{n+1} = -\frac{1}{\varepsilon^2} g^{n+1} \tag{21b}$$

Formally as $\varepsilon \ll 1$

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \langle v g^n \rangle = 0 \tag{22a}$$

$$g^{n+1} = -v \partial_x \rho^{n+1} + O(\varepsilon) \tag{22b}$$

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3. Spatial Discretization: DG method

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \langle v g^n \rangle = 0$$

$$\frac{g^{n+1} - g^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x (v g^n) + \frac{1}{\varepsilon^2} v \partial_x \rho^{n+1} = -\frac{1}{\varepsilon^2} g^{n+1}$$

Discrete space and numerical fluxes

- $U_h = U_h^k$: piecewise polynomials of degree up to k
- vg: upwind flux
- $\langle vg \rangle$ and ρ: alternating flux

$$\langle \widehat{vg} \rangle = \langle vg \rangle^-, \quad \hat{\rho} = \rho^+; \qquad \text{or} \quad \langle \widehat{vg} \rangle = \langle vg \rangle^+, \quad \hat{\rho} = \rho^-$$

Fully-discrete method: look for $\rho_h^{n+1}(\cdot), g_h^{n+1}(\cdot, v) \in U_h$

DG-IMEX1 method

$$\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} + \mathcal{D}_h^{(g)} \langle v g_h^n \rangle = 0 \tag{23a}$$

$$\frac{g_h^{n+1} - g_h^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\mathcal{D}_{h,v}^{(up)} g_h^n + \frac{v}{\varepsilon^2}\mathcal{D}_h^{(\rho)} \rho_h^{n+1} = -\frac{1}{\varepsilon^2} g_h^{n+1} \tag{23b}$$

- $\rho_h^n(x) \approx \rho(x, t_n)$, $g_h^n(x, v) \approx g(x, v, t_n)$
- Computational complexity: *to implement, one first solves* ρ_h^{n+1} *in* (23a), *and then* g_h^{n+1} *in* (23b), *by solving (block-)diagonal linear systems.*

Theoretical Results

Continuous level: (periodic in x)

$$\begin{aligned}
&\varepsilon\partial_t f + v\partial_x f = \frac{1}{\varepsilon}(\langle f\rangle - f) \\
\Rightarrow &\frac{\varepsilon}{2}\frac{d}{dt}\int_{\Omega_x\times\Omega_v} f^2 dxdv = -\frac{1}{\varepsilon}\int_{\Omega_x\times\Omega_v}(f-\langle f\rangle)^2 dxdv \\
\Leftrightarrow &\frac{1}{2}\frac{d}{dt}\left(||\rho||^2(t) + \varepsilon^2|||g|||^2(t)\right) = -|||g|||(t)^2 \le 0 \qquad (24)
\end{aligned}$$

- $||\phi|| = (\int_{\Omega_x} \phi^2(x)dx)^{1/2}$
- $|||\psi||| = (\int_{\Omega_x\times\Omega_v} \psi^2(x,v)dxdv)^{1/2}$

Theorem (numerical stability for DG-IMEX1 method)

With periodic boundary condition, the following holds for the DG-IMEX1 method with the discrete space $U_h = U_h^k$:

$$||\rho_h^{n+1}||^2 + \varepsilon^2|||g_h^n|||^2 \le ||\rho_h^n||^2 + \varepsilon^2|||g_h^{n-1}|||^2, \ \forall n \qquad (25)$$

under the time step condition

$$\Delta t \le \Delta t_{stab} = \begin{cases} \frac{2h}{\alpha_2\alpha_3}(h+\alpha_3\varepsilon) = c_1\varepsilon h + c_2 h^2, & \text{for } k=0 \\ \frac{h}{\alpha_1+\alpha_2\alpha_3}(h+\min(\varepsilon, \frac{\alpha_2 h}{\alpha_1})\alpha_3) \le c_3 h^2, & \text{for } k \ge 1 \end{cases} \qquad (26)$$

- ⋄ Uniform stability in ε
- ⋄ Here $\alpha_i, i = 1, 2, 3$ are computable constants, dependent of Ω_v and k.

