Stability analysis and error estimates of Runge-Kutta discontinuous Galerkin methods for linear hyperbolic equations

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Outline

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The discontinuous finite element space

Consider the 1d linear hyperbolic equation

$$U_t + \beta U_x = 0, \quad x \in (0,1), \quad t \in (0,T], \tag{1}$$

equipped with the periodic boundary condition. Here $\beta \neq 0$ is a constant, and the initial solution is $U(x,0) = U_0(x)$.

- Let $\{I_i\}_{i=1}^N$ be the quasi-uniform partition, with the maximum length h.
- The discontinuous finite element space V_h is defined as the piecewise polynomials of degree at most $k \geq 0$.
- jump: $[\![v]\!] = v^+ - v^-$
- and weighted average: $\{\!\{v\}\!\}^{(\theta)} = \theta v^- + (1-\theta)v^+$

The semi-discrete DG method

- Find the map $u\colon [0,T] \to V_h$ such that

$$(u_t, v) = \mathcal{H}(u, v), \quad \forall\, v \in V_h, \quad t \in (0, T], \tag{2}$$

  with the initial solution $u(x, 0) \in V_h$.
- The spatial DG discretization is given in the form

$$\mathcal{H}(u, v) = \sum_{1 \le i \le N} \left[\int_{I_i} \beta u v_x \,\mathrm{d}x + \beta \{\!\{u\}\!\}^{(\theta)}_{i+\frac{1}{2}} [\![v]\!]_{i+\frac{1}{2}} \right], \tag{3}$$

  in which $\beta(\theta - 1/2) > 0$ provides the upwind-biased numerical flux.
- After choosing the basis functions of V_h, the above semi-discrete DG method can be written into an ODEs of $N(k+1)$ order

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \mathbb{L}_h \vec{u},$$

  where $\vec{u}$ is the vector-valued function made up of all freedoms of the numerical solution, and $\mathbb{L}_h$ is a constant matrix.

The fully discrete RKDG method

- Consider the widely-used RKDG(s, r, k) method:
  - the explicit RK algorithm of s-stages and r-th order,
  - the DG spatial discretization with piecewise polynomials of degree at most k.
- Discretize the time by $t^n = n\tau$ with the time step τ. The single-step time marching is generally given in the Shu-Osher representation:

> - $u^{n,0} = u^n$;
> - For stage number $\ell = 0, 1, \ldots, s-1$, successively seek the stage solution $u^{n,\ell+1} \in V_h$ by the variational form
>
> $$(u^{n,\ell+1}, v) = \sum_{0\le\kappa\le\ell} \Big[c_{\ell\kappa}(u^{n,\kappa}, v) + d_{\ell\kappa}\tau\mathcal{H}(u^{n,\kappa}, v) \Big], \quad \forall v \in V_h;$$
>
> - $u^{n+1} = u^{n,s}$.

- All parameters $c_{\ell\kappa}$ and $d_{\ell\kappa}$ are given by the used RK algorithm;
- Note that $d_{\ell\ell} \neq 0$;

The purpose of this talk

- It is well known for the semi-discrete DG method that the L^2-norm of numerical solution does not increase with time.
- What about the L^2-norm stability of the fully-discrete RKDG method?
  - temporal-spatial condition? or the restriction on the CFL number
  $$\lambda = |\beta|\tau h^{-1}?$$
  - stability performance?
- New observation and explanation!
  - energy technique, not Fourier method!
  - easy extension to linear varying-coefficient problem and nonlinear problem.
  - flexible application for error estimates, superconvergence analysis, ...

Outline

0. Related works on the L^2-norm stability

Non SSP framework

- SSP framework
  - Levy and Tadmor (SR98) proved monotonicity stability for some fully discrete schemes with RK time-marching of third order and fourth order, when solving coercive problems.
  - Gottlieb (SR01) extended the above results to general RK algorithms, and utilized the strong-stability-preserving (SSP) framework.
- However, hyperbolic equation is not coercive, and the RKDG method does not satisfy the elemental assumption in SSP framework:

> the Euler-forward time-marching in each stage evolution is stable under the standard CFL condition.

- New proof line is needed!

ODEs with semi-negative linear spatial operator

- Tadmor (SINUM02) proved monotonicity stability for the three-stages and third-order RK time-marching, and present an open problem:

> Does monotonicity stability hold for the four-stages and fourth order RK time discretization?

- Along the same line, Sun and Shu (AMSA17) answered this problem:
  - presented a counter-example (not a real DG operator) which does not always have monotonicity stability;
  - proved that the L^2-norm does not increase every two steps.
- Sun and Shu (SINUM19) then extended the above work and proposed a framework to investigate the L^2-norm stability performance for arbitrary RK time discretization.
- The above works ignore the particular effect of DG discretization, and are not good at carrying out the error estimates.

The lower order RKDG methods

- In the L^2-norm error estimates, some stability analysis have been done for lower order RKDG methods.
  - Zhang and Shu (SINUM04) have implicitly showed for RKDG(2,2,k) method that the monotonicity stability holds only for $k = 1$.
  - Zhang and Shu (SINUM10) have proved for RKDG$(3, 3, k)$ method that the monotonicity stability holds for arbitrary k.
  - Similar result on the third order RKDG methods was given by Burman and Ern (SINUM10), with different energy equation.
- Based on our experience on error estimate, Y. Xu and e.t.c. (SINUM19) independently proposed an uniform framework to judge the L^2-norm stability performance for arbitrary RKDG method.
- The important contribution is the *wonderful* energy equation, which is expressed by the temporal difference of stage solutions. This purpose can be automatically obtained by an matrix transform process.

1. Concepts on stability

Three concepts on the L^2-norm stability

Definition 1

Weak(γ) stability: there exist two constants $C > 0$ and $\gamma \geq 2$, such that

$$\|u^{n+1}\|^2 \leq (1 + C\lambda^\gamma)\|u^n\|^2, \quad n \geq 0.$$

- This implies the general stability with exponent constant, namely

  $$\|u^n\| \leq e^{CT}\|u^0\|, \quad n \geq 0,$$

  when

  $$\lambda^\gamma/\tau \text{ is bounded} \Leftrightarrow \tau = \mathcal{O}(h^{\frac{\gamma}{\gamma-1}}).$$

- Note that a more stronger temporal-spatial condition is demanded, which is not generally acceptable in practice.

Three concepts on the L^2-norm stability

Definition 2

Strong (boundedness) stability: there exists an integer $n_* \geq 1$, such that

$$\|u^n\| \leq \|u^0\|, \quad n \geq n_*,$$

if the CLF number λ is fixed and small enough.

Definition 3

Monotonicity stability: (implies the strong (boundedness) stability)

$$\|u^{n+1}\| \leq \|u^n\|, \quad n \geq 0,$$

if the CFL number λ is fixed and small enough.

- Strong (boundedness) stability is a new concept.
- Monotonicity stabilityis often called strong stability in many literature.

What is the strong stability?

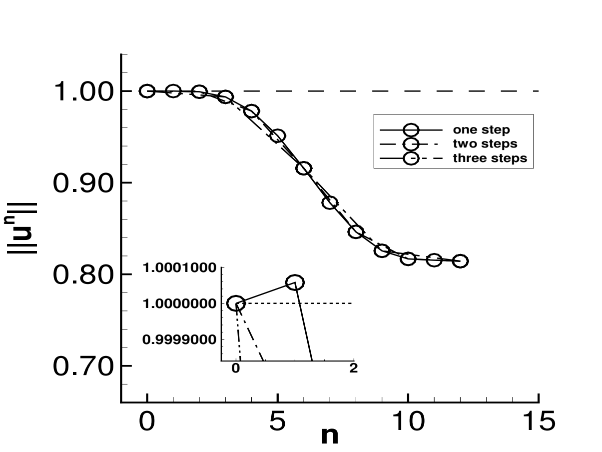


Figure: An example of strong (boundedness) stability. RKDG$(4, 4, 3)$ method: $\theta = 1$, $J = 64$, $\lambda = 0.05$.

Stability of some RKDG methods

Theorem 2.1 ($s = r$ or $s = r + 1$)

For these RKDG(s, r, k) method, we have

1. *arbitrary degree: the L^2-norm stability performance depends on the remainder when r is divided by 4, namely*

r	4,8,12	1,5,9	2,6,10	3,7,11
stability	strong	weak($r+1$)	weak($r+2$)	monotonicity

2. *lower degree: there has a better stability performance*

	r	1	2	3	4	5	6	7	8	9	10	11	12
monotonicity	$k \leq$	0	1	∞	1	2	3	∞	3	4	5	∞	5
strong	$k \leq$	0	1	∞	∞	2	3	∞	∞	4	5	∞	∞

- The above RK algorithms can be found in the works of S. Gottlieb, for example

Strong stability preserving time discretizations: a review.
In: ICOSAHOM 2014, Lecture Notes Computer Science Engineering, 106 (2015), 17–30.

Framework on energy analysis

1. This theorem is proved by the matrix transferring process, independently introduced by the investigation of the construct in time discretization and the properties of DG discretization.

2. The matrix transferring process can be done by the computer-aided manipulations, as in

Y. Xu, Q. Zhang, C.- W. Shu, and H. J. Wang,
The L^2-norm stability analysis of Runge-Kutta discontinuous Galerkin methods for linear hyperbolic equations, SIAM J. Numer. Anal. **57** (2019), no. 4, 1574–1601.

3. Actually, this process can be totally hidden in theory analysis.

Y. Xu, C. -W. Shu, and Q. Zhang,
Error estimate of the fourth-order Runge–Kutta discontinuous Galerkin methods for linear hyperbolic equations, accepted by SINUM (2020)

Y. Xu, X. Meng, C. -W. Shu, and Q. Zhang,
Superconvergence analysis of Runge-Kutta Discontinuous Galerkin Method for Linear Hyperbolic Equation, online J. Sci. Comput. (2020)

Framework on energy analysis

1. The temporal differences of stage solutions are related to the time derivatives of different orders, in some sense.
   - easily defined by induction;
   - not limited to one-step time marching, also works well for the multiple-steps time marching;
   - not limited to the uniform time step, also works well for different time step sizes.
2. The matrix transferring process provides a uniform manipulation to get an energy equation, which is nice to show the stability performance.
3. The stability mechanism of RKDG method is explicitly shown by
   - the temporal discretization: the termination index, and the sign of the central objective;
   - the interaction of the temporal discretization and the spatial discretization: the contribution index;
   - the spatial discretization.
4. Deep investigation on the relationships of the temporal differences.

2. Matrix transferring process

Temporal differences of stage solutions

- Let m be the number of multiple-steps (1 means one-step) time marching, i.e., updating the solution from t^n to t^{n+m}.
- Generalized notation: $u^{n,sa+b} = u^{n+a,b}$, where $a \geq 0$ and $0 \leq b < s$.

Definition 4 (temporal difference of stage solutions)

Let $\mathbb{D}_0(m)u^n = u^n$; then recursively define temporal difference of stage solutions

$$\mathbb{D}_\kappa(m)u^n = \sum_{0\leq\ell\leq\kappa} \sigma_{\kappa\ell}(m)u^{n,\ell}, \quad \kappa \geq 1,$$

to satisfy the kernel construction[a]

$$(\mathbb{D}_\kappa(m)u^n, v) = \tau\mathcal{H}(\mathbb{D}_{\kappa-1}(m)u^n, v), \quad \forall\, v \in V_h. \tag{4}$$

[a] used in: Y. Xu and e.t.c., SINUM2019, 1574–1601.

- They are easily obtained by linear combinations of numerical schemes, not depending on the specific definition of spatial discretization.

The equivalent representation of RKDG method

- At the same time when defining the temporal differences

$$\begin{bmatrix} \mathbb{D}_0(m)u^n \\ \mathbb{D}_1(m)u^n \\ \vdots \\ \mathbb{D}_{ms}(m)u^n \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \sigma_{10} & \sigma_{11} & & \\ \vdots & \vdots & \ddots & \\ \sigma_{ms,0} & \sigma_{ms,1} & \dots & \sigma_{ms,ms} \end{bmatrix} (m) \begin{bmatrix} u^n \\ u^{n,1} \\ \vdots \\ u^{n+m} \end{bmatrix},$$

we also achieve the evolution equation ($\alpha_0(m) > 0$ only for scaling)

$$\alpha_0(m)u^{n+m} = \sum_{0 \le i \le ms} \alpha_i(m)\mathbb{D}_i(m)u^n. \tag{5}$$

- Note that $\sum_{0\le\ell\le\kappa} \sigma_{\kappa\ell}(m) = 0$ and $\sigma_{\kappa\kappa}(m) \neq 0$.
- The evolution equation can be denoted by the evolution vector

$$\boldsymbol{\alpha}(m) = (\alpha_0(m), \alpha_1(m), \dots, \alpha_{ms}(m)). \tag{6}$$

If needed, define $\alpha_i(m) = 0$ for $i > ms$.

Matrix transferring process

- This process is able to automatically provide a good energy equation

$$\alpha_0^2(m)\left[\|u^{n+m}\|^2 - \|u^n\|^2\right] = SP(\ell) + TM(\ell), \tag{7}$$

for $\ell = 0, 1, \ldots$, with

temporal information: $$TM(\ell) = \sum_{0\le i,j\le ms} a_{ij}^{(\ell)}(m)(\mathbb{D}_i(m)u^n, \mathbb{D}_j(m)u^n),$$

spatial information: $$SP(\ell) = \sum_{0\le i,j\le ms} b_{ij}^{(\ell)}(m)\tau\mathcal{H}(\mathbb{D}_i(m)u^n, \mathbb{D}_j(m)u^n).$$

- The formulations of SP(ℓ) and TM(ℓ) can be respectively expressed by the $(ms+1)$th order symmetric matrices:

$$\mathbb{A}^{(\ell)}(m) = \{a_{ij}^{(\ell)}(m)\}, \quad \mathbb{B}^{(\ell)}(m) = \{b_{ij}^{(\ell)}(m)\}.$$

Note that the row/column indices i and j are taken from $\{0, 1, \ldots, ms\}$.

Matrix transferring process

- It is easy to see $\mathbb{A}^{(0)}(m) = \{a_{ij}^{(0)}(m)\}$ and $\mathbb{B}^{(0)} = \mathbb{O}$ for the initial situation, i.e.,

$$a_{00}^{(0)}(m) = \begin{cases} 0 & i = j = 0, \\ \alpha_i(m)\alpha_j(m), & \text{otherwise.} \end{cases} \qquad b_{ij}^{(0)}(m) = 0.$$

- The initial energy equation does not reflect the contribution of spatial discretization. Transform is needed!

- The symmetric demand on $\mathbb{B}^{\ell}(m)$ comes from the following observation, which is the starting point of the matrix transferring process.

Lemma 2.1

There holds the approximating skew-symmetric property

$$\mathcal{H}(w,v) + \mathcal{H}(v,w) = -2\beta(\theta - 1/2) \sum_{1\le j\le N} [\![w]\!]_{i+\frac{1}{2}} [\![v]\!]_{i+\frac{1}{2}}, \quad \forall\, w, v \in V_h.$$

Matrix transferring process

- The purpose in the ℓ-th matrix transform is very simple:
  1. move out the lower-order temporal information

     > Eliminating the entries at the ℓ-th row/column of $\mathbb{A}^{(\ell)}(m)$, at the left-top corner,

  2. and turn them into the spatial information

     > Exporting the entries at the ℓ-th row/column of $\mathbb{B}^{(\ell+1)}(m)$, at the right-bottom corner.

- The above actions employ two facts:
  1. the relationship of the temporal differences of stage solutions

$$(\mathbb{D}_\kappa(m)u^n, v) = \tau\mathcal{H}(\mathbb{D}_{\kappa-1}(m)u^n, v), \quad \forall\, v \in V_h.$$

  2. the symmetrical demand of all matrices.
- Hence, some high-order temporal information must be changed, due to the approximating skew-symmetric property (Lemma 2.1).

Description of matrix transferring process

- Since $a_{00}^{(0)}(m) = 0$ forever, we carry out the following transferring.
- Looking at the $(1,0)$ position of matrix (drop (m) for simplicity)

$$\mathbb{A}^{(0)} = \left[\begin{array}{c|cccc} 0 & \boxed{a_{01}^{(0)}} & a_{02}^{(0)} & \cdots & a_{0,ms}^{(0)} \\ \hline \boxed{a_{10}^{(0)}} & a_{11}^{(0)} & a_{12}^{(0)} & \cdots & a_{1,ms}^{(0)} \\ a_{20}^{(0)} & a_{21}^{(0)} & a_{22}^{(0)} & \cdots & a_{2,ms}^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{ms,0}^{(0)} & a_{ms,1}^{(0)} & a_{ms,2}^{(0)} & \cdots & a_{ms,ms}^{(0)} \end{array}\right],$$

- Noticing the symmetry of matrix, we have

$$(\mathbb{D}_0 u^n, \mathbb{D}_1 u^n) + (\mathbb{D}_1 u^n, \mathbb{D}_0 u^n) = 2\tau\mathcal{H}(\mathbb{D}_0 u^n, \mathbb{D}_0 u^n).$$

- This eliminates $a_{10}^{(0)}$ such that $a_{10}^{(1)} = 0$ and $b_{00}^{(1)} = 2a_{10}^{(0)}$.