Stability mechanisms:

- Implicit part in time discretization:

$$\left(\frac{\phi^{n+1}-\phi^n}{\Delta t}, \phi^{n+1}\right) = \frac{1}{2\Delta t}\left(||\phi^{n+1}||^2 - ||\phi^n||^2 + ||\phi^{n+1}-\phi^n||^2\right) \quad (27)$$

- Upwind stabilization
- Damping due to the scattering process

Theorem (error estimates for DG-IMEX1 method)

With periodic boundary condition, the following error estimates hold for the DG-IMEX1 method with the discrete space $U_h = U_h^k$ and sufficiently smooth exact solutions:

$$||\rho^n - \rho_h^n||^2 + \varepsilon^2|||g^{n-1} - g_h^{n-1}|||^2 \\ \leq C_* \left((1+\varepsilon^2)h^{2k+2} + \varepsilon h^{2k+1} + (1+\varepsilon^4)\Delta t^2\right)$$

for $n : n\Delta t \leq T$ under the condition $\Delta t < \min(\Delta t_{\text{stab}}, \frac{1}{2})$. And C_* depends on the exact solution, T, k, and Ω_v.

⋄ Proof: *Stability + approximation property of the space + local truncation error in time*

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⋄ Proof: *Stability* + *approximation property of the space* + *local truncation error in time*

$$\begin{array}{ccc} \mathcal{F}^{\varepsilon} & \xrightarrow{\varepsilon \to 0} & \mathcal{F}^{0} \\ (\delta \to 0) \uparrow & & \uparrow (\delta \to 0) \\ \mathcal{F}_{\delta}^{\varepsilon} & \xrightarrow{\varepsilon \to 0} & \mathcal{F}_{\delta}^{0} \end{array}$$

What about the asymptotic behavior on under-resolved meshes?

Formal asymptotic analysis:

- ($\mathcal{F}^{\varepsilon}$) The equation in its micro-macro formulation:

$$\partial_t \rho + \partial_x \langle v g \rangle = 0 \tag{28a}$$

$$\partial_t g + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}(v \partial_x g) + \frac{1}{\varepsilon^2} v \partial_x \rho = -\frac{1}{\varepsilon^2} g \tag{28b}$$

- ($\mathcal{F}^{\varepsilon}_{\delta}$) DG-IMEX1 scheme:

$$\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} + \mathcal{D}_h^{(g)} \langle v g_h^n \rangle = 0 \tag{29a}$$

$$\frac{g_h^{n+1} - g_h^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\} \mathcal{D}_{h,v}^{(up)} g_h^n + \frac{1}{\varepsilon^2} v \mathcal{D}_h^{(\rho)} \rho_h^{n+1} = -\frac{1}{\varepsilon^2} g_h^{n+1} \tag{29b}$$

With fixed Δt and h, $\kappa = \langle v^2 \rangle$, when $\varepsilon \to 0$,

- ($\mathcal{F}^0$) The limiting equations: with $q = \langle vg \rangle$,

$$\partial_t \rho + \partial_x q = 0 \tag{30a}$$
$$-\kappa \partial_x \rho = q \tag{30b}$$

- ($\mathcal{F}_\delta^0$) DG-IMEX1 scheme in the limit: with $q_h = \langle v g_h \rangle$,

$$\frac{\rho_h^{n+1} - \rho_h^n}{\Delta t} + \mathcal{D}_h^{(g)} q_h^n = 0$$
$$-\kappa \mathcal{D}_h^{(\rho)} \rho_h^{n+1} = q_h^{n+1}$$

Remarks:

(1) In the limit of $\varepsilon \to 0$, with the well-prepared initial data (i.e. $\langle vg \rangle = -\kappa \partial_x \rho$ at $t = 0$), the proposed scheme becomes a consistent and stable discretization for the limiting heat equation on fixed mesh.

(2) The limiting scheme combines a local DG method with the alternating flux in space and (essentially) the forward Euler method in time. It is stable.