Description of matrix transferring process

- Looking at the $(2,0)$ position of matrix

$$\mathbb{A}^{(0)} = \left[\begin{array}{c|cccc} 0 & 0 & \boxed{a_{02}^{(0)}} & \cdots & a_{0,ms}^{(0)} \\ \hline 0 & \boxed{a_{11}^{(0)}} & a_{12}^{(0)} & \cdots & a_{1,ms}^{(0)} \\ \boxed{a_{20}^{(0)}} & a_{21}^{(0)} & a_{22}^{(0)} & \cdots & a_{2,ms}^{(0)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{ms,0}^{(0)} & a_{ms,1}^{(0)} & a_{ms,2}^{(0)} & \cdots & a_{ms,ms}^{(0)} \end{array}\right].$$

- To eliminate $a_{20}^{(0)}$ and preserve the symmetry of $\mathbb{B}^{(1)}$, we need the help of the time information at the $(1,1)$ position

$$(\mathbb{D}_2 u^n, \mathbb{D}_0 u^n) + (\mathbb{D}_1 u^n, \mathbb{D}_1 u^n) = \tau\mathcal{H}(\mathbb{D}_1 u^n, \mathbb{D}_0 u^n) + \tau\mathcal{H}(\mathbb{D}_0 u^n, \mathbb{D}_1 u^n),$$

This leads into $a_{20}^{(1)} = 0$, $a_{11}^{(1)} = a_{11}^{(0)} - 2a_{20}^{(0)}$, and $b_{10}^{(1)} = 2a_{20}^{(0)}$.

Description of matrix transferring process

- Looking at the $(3,0)$ position of matrix

$$\mathbb{A}^{(0)} = \left[\begin{array}{c|cccc} 0 & 0 & 0 & \boxed{a^{(0)}_{0,3}} & \rightarrow \\ \hline 0 & a^{(0)}_{11} & \boxed{a^{(0)}_{12}} & \rightarrow & \cdots \\ 0 & \boxed{a^{(0)}_{21}} & a^{(0)}_{22} & a^{(0)}_{23} & \cdots \\ \boxed{a^{(0)}_{30}} & \downarrow & a^{(0)}_{32} & a^{(0)}_{33} & \cdots \\ \downarrow & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

- To eliminate $a^{(0)}_{30}$ and preserve the symmetry of $\mathbb{B}^{(1)}$, we need the help of the time information at the $(2,1)$ position

$$(\mathbb{D}_3 u^n, \mathbb{D}_0 u^n) + (\mathbb{D}_1 u^n, \mathbb{D}_2 u^n) = \tau\mathcal{H}(\mathbb{D}_2 u^n, \mathbb{D}_0 u^n) + \tau\mathcal{H}(\mathbb{D}_0 u^n, \mathbb{D}_2 u^n),$$

This leads to $a^{(1)}_{30} = 0$, $a^{(1)}_{21} = a^{(0)}_{21} - a^{(0)}_{20}$, and $b^{(1)}_{20} = 2a^{(0)}_{30}$.

Formulations of matrix transferring process

- Let $\ell \geq 1$ be the number of matrix transform, and assume

$$\mathbb{A}^{(\ell-1)} = \{a_{ij}^{(\ell-1)}\}_{i,j\geq 0} = \left[\begin{array}{c:c:cccc} \mathbb{O} & \mathbb{O} & \mathbb{O} & \cdots & \mathbb{O} \\ \hdashline \mathbb{O} & a_{\ell-1,\ell-1}^{(\ell-1)} & a_{\ell-1,\ell}^{(\ell-1)} & \cdots & a_{\ell-1,ms}^{(\ell-1)} \\ \hdashline \mathbb{O} & a_{\ell,\ell-1}^{(\ell-1)} & a_{\ell\ell}^{(\ell-1)} & \cdots & a_{\ell,ms}^{(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & a_{ms,\ell-1}^{(\ell-1)} & a_{ms,\ell}^{(\ell-1)} & \cdots & a_{ms,ms}^{(\ell-1)} \end{array}\right],$$

$$\mathbb{B}^{(\ell-1)} = \{b_{ij}^{(\ell-1)}\}_{i,j\geq 0} = \left[\begin{array}{c:c:cccc} \star & \star & \star & \cdots & \star \\ \hdashline \star & b_{\ell-1,\ell-1}^{(\ell-1)} & b_{\ell-1,\ell}^{(\ell-1)} & \cdots & b_{\ell-1,ms}^{(\ell-1)} \\ \hdashline \star & b_{\ell,\ell-1}^{(\ell-1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & b_{ms,\ell-1}^{(\ell-1)} & 0 & \cdots & 0 \end{array}\right].$$

- In $\mathbb{B}^{(\ell-1)}$, only the entries marked by $\star$ are allowed to be nonzero.

Formulations of matrix transferring process

- If $a^{(\ell-1)}_{\ell-1,\ell-1}(m) \neq 0$, carry out the following manipulations[1]

$$a^{(\ell)}_{ij}(m) = \begin{cases} 0, & j=\ell-1, \\ a^{(\ell-1)}_{ij}(m) - 2a^{(\ell-1)}_{i+1,j-1}(m), & i=\ell \text{ and } j=\ell, \\ a^{(\ell-1)}_{ij}(m) - a^{(\ell-1)}_{i+1,j-1}(m), & \ell+1\leq i\leq ms-1 \text{ and } j=\ell, \\ a^{(\ell-1)}_{ij}(m), & \text{otherwise}, \end{cases}$$

$$b^{(\ell)}_{ij}(m) = \begin{cases} 2a^{(\ell-1)}_{i+1,j}(m), & \ell-1\leq i\leq ms-1 \text{ and } j=\ell-1, \\ b^{(\ell-1)}_{ij}(m), & \text{otherwise}, \end{cases}$$

- Otherwise, this nonzero diagonal entry is called the central objective, and the matrix transferring process is terminated.

[1] Since the matrices are symmetrical, only the lower-triangular are presented.

Algorithm of matrix transferring process

Algorithm 1 The matrix transferring process

Require: $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{ms})$
Ensure: $\mathbb{A}$ and $\mathbb{B}$

$\mathbb{A} \leftarrow \boldsymbol{\alpha}^{\perp}\boldsymbol{\alpha}$, $\mathbb{B} \leftarrow \mathbb{O}$
for $j \leftarrow 0$ to ms **do**
    if $a_{jj} \neq 0$ **then**
        Break
    end if
    $b_{jj} \leftarrow 2a_{j+1,j}$
    $a_{j+1,j} \leftarrow 0$
    for $i \leftarrow j+1$ to ms **do**
        $b_{ij} \leftarrow 2a_{i+1,j}$
        $b_{ji} \leftarrow 2a_{j,i+1}$
        $a_{i,j+1} \leftarrow a_{i,j+1} - a_{i+1,j}$
        $a_{j+1,i} \leftarrow a_{j+1,i} - a_{j,i+1}$
        $a_{i+1,j} \leftarrow 0$
        $a_{j,i+1} \leftarrow 0$
    end for
end for

Two important indices

Definition 5

Assume the matrix transferring process stop when $a_{\ell\ell}^{(\ell)}(m) \neq 0$. Denote

$$\zeta(m) = \ell, \tag{8}$$

and this number is defined as the termination index.

Definition 6

The contribution index of the spatial DG discretization is defined by

$$\rho(m) = \min\{\kappa : \kappa \in BI(m) \cup \{\zeta(m)\}\}. \tag{9}$$

where $\zeta(m)$ is the termination index and

$$BI(m) = \left\{\kappa \colon 0 \leq \kappa \leq \zeta(m) - 1 \text{ and } \det\left\{b_{ij}^{(\zeta(m))}(m)\right\}_{0\leq i,j\leq\kappa} \leq 0\right\}.$$

- $\rho(m) \leq \zeta(m)$ is the maximal order of SPD submatrix at the left-top corner.

3. energy analysis

Elemental inequality

Lemma 2.2 (denote $\zeta = \zeta(m)$ and $\rho = \rho(m)$ for simiplicity)

Let $\varepsilon = \varepsilon(m) > 0$ *be the smallest eigenvalue of the left-top SPD submatrix*

$$\mathbb{B}_{\rho}^{(\zeta)}(m) = \{b_{ij}^{(\zeta)}\}_{0\le i,j\le \rho-1}.$$

There holds the inequality

$$\alpha_0^2(m)\left[\|u^{n+m}\|^2 - \|u^n\|^2\right] \le \mathcal{Y}_1(m) + \mathcal{Y}_2(m), \tag{10}$$

where λ *is the CFL number and*

$$\mathcal{Y}_1(m) = \left[a_{\zeta\zeta}^{(\zeta)}(m) + \lambda\mathcal{Q}_1(\lambda) + \lambda\mathcal{Q}_2(\lambda)\right]\|\mathbb{D}_{\zeta}(m)u^n\|^2 + \lambda\mathcal{Q}_2(\lambda)\|\mathbb{D}_{\rho}u^n\|^2, \tag{11}$$

$$\mathcal{Y}_2(m) = -\frac{1}{2}\varepsilon(m)\beta(\theta - \frac{1}{2})\tau \sum_{0\le i\le \rho-1} \|[\![\mathbb{D}_i(m)u^n]\!]\|_{\Gamma_h}^2. \tag{12}$$

Here $\mathcal{Q}_1(\cdot)$ *and* $\mathcal{Q}_2(\cdot)$ *are generic polynomials with nonnegative coefficients.*

Remarks on Lemma 2.2

- The stability effects of time-marching are explicitly shown by the term $\mathcal{Y}_1$, where
  - $\mathcal{Q}_1(\cdot)$: the negative effects due to the high order temporal differences;
  - $\mathcal{Q}_2(\cdot)$: the negative effects due to the skew-symmetric property of spatial discretization.
  - The sign of the central objective $a_{\zeta\zeta}^{(\zeta)}(m)$ is very important.
- The stability mechanism, inherited from semi-discrete DG method, is explicitly showed by $\mathcal{Y}_2 \leq 0$.
- The proof is trivial.

Proof of Lemma 2.2 (cont.)

- Recall some elemental properties of the DG discretization:
- The non-positive property (numerical viscosity):

$$\mathcal{H}(w,w) = -\beta(\theta - 1/2)\|[\![w]\!]\|^2_{\Gamma_h}, \quad w \in V_h.$$

- A development on the non-positive property:

Lemma 2.3 (the positive contribution of interface jumps)

Given arbitrary (row and column) index set $\mathcal{G}$. Assume $\mathbb{G} = \{g_{ij}\}_{i,j\in\mathcal{G}}$ form a symmetric positive semidefinite matrix, then there holds

$$\sum_{i\in\mathcal{G}}\sum_{j\in\mathcal{G}} g_{ij}\mathcal{H}(w_i, w_j) \leq 0,$$

for any function sequence $\{w_i \in V_h : i \in \mathcal{G}\}$.

- "weak" boundedness

$$|\mathcal{H}(w,v)| \leq C|\beta|h^{-1}\|w\|\|v\|, \quad \forall\, w, v \in V_h.$$

Proof of Lemma 2.2 (cont.)

- By the matrix transferring process, we have gotten

$$\alpha_0^2(m)\left[\|u^{n+m}\|^2 - \|u^n\|^2\right] = SP(\zeta) + TM(\zeta), \tag{13}$$

where $\zeta = \zeta(m)$ is the termination index, and

$$\begin{aligned} TM(\zeta) &= \sum_{\zeta \le i,j \le ms} a_{ij}^{(\zeta)}(m)(\mathbb{D}_i(m)u^n, \mathbb{D}_j(m)u^n), \\ SP(\zeta) &= \sum_{0 \le i,j \le ms} b_{ij}^{(\zeta)}(m)\tau\mathcal{H}(\mathbb{D}_i(m)u^n, \mathbb{D}_j(m)u^n). \end{aligned} \tag{14}$$

- It is easy to see from the "weak" boundedness that

$$TM(\zeta) \le \left[a_{\zeta\zeta}^{(\zeta)} + \lambda\mathcal{Q}_1(\lambda)\right]\|\mathbb{D}_\zeta u^n\|^2.$$

Proof of Lemma 2.2 (cont.)

- Drop the index m below.
- Split the index set $\{0, 1, \dots, ms\}$ into three subsets

$$\pi_1 = \{0, \dots, \rho - 1\}, \quad \pi_2 = \{\rho, \dots, \zeta - 1\}, \quad \pi_3 = \{\zeta, \dots, ms\}.$$

  Note that $\pi_1 = \emptyset$ if $\rho = 0$, and $\pi_2 = \emptyset$ if $\rho = \zeta$.
- The second term in RHS can be written in the form

$$SP(\zeta) = \sum_{\xi,\eta \in \{1,2,3\}} \underbrace{\sum_{i \in \pi_\xi, j \in \pi_\eta} \tau b_{ij}^{(\zeta)} \mathcal{H}(\mathbb{D}_i u^n, \mathbb{D}_j u^n)}_{T_{\xi\eta}}.$$

- If the index subset is empty, the corresponding term is equal to zero.

Proof of Lemma 2.2 (cont.)

- The submatrix $\mathbb{B}_{\rho-1}^{(\zeta)}$ is symmetric positive definite.
- Hence the negative property of DG discretization yields

$$T_{11} \leq -\varepsilon\beta(\theta - \frac{1}{2})\tau \sum_{i\in\pi_1} \|[\![\mathbb{D}_i u^n]\!]\|_{\Gamma_h}^2.$$

- By the approximate skew-symmetric property, the relationship among temporal differences and the weak boundedness, the inverse inequality, and Young's inequality, we have

$$\begin{aligned} T_{12} + T_{21} = & -\beta(\theta - \frac{1}{2})\tau \sum_{i\in\pi_1, j\in\pi_2} b_{ij}^{(\zeta)} \|[\![\mathbb{D}_i u^n]\!]\|_{\Gamma_h} \|[\![\mathbb{D}_j u^n]\!]\|_{\Gamma_h} \\ \leq & \frac{1}{4}\varepsilon\beta(\theta - \frac{1}{2})\tau \sum_{i\in\pi_1} \|[\![\mathbb{D}_i u^n]\!]\|_{\Gamma_h}^2 + C_\varepsilon \tau \sum_{j\in\pi_2} \|[\![\mathbb{D}_j u^n]\!]\|_{\Gamma_h}^2 \\ \leq & \frac{1}{4}\varepsilon\beta(\theta - \frac{1}{2})\tau \sum_{i\in\pi_1} \|[\![\mathbb{D}_i u^n]\!]\|_{\Gamma_h}^2 + \lambda \mathcal{Q}_2(\lambda) \|\mathbb{D}_\rho u^n\|^2. \end{aligned}$$

Proof of Lemma 2.2

- Similarly, we also have

$$T_{22} + T_{23} + T_{32} \le \lambda \mathcal{Q}_2(\lambda)\|\mathbb{D}_\rho u^n\|^2 + \lambda \mathcal{Q}_2(\lambda)\|\mathbb{D}_\varsigma u^n\|^2.$$

- Along the same line, we have

$$T_{13} + T_{31} \le \frac{1}{4}\varepsilon\beta(\theta - \frac{1}{2})\tau \sum_{i\in\pi_1} \|[\![\mathbb{D}_i u^n]\!]\|_{\Gamma_h}^2 + \lambda \mathcal{Q}_1(\lambda)\|\mathbb{D}_\varsigma u^n\|^2.$$

- It is trivial to see that $T_{33} = 0$, since all related coefficients are zero.
- Collecting the above estimates, we complete the proof.

Main results on stability

Theorem 2.2 (stated for $m = 1$)

For the piecewise polynomials of arbitrary degree $k \geq 0$, the RKDG method has the following stability performance at least:

1. $a_{\zeta\zeta}^{(\zeta)} < 0$ *and* $\rho = \zeta$*: monotonicity stability;*
2. $a_{\zeta\zeta}^{(\zeta)} < 0$ *and* $\rho < \zeta$*: weak(γ) stability with* $\gamma = 2\rho + 1$*;*
3. $a_{\zeta\zeta}^{(\zeta)} > 0$*: weak($\gamma$) stability with* $\gamma = \min(2\zeta, 2\rho + 1)$.

- The proof is easy, as a corollary of Lemma 2.2.
- The contribute of $\mathcal{Y}_2$ is not considered yet.
- The strong stability is implied by the multi-step marching, namely $m > 1$.

Discussions on Theorem 2.2

1. The first conclusion:
   - neither the temporal nor the spatial discretization produces any anti-dissipative energy.

   Example: RKDG$(3,3,k)$ scheme.

2. The second conclusion is pointed to an intermediate state:
   - The temporal discretization provides dissipative energy, but the spatial discretization causes some anti-dissipative modes that must be controlled by reducing the time step.
   - This trouble results from the approximate skew-symmetric property.

   Example: RKDG$(4,4,k)$ scheme.

3. The third conclusion:
   - the temporal discretization has an anti-dissipative energy that can only be controlled through a time-step reduction.

   Example: RKDG$(1,1,k)$ and RKDG$(2,2,k)$ scheme.

More discussions on the second conclusion

- When there holds the strictly skew-symmetric property

$$\mathcal{H}(w,v)+\mathcal{H}(v,w)=0,\quad \forall\, w,v\in V_h,$$

for example,

  - the functions in V_h are restricted to be continuous (the DG method degenerates to the standard finite element method),
  - or the central numerical flux (i.e., $\theta = 1/2$) is used,

the spatial discretization does not cause any trouble in the L^2-norm stability of fully-discrete scheme.

- As a result, the scheme has the monotonicity stability.

4. Examples

Example: the RKDG(3,3,k) method

Example 2.1 (Zhang and Shu, SINUM2010)

The RKDG($3,3,k$) method

$$
\begin{aligned}
(u^{n,1}, v) &= (u^n, v) + \tau\mathcal{H}(u^n, v),\\
(u^{n,2}, v) &= (u^{n,1}, v) + \tau\mathcal{H}(u^{n,1}, v),\\
(u^{n+1}, v) &= \frac{1}{3}(u^n, v) + \frac{1}{2}(u^{n,1}, v) + \frac{1}{6}(u^{n,2}, v) + \frac{\tau}{6}\mathcal{H}(u^{n,2}, v).
\end{aligned}
$$

has the monotonicity stability for any degree k.

Example: the RKDG(3,3,k) method

- First transferring:

$$\mathbb{A}^{(1)} = \left[\begin{array}{c:ccc} 0 & & & \\ \hdashline & 0 & 12 & 6 \\ & 12 & 9 & 3 \\ & 6 & 3 & 1 \end{array}\right], \quad \mathbb{B}^{(1)} = \left[\begin{array}{c:ccc} 72 & 36 & 12 & 0 \\ \hdashline 36 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Example: the RKDG(3,3,k) method

- Second transferring:

$$\mathbb{A}^{(2)} = \left[\begin{array}{cc:cc} 0 & & & \\ & 0 & & \\ \hdashline & & -3 & 3 \\ & & 3 & 1 \end{array}\right], \quad \mathbb{B}^{(2)} = \left[\begin{array}{cc:cc} 72 & 36 & 12 & 0 \\ 36 & 24 & 12 & 0 \\ \hdashline 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{22}^{(2)} = -3$, the transferring process is terminated with $\zeta = 2$.

Example: the RKDG(3,3,k) method

- Second transferring:

$$\mathbb{A}^{(2)} = \left[\begin{array}{cc:cc} 0 & & & \\ & 0 & & \\ \hdashline & & -3 & 3 \\ & & 3 & 1 \end{array}\right], \quad \mathbb{B}^{(2)} = \left[\begin{array}{cc:cc} 72 & 36 & 12 & 0 \\ 36 & 24 & 12 & 0 \\ \hdashline 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{22}^{(2)} = -3$, the transferring process is terminated with $\zeta = 2$.
- Furthermore we get $\rho = 2$, since the former two leading principal minors are respectively equal to 72 and 432.

Example: the RKDG(3,3,k) method

- Second transferring:

$$\mathbb{A}^{(2)} = \left[\begin{array}{cc:cc} 0 & & & \\ & 0 & & \\ \hdashline & & -3 & 3 \\ & & 3 & 1 \end{array}\right], \quad \mathbb{B}^{(2)} = \left[\begin{array}{cc:cc} 72 & 36 & 12 & 0 \\ 36 & 24 & 12 & 0 \\ \hdashline 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{22}^{(2)} = -3$, the transferring process is terminated with $\zeta = 2$.
- Furthermore we get $\rho = 2$, since the former two leading principal minors are respectively equal to 72 and 432.
- Since $a_{22}^{(2)} = -3 < 0$ and $a_{22}^{(2)}\|\mathbb{D}_2 u^n\|^2$ provides an additional stability mechanism, the monotonicity stability is proved.

Example: the RKDG(3,3,k) method

- Second transferring:

$$\mathbb{A}^{(2)} = \left[\begin{array}{cc:cc} 0 & & & \\ & 0 & & \\ \hdashline & & -3 & 3 \\ & & 3 & 1 \end{array}\right], \quad \mathbb{B}^{(2)} = \left[\begin{array}{cc:cc} 72 & 36 & 12 & 0 \\ 36 & 24 & 12 & 0 \\ \hdashline 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{22}^{(2)} = -3$, the transferring process is terminated with $\zeta = 2$.
- Furthermore we get $\rho = 2$, since the former two leading principal minors are respectively equal to 72 and 432.
- Since $a_{22}^{(2)} = -3 < 0$ and $a_{22}^{(2)}\|\mathbb{D}_2 u^n\|^2$ provides an additional stability mechanism, the monotonicity stability is proved.

Example: the RKDG(4,4,k) method

Example 2.2 (classical RK algorithm)

The RKDG$(4,4,k)$ *method*

$$\begin{aligned}
(u^{n,1},v) &= (u^n,v) + \tau\mathcal{H}(u^n,v),\\
(u^{n,2},v) &= (u^{n,1},v) + \tau\mathcal{H}(u^{n,1},v),\\
(u^{n,3},v) &= (u^{n,2},v) + \tau\mathcal{H}(u^{n,2},v),\\
(u^{n+1},v) &= \frac{3}{8}(u^n,v) + \frac{1}{3}(u^{n,1},v) + \frac{1}{4}(u^{n,2},v) + \frac{1}{24}(u^{n,3},v) + \frac{1}{24}\tau\mathcal{H}(u^{n,3},v).
\end{aligned}$$

has the strong stability with $n_\star = 2$.

Example: the RKDG(4,4,k) method

- Define the temporal differences

$$\begin{bmatrix} \mathbb{D}_0 u^n \\ \mathbb{D}_1 u^n \\ \mathbb{D}_2 u^n \\ \mathbb{D}_3 u^n \\ \mathbb{D}_4 u^n \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ -8 & -12 & 0 & -4 & 24 \end{bmatrix} \begin{bmatrix} u^n \\ u^{n,1} \\ u^{n,2} \\ u^{n,3} \\ u^{n+1} \end{bmatrix}$$

- Obtain the evolution identity

$$24u^{n+1} = 24\mathbb{D}_0 u^n + 24\mathbb{D}_1 u^n + 12\mathbb{D}_2 u^n + 4\mathbb{D}_3 u^n + \mathbb{D}_4 u^n,$$

with $\boldsymbol{\alpha} = (24, 24, 12, 4, 1)$.

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(0)} = \begin{bmatrix} 0 & 576 & 288 & 96 & 24 \\ 576 & 576 & 288 & 96 & 24 \\ 288 & 288 & 144 & 48 & 12 \\ 96 & 96 & 48 & 16 & 4 \\ 24 & 24 & 12 & 4 & 1 \end{bmatrix}, \quad \mathbb{B}^{(0)} = \mathbb{O}.$$

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(1)} = \left[\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 192 & 72 & 24 \\ & 192 & 144 & 48 & 12 \\ & 72 & 48 & 16 & 4 \\ & 24 & 12 & 4 & 1 \end{array}\right], \quad \mathbb{B}^{(1)} = \left[\begin{array}{c|cccc} 1152 & 576 & 192 & 48 & 0 \\ \hline 576 & 0 & 0 & 0 & 0 \\ 192 & 0 & 0 & 0 & 0 \\ 48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(2)} = \left[\begin{array}{cc:ccc} 0 & & & & \\ & 0 & & & \\ \hdashline & & 0 & 24 & 12 \\ & & 24 & 16 & 4 \\ & & 12 & 4 & 1 \end{array}\right], \quad \mathbb{B}^{(2)} = \left[\begin{array}{cc:ccc} 1152 & 576 & 192 & 48 & 0 \\ 576 & 384 & 144 & 48 & 0 \\ \hdashline 192 & 144 & 0 & 0 & 0 \\ 48 & 48 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(3)} = \left[\begin{array}{ccc:cc} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ \hdashline & & & -8 & 4 \\ & & & 4 & 1 \end{array}\right], \quad \mathbb{B}^{(3)} = \left[\begin{array}{ccc:cc} 1152 & 576 & 192 & 48 & 0 \\ 576 & 384 & 144 & 48 & 0 \\ 192 & 144 & 48 & 24 & 0 \\ \hdashline 48 & 48 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(3)} = \left[\begin{array}{ccc|cc} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ \hline & & & -8 & 4 \\ & & & 4 & 1 \end{array}\right], \quad \mathbb{B}^{(3)} = \left[\begin{array}{ccc|cc} 1152 & 576 & 192 & 48 & 0 \\ 576 & 384 & 144 & 48 & 0 \\ 192 & 144 & 48 & 24 & 0 \\ \hline 48 & 48 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{33}^{(3)} = -8 < 0$, we stop the transferring and obtain $\zeta = 3$.

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(3)} = \left[\begin{array}{ccc:cc} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ \hdashline & & & -8 & 4 \\ & & & 4 & 1 \end{array}\right], \quad \mathbb{B}^{(3)} = \left[\begin{array}{ccc:cc} 1152 & 576 & 192 & 48 & 0 \\ 576 & 384 & 144 & 48 & 0 \\ 192 & 144 & 48 & 24 & 0 \\ \hdashline 48 & 48 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{33}^{(3)} = -8 < 0$, we stop the transferring and obtain $\zeta = 3$.
- It is easy to see $\rho = 2 = \zeta - 1$, since the former three leading principal minors in order of

$$\begin{bmatrix} 1152 & 576 & 192 \\ 576 & 384 & 144 \\ 192 & 144 & 48 \end{bmatrix}.$$

are 1152, 110592, and -884736.

Example: the RKDG(4,4,k) method

- Matrix transferring process

$$\mathbb{A}^{(3)} = \left[\begin{array}{ccc|cc} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ \hline & & & -8 & 4 \\ & & & 4 & 1 \end{array}\right], \quad \mathbb{B}^{(3)} = \left[\begin{array}{ccc|cc} 1152 & 576 & 192 & 48 & 0 \\ 576 & 384 & 144 & 48 & 0 \\ 192 & 144 & 48 & 24 & 0 \\ \hline 48 & 48 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

- Since $a_{33}^{(3)} = -8 < 0$, we stop the transferring and obtain $\zeta = 3$.
- It is easy to see $\rho = 2 = \zeta - 1$, since the former three leading principal minors in order of

$$\begin{bmatrix} 1152 & 576 & 192 \\ 576 & 384 & 144 \\ 192 & 144 & 48 \end{bmatrix}.$$

  are 1152, 110592, and -884736.
- Hence the RKDG$(4,4,k)$ scheme with one-step time-marching has the weak(5) stability at least.

Example: Two-steps of RKDG(4,4,k) method

- Two-steps time-marching of RKDG$(4,4,k)$ method can be looked upon as one-step time-marching of an RKDG$(8,4,k)$ method, in the form

$$
\begin{aligned}
(u^{n,\ell+1}, v) &= (u^{n,\ell}, v) + \tau\mathcal{H}(u^{n,\ell}, v), \qquad \ell = 0,1,2,\\
(u^{n,4}, v) &= \Big(\frac{3}{8}u^{n} + \frac{1}{3}u^{n,1} + \frac{1}{4}u^{n,2} + \frac{1}{24}u^{n,3}, v\Big) + \frac{1}{24}\tau\mathcal{H}(u^{n,3}, v),\\
(u^{n,\ell+1}, v) &= (u^{n,\ell}, v) + \tau\mathcal{H}(u^{n,\ell}, v), \qquad \ell = 4,5,6,\\
(u^{n+2}, v) &= \Big(\frac{3}{8}u^{n,4} + \frac{1}{3}u^{n,5} + \frac{1}{4}u^{n,6} + \frac{1}{24}u^{n,7}, v\Big) + \frac{1}{24}\tau\mathcal{H}(u^{n,7}, v).
\end{aligned}
$$

- Notations: $u^{n,4} = u^{n+1}, u^{n,5} = u^{n+1,1}, u^{n,6} = u^{n+1,2}, \ldots$
- The related discussion on the multi-steps RKDG(4,4,k) method is cited from the paper

Y. Xu, SIAM J. Numer. Anal. **57** (2019), no. 4, 1574–1601.

Example: Two-steps of RKDG(4,4,k) method

- Repeating the above process, we can define temporal differences:

$$\begin{bmatrix} \mathbb{D}_0 u^n \\ \mathbb{D}_1 u^n \\ \mathbb{D}_2 u^n \\ \mathbb{D}_3 u^n \\ \mathbb{D}_4 u^n \\ \mathbb{D}_5 u^n \\ \mathbb{D}_6 u^n \\ \mathbb{D}_7 u^n \\ \mathbb{D}_8 u^n \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & & \\ -1 & 1 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ -1 & 3 & -3 & 1 & & & & & \\ -8 & -12 & 0 & -4 & 24 & & & & \\ 44 & 36 & 12 & 4 & -120 & 24 & & & \\ -80 & -24 & 0 & 8 & 216 & -144 & 24 & & \\ 8 & -120 & -72 & -8 & -24 & 360 & -168 & 24 & \\ 64 & 192 & 0 & -64 & -384 & -576 & 384 & -192 & 576 \end{bmatrix} \begin{bmatrix} u^n \\ u^{n,1} \\ u^{n,2} \\ u^{n,3} \\ u^{n,4} \\ u^{n,5} \\ u^{n,6} \\ u^{n,7} \\ u^{n+2} \end{bmatrix}$$

- The former five lines are the same as those for one-step time-marching.
- Obtain the evolution identity

$$\alpha_0 u^{n+2} = \sum_{0\le i\le 8} \alpha_i \mathbb{D}_i u^n,$$

with $\alpha = (576, 1152, 1152, 768, 384, 144, 40, 8, 1)$.

Example: Two-steps of RKDG(4,4,k) method

- The process stops at the third transferring

$$
\mathbb{A}^{(3)} = \left[\begin{array}{ccc|cccccc}
0 & & & & & & & & \\
& 0 & & & & & & & \\
& & 0 & & & & & & \\
\hline
& & & -9216 & 170496 & 73152 & 22656 & 4992 & 768 \\
& & & 170496 & 147456 & 55296 & 15360 & 3072 & 384 \\
& & & 73152 & 55296 & 20736 & 5760 & 1152 & 144 \\
& & & 22656 & 15360 & 5760 & 1600 & 320 & 40 \\
& & & 4992 & 3072 & 1152 & 320 & 64 & 8 \\
& & & 768 & 384 & 144 & 40 & 8 & 1
\end{array}\right],
$$

$$
\mathbb{B}^{(3)} = \left[\begin{array}{ccc|cccccc}
1327104 & 1327104 & 884736 & 442368 & 165888 & 46080 & 9216 & 1152 & 0 \\
1327104 & 1769472 & 1327104 & 718848 & 285696 & 82944 & 17280 & 2304 & 0 \\
884736 & 1327104 & 1050624 & 599040 & 248832 & 74880 & 16128 & 2304 & 0 \\
\hline
442368 & 718848 & 599040 & 0 & 0 & 0 & 0 & 0 & 0 \\
165888 & 285696 & 248832 & 0 & 0 & 0 & 0 & 0 & 0 \\
46080 & 82944 & 74880 & 0 & 0 & 0 & 0 & 0 & 0 \\
9216 & 17280 & 16128 & 0 & 0 & 0 & 0 & 0 & 0 \\
1152 & 2304 & 2304 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

- Obtain $\zeta = 3$, and $a_{33}^{(3)} = -9216 < 0$.
- It is easy to see $\rho = 3$, since the three leading principal minors are 1327104, 587068342272 and $1082084368475750 4$ respectively.
- Hence the monotonicity stability is proved for two-steps time-marching.

Example: Three-steps of of RKDG(4,4,k) method

- Three-steps time-marching of the RKDG($4,4,k$) method can be looked upon as one-step time-marching of an RKDG($12,4,k$) method.
- Define temporal differences, and obtain the evolution identity with

$$\begin{aligned}\boldsymbol{\alpha} =&(13824, 41472, 62208, 62208, 46656,\\ &27648, 13248, 5184, 1656, 424, 84, 12, 1).\end{aligned}$$

- The matrix transferring stops with $\zeta = 3$, and $a_{33}^{(3)} = -7962624 < 0$.
- Also we have $\rho = 3$, since three leading principal minors in

$$\begin{bmatrix} 1146617856 & 1719926784 & 1719926784 \\ 1719926784 & 3439853568 & 3869835264 \\ 1719926784 & 3869835264 & 4634247168 \end{bmatrix}.$$

  are $1146617856, 986049380773527552$ and $1177731069679864357532467\,20$.
- Hence the monotonicity stability is proved for three-steps time-marching.