Theorem (AP property)
(1) Under the assumptions on the initial data regarding the *weak convergence, well-preparedness, boundedness*

$$(\mathcal{A}1)\quad \rho_\varepsilon \rightharpoonup \rho_0, \quad \langle w g_\varepsilon\rangle \rightharpoonup \langle w g_0\rangle \text{ in } L^2(\Omega_x) \text{ with } \forall w \in L^2(\Omega_v), \quad \text{as } \varepsilon \to 0,$$

$$(\mathcal{A}2)\quad \langle v(g_\varepsilon + v\partial_x \rho_\varepsilon)\rangle \rightharpoonup 0 \text{ in } L^2(\Omega), \quad \text{as } \varepsilon \to 0,$$

$$(\mathcal{A}3)\quad \sup_\varepsilon(||\rho_\varepsilon||)|_{t=0} < \infty, \qquad \sup_\varepsilon(|||g_\varepsilon|||)|_{t=0} < \infty,$$

(2) and under the condition $\Delta t < \Delta t_{stab}$,

- there exist $\rho^n_{\Delta t,h}, q^n_{\Delta t,h} \in U_h$, such that

$$\lim_{\varepsilon\to 0} \rho^n_{\varepsilon,\Delta t,h} = \rho^n_{\Delta t,h}, \quad \lim_{\varepsilon\to 0} q^n_{\varepsilon,\Delta t,h} = q^n_{\Delta t,h} \quad \forall n \geq 0,$$

- and the limits satisfy the limiting scheme on the previous slide, with the initial data

$$\rho^0_{\Delta t,h} = \pi_h \rho_0, \qquad q^0_{\Delta t,h} = -\kappa \pi_h \partial_x \rho_0.$$

Here π_h is the L^2 projection onto U_h.

◇ Proof: *uniform boundedness + compactness argument*

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High Order Accuracy in Time

An s-stage implicit-explicit (IMEX) Runge-Kutta (RK) scheme:

$$\begin{array}{c|c} \tilde{c} & \tilde{\mathcal{A}} \\ \hline & \tilde{b}^T \end{array} \qquad \begin{array}{c|c} c & \mathcal{A} \\ \hline & b^T \end{array} \tag{31}$$

- $\tilde{\mathcal{A}} = (\tilde{a}_{ij}), \mathcal{A} = (a_{ij}) \in \mathbb{R}^{s\times s}, \tilde{c} = (\tilde{c}_j), c = (c_j) \in \mathbb{R}^s,$ $\tilde{b} = (\tilde{b}_j), b = (b_j) \in \mathbb{R}^s; \tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}, c_i = \sum_{j=1}^{i} a_{ij}$
- $\tilde{\mathcal{A}}$: lower triangular with zero diagonal entries; $\mathcal{A}$: lower triangular;

Globally stiffly accurate (*Boscarino-Pareschi-Russo 2013*)

$$c_s = \tilde{c}_s = 1, \text{ and } a_{sj} = b_j, \tilde{a}_{sj} = \tilde{b}_j, \forall j = 1, \cdots, s. \tag{32}$$

In time, we apply high order globally stiffly accurate IMEX-RK methods of type ARS.

- Globally stiffly accurate: *to ensure the numerical solutions from both inner and full RK stages to stay close to the local equilibrium $g + v\partial_x \rho = 0$ for the well-prepared initial data*

1st order: IMEX1 /ARS(1, 1, 1)

0	0	0
1	1	0
	1	0

0	0	0
1	0	1
1	0	1

3rd order: ARS(4, 4, 3)

0	0	0	0	0	0
1/2	1/2	0	0	0	0
2/3	11/18	1/18	0	0	0
1/2	5/6	−5/6	1/2	0	0
1	1/4	7/4	3/4	−7/4	0
	1/4	7/4	3/4	−7/4	0

0	0	0	0	0	0
1/2	0	1/2	0	0	0
2/3	0	1/6	1/2	0	0
1/2	0	−1/2	1/2	1/2	0
1	0	3/2	−3/2	1/2	1/2
	0	3/2	−3/2	1/2	1/2

- Methods: P^{k-1} DG-IMEXk, also denoted as DGk-IMEXk
- Formal asymptotic analysis is carried out. *Under the assumption that initial data is well-prepared*, the limiting scheme is the explicit part of the IMEX scheme in time, combined with an local DG method in space.

Numerical stability based on Fourier analysis: consider the telegraph equation with $\Omega_v = \{-1, 1\}$, the mesh is uniform with periodic boundary conditions.

- Let the numerical solution

$$\rho_h^n(x) = \sum_{l=0}^{k-1} \rho_{ml}^n \phi^l\left(\frac{x - x_m}{h_m/2}\right) = \boldsymbol{\rho}_m^n \cdot \Phi\left(\frac{x - x_m}{h_m/2}\right), \forall x \in I_m$$

  Here ϕ^l being the l-th Legendre polynomial on $[-1, 1]$.