Example: more applications

- The above analysis framework works well for many fully-discrete RKDG methods, even $s \neq r$ and non-uniform time step.
- For example, consider the RKDG$(10,4,k)$ method [Gottlieb, 2009]

$$
\begin{aligned}
(u^{n,i+1},v) &= (u^{n,i},v) + \frac{1}{6}\tau\mathcal{H}(u^{n,i},v), && i=0,1,2,3,\\
(u^{n,5},v) &= (\frac{3}{5}u^{n} + \frac{2}{5}u^{n,4},v) + \frac{1}{15}\tau\mathcal{H}(u^{n,4},v), &&\\
(u^{n,i+1},v) &= (u^{n,i},v) + \frac{1}{6}\tau\mathcal{H}(u^{n,i},v), && i=5,6,7,8,\\
(u^{n+1},v) &= (\frac{1}{25}u^{n} + \frac{9}{25}u^{n,4} + \frac{3}{5}u^{n,9},v) + \frac{3}{50}\tau\mathcal{H}(u^{n,4},v) + \frac{1}{10}\tau\mathcal{H}(u^{n,9},v). &&
\end{aligned}
$$

  It can be proved to have the monotonicity stability.
- A deep discussion on temporal differences can yield the monotonicity stability for lower-degree polynomials, as expected.

5. Development

Remarks on the previous analysis framework

- At the previous study, three important quantities
  1. the termination index;
  2. the contribution index of spatial discretization;
  3. the sign of the central objective

  are computed through the matrix transferring process.
- However,
  - The above process may be carried out again and again for different number $m = 1, 2, \ldots$.
  - The above heavy manipulations are implemented by the help of computer. By introduction of $\alpha_0(m)$, all arithmetic is accurate!
- Some development:

  How to get rid of (or reduce) the aid of computer and set up the theory results in a uniform theory framework?

Solutions on this purpose

- In fact, we are able to find out the essential property hidden in the matrix transferring process, by making two minor modifications on the previous works:
  - new expression for the m-step time marching

$$(u^{n,\ell+1}, v) = \sum_{0\leq\kappa\leq\ell} \Big[c_{\ell\kappa}(m)(u^{n,\kappa}, v) + m\tau d_{\ell\kappa}(m)\mathcal{H}(u^{n,\kappa}, v) \Big].$$

  - new kernel construction

$$(\mathbb{D}_\kappa(m)u^n, v) = m\tau\mathcal{H}(\mathbb{D}_{\kappa-1}(m)u^n, v), \quad \forall\, v \in V_h, \tag{15}$$

    when defining the temporal difference of stage solutions.
- The others are the same.
- The analysis process is long and complex, although the proof line is the same.

The matrix transform

Proposition 2.1 (at most change once)

If integers κ_1 and κ_2 strictly stand on the same side of $\min(i,j)$, then

$$a_{ij}^{(\kappa_1)}(m) = a_{ij}^{(\kappa_2)}(m). \tag{16}$$

Proposition 2.2 (directly expressed by the evolution vector)

For $0 \leq \ell \leq \zeta(m)$, we have

$$a_{i\ell}^{(\ell)}(m) = \begin{cases} \sum\limits_{0\leq\kappa\leq\ell} (-1)^\kappa \alpha_{i+\kappa}(m)\alpha_{\ell-\kappa}(m), & \ell < i \leq ms, \\ \sum\limits_{-\ell\leq\kappa\leq\ell} (-1)^\kappa \alpha_{\ell+\kappa}(m)\alpha_{\ell-\kappa}(m), & i = \ell \neq 0, \\ 0, & i = \ell = 0. \end{cases} \tag{17}$$

The evolution vector with different multi-steps

Proposition 2.3 (main relationship)

Let $m \geq 1$ *be the step of multi-steps. There holds*

$$\frac{\alpha_i(m)}{\alpha_0(m)} = \frac{1}{m^i} \underbrace{\sum \sum \cdots \sum}_{\substack{i_1+i_2+\ldots+i_m=i \\ 0\leq i_1,i_2,\ldots,i_m\leq s}} \frac{\alpha_{i_1}(1)}{\alpha_0(1)} \frac{\alpha_{i_2}(1)}{\alpha_0(1)} \cdots \frac{\alpha_{i_m}(1)}{\alpha_0(1)}, \tag{18}$$

for $i = 0, 1, 2, \ldots, ms$.

Proposition 2.4 (time order)

For any $m \geq 1$*, there holds* $\alpha_i(m) = 1/i!$ *if* $0 \leq i \leq r$.

Remarks on the above conclusion

- By an induction process.
- Associate with $\alpha(m)$, define the generating polynomial

$$p^{(m)}(z) = \sum_{0\le i\le ms} \frac{\alpha_i(m)}{\alpha_0(m)} z^i \tag{19}$$

The conclusion in Proposition 2.3 can be expressed by the identity

$$p^{(m)}(z) = \left[p^{(1)}\left(\frac{z}{m}\right)\right]^m. \tag{20}$$

- The value of $\alpha_i(m)$ for $i > r$ depends on m and s.
- As a result, some important information in the matrix transferring process can be shown by the evolution vector related to the single-step.

General conclusions on three important quantities

Proposition 2.5 (dependence on the time order r)

1. *The termination index satisfies*

$$\zeta(m) \equiv \zeta \geq \lfloor (r+2)/2 \rfloor.$$

and the central objectives have the same sign, namely

$$a_{\zeta\zeta}^{(\zeta)}(m) a_{\zeta\zeta}^{(\zeta)}(1) > 0.$$

2. *The contribution index satisfies*

$$\rho(m) \geq \lfloor (r+1)/2 \rfloor.$$

3. *There exists an integer $n_\star \geq 1$ such that*

$$\rho(m) = \zeta(m), \quad \textit{if} \quad m \geq n_\star.$$

A quantity for quick statement

- The stability performance of RKDG method can be quickly judged by the following quantity

$$\psi_r = \begin{cases} \tilde{\alpha}_{r+2}(1) - \tilde{\alpha}_{r+1}(1) & \text{if } r \text{ is even}, \\ \tilde{\alpha}_{r+1}(1), & \text{if } r \text{ is odd}, \end{cases} \tag{21}$$

when it is not equal to zero. Here $\tilde{\alpha}_i(1) = \alpha_i(1)/\alpha_0(1) - 1/i!$.

- Based on the conclusion on the previous page, we have

Proposition 2.6

- $\zeta(m) \equiv \zeta = \lfloor (r+2)/2 \rfloor$, *and the sign of* $a_{\zeta\zeta}^{(\zeta)}(m)$ *is the same as* $(-1)^\zeta \psi_r$.
- *For odd* r *we have* $n_\star = 1$, *and for even* r *we have*

$$n_\star = \min\left\{ m \colon m^r + (-1)^{\frac{r}{2}} \frac{r!(r+1)!}{[(r/2)!]^2} \tilde{\alpha}_{r+1}(1) > 0 \right\}.$$

Stability conclusion

Theorem 2.3

The sign of $(-1)^{\lfloor r/2 \rfloor}\psi_r$ *is very important:*

- *if it is postive, then the RKDG*(s, r, k) *method has*
  - *the strong (boundedness) stability, at least;*
  - *the monotonicity stability, if* $n_\star = 1$ *is admitted.*
- *Assume it is negative, then the RKDG*(s, r, k) *method has*
  - *the weak*(γ) *stability with* $\gamma = 2\lfloor (r+2)/2 \rfloor$.

- strong (boundedness) stability: RKDG($3, 3, k$).
- monotonicity stability: RKDG($4, 4, k$).
- weak(4) stability: RKDG($2, 2, k$).
- weak(6) stability: RKDG($5, 5, k$).

6. Remarks on the lower degree of piecewise polynomials

Better stability for lower degree polynomials

Theorem 2.4 ($m = 1$: monotonicity; otherwise, strong)

If $\rho(m) = \zeta(m)$, then the RKDG(s, r, k) method has the m-steps monotonicity stability for piecewise polynomials with degree at most $\rho(m) - 1$.

- By recursively using Lemma 2.4 (next page), we have

$$\|\mathbb{D}_\kappa u^n\|^2 \le C\|\partial_x^\ell(\mathbb{D}_{\kappa-\ell}u^n)\|^2 + \lambda\mathcal{Q}_3(\lambda)\tau\sum_{1\le i\le \ell}\|[\![\mathbb{D}_{\kappa-i}u^n]\!]\|_{\Gamma_h}^2. \tag{22}$$

- Taking $\kappa = \ell = \rho$, we have for the lower degree piecewise polynomials

$$\|\mathbb{D}_\rho u^n\|^2 \le \lambda\mathcal{Q}_3(\lambda)\tau\sum_{0\le i\le \rho-1}\|[\![\mathbb{D}_i u^n]\!]\|_{\Gamma_h}^2.$$

- Note that $\|\mathbb{D}_\zeta u^n\| \le C\|\mathbb{D}_\rho u^n\|$, since $\rho \le \zeta$.
- Substituting the above results into Lemma 2.2, we have $\mathcal{Y}_1 + \mathcal{Y}_2 \le 0$ for small CFL number, and get the m-steps monotonicity stability.

Lemma to get (22)

Lemma 2.4 (drop m)

There exists a constant $C = C(m, \theta, i, \mu) > 0$, such that

$$\|\partial_x^i(\mathbb{D}_\ell u^n)\| \le \tau|\beta|\|\partial_x^{i+1}(\mathbb{D}_{\ell-1}u^n)\| + C\tau|\beta|h^{-i-1/2}\|[\![\mathbb{D}_{\ell-1}u^n]\!]\|_{\Gamma_h}.$$

- The proof is trivial by an induction.
- Denote $\mathcal{S} = \mathbb{D}_\ell u^n + \tau\beta\partial_x(\mathbb{D}_{\ell-1}u^n)$. Integrating by parts yields

$$(\mathcal{S}, v) = -\tau\beta \sum_{1\le j\le J} [\![\mathbb{D}_{\ell-1}u^n]\!]_{j+\frac{1}{2}} \{\!\{v\}\!\}_{j+\frac{1}{2}}^{(1-\theta)}, \quad \forall\, v \in V_h. \tag{23}$$

- Taking $v = \mathcal{S}$ in (23) and using the inverse inequality, we have

$$(\mathcal{S}, \mathcal{S}) = -\tau\beta \sum_{1\le j\le J} [\![\mathbb{D}_{\ell-1}u^n]\!]_{j+\frac{1}{2}} \{\!\{\mathcal{S}\}\!\}_{j+\frac{1}{2}}^{(1-\theta)} \le C\tau|\beta|h^{-\frac{1}{2}}\|[\![\mathbb{D}_{\ell-1}u^n]\!]\|_{\Gamma_h}\|\mathcal{S}\|, \tag{24}$$

  which implies this lemma for $i = 0$.

Proof of Lemma 2.4

- Let $i \geq 1$, and take $v = \partial_x^{2i}\mathcal{S}$ in (23).
- Integrating by parts for i times to deal with $(\mathcal{S}, \partial_x^{2i}\mathcal{S})$, we have

$$
\begin{aligned}
&(-1)^i\|\partial_x^i\mathcal{S}\|^2 + \sum_{0\leq i'<i}\sum_{1\leq j\leq J}(-1)^{i-i'}[\![\partial_x^{i+i'}\mathcal{S}\partial_x^{i-i'-1}\mathcal{S}]\!]_{j+\frac{1}{2}} \\
= &-\tau\beta\sum_{1\leq j\leq J}[\![\mathbb{D}_{\ell-1}u^n]\!]_{j+\frac{1}{2}}\{\!\{\partial_x^{2i}\mathcal{S}\}\!\}_{j+\frac{1}{2}}^{(1-\theta)}.
\end{aligned}
$$

- The inverse inequality implies that

$$
\begin{aligned}
\|\partial_x^i\mathcal{S}\|^2 &\leq Ch^{-1}\sum_{0\leq i'<i}\|\partial_x^{i+i'}\mathcal{S}\|\|\partial_x^{i-i'-1}\mathcal{S}\| + C\tau|\beta|h^{-1/2}\|[\![\mathbb{D}_{\ell-1}u^n]\!]\|_{\Gamma_h}\|\partial_x^{2i}\mathcal{S}\| \\
&\leq Ch^{-i}\|\partial_x^i\mathcal{S}\|\|\mathcal{S}\| + C\tau|\beta|h^{-1/2-i}\|[\![\mathbb{D}_{\ell-1}u^n]\!]\|_{\Gamma_h}\|\partial_x^i\mathcal{S}\|.
\end{aligned}
$$

- Substituting the estimate of $\mathcal{S}$, and we complete the proof of this lemma.

7. Numerical experiments

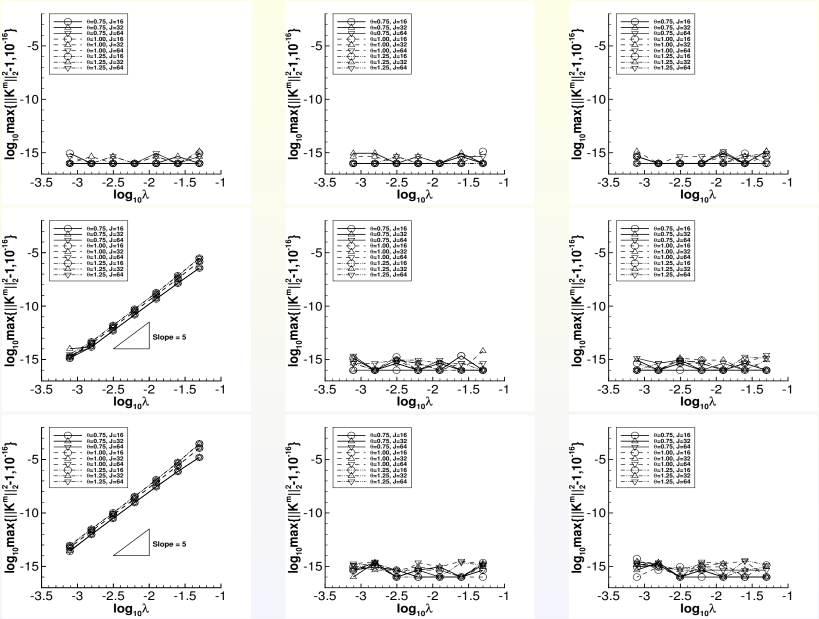
Numerical experiments: RKDG$(4, 4, k)$

- Consider $U_t + U_x = 0$, and use the uniform meshes with $J = 16, 32, 64$.
- The RKDG method can be written into

$$\mathbf{u}^{n+1} = \mathbb{K}\mathbf{u}^n,$$

and $\|\mathbb{K}^m\|_2$ shows the L^2-norm amplification of solution every m-steps.

- Plot the picture of $\|\mathbb{K}^m\|_2^2 - 1$ v.s. the CFL number λ, where
  - $k = 1, 2, 3$ from top to bottom;
  - $m = 1, 2, 3$ from left to right.

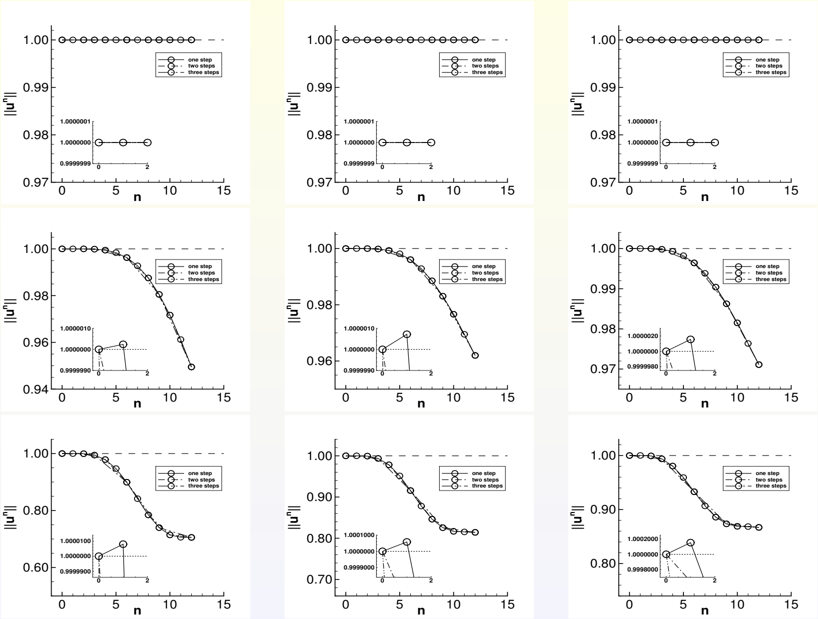

θ=0.75, J=16
θ=0.75, J=32
θ=0.75, J=64
θ=1.00, J=16
θ=1.00, J=32
θ=1.00, J=64
θ=1.25, J=16
θ=1.25, J=32
θ=1.25, J=64
$\log_{10}\max\{\|K^m\|_2^2-1,10^{-16}\}$
$\log_{10}\lambda$
Slope = 5

Numerical experiments: RKDG$(4, 4, k)$

- Also plot the evolution of the L^2-norm for $0 \leq n \leq 12$.
- $\lambda = 0.05$, and $J = 64$.
- Let the initial solution u^0 be the unit singular vector with respect to the largest singular value of $\mathbb{K}$.
- $k = 1, 2, 3$ from top to bottom, and
- $\theta = 0.75, 1.00, 1.25$ from left to right.