- Take the ansatz $\boldsymbol{\rho}_m^n = \widehat{\boldsymbol{\rho}}^n \exp(\mathcal{I} \mu x_m)$, with μ as the wavenumber and $\mathcal{I}^2 = -1$.
- Similar steps are taken to g_h^n, and one further reduces the variables based on $\langle g_h^n \rangle = 0$.
- With $\xi = \mu h \in [-\pi, \pi]$,

$$\begin{pmatrix} \widehat{\boldsymbol{\rho}}^{n+1} \\ \widehat{\boldsymbol{g}}^{n+1} \end{pmatrix} = \boldsymbol{G}(\varepsilon, h, \Delta t; \xi) \begin{pmatrix} \widehat{\boldsymbol{\rho}}^{n} \\ \widehat{\boldsymbol{g}}^{n} \end{pmatrix} \tag{33}$$

  $\boldsymbol{G}(\varepsilon, h, \Delta t; \xi)$: amplification matrix

Theorem (invariant property of G)

$\boldsymbol{G}(\varepsilon, h, \Delta t; \xi)$ is similar to $\hat{\boldsymbol{G}}(\frac{\varepsilon}{h}, \frac{\varepsilon^2}{\Delta t}; \xi)$, and equivalently, it is similar to $\tilde{\boldsymbol{G}}(\frac{\varepsilon}{h}, \frac{\Delta t}{\varepsilon h}; \xi)$, given $\frac{\varepsilon^2}{\Delta t} = \frac{\varepsilon/h}{\Delta t/(\varepsilon h)}$.

Principle for Numerical Stability: *For any given $\varepsilon, h, \Delta t$, let the eigenvalues of $\mathbf{G}$ be $\lambda_i(\xi)$, $i = 1, \ldots, 2k$. Our scheme is "stable", if for all $\xi \in [-\pi, \pi]$, it satisfies either*

$$(*) \qquad \max_{i=1,\ldots,2k} \{|\lambda_i(\xi)|\} < 1, \quad \text{or} \tag{34}$$

$$(*) \qquad \max_{i=1,\ldots,2k} \{|\lambda_i(\xi)|\} = 1 \quad \text{and} \quad \mathbf{G} \quad \text{is diagonalizable.} \tag{35}$$

Implication: *numerical stability will depend on $\varepsilon, h, \Delta t$ only in terms of ε/h and $\frac{\Delta t}{\varepsilon h}$.*

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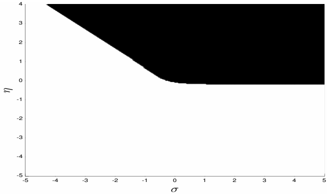

Figure: Stability regions of the DG1-IMEX1 method. White: stable.

- $\sigma = \log_{10}(\frac{\varepsilon}{h})$, $\eta = \log_{10}(\frac{\Delta t}{\varepsilon h})$, and scheme is stable when $\eta \leq \mathcal{F}(\sigma)$ for some function $\mathcal{F}$.
- Kinetic / transport regime: when $\varepsilon = O(1)$ (with relatively large σ), the scheme is stable if $\mathcal{F}(\sigma) \approx O(1)$, that is

$$\eta < O(1) \Rightarrow \Delta t = O(\varepsilon h)$$

- Diffusive regime: when $\varepsilon/h \ll 1$ (with relatively small σ), the scheme is stable if $\mathcal{F}(\sigma) \approx -\sigma + C$, that is,

$$\log_{10}(\frac{\Delta t}{\varepsilon h}) < -\log_{10}(\frac{\varepsilon}{h}) + C \Rightarrow \Delta t = O(h^2).$$

- Recall the analytical form (by energy stability analysis): $\Delta t \leq \frac{1}{4}h^2 + \frac{1}{2}\varepsilon h$

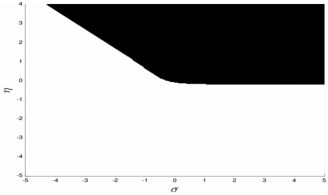

Figure: Stability regions of the DG1-IMEX1 method. White: stable.

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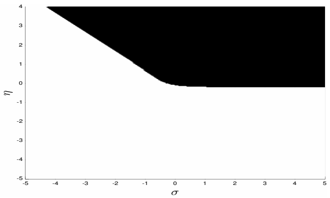
- Kinetic / transport regime: when $\varepsilon = O(1)$ (with relatively large σ), the scheme is stable if $\mathcal{F}(\sigma) \approx O(1)$, that is

$$\eta < O(1) \Rightarrow \Delta t = O(\varepsilon h)$$

- Diffusive regime: when $\varepsilon/h \ll 1$ (with relatively small σ), the scheme is stable if $\mathcal{F}(\sigma) \approx -\sigma + C$, that is,

$$\log_{10}(\frac{\Delta t}{\varepsilon h}) < -\log_{10}(\frac{\varepsilon}{h}) + C \Rightarrow \Delta t = O(h^2).$$

- Recall the analytical form (by energy stability analysis): $\Delta t \leq \frac{1}{4}h^2 + \frac{1}{2}\varepsilon h$



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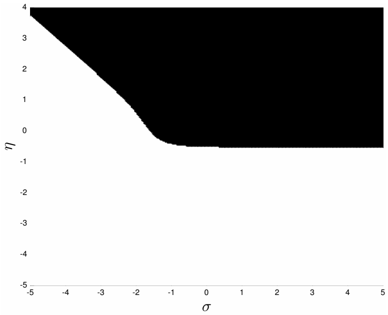
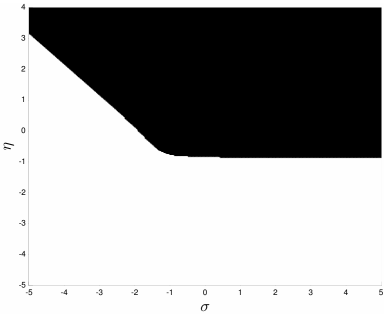


(a) DG2-IMEX2



(b) DG3-IMEX3

Figure: Stability regions of the DGk-IMEXk methods, $k = 2, 3$. White: stable.

Fourier analysis shows similar stability behavior of the second and third order methods, DGk-IMEXk ($k = 2, 3$)

$$\Delta t = \begin{cases} O(\varepsilon h) & \text{for } \varepsilon = O(1), \quad (\textit{hyperbolic type}) \\ O(h^2) & \text{for } \varepsilon/h \ll 1, \quad (\textit{parabolic type}) \end{cases} \tag{36}$$

Numerical Examples

Example 1 (one-group transport in slab geometry)

- Domain: $\Omega_x = [-\pi, \pi]$, $\Omega_v = [-1, 1]$
- Initial condition:

$$\begin{cases} \rho(x,0) = 2 + \sin(x) \\ g(x,v,0) = -v\cos(x) \end{cases}$$

- Periodic boundary condition
- Final time: $T = 0.1$
- v-direction: 16 Gaussian points
- x-direction: uniform mesh with N elements

L^2 errors and orders of ρ and $q = \langle vg \rangle$ by P^2 DG - IMEX3 method

	N	error in ρ	order	error in q	order
$\varepsilon = 0.5$	10	6.21E-04	–	3.79E-04	–
	20	9.14E-05	2.76	3.80E-05	3.32
	40	1.30E-05	2.82	4.49E-06	3.08
	80	1.64E-06	2.98	5.48E-07	3.03
$\varepsilon = 10^{-2}$	10	7.79E-04	–	2.82E-04	–
	20	1.02E-04	2.94	3.39E-05	3.06
	40	1.27E-05	3.00	4.24E-06	3.00
	80	1.59E-06	3.00	5.30E-07	3.00
$\varepsilon = 10^{-6}$	10	7.81E-04	–	2.80E-04	–
	20	1.02E-04	2.94	3.39E-05	3.04
	40	1.27E-05	3.00	4.24E-06	3.00
	80	1.59E-06	3.00	5.30E-07	3.00

Example 2 (one-group transport in slab geometry): two-material problem at stationary state

$$\partial_t f + v \cdot \nabla_x f = \sigma_S(x)(\langle f \rangle - f) - \sigma_A(x) f \tag{37}$$