Results:

- For $k = 1$, the monotonicity stability is clearly observed.
- For $k \geq 2$, the monotonicity stability does not hold, however the multi-steps monotonicity stability is observed.
- Hence, there only holds the strong stability but not monotonicity stability for high order piecewise polynomials.



||u^n||
n
one step
two steps
three steps

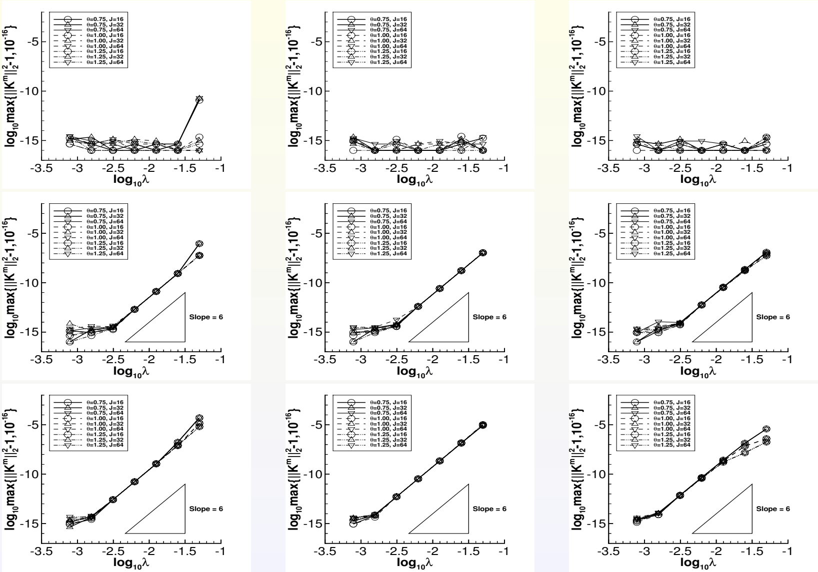
Numerical experiments: RKDG$(5, 5, k)$

- Take $\beta = 1$ and use uniform meshes with $J = 16, 32, 64$;
- The RKDG method with $\theta = 0.75, 1.00, 1.25$.
- Plot the picture of $\|\mathbb{K}^m\|_2^2 - 1$ with respect to λ, where
  - $k = 2, 3, 4$ from top to bottom, and
  - $m = 1, 2, 3$ from left to right.

Results:

- For $k = 2$ and $m = 1, 2, 3$, the quantity is very close to machine-precision, which numerically verifies the monotonicity stability for lower-degree piecewise polynomials.
- For $k = 3$ and $k = 4$, the quantity strongly depends on λ, with slope 6 in the logarithmic coordinates, for $m = 1, 2, 3$.
- This performance is different to the RKDG$(4, 4, k)$ method.

The RKDG($5, 5, k$) method



Linearly unstable?

- Let u^0 be the L^2-projection of

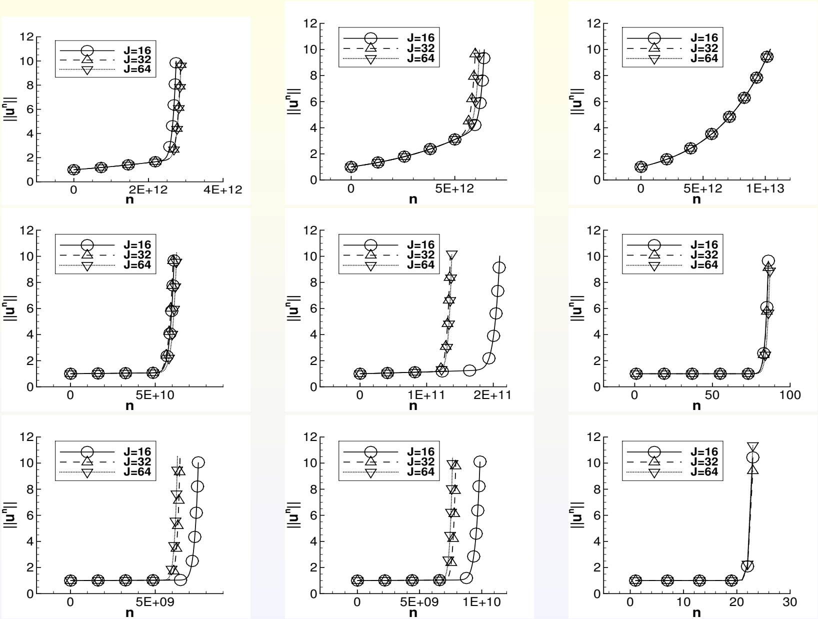
$$u(x,0)=\sqrt{2}\sin(\frac{J}{16}2\pi x)$$

with $J=16,32,64$.

- Plot the evolution performance of the L^2-norm, with $\lambda=0.06,0.08,0.10$ (from top to bottom), and $\theta=0.75,1.00,1.25$ (from left to right).

Result:

- The L^2-norms of the solution are shown to be exponentially increases after an extremely large number of time steps (for most cases), and
- this phenomenon is independent of the mesh size.
- Maybe linearly instability!


12
10
8
6
4
2
0
||u^n||
J=16
J=32
J=64
0
2E+12
4E+12
n
5E+12
n
5E+12
1E+13
n
5E+10
n
1E+11
2E+11
n
50
100
n
5E+09
n
5E+09
1E+10
n
10
20
30
n

Outline

1. Main conclusion and proof

error estimate

Theorem 3.1

Suppose the time step τ ensures the L^2-norm stability of the RKDG(s, r, k) method. Then we have

$$\|U(t^n) - u^n\| \leq C\|U_0\|_{H^{\max(k+2,r+1)}(I)}(h^{k+1} + \tau^r), \quad \forall n.$$

if the initial solution ensures (L^2/GGR projection and so on)

$$\|U(0) - u^0\| \leq Ch^{k+1}.$$

- The assumption on smoothness is weak!
- The proof is trivial, including the following technique:
  - stability analysis for the nonhomogeneous scheme;
  - reference functions at every time stage, with the cutting-off trick;
  - application of the GGR projection.

Stability result for the nonhomogeneous scheme

- At each time-marching there holds

$$(u^{n,\ell+1}, v) = \sum_{0\le\kappa\le\ell} \left\{ c_{\ell\kappa}(u^{n,\kappa}, v) + \tau d_{\ell\kappa}\Big[\mathcal{H}(u^{n,\kappa}, v) + (f^{n,\kappa}, v)\Big] \right\}, \quad \forall v \in V_h,$$

  for $\ell = 0, 1, \ldots, s-1$. Here $f^{n,\ell}$ is the given source term.
- Similarly as the previous analysis, we have

Lemma 3.1

Under the temporal-spatial condition as stated before, there holds the general stability

$$\|u^n\|^2 \le C \left\{ \|u^0\|^2 + \tau \sum_{0\le\kappa<n} \sum_{0\le\ell<s} \|f^{\kappa,\ell}\|^2 \right\}, \tag{25}$$

where the bounding constant $C > 0$ is independent of n, h, τ, and u, but may depends on the final time T.

Reference functions at every time stage

- Given an integer $1 \leq \sigma \leq r$, and let $U_{[\sigma]}^{(0)}(t) = U(t)$.
- The other reference functions are defined by an induction process:
  - Suppose for an integer $\ell \geq 0$ that the previous $\ell + 1$ reference functions have been defined well, and been expanded in the form

$$U_{[\sigma]}^{(\kappa)} = \sum_{0 \leq i \leq \min(\sigma,\kappa)} \gamma_{i[\sigma]}^{(\kappa)} \tau^i \partial_t^i U, \quad 0 \leq \kappa \leq \ell. \tag{26}$$

  - According to the stage marching of RKDG method, define an auxiliary reference function and then expand it in the form

$$\widetilde{U}_{[\sigma]}^{(\ell+1)} = \sum_{0 \leq \kappa \leq \ell} \left[c_{\ell\kappa} U_{[\sigma]}^{(\kappa)} - \tau d_{\ell\kappa} \beta \partial_x U_{[\sigma]}^{(\kappa)} \right] = \sum_{0 \leq i \leq \min(\sigma+1,\ell+1)} \gamma_{i[\sigma]}^{(\ell+1)} \tau^i \partial_t^i U.$$

  - By cutting off the term involving the $(\sigma + 1)$th order time derivative – if exists – the new reference function is defined as

$$U_{[\sigma]}^{(\ell+1)} = \sum_{0 \leq i \leq \min(\sigma,\ell+1)} \gamma_{i[\sigma]}^{(\ell+1)} \tau^i \partial_t^i U. \tag{27}$$

- For the convenience of notations, denote $U_{[\sigma]}^{(s)}(x,t) = U(x, t + \tau)$.

Reference functions at every time stage

- After some manipulations that all Taylor expansions in time are only done up to the $(\sigma+1)$-th time derivatives, it is easy to see that

$$U_{[\sigma]}^{(\ell+1)} = \sum_{0\le\kappa\le\ell}\left[c_{\ell\kappa}U_{[\sigma]}^{(\kappa)} - \tau d_{\ell\kappa}\beta\partial_x U_{[\sigma]}^{(\kappa)}\right] + \tau\varrho_{[\sigma]}^{(\ell)}, \quad 0\le\ell\le s-1, \tag{28}$$

where $\varrho_{[\sigma]}^{(\ell)}$ are the truncation errors in time, bounded by

$$\|\varrho_{[\sigma]}^{(\ell)}\|_{L^\infty(H^R(I))} \le C\|\partial_t^{\sigma+1}U\|_{L^\infty(H^R(I))}\tau^\sigma \le C\|U_0\|_{H^{i+\sigma+1}(I)}\tau^\sigma, \quad R\ge 0. \tag{29}$$

- Here $L^\infty(H^R(I))$ denotes the space-time Sobolev space in which the function's $H^R(I)$-norm at any time $t\in[0,T]$ is uniformly bounded.

- Actually, there holds $\varrho_{[\sigma]}^{(\ell)} = 0$ for $\ell\le\min(\sigma-1, s-2)$.

Standard analysis process: skectch

- Denote by $e^{n,\ell} = u^{n,\ell} - U^{n,\ell}$ the stage error, where

$$U^{n,\ell} = U_{[r]}^{(\ell)}(x, t^n), \quad 0 \le \ell \le s-1, \tag{30}$$

  is the reference function at each time stage.

- Let $\chi^{n,\ell} \in V_h$ be arbitrary series of stage functions. Consider the error decomposition

$$e^{n,\ell} = \xi^{n,\ell} - \eta^{n,\ell}, \tag{31}$$

  where
  - the error in the finite element space: $\xi^{n,\ell} = u^{n,\ell} - \chi^{n,\ell} \in V_h$,
  - and the approximation of reference solution: $\eta^{n,\ell} = U^{n,\ell} - \chi^{n,\ell}$.

- In general, $\eta^{n,\ell}$ is easily bounded by the help of some projection and other techniques.
- Estimate $\xi^{n,\ell}$ by its error equation and the obtained stability result.
- Application of the triangular inequality.

Standard analysis process: error equation

- Letting $t = t^n$ in (28), we can get a group of variational forms similar as in the RKDG method.
- Subtracting them from each other and using the error decomposition, we can achieve the error equation for $\ell = 0, 1, \ldots, s-1$,

$$(\xi^{n,\ell+1}, v) = \sum_{0\le\kappa\le\ell} \Big\{ c_{\ell\kappa}(\xi^{n,\kappa}, v) + \tau d_{\ell\kappa}\Big[\mathcal{H}(\xi^{n,\kappa}, v) + (F^{n,\kappa}, v)\Big]\Big\}, \tag{32}$$

for any $v \in V_h$. Here $(F^{n,\ell}, v)$ is the residual functional at every time stage, recursively defined by

$$d_{\ell\ell}(F^{n,\ell}, v) = \underbrace{(\eta_{\mathsf{c}}^{n,\ell}, v) - \mathcal{H}(\eta_{\mathsf{d}}^{n,\ell}, v) - (\varrho_{[r]}^{n,\ell}, v)}_{\mathcal{Z}^{n,\ell}(v)} - \sum_{0\le\kappa\le\ell-1} d_{\ell\kappa}(F^{n,\kappa}, v), \tag{33}$$

with the compact notation

$$\eta_{\mathsf{c}}^{n,\ell} = \frac{1}{\tau}\Big[\eta^{n,\ell+1} - \sum_{0\le\kappa\le\ell} c_{\ell\kappa}\eta^{n,\kappa}\Big], \quad \eta_{\mathsf{d}}^{n,\ell} = \sum_{0\le\kappa\le\ell} d_{\ell\kappa}\eta^{n,\kappa}, \tag{34}$$

- Note that $\varrho_{[r]}^{n,\ell} = \varrho_{[r]}^{(\ell)}(t^n)$, and the summation in (33) is zero if $\ell = 0$.

Standard analysis process: error equation

- The above error equations are very important in the error estimates for the RKDG method, due to the following conclusion

Lemma 3.2 (application of Lemma 3.1)

Assume that the RKDG(s, r, k) method has the L^2-norm stability under suitable temporal-spatial condition. Then we have

$$\|\xi^n\|_{L^2(I)}^2 \leq C\Big\{\|\xi^0\|_{L^2(I)}^2 + \tau \sum_{0\leq\kappa<n} \sum_{0\leq\ell<s} \|\mathcal{Z}^{\kappa,\ell}\|^2\Big\}, \tag{35}$$

where $\|\mathcal{Z}^{\kappa,\ell}\| = \sup_{0\neq v\in V_h} \mathcal{Z}^{\kappa,\ell}(v)/\|v\|_{L^2(I)}$.

- For the optimal error estimate, it is good enough to take the GGR projection of the reference functions

$$\chi^{n,\ell} = \mathbb{G}_h^\theta U^{n,\ell} = \mathbb{G}_h^\theta U_{[r]}^{n,\ell} \in V_h, \tag{36}$$

such that the regularity assumption is independent of the stage number.

2. Numerical experiments

Numerical experiments

- The above results also hold for multi-dimensional problem.
- Below we present the numerical experiments of the RKDG(4,4,k) method and the RKDG(10,4,k) method to solve

$$U_t + U_x + U_y = 0.$$

- Test on accuracy order:

$$U(x,y,t) = \sin(x+y-2t).$$

- Test on regularity assumption:

$$U(x,y,t) = \sin^{q+\frac{2}{3}}[2\pi(x+y-2t)]$$

with different integer q.