- Domain: $\Omega_x = [0, 11], \quad \Omega_v = [-1, 1]$
- Initial condition: $f(x, v, t = 0) = 0$
- Boundary condition:
  $f(x = 0, v, t) = 5, v \geq 0, \quad f(x = 11, v, t) = 0, v \leq 0$
- Parameters:
  $\sigma_S(x) = 0, \quad \sigma_A(x) = 1, \quad$ for $x \in [0, 1]$
  $\sigma_S(x) = 100, \quad \sigma_A(x) = 0, \quad$ for $x \in [1, 11]$

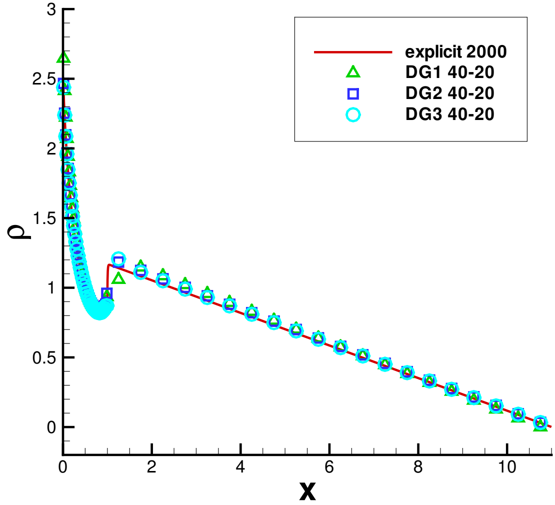


Figure: The computed ρ by P^{k-1} DG-IMEXk, $k = 1, 2, 3$ at $t = 20000$. $\Delta x = h = 0.025$ in $[0, 1]$ and $\Delta x = h = 0.5$ in $[1, 11]$.

Example 3 (telegraph equation): Riemann problem

- Domain: $\Omega_x = [-1, 1], \quad \Omega_v = \{-1, 1\}$
- Initial condition: $\rho = \langle f \rangle$, $J = \frac{1}{\varepsilon}\langle v f \rangle$

$$(\rho, J) = \begin{cases} (2.0, 1.0), & -1 < x < 0 \\ (1.0, 0.0), & 0 < x < 1 \end{cases}$$

- P^2 DG-IMEX3, with $\Delta x = h = 0.02$

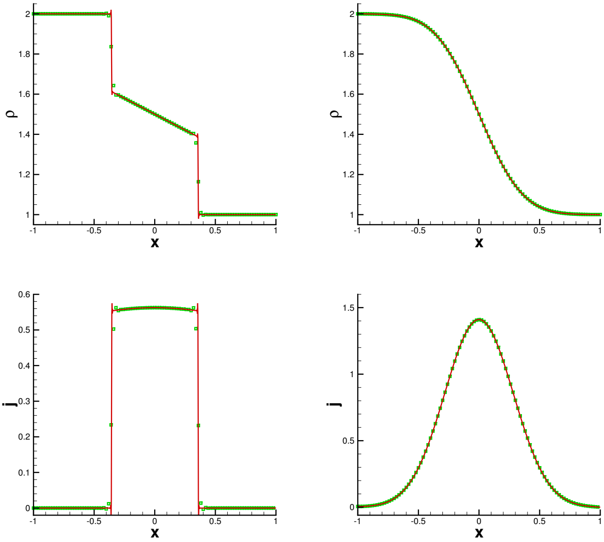


Figure: Left: $\varepsilon = 0.7$ at $t = 0.25$; Right: $\varepsilon = 10^{-6}$ at $t = 0.04$. Top: ρ; Bottom: J. No limiter.

What if there is initial layer as $\varepsilon \ll 1$?

- Recall $f = \rho + \varepsilon g$, and g can be of $O(\frac{1}{\varepsilon})$ at $t = 0$. Away from the $O(\varepsilon^2)$-width initial layer, both ρ and g should be of $O(1)$.
- The proposed methods may suffer from order reduction or poor accuracy.

Implicit-explicit strategy (original)

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \langle v g^n \rangle = 0 \tag{38}$$

$$\frac{g^{n+1} - g^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x(v g^n) + \frac{1}{\varepsilon^2} v \partial_x \rho^{n+1} = -\frac{1}{\varepsilon^2} g^{n+1}$$

Modified implicit-explicit strategy (used in the first step)

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \langle v g^{n+1} \rangle = 0 \tag{39}$$

$$\frac{g^{n+1} - g^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x(v g^n) + \frac{1}{\varepsilon^2} v \partial_x \rho^{n} = -\frac{1}{\varepsilon^2} g^{n+1}$$

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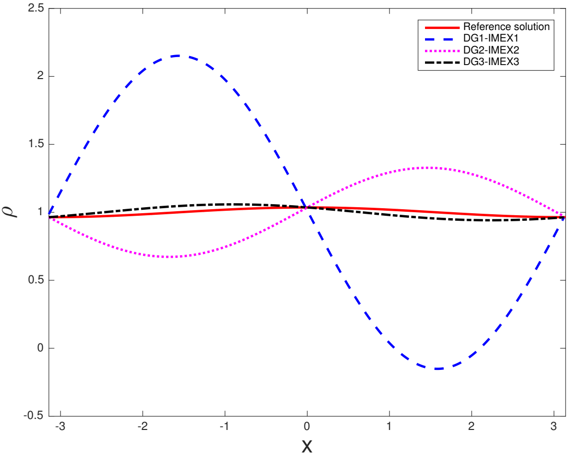
An example

- Setting: $\Omega_x = [0, 2\pi]$, $\Omega_v = [-1, 1]$ and $\sigma(x) = 1$; periodic boundary condition and $T = 1$; $\varepsilon = 10^{-6}$; $N = 320$
- Initial data: $f(x, v, 0) = (1 + (v - 0.5)^2)(1 + 0.05 \cos x)$



L^1 errors and orders in ρ and $j = \langle vg \rangle$ at $T = 1$ by P^{k-1} DG-IMEXk method. $\varepsilon = 10^{-6}$ with an accuracy recovering fixing strategy.

	N	error of ρ	order	error of j	order
$k = 1$	10	2.93E-03	-	5.71E-04	-
	20	1.70E-03	0.78	2.69E-04	1.08
	40	8.85E-04	0.95	1.34E-04	1.01
	80	4.46E-04	0.99	6.69E-05	1.00
	160	2.23E-04	1.00	3.34E-05	1.00
$k = 2$	10	1.04E-03	-	8.20E-05	-
	20	2.68E-04	1.97	2.01E-05	2.03
	40	6.68E-05	2.00	5.00E-06	2.01
	80	1.67E-05	2.00	1.24E-06	2.00
	160	4.71E-06	2.00	3.12E-07	2.00
$k = 3$	10	7.93E-05	-	6.01E-06	-
	20	1.01E-05	2.96	7.60E-07	2.98
	40	1.29E-06	2.98	9.66E-08	2.98
	80	1.62E-07	2.99	1.21E-08	2.99
	160	2.03E-08	3.00	1.52E-09	2.99

To overcome the parabolic time step condition $\Delta t = O(h^2)$ when $\varepsilon \ll 1$

One more reformulation: *by adding/subtracting a weighted diffusive term*[8]

$$\partial_t \rho + \partial_x \langle vg \rangle + \omega \langle v^2 \rangle \partial_{xx} \rho = \omega \langle v^2 \rangle \partial_{xx} \rho \tag{40a}$$

$$\partial_t g + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x (vg) + \frac{1}{\varepsilon^2} v \partial_x \rho = -\frac{1}{\varepsilon^2} g \tag{40b}$$

implicit-explicit strategy

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \langle v(g^n + \omega v \partial_x \rho^n) \rangle = \omega \langle v^2 \rangle \partial_{xx} \rho^{n+1} \tag{41}$$

$$\frac{g^{n+1} - g^n}{\Delta t} + \frac{1}{\varepsilon}\{\mathbf{I} - \Pi\}\partial_x (vg^n) + \frac{1}{\varepsilon^2} v \partial_x \rho^{n+1} = -\frac{1}{\varepsilon^2} g^{n+1} \tag{42}$$

Formally as $\varepsilon \to 0$, we obtain an IMPLICIT scheme,

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = \langle v^2 \rangle \partial_{xx} \rho^{n+1}, \qquad g^{n+1} = -v \partial_x \rho^{n+1} \tag{43}$$

[8] *Boscarino-Pareschi-Russo 2013*, Weight function ω: non-negative, independent of x, satisfying

$$\omega \to 1 \quad \text{as} \quad \varepsilon \to 0$$

To overcome the parabolic time step condition $\Delta t = O(h^2)$ when $\varepsilon \ll 1$