Test on accuracy order: RKDG(4,4,k)

	$I_x \times J_y$	$\theta = 0.75$		$\theta = 1$		$\theta = 1.25$	
		error	order	error	order	error	order
$k = 1$	40×40	2.52E-03		1.50E-03		1.21E-03	
	80×80	6.35E-04	1.99	3.75E-04	2.00	3.03E-04	2.00
	120×120	2.83E-04	2.00	1.67E-04	2.00	1.35E-04	2.00
	160×160	1.59E-04	2.00	9.38E-05	2.00	7.58E-05	2.00
	200×200	1.02E-04	2.00	6.01E-05	2.00	4.85E-05	2.00
$k = 2$	40×40	1.42E-05		1.89E-05		2.48E-05	
	80×80	1.77E-06	3.00	2.36E-06	3.00	3.11E-06	3.00
	120×120	5.25E-07	3.00	7.00E-07	3.00	9.21E-07	3.00
	160×160	2.22E-07	3.00	2.95E-07	3.00	3.88E-07	3.00
	200×200	1.13E-07	3.00	1.51E-07	3.00	1.99E-07	3.00
$k = 3$	40×40	2.97E-07		1.83E-07		1.52E-07	
	80×80	1.87E-08	3.99	1.14E-08	4.00	9.47E-09	4.00
	120×120	3.69E-09	4.00	2.25E-09	4.00	1.87E-09	4.00
	160×160	1.17E-09	4.00	7.13E-10	4.00	5.92E-10	4.00
	200×200	4.79E-10	4.00	2.92E-10	4.00	2.42E-10	4.00

Test on accuracy order: RKDG(10,4,k)

	$I_x \times J_y$	$\theta = 0.75$		$\theta = 1$		$\theta = 1.25$	
		error	order	error	order	error	order
$k=1$	40×40	2.52E-03		1.50E-03		1.21E-03	
	80×80	6.35E-04	1.99	3.75E-04	2.00	3.03E-04	2.00
	120×120	2.83E-04	2.00	1.67E-04	2.00	1.35E-04	2.00
	160×160	1.59E-04	2.00	9.38E-05	2.00	7.58E-05	2.00
	200×200	1.02E-04	2.00	6.01E-05	2.00	4.85E-05	2.00
$k=2$	40×40	1.42E-05		1.89E-05		2.48E-05	
	80×80	1.77E-06	3.00	2.36E-06	3.00	3.11E-06	3.00
	120×120	5.25E-07	3.00	7.00E-07	3.00	9.21E-07	3.00
	160×160	2.22E-07	3.00	2.95E-07	3.00	3.88E-07	3.00
	200×200	1.13E-07	3.00	1.51E-07	3.00	1.99E-07	3.00
$k=3$	40×40	2.97E-07		1.83E-07		1.52E-07	
	80×80	1.87E-08	3.99	1.14E-08	4.00	9.47E-09	4.00
	120×120	3.69E-09	4.00	2.25E-09	4.00	1.87E-09	4.00
	160×160	1.17E-09	4.00	7.13E-10	4.00	5.92E-10	4.00
	200×200	4.79E-10	4.00	2.92E-10	4.00	2.42E-10	4.00

Test on regularity assumption: independent of s

	$I_x \times J_y$	$\theta = 0.75$		$\theta = 1$		$\theta = 1.25$	
		error	order	error	order	error	order
$s = 4$, $q = 3$	40×40	9.38E-06		9.71E-06		1.11E-05	
	80×80	6.95E-07	3.76	7.34E-07	3.72	8.47E-07	3.71
	120×120	1.52E-07	3.75	1.64E-07	3.69	1.90E-07	3.69
	160×160	5.18E-08	3.73	5.71E-08	3.68	6.57E-08	3.68
	200×200	2.26E-08	3.71	2.52E-08	3.67	2.89E-08	3.68
$s = 4$, $q = 4$	40×40	1.07E-05		7.82E-06		6.96E-06	
	80×80	7.43E-07	3.84	4.82E-07	4.02	4.10E-07	4.09
	120×120	1.51E-07	3.94	9.47E-08	4.01	7.98E-08	4.04
	160×160	4.81E-08	3.97	2.99E-08	4.01	2.51E-08	4.02
	200×200	1.98E-08	3.98	1.22E-08	4.01	1.02E-08	4.02
$s = 10$, $q = 3$	40×40	9.40E-06		9.73E-06		1.11E-05	
	80×80	6.98E-07	3.75	7.37E-07	3.72	8.47E-07	3.71
	120×120	1.53E-07	3.75	1.65E-07	3.69	1.90E-07	3.69
	160×160	5.24E-08	3.72	5.73E-08	3.68	6.56E-08	3.69
	200×200	2.29E-08	3.70	2.53E-08	3.67	2.88E-08	3.68
$s = 10$, $q = 4$	40×40	1.07E-05		7.82E-06		6.95E-06	
	80×80	7.43E-07	3.84	4.81E-07	4.02	4.09E-07	4.09
	120×120	1.51E-07	3.94	9.46E-08	4.01	7.96E-08	4.04
	160×160	4.81E-08	3.97	2.99E-08	4.01	2.50E-08	4.02
	200×200	1.98E-08	3.98	1.22E-08	4.01	1.02E-08	4.02

Outline

Motivation and difficulties

- Many theoretical analysis have been presented for the semi-discrete DG method. However, the numerical experiments are gotten by virtue of some suitable time-marching.
- In some sense, the superconvergence results of RKDG(s, r, k) method is stated as follows:

> The superconveregence results of the semi-discrete DG method are preserved, and the time order is provided additionally.

- The difficulties in analysis:
  - stability analysis of high order RKDG method;
  - reference function in the stage time;
  - the incomplete correction function;
  - Deep investigation on the derivation of error.

1. Incomplete correction of the reference functions

Technique of correction function

- Cited from the following paper for the semi-discrete DG method, however, with minor modification

  W. X. Cao, and e.t.c, ESAIM **51** (2017), 467-486

- Given any integer $p \geq 0$. For any function $w \in H^1(\mathcal{T}_h)$, the p-th correction function is defined by

$$\mathcal{F}_p w = (-\mathbb{G}_h^\theta \mathbb{D}_h^{-1})^p (\mathbb{P}_h - \mathbb{G}_h^\theta) w \in V_h. \tag{37}$$

  Here
  - $\mathbb{P}_h$ and $\mathbb{G}_h^\theta$ are the L^2 projection and the GGR projection, respectively;
  - $\mathbb{D}_h^{-1}$ is the antiderivative in each element, defined by

$$\mathbb{D}_h^{-1} z(x) = \int_{x_{i-1/2}}^{x} z(x')\mathrm{d}x', \quad x \in I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}). \tag{38}$$

Technique of correction function

Lemma 4.1

- *Let* $0 \le p \le k$ *and* $w \in H^1(\mathcal{T}_h)$.
- *There exists a constant* $C > 0$ *independent of* h *and* w*, such that*

$$\|\mathcal{F}_p w\|_{L^2(I)} \le Ch^p \|(\mathbb{P}_h - \mathbb{G}_h^\theta)w\|_{L^2(I)}. \tag{39}$$

- *As a corollary, the correction operator* $\mathcal{F}_p$ *is linear and continuous from* $H^1(\mathcal{T}_h)$ *to* V_h.

Technique of correction function

Lemma 4.2

- *Let* $1 \leq p \leq k$ *and* $w \in H^1(\mathcal{T}_h)$.
- *There holds*
  1. *the exact collocation of the numerical flux, namely,*

$$\{\!\!\{\mathcal{F}_p w\}\!\!\}_{i+\frac{1}{2}}^{(\theta)} = 0, \quad i = 1, 2, \ldots, N.$$

  2. *the recurrence relationship, namely,*

$$(\mathcal{F}_p w, v_x) = (\mathcal{F}_{p-1} w, v), \quad \forall v \in V_h.$$

- *As a corollary of the above results, we have*

$$\mathcal{H}(\mathcal{F}_p w, v) = \beta(\mathcal{F}_{p-1} w, v), \quad \forall v \in V_h.$$

Technique of correction function

Lemma 4.3

Let $\underline{1 \leq p \leq k}$ *and* $w \in H^1(\mathcal{T}_h)$*. There holds*

$$(\mathcal{F}_{p-1}w, v) = 0, \quad \forall v \in V_h^{k-p}.$$

Incomplete correction function

- Let $1 \le q \le k$ be the total number of correction manipulation. Take

$$\chi^{n,\ell} = \mathbb{G}_h^{\theta} U^{n,\ell} - \sum_{1\le p\le q} \mathcal{F}_p(-\partial_x)^p W^{n,\ell} \in V_h. \tag{40}$$

- The word "incomplete" comes from the fact that

$$W^{n,\ell} = U^{(\ell)}_{[\min(q,r)]}(t^n)$$

is truncated from the reference function $U^{n,\ell} = U^{(\ell)}_{[r]}(t^n)$.

- Based on the above lemmas, we have the following estimate for the RHS term in the error equation of ξ.

Lemma 4.4 (trivial proof but a little complex)

Assume τ/h is upper bounded. With the choice (40), *there holds*

$$\|\mathcal{Z}^{n,\ell}\| \le C\|U_0\|_{H^{\max(k+q+2,r+1)}(I)}(h^{k+q+1} + \tau^r), \tag{41}$$

for $\ell = 0, 1, \ldots, s-1$.

2. Supraconvergence analysis

Supracongergence results

Theorem 4.1

- *Suppose here and below that the time step τ is taken to ensure the L^2-norm stability of the RKDG(s, r, k) method.*
- *Let $1 \leq q \leq k$ and take the initial solution*

$$u^0 = \mathbb{G}_h^\theta U_0 - \sum_{1 \leq p \leq q_{\mathrm{nt}}} \mathcal{F}_p (-\partial_x)^p U_0, \tag{42}$$

  where q_{nt} is a given integer.
- *Then there holds the following supraconvergence results:*
  - *Let $q - 1 \leq q_{\mathrm{nt}} \leq k$, then we have*

$$\|\xi^n\| \leq C \|U_0\|_{H^{\max(k+q+2, r+1)}(I)} (h^{k+q+1} + \tau^r), \tag{43}$$

  - *Let $q \leq q_{\mathrm{nt}} \leq k$, then we have*

$$\|\xi_x^n\| \leq C \|U_0\|_{H^{\max(k+q+3, r+2)}(I)} (h^{k+q+1} + \tau^r), \tag{44}$$

Proof of Theorem 4.1 (cont.)

- The proof about the solution is trivial, by using Lemmas 3.2 and 4.4, together with the estimate for the initial error

$$\begin{aligned}\|\xi^0\|_{L^2(I)} &\le \sum_{q\le p\le k} \|\mathcal{F}_p(-\partial_x)^p U_0\|_{L^2(I)} \\ &\le \sum_{q\le p\le k} Ch^p h^{k+1+q-p} \|U_0\|_{H^{p+k+1+q-p}(I)} \\ &\le Ch^{k+1+q}\|U_0\|_{H^{k+1+q}(I)}.\end{aligned}$$

  At the second step, we have used Lemma 4.1 and the approximation properties of the two projections.
- The proof about the derivative is a little complex. A new PDE:
  - Let $\Pi = -\beta U_x$.
  - Obviously, it satisfies the auxiliary (same-form) problem

$$\Pi_t + \beta \Pi_x = 0, \quad x \in I = (0,1), \quad t \in (0,T], \tag{45}$$

    equipped with the periodic boundary condition and the initial solution $\Pi(x,0) = \Pi_0(x)$.

Proof of Theorem 4.1 (cont.)

- For any function $w \in V_h$, there exists a unique function $\tilde{w} \in V_h$ such that

$$(\tilde{w}, v) = \mathcal{H}(w, v), \quad \forall v \in V_h.$$

  Define $\tilde{w} = \mathcal{H}_h w$, which forms a linear map from V_h to itself.
- Let $\tilde{u}^{n,\ell} = \mathcal{H}_h u^{n,\ell}$.
- It follows from the RKDG method that

$$u^{n,\ell+1} = \sum_{0 \le \kappa \le \ell} [c_{\ell\kappa} u^{n,\kappa} + \tau d_{\ell\kappa} \tilde{u}^{n,\kappa}].$$

  Making a left-multiplication of $\mathcal{H}_h$ yields for n and ℓ, and make an L^2 inner product, we will have

$$(\tilde{u}^{n,\ell+1}, v) = \sum_{0 \le \kappa \le \ell} \Big[c_{\ell\kappa} (\tilde{u}^{n,\kappa}, v) + \tau d_{\ell\kappa} \mathcal{H}(\tilde{u}^{n,\kappa}, v) \Big]. \tag{46}$$

Proof of Theorem 4.1 (cont.)

- These formulations can be viewed as the RKDG(s, r, k) method to approximate the solution of (45), with the initial solution

$$\tilde{u}^0 = \mathcal{H}_h u^0. \tag{47}$$

- Along the same line as the previous discussion, we analogously define

$$\tilde{\xi}^{n,\ell} = \tilde{u}^{n,\ell} - \mathbb{G}_h^\theta \Pi_{[r]}^{n,\ell} + \sum_{1\le p\le q} \mathcal{F}_p(-\partial_x)^p \Pi_{[\min(q,r)]}^{n,\ell} \in V_h, \tag{48}$$

  and achieve the supraconvergence result

$$\|\tilde{\xi}^n\|_{L^2(I)} \le C\|\tilde{\xi}^0\|_{L^2(I)} + C\|\Pi_0\|_{H^{\max(k+q+2,r+1)}(I)}(h^{k+q+1} + \tau^r). \tag{49}$$

- By a tedious manipulation, we can yield (later)

$$\|\tilde{\xi}^0\|_{L^2(I)} \le Ch^{k+1+q}\|U_0\|_{H^{k+q+2}(I)}. \tag{50}$$

Proof of Theorem 4.1 (cont.)

- It is no harm to assume $q \geq 1$.
- Recalling the definition of $\xi^{n,\ell}$, we know that

$$\mathbb{D}_1(1)\xi^n = \mathbb{D}_1(1)u^n - \mathbb{G}_h\left(\mathbb{D}_1(1)U^n_{[r]}\right) + \sum_{1\leq p\leq q}\mathcal{F}_p(-\partial_x)^p\left(\mathbb{D}_1(1)U^n_{[\min(q,r)]}\right).$$

  Each term satisfies:
  - By the definition of temporal difference, there holds

$$\mathbb{D}_1(1)u^n = \tau\mathcal{H}_h u^n = \tau\tilde{u}^n.$$

  - By the definition of the reference functions, we have

$$\mathbb{D}_1(1)U^n_{[r]} = -\tau\beta U^n_x = \mathbb{D}_1(1)U^n_{[\min(q,r)]}.$$

- A comparison with this formulation and (48) yields for that

$$\tau^{-1}\mathbb{D}_1(1)\xi^n = \tilde{\xi}^n. \tag{51}$$

- Note that (51) also holds for $q = 0$, since the summations vanish.

Proof of Theorem 4.1

- Substituting identity (51) into error equation (32) with $\ell = 0$, we have

$$\mathcal{H}(\xi^n, v) = (\tilde{\xi}^n, v) - d_{00}^{-1} \mathcal{Z}^{n,0}(v), \quad \forall v \in V_h. \tag{52}$$

  Here d_{00} is the given parameter in the RKDG method.

- Similar as Lemma 2.3 in the paper

  H. J.Wang, Q. Zhang and C. W. Shu, JSC 81(2019), 2080-2114

  we have that

$$\|\xi_x^n\|_{L^2(I)} \leq C\|\tilde{\xi}^n\|_{L^2(I)} + C\|\mathcal{Z}^{n,0}\|. \tag{53}$$

- Together with (49), (50) and Lemma 4.4, this inequality completes the proof of this theorem.

Supplement: proof of inequality (50)

- Two simple facts (no harm in assuming that $q \geq 1$):
  - Since $U_0 \in H^1(I)$ is continuous, the definitions of the two projections imply

$$(\mathcal{H}_h \mathbb{G}_h^\theta U_0, v) = \mathcal{H}(\mathbb{G}_h^\theta U_0, v) = \mathcal{H}(U_0, v) = -\beta((U_0)_x, v) = -\beta(\mathbb{P}_h (U_0)_x, v),$$

  holds for any $v \in V_h$. Hence

$$\mathcal{H}_h \mathbb{G}_h^\theta U_0 = -\beta \mathbb{P}_h (U_0)_x = -\beta \mathbb{G}_h^\theta (U_0)_x + \beta \mathcal{F}_0(-\partial_x) U_0. \tag{54}$$

  - Similarly, due to Lemma 4.2, there holds

$$(\mathcal{H}_h \mathcal{F}_p (-\partial_x)^p U_0, v) = \mathcal{H}(\mathcal{F}_p (-\partial_x)^p U_0, v) = \beta(\mathcal{F}_{p-1}(-\partial_x)^p U_0, v)$$

  for any $v \in V_h$. Hence,

$$\mathcal{H}_h \mathcal{F}_p (-\partial_x)^p U_0 = \beta \mathcal{F}_{p-1}(-\partial_x)^p U_0. \tag{55}$$

- Hence, it follows from the initial setting (42) that

$$\begin{aligned} \tilde{u}^0 = \mathcal{H}_h u^0 &= \mathcal{H}_h \Big(\mathbb{G}_h U_0 - \sum_{1 \leq p \leq q_{\rm nt}} \mathcal{F}_p (-\partial_x)^p U_0 \Big) \\ &= \mathbb{G}_h^\theta \Pi_0 - \beta \sum_{2 \leq p \leq q_{\rm nt}} \mathcal{F}_{p-1}(-\partial_x)^p U_0. \end{aligned}$$

Supplement: proof of inequality (50)

- It is no harm in assuming that $q \geq 1$.
- Substituting the initial setting (42) into the definition of $\tilde{\xi}^0$. Together with the above facts, we have

$$\begin{aligned}\tilde{\xi}^0 &= \tilde{u}^0 - \Big(\mathbb{G}_h\Pi_0 - \sum_{1\leq p\leq q} \mathcal{F}_p(-\partial_x)^p\Pi_0\Big)\\ &= -\beta \sum_{1\leq p\leq q_{\mathrm{nt}}-1} \mathcal{F}_p(-\partial_x)^{p+1}U_0 + \beta\sum_{1\leq p\leq q}\mathcal{F}_p(-\partial_x)^{p+1}U_0.\end{aligned}$$

- Since $q \leq q_{\mathrm{nt}} \leq k$, we can get (50).
- A supplement is given for $q = 0$.
  - Since the summation is equal to zero if the index set is empty, the above formula also holds for $q = 0$ and $q_{\mathrm{nt}} \geq 1$.
  - If $q = q_{\mathrm{nt}} = 0$, a direct manipulation shows $\tilde{\xi}^0 = -\beta\mathcal{F}_0(U_0)_x$.
  - For these special cases, it is easy to see that (50) holds.

3. Superconvergence analysis

Superconvergence results

Theorem 4.2

1. *Let $k-1 \leq q_{\text{init}} \leq k$, then the numerical fluxes and the cell averages are superconvergent, namely,*

$$\|\{\!\{e^n\}\!\}^{(\theta)}\|_{L^2(S_h^B)} + \|\bar{e}^n\|_{L^2(S_h^E)} \leq C\|U_0\|_{H^{\max(2k+2,r+1)}(I)}(h^{2k+1} + \tau^r).$$

2. *Let $0 \leq q_{\text{init}} \leq k$, then the solution is superconvergent at the roots namely,*

$$\|e^n\|_{L^2(S_h^R)} \leq C\|U_0\|_{H^{\max(k+3,r+1)}(I)}(h^{k+2} + \tau^r).$$

   and the derivative is superconvergent at the extremums, namely,

$$\|e_x^n\|_{L^2(S_h^L)} \leq C\|U_0\|_{H^{\max(k+3,r+2)}(I)}(h^{k+1} + \tau^r).$$

3. *The above roots and extremums in each element are both related to the parameter-dependent Radau polynomial, stated as follows.*

The parameter-dependent Radau polynomial

- Let $L_i(\hat{x})$ be the standard Legendre polynomial of degree i on $[-1,1]$, and thus
$$L_{j,i}(x) = L_i(\hat{x}) = L_i\left(2(x-x_j)/h_j\right), \quad i \geq 0,$$
is the Legendre polynomial of degree i in I_j.

- Associated with the mesh and the upwind-biased parameter, we are able to seek a group of parameters $\{\vartheta_j\}_{1\leq j\leq J}$ by the following system of linear equations
$$\theta h_j^{k+1}\vartheta_j + (-1)^k(1-\theta)h_{j+1}^{k+1}\vartheta_{j+1} = \theta h_j^{k+1} - (-1)^k(1-\theta)h_{j+1}^{k+1}, \tag{56}$$
where $j = 1, 2, \ldots, N$.

- The existence and uniqueness can be verified since the determinant is not equal to zero, due to $\theta \neq 1/2$.

The parameter-dependent Radau polynomial

- The parameter-dependent Radau polynomial is defined element by element, namely

$$R_{j,k+1}(x) = L_{j,k+1}(x) - \vartheta_j L_{j,k}(x), \quad x \in I_j.$$

- The roots in I_j are denoted by r_{ij} for $1 \le i \le n_j^{\mathrm{R}}$, where

$$n_j^{\mathrm{R}} = \begin{cases} k+1, & \text{if } |\vartheta_j| < 1, \\ k, & \text{otherwise} \end{cases}$$

  The set S_h^{R} includes all of the above roots.

- The extrema in I_j are denoted by l_{ij} for $1 \le i \le n_j^{\mathrm{L}}$, where

$$n_j^{\mathrm{L}} \ge n_j^{\mathrm{R}} - 1.$$

  The set S_h^{L} includes all of the above extrema.

Remarks

- For the purely upwind flux ($\theta = 1$), there always holds $\vartheta_j = 1$ and hence $R_{j,k+1}(x)$ is the right Radau polynomial.
- For the upwind-biased flux ($\theta \neq 1$):
  - if the uniform mesh is used, we have

$$\vartheta_j \equiv \frac{\theta - (-1)^k(1-\theta)}{\theta + (-1)^k(1-\theta)} > 0.$$

  - However, it may happen $\vartheta_j \leq 0$ for thee non-uniform mesh.
- To show that, we give a numerical example.

N	1000	2000	4000	8000	16000
10% perturbation	9.318%	9.022%	9.199%	9.085%	9.139%
20% perturbation	25.513%	25.477%	25.449%	25.510%	25.471%

Here $k = 2$ and $\theta = 0.75$. The non-uniform mesh is gotten by random perturbations of a uniform mesh with N elements.

Proof for the numerical flux

- Apply the definition of GGR projection.
- Take $q = k$ in (40), and get

$$\{\!\{e^n\}\!\}^{(\theta)} = \{\!\{\xi^n\}\!\}^{(\theta)} - \{\!\{(\mathbb{G}_h^\theta)^\perp U^n\}\!\}^{(\theta)} - \sum_{1\le p\le k} \{\!\{\mathcal{F}_p(-\partial_x)^p U^n\}\!\}^{(\theta)} = \{\!\{\xi^n\}\!\}^{(\theta)},$$

  due to the first property in Lemma 4.2.
- Applying the inverse inequality and Theorem 4.1 we have

$$\begin{aligned} \|\{\!\{e^n\}\!\}^{(\theta)}\|_{L^2(S_h^{\mathrm{B}})} &\le C\|\xi^n\|_{L^2(I)} \\ &\le C\|U_0\|_{H^{\max(2k+2,r+1)}(I)}(h^{2k+1} + \tau^r). \end{aligned}$$

Proof for the average

- Analogously we have

$$\bar{e}^n = \bar{\xi}^n - \overline{\mathbb{G}_h^{\perp} U^n} - \sum_{1\le p\le k} \overline{\mathcal{F}_p(-\partial_x)^p U^n} = \bar{\xi}^n - \overline{\mathcal{F}_k(-\partial_x)^k U^n},$$

  due to Lemma 4.3.
- Applying the triangle inequality and the Holder's inequality, we obtain

$$\begin{aligned} \|\bar{e}^n\|_{L^2(S_h^{\mathrm{E}})} &\le C\|\xi^n\|_{L^2(I)} + C\|\mathcal{F}_k(-\partial_x)^k U^n\|_{L^2(I)} \\ &\le C\|U_0\|_{H^{\max(2k+2,r+1)}(I)}(h^{2k+1} + \tau^r). \end{aligned}$$

  where
  - Theorem 4.1 is used for the first term, and
  - Lemma 4.1 and the approximation properties of the two projections are used for the second term.

Proof for the roots and extrema (cont.)

- A local projection is introduced with minor modification, which is cited from the work of W. X. Cao.

Definition 7

- Let $w \in H^1(\mathcal{T}_h)$ be any given function.
- The projection $\mathbb{C}_h w \in V_h$ is defined element by element, and depends on the parameter ϑ_j.
- If $\vartheta_j \neq 0$, it satisfies (here $\mathbb{C}_h^{\perp} w = w - \mathbb{C}_h w$ is the projection error)

$$\int_{I_j} (\mathbb{C}_h^{\perp} w) v \, \mathrm{d}x = 0, \quad \forall v \in \mathcal{P}^{k-1}(I_j),$$

$$\theta_j (\mathbb{C}_h^{\perp} w)^{-}_{j+\frac{1}{2}} + (1-\theta_j)(\mathbb{C}_h^{\perp} w)^{+}_{j-\frac{1}{2}} = 0,$$

  where $\theta_j = (\vartheta_j + 1)/2$ for even k, and $\theta_j = (\vartheta_j^{-1} + 1)/2$ for odd k.
- If $\vartheta_j = 0$, it is defined as the standard L^2-projection $\mathbb{P}_h w$ in this element.

Proof for the roots and extrema

Lemma 4.5 (on weak regularity assumption)

There exists a bounding constant $C > 0$ *independent of* j, h_j *and* w*, such that*

$$\left|\mathbb{C}_h^{\perp} w(r_{ij})\right| + h_j \left|(\mathbb{C}_h^{\perp} w)_x(l_{ij})\right| \leq C h_j^{k+\frac{3}{2}} \|w\|_{H^{k+2}(I_j)}. \quad (57)$$

Lemma 4.6 (key conclusion for superconvergence)

There exists a bounding constant $C > 0$ *independent of* h *and* w*, such that*

$$\|\mathbb{G}_h^{\theta} w - \mathbb{C}_h w\|_{L^2(I)} \leq C h^{k+2} \|w\|_{H^{k+2}(I)}. \quad (58)$$

Lemma 4.7 (corollary of above conclusions)

There exists a bounding constant $C > 0$ *independent of* h *and* w*, such that*

$$\|(\mathbb{G}_h^{\theta})^{\perp} w\|_{L^2(S_h^{\mathrm{R}})} + h\|((\mathbb{G}_h^{\theta})^{\perp} w)_x\|_{L^2(S_h^{\mathrm{L}})} \leq C h^{k+2} \|w\|_{H^{k+2}(I)}. \quad (59)$$

Proof for the roots and extrema

- Taking $q = 1$ in (40), we have

$$\|e^n\|_{L^2(S_h^{\mathrm{R}})} \le \|\xi^n\|_{L^2(S_h^{\mathrm{R}})} + \|(\mathbb{G}_h^\theta)^\perp U^n\|_{L^2(S_h^{\mathrm{R}})} + \|\mathcal{F}_1(-\partial_x)U^n\|_{L^2(S_h^{\mathrm{R}})}.$$

- Using the inverse inequality, together with Theorem 4.1 and Lemma 4.1, we obtain

$$\|\xi^n\|_{L^2(S_h^{\mathrm{R}})} \le C\|\xi^n\|_{L^2(I)} \le C(h^{k+2} + \tau^r)\|U_0\|_{H^{\max(k+3,r+1)}(I)},$$
$$\|\mathcal{F}_1(-\partial_x)U^n\|_{L^2(S_h^{\mathrm{R}})} \le C\|\mathcal{F}_1(-\partial_x)U^n\|_{L^2(I)} \le Ch^{k+2}\|U_0\|_{H^{k+2}(I)},$$

  where the approximation property of the two projections are also used.
- It follows from Lemma 4.7 that

$$\|(\mathbb{G}_h^\theta)^\perp U^n\|_{L^2(S_h^{\mathrm{R}})} \le Ch^{k+2}\|U^n\|_{H^{k+2}(I)} \le Ch^{k+2}\|U_0\|_{H^{k+2}(I)}.$$

- Collecting up the above conclusions, we prove the estimate for roots in Theorem 4.2.
- Along the similar line, we can bound $\|e_x^n\|_{L^2(S_h^{\mathrm{L}})}$ by taking $q = 0$ in (40),.

4. Accuracy enhancement of post-processed solution

Accuracy enhancement of post-processed solution

- The filter on a uniform mesh is implemented by the convolution of the numerical solution with the kernel function

$$K_h^{2k+1,k+1}(x) = \frac{1}{h} \sum_{-k\leq\gamma\leq k} c_\gamma^{2k+1,k+1} \varphi^{(k+1)} \left(\frac{x}{h} - \gamma\right), \tag{60}$$

  where $\varphi^{(k+1)}$ is the B-spline function of order $k+1$.
- As a byproduct of the above superconvergence analysis, we have the following conclusion.

Theorem 4.3

Assume that the RKDG(s, r, k) method is of the L^2-norm stability under suitable temporal-spatial condition. Let (42) *be the initial solution with* $q_{\rm nt} = k$*, and* $M\tau = T$ *is the final time, then we have that*

$$\|U(M\tau) - K_h^{2k+1,k+1} \star u^M\| \leq C\|U_0\|_{H^{\max(2k+2,r+1)}(I)}(h^{2k+1} + \tau^r). \tag{61}$$

Here $\star$ *denotes the convolution.*

Proof of Theorem 4.3 (cont.)

- For the obtained numerical solution u^M, the post-processed solution satisfies the well-known conclusion[2]

$$\text{LHS} \le Ch^{2k+1}\|U^M\|_{H^{2k+1}(I)} + C\sum_{0\le\ell\le k+1}\|\partial_h^\ell e^M\|_{H^{-(k+1)}(I)}, \tag{62}$$

  where the bounding constant $C>0$ solely depends on k.

- Here $\partial_h^\ell e^M$ is the ℓ-th order divided difference of the numerical error.
- It is sufficient to prove this theorem by showing

$$(\partial_h^\ell e^M, \Phi) \le C\|U_0\|_{H^{\max(2k+2,r+1)}(I)}(h^{2k+1}+\tau^r)\|\Phi\|_{H^{k+1}(I)}, \tag{63}$$

  for any $\Phi \in C_0^\infty(I)$ and $0\le\ell\le k+1$.

- This purpose can be achieved with the previous superconvergence results.

[2] B. Cockburn, M. Luskin, C. -W. Shu and E. Süli, Math. Comp., 72 (2003), 577-606

Proof of Theorem 4.3 (cont.)

- Let w be any piecewise smooth function on $\mathcal{T}_h$. The divided difference is recursively defined by

$$\partial_h^\ell w(x) = h^{-1}\left[\partial_h^{\ell-1} w(x+h/2) - \partial_h^{\ell-1} w(x-h/2)\right],$$

  for $\ell \geq 1$, where $\partial_h^0 w(x) = w(x)$.
- By Holder's inequality, one can see that

$$\|\partial_h^\ell w\|_{L^2(I)} \leq \|\partial_x^\ell w\|_{L^2(I)}. \tag{64}$$

- If w and v are periodic, there holds

$$(\partial_h^\ell w, v) = (-1)^\ell (w, \partial_h^\ell v), \quad \ell \geq 0. \tag{65}$$

- ∂_h^ℓ commutes with $\mathbb{G}_h$, $\partial_x, \mathcal{F}_p$, and many operators.
- The above manipulations should be understood on the correspondingly shifted meshes.

Proof of Theorem 4.3 (cont.)

- Take $q = k$ in (40), and define for $0 \le \kappa \le s-1$,

$$\chi^{n,\kappa} = \mathbb{G}_h^{\theta} U^{n,\kappa} - \sum_{1\le p\le k} \mathcal{F}_p (-\partial_x)^p W^{n,\kappa} \in V_h.$$

  Recalling the error decomposition $e^{n,\kappa} = \xi^{n,\kappa} - \eta^{n,\kappa}$ with $\xi^{n,\kappa} = u^{n,\kappa} - \chi^{n,\kappa}$, we have

$$(\partial_h^{\ell} e, \Phi) = (\partial_h^{\ell} \xi, \Phi) - (\partial_h^{\ell} \eta, \Phi).$$

  Here and below $e = e^M$, i.e., the superscript M is dropped for simplicity of notations.

- Since $0 \le \ell \le k+1$, by (65) and (64) we have

$$(\partial_h^{\ell} \xi, \Phi) = (-1)^{\ell} (\xi, \partial_h^{\ell} \Phi) \le \|\xi\|_{L^2(I)} \|\Phi\|_{H^{k+1}(I)}.$$

  Hence, due to Theorem 4.1, we have

$$(\partial_h^{\ell} \xi, \Phi) \le C(h^{2k+1} + \tau^r) \|U_0\|_{H^{\max(2k+2,r+1)}(I)} \|\Phi\|_{H^{k+1}(I)}.$$

Proof of Theorem 4.3 (cont.)

- By the definition of the correction function, $(\partial_h^\ell \eta, \Phi)$ is equal to

$$(\partial_h^\ell (\mathbb{G}_h^\theta)^\perp U, \Phi) + \sum_{1\le p\le k-1} \big(\partial_h^\ell \mathcal{F}_p (-\partial_x)^p U, \Phi\big) + \big(\partial_h^\ell \mathcal{F}_k (-\partial_x)^k U, \Phi\big).$$

  Each tern will be separately estimated below.
- The first term:
  - The GGR projection implies

$$(\partial_h^\ell (\mathbb{G}_h^\theta)^\perp U, \Phi) = ((\mathbb{G}_h^\theta)^\perp \partial_h^\ell U, \Phi) = ((\mathbb{G}_h^\theta)^\perp \partial_h^\ell U, \Phi - \mathbb{P}_h^{k-1}\Phi),$$

    where $\mathbb{P}_h^{k-1}$ is the local L^2-projection onto V_h^{k-1}.
  - The approximation properties of the two projections lead to

$$\begin{aligned}(\partial_h^\ell (\mathbb{G}_h^\theta)^\perp U, \Phi) &\le Ch^{2k+1} \|\partial_h^\ell U\|_{H^{k+1}(I)} \|\Phi\|_{H^k(I)}\\ &\le Ch^{2k+1} \|U\|_{H^{2k+2}(I)} \|\Phi\|_{H^k(I)}.\end{aligned}$$

Proof of Theorem 4.3 (cont.)

- The second term: assume $k \geq 2$, since it vanishes when $k = 1$.
  - Depending on ℓ, we split the summation index into two sets.
  - There is no harm in assuming $1 \leq p \leq \min(\ell, k-1)$.
  - By using (65), the commutative property, and Lemma 4.3, we have

$$\begin{aligned}(\partial_h^\ell \mathcal{F}_p(-\partial_x)^p U, \Phi) &= (-1)^p (\partial_h^{\ell-p} \mathcal{F}_p(-\partial_x)^p U, \partial_h^p \Phi) \\ &= (-1)^p (\mathcal{F}_p(-\partial_x)^p \partial_h^{\ell-p} U, \partial_h^p \Phi) \\ &= (-1)^p (\mathcal{F}_p(-\partial_x)^p \partial_h^{\ell-p} U, \partial_h^p \Phi - \mathbb{P}_h^{k-1-p} \partial_h^p \Phi) \\ &\leq C h^p \cdot h^{k+1} \|\partial_h^{\ell-p} U\|_{H^{k+1+p}(I)} \cdot h^{k-p} \|\partial_h^p \Phi\|_{H^{k-p}(I)} \\ &\leq C h^{2k+1} \|U\|_{H^{2k+2}(I)} \|\Phi\|_{H^k(I)},\end{aligned}$$

    where Lemma 4.1, the approximation property of the two projections, and (64) are used.
  - Along the same line we can get the same boundedness for $\ell < p \leq k-1$.
  - Hence, the second term is bounded by

$$\sum_{1 \leq p \leq k-1} \left(\partial_h^\ell \mathcal{F}_p(-\partial_x)^p U, \Phi\right) \leq C h^{2k+1} \|U\|_{H^{2k+2}(I)} \|\Phi\|_{H^k(I)}.$$

Proof of Theorem 4.3

- Similarly, the third term can be bounded in the form

$$
\begin{aligned}
(\partial_h^\ell \mathcal{F}_k(-\partial_x)^k U, \Phi) &= (-1)^\ell (\mathcal{F}_k(-\partial_x)^k U, \partial_h^\ell \Phi) \\
&\le Ch^{2k+1} \|U\|_{H^{2k+1}(I)} \|\partial_h^\ell \Phi\|_{L^2(I)} \\
&\le Ch^{2k+1} \|U\|_{H^{2k+1}(I)} \|\Phi\|_{H^{k+1}(I)}.
\end{aligned}
$$

- Collecting up the above estimates and noticing that $U = U_0(x - \beta T)$, we can obtain (63) and then prove this theorem.