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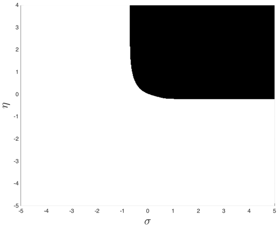
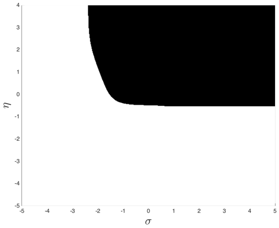
$$\omega \to 1 \quad \text{as} \quad \varepsilon \to 0$$

- More about discretizations
  - Local DG methods in space
  - High order globally stiffly accurate IMEX-RK methods in time
  - Weight function: $\omega = \omega(\varepsilon/h, \Delta t/(\varepsilon h))$
- Energy stability for IMEX1-LDG methods
- Fourier type stability analysis: with $\omega = \exp(-\varepsilon/h)$, the time step condition for the stability of the IMEXp-LDGp scheme ($p = 1, 2, 3$) is

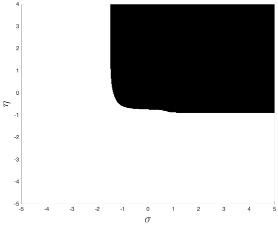
$$\Delta t \leq \begin{cases} O(\varepsilon h) & \text{for } \varepsilon = O(1), \quad (\textit{hyperbolic type}) \\ \infty & \text{for } \frac{\varepsilon}{h} \ll 1, \quad (\textit{unconditionally stable}) \end{cases}$$



(a) IMEX1-LDG1



(b) IMEX2-LDG2



(c) IMEX3-LDG3

Figure: Stability regions of the IMEXk-LDGk methods with the weight function $\omega = \exp(-\varepsilon/h)$. $\sigma = \log_{10}(\frac{\varepsilon}{h})$ and $\eta = \log_{10}(\frac{\Delta t}{\varepsilon h})$. White: stable

How about the computational cost?

- *a discrete Poisson to solve for each inner stage of one RK step*

Another idea: to overcome the parabolic time step condition when $\varepsilon \ll 1$

implicit-explicit strategy

$$\frac{\rho^{n+1}-\rho^{n}}{\Delta t}+\partial_x\langle v g^{n+1}\rangle=0 \qquad (44)$$

$$\frac{g^{n+1}-g^{n}}{\Delta t}+\frac{1}{\varepsilon}\{\mathbf{I}-\Pi\}\partial_x(v g^{n})+\frac{1}{\varepsilon^2}v\partial_x\rho^{n+1}=-\frac{1}{\varepsilon^2}g^{n+1}$$

- Different implicit-explicit strategy
- Weight-free
- Schur complement at the algebraic level: *a discrete Poisson to solve for each inner stage of one RK step*

Concluding Remarks

- A linear kinetic transport equation model is considered under the diffusive scaling.
- High order asymptotic preserving discontinuous Galerkin methods are designed based on reformulation(s).
  - Uniform numerical stability
  - The methods can be extended to more general model: $\sigma_s(x)$, $\sigma_a(x)$.
  - Reasonable computational complexity

References

- J. Jang, F. Li, J.-M. Qiu, T. Xiong, Analysis of asymptotic preserving DG-IMEX schemes for linear kinetic transport equations in a diffusive scaling, SIAM Journal on Numerical Analysis, v52 (2014), pp. 1497-2206
- J. Jang, F. Li, J.-M. Qiu, T. Xiong, High order asymptotic preserving DG-IMEX schemes for discrete-velocity kinetic equations in a diffusive scaling, Journal of Computational Physics, v281 (2015), pp.199-224
- Z. Peng, Y. Cheng, J.-M. Qiu, F. Li, Stability-enhanced AP IMEX-LDG schemes for linear kinetic transport equations under a diffusive scaling, Journal of Computational Physics, v415 (2020), pp.109485
- Z. Peng, Y. Cheng, J.-M. Qiu, F. Li, Stability-enhanced AP IMEX1-LDG method: energy-based stability and rigorous AP property, arXiv:2005.05454 (2020)
- Z. Peng and F. Li, Asymptotic preserving IMEX-DG-S schemes for linear kinetic transport equations based on Schur complement, arXiv:2006.07497 (2020)