Remark 4.1

- *Theorem 4.3 requires a special setting on the initial solution, which is inherited from the supraconvergence study.*
- *In practice, the L^2-projection setting (not included in this theorem) still works well to obtain the accuracy enhancement; see the numerical experiment below.*

5. Numerical experiments

Numerical experiments

- Carry out the RKDG$(r, r, 2)$ method with the upwind-biased parameter $\theta = 0.75$, to solve the model problem (1) with $\beta = 1$ and $T = 1$.
- The non-uniform meshes are obtained by a random perturbation of the equidistance nodes by at most 10%.
- The uniform mesh is used for the post-processing.
- The time step is taken $\tau = 0.2h_{\min}$, where $h_{\min}$ is the minimum of all element lengths.
- Two tests:
  - supraconvergence/superconvergence order:
  $$U_0 = \sin(2\pi x).$$
  - the sharpness of the regularity assumption:
  $$U_0 = \sin^{\epsilon+2/3}(2\pi x)$$
  with a positive integer ϵ.

Superconvergence order: solution and derivative (1)

$q_{\rm nt}=1$	N	$\|\|e\|\|_{L^2(S_h^{\rm R})}$		$\|\|e\|\|_{L^\infty(S_h^{\rm R})}$		$\|\|e_x\|\|_{L^2(S_h^{\rm L})}$		$\|\|e_x\|\|_{L^\infty(S_h^{\rm L})}$	
$r=3$	1000	1.89E-10		2.79E-10		1.20E-08		7.30E-08	
	2000	2.45E-11	2.95	3.53E-11	2.98	1.41E-09	3.09	7.97E-09	3.19
	4000	2.98E-12	3.04	4.23E-12	3.06	1.71E-10	3.04	1.12E-09	2.83
	8000	3.73E-13	3.00	5.30E-13	3.00	2.11E-11	3.01	1.33E-10	3.07
	16000	4.62E-14	3.01	6.54E-14	3.02	2.68E-12	2.98	1.83E-11	2.87
$r=4$	1000	1.83E-12		8.22E-12		1.05E-08		5.02E-08	
	2000	1.20E-13	3.94	6.85E-13	3.59	1.39E-09	2.91	7.76E-09	2.69
	4000	7.31E-15	4.04	5.48E-14	3.64	1.69E-10	3.05	1.14E-09	2.77
	8000	4.60E-16	3.99	3.38E-15	4.02	2.12E-11	2.99	1.43E-10	2.99
	16000	2.90E-17	3.99	2.17E-16	3.96	2.68E-12	2.98	1.79E-11	3.00
$r=5$	1000	1.90E-12		8.44E-12		1.10E-08		5.10E-08	
	2000	1.16E-13	4.03	6.26E-13	3.75	1.33E-09	3.05	7.25E-09	2.81
	4000	7.35E-15	3.98	4.03E-14	3.96	1.70E-10	2.97	9.23E-10	2.97
	8000	4.63E-16	3.99	3.28E-15	3.62	2.14E-11	2.99	1.38E-10	2.74
	16000	2.92E-17	3.98	2.51E-16	3.71	2.72E-12	2.98	2.12E-11	2.70
$r=6$	1000	1.91E-12		1.00E-11		1.11E-08		5.84E-08	
	2000	1.18E-13	4.02	7.79E-13	3.69	1.36E-09	3.03	8.65E-09	2.75
	4000	7.29E-15	4.01	4.17E-14	4.22	1.68E-10	3.01	9.12E-10	3.25
	8000	4.60E-16	3.99	3.51E-15	3.57	2.12E-11	2.99	1.49E-10	2.62
	16000	2.88E-17	4.00	2.35E-16	3.90	2.66E-12	3.00	1.94E-11	2.94

Superconvergence order: solution and derivative (2)

$q_{\mathrm{nt}} = 0$	N	$\lVert e \rVert_{L^2(S_h^{\mathrm{R}})}$		$\lVert e \rVert_{L^\infty(S_h^{\mathrm{R}})}$		$\lVert e_x \rVert_{L^2(S_h^{\mathrm{L}})}$		$\lVert e_x \rVert_{L^\infty(S_h^{\mathrm{L}})}$	
$r = 3$	1000	1.96E-10		2.89E-10		1.12E-08		6.70E-08	
	2000	2.40E-11	3.03	3.48E-11	3.06	1.39E-09	3.02	8.61E-09	2.96
	4000	2.98E-12	3.01	4.24E-12	3.03	1.74E-10	2.99	1.14E-09	2.91
	8000	3.71E-13	3.01	5.27E-13	3.01	2.16E-11	3.01	1.53E-10	2.90
	16000	4.61E-14	3.01	6.54E-14	3.01	2.70E-12	3.00	1.87E-11	3.04
$r = 4$	1000	1.96E-12		1.16E-11		1.14E-08		6.00E-08	
	2000	1.19E-13	4.03	7.46E-13	3.96	1.39E-09	3.04	8.18E-09	2.88
	4000	7.27E-15	4.04	4.85E-14	3.94	1.67E-10	3.06	9.84E-10	3.06
	8000	4.62E-16	3.98	3.84E-15	3.66	2.13E-11	2.97	1.57E-10	2.65
	16000	2.88E-17	4.00	2.19E-16	4.13	2.66E-12	3.00	2.03E-11	2.95
$r = 5$	1000	1.93E-12		8.61E-12		1.13E-08		5.24E-08	
	2000	1.14E-13	4.08	6.47E-13	3.73	1.30E-09	3.12	7.42E-09	2.82
	4000	7.52E-15	3.92	5.59E-14	3.53	1.75E-10	2.90	1.13E-09	2.72
	8000	4.58E-16	4.04	2.74E-15	4.35	2.11E-11	3.05	1.48E-10	2.94
	16000	2.87E-17	4.00	2.53E-16	3.44	2.65E-12	3.00	2.04E-11	2.86
$r = 6$	1000	1.91E-12		1.34E-11		1.11E-08		7.20E-08	
	2000	1.18E-13	4.02	8.20E-13	4.03	1.35E-09	3.03	8.81E-09	3.03
	4000	7.37E-15	4.00	4.81E-14	4.09	1.70E-10	2.99	1.06E-09	3.05
	8000	4.55E-16	4.02	3.80E-15	3.66	2.09E-11	3.02	1.53E-10	2.79
	16000	2.88E-17	3.98	2.35E-16	4.01	2.66E-12	2.98	1.95E-11	2.98

Superconvergence order: flux and average (1)

$q_{\rm nt}=k$	N	$\|\{\!\{e\}\!\}^{(\theta)}\|_{L^2(S_h^{\rm B})}$		$\|\{\!\{e\}\!\}^{(\theta)}\|_{L^\infty(S_h^{\rm B})}$		$\|\bar{e}\|_{L^2(S_h^{\rm E})}$		$\|\bar{e}\|_{L^\infty(S_h^{\rm E})}$	
$r=3$	1000	1.92E-10		2.71E-10		1.92E-10		2.71E-10	
	2000	2.44E-11	2.97	3.45E-11	2.97	2.44E-11	2.97	3.45E-11	2.97
	4000	2.98E-12	3.03	4.21E-12	3.03	2.98E-12	3.03	4.21E-12	3.03
	8000	3.72E-13	3.00	5.27E-13	3.00	3.72E-13	3.00	5.27E-13	3.00
	16000	4.60E-14	3.02	6.51E-14	3.02	4.60E-14	3.02	6.51E-14	3.02
$r=4$	1000	3.90E-14		5.51E-14		3.95E-14		5.77E-14	
	2000	2.40E-15	4.02	3.39E-15	4.02	2.42E-15	4.03	3.48E-15	4.05
	4000	1.53E-16	3.97	2.17E-16	3.97	1.54E-16	3.97	2.19E-16	3.99
	8000	9.35E-18	4.03	1.32E-17	4.03	9.37E-18	4.04	1.33E-17	4.04
	16000	5.85E-19	4.00	8.27E-19	4.00	5.85E-19	4.00	8.30E-19	4.00
$r=5$	1000	3.62E-15		5.12E-15		3.69E-15		5.63E-15	
	2000	1.13E-16	5.00	1.60E-16	5.00	1.15E-16	5.00	1.79E-16	4.97
	4000	3.51E-18	5.01	4.97E-18	5.01	3.58E-18	5.00	5.66E-18	4.99
	8000	1.11E-19	4.99	1.57E-19	4.99	1.13E-19	4.99	1.79E-19	4.98
	16000	3.45E-21	5.00	4.88E-21	5.00	3.52E-21	5.00	5.64E-21	4.99
$r=6$	1000	3.65E-15		5.16E-15		3.72E-15		5.79E-15	
	2000	1.14E-16	5.00	1.61E-16	5.00	1.16E-16	5.00	1.87E-16	4.96
	4000	3.54E-18	5.01	5.01E-18	5.01	3.61E-18	5.01	5.79E-18	5.01
	8000	1.11E-19	5.00	1.57E-19	5.00	1.13E-19	5.00	1.80E-19	5.01
	16000	3.47E-21	5.00	4.91E-21	5.00	3.54E-21	5.00	5.57E-21	5.02

Superconvergence order: flux and average (2)

$q_{\text{nt}} = k - 1$	N	$\lVert \{\!\{e\}\!\}^{(\theta)} \rVert_{L^2(S_h^{\mathrm{B}})}$		$\lVert \{\!\{e\}\!\}^{(\theta)} \rVert_{L^\infty(S_h^{\mathrm{B}})}$		$\lVert \bar{e} \rVert_{L^2(S_h^{\mathrm{E}})}$		$\lVert \bar{e} \rVert_{L^\infty(S_h^{\mathrm{E}})}$	
	1000	1.89E-10		2.67E-10		1.89E-10		2.67E-10	
	2000	2.45E-11	2.94	3.47E-11	2.94	2.45E-11	2.94	3.47E-11	2.94
$r = 3$	4000	2.98E-12	3.04	4.21E-12	3.04	2.98E-12	3.04	4.21E-12	3.04
	8000	3.73E-13	3.00	5.27E-13	3.00	3.73E-13	3.00	5.27E-13	3.00
	16000	4.62E-14	3.01	6.53E-14	3.01	4.62E-14	3.01	6.53E-14	3.01
	1000	3.95E-14		5.62E-14		4.01E-14		5.77E-14	
	2000	2.41E-15	4.04	3.41E-15	4.04	2.42E-15	4.05	3.47E-15	4.06
$r = 4$	4000	1.49E-16	4.02	2.11E-16	4.02	1.49E-16	4.02	2.12E-16	4.03
	8000	9.29E-18	4.00	1.31E-17	4.00	9.31E-18	4.00	1.32E-17	4.01
	16000	5.81E-19	4.00	8.22E-19	4.00	5.82E-19	4.00	8.24E-19	4.00
	1000	3.70E-15		5.26E-15		3.67E-15		5.33E-15	
	2000	1.14E-16	5.01	1.64E-16	5.00	1.14E-16	5.02	1.66E-16	5.00
$r = 5$	4000	3.61E-18	4.99	5.21E-18	4.98	3.59E-18	4.98	5.29E-18	4.97
	8000	1.12E-19	5.01	1.61E-19	5.01	1.12E-19	5.01	1.65E-19	5.00
	16000	3.52E-21	5.00	5.07E-21	4.99	3.50E-21	4.99	5.19E-21	4.99
	1000	3.73E-15		5.34E-15		3.71E-15		5.41E-15	
	2000	1.15E-16	5.02	1.66E-16	5.01	1.14E-16	5.02	1.69E-16	5.00
$r = 6$	4000	3.58E-18	5.00	5.12E-18	5.02	3.56E-18	5.00	5.24E-18	5.01
	8000	1.12E-19	4.99	1.62E-19	4.99	1.12E-19	4.99	1.65E-19	4.98
	16000	3.51E-21	5.00	5.05E-21	5.00	3.49E-21	5.00	5.19E-21	5.00

Test on supracongergence order

$q_{\rm nt} = k$	N	$\|\xi\|_{L^2(I)}$		$\|\xi_x\|_{L^2(I)}$		$\|\xi_{xx}\|_{L^2(I)}$	
$r = 3$	1000	1.92E-10		1.20E-09		7.57E-09	
	2000	2.44E-11	2.97	1.53E-10	2.97	9.63E-10	2.97
	4000	2.98E-12	3.03	1.87E-11	3.03	1.18E-10	3.03
	8000	3.72E-13	3.00	2.34E-12	3.00	1.47E-11	3.00
	16000	4.60E-14	3.02	2.89E-13	3.02	1.82E-12	3.02
$r = 4$	1000	3.90E-14		2.45E-13		7.97E-12	
	2000	2.40E-15	4.02	1.51E-14	4.02	4.79E-13	4.06
	4000	1.53E-16	3.97	9.63E-16	3.97	3.02E-14	3.99
	8000	9.35E-18	4.03	5.88E-17	4.03	1.93E-15	3.96
	16000	5.85E-19	4.00	3.67E-18	4.00	1.19E-16	4.02
$r = 5$	1000	3.62E-15		2.30E-14		7.62E-12	
	2000	1.13E-16	5.00	7.18E-16	5.00	4.78E-13	3.99
	4000	3.51E-18	5.01	2.23E-17	5.01	2.95E-14	4.02
	8000	1.11E-19	4.99	7.06E-19	4.98	1.92E-15	3.95
	16000	3.45E-21	5.00	2.20E-20	5.01	1.17E-16	4.04
$r = 6$	1000	3.65E-15		2.32E-14		7.90E-12	
	2000	1.14E-16	5.00	7.24E-16	5.00	4.77E-13	4.05
	4000	3.54E-18	5.01	2.25E-17	5.01	3.00E-14	3.99
	8000	1.11E-19	5.00	7.06E-19	4.99	1.89E-15	3.99
	16000	3.47E-21	5.00	2.21E-20	5.00	1.18E-16	4.00

Supraconvergence order (1)

$q_{\text{nt}} = k-1$	N	$\|\xi\|_{L^2(I)}$		$\|\xi_x\|_{L^2(I)}$		$\|\xi_{xx}\|_{L^2(I)}$	
$r=3$	1000	1.89E-10		1.19E-09		7.45E-09	
	2000	2.45E-11	2.94	1.54E-10	2.94	9.68E-10	2.94
	4000	2.98E-12	3.04	1.87E-11	3.04	1.17E-10	3.04
	8000	3.73E-13	3.00	2.34E-12	3.00	1.47E-11	3.00
	16000	4.62E-14	3.01	2.90E-13	3.01	1.82E-12	3.01
$r=4$	1000	3.95E-14		2.77E-13		1.34E-10	
	2000	2.41E-15	4.04	1.61E-14	4.11	1.04E-11	3.69
	4000	1.49E-16	4.02	9.72E-16	4.05	9.38E-13	3.47
	8000	9.29E-18	4.00	5.98E-17	4.02	8.10E-14	3.53
	16000	5.81E-19	4.00	3.72E-18	4.01	7.53E-15	3.43
$r=5$	1000	3.70E-15		9.76E-14		1.05E-10	
	2000	1.14E-16	5.01	4.91E-15	4.31	9.65E-12	3.44
	4000	3.61E-18	4.99	2.72E-16	4.17	9.42E-13	3.36
	8000	1.12E-19	5.01	1.33E-17	4.36	8.24E-14	3.51
	16000	3.52E-21	5.00	6.96E-19	4.25	7.70E-15	3.42
$r=6$	1000	3.73E-15		1.08E-13		1.14E-10	
	2000	1.15E-16	5.02	5.20E-15	4.38	1.03E-11	3.47
	4000	3.58E-18	5.00	2.71E-16	4.26	9.57E-13	3.43
	8000	1.12E-19	4.99	1.38E-17	4.30	8.54E-14	3.49
	16000	3.51E-21	5.00	6.64E-19	4.37	7.38E-15	3.53

Supraconvergence order (2)

$q_{\mathrm{nt}} = k - 2$	N	$\|\xi\|_{L^2(I)}$		$\|\xi_x\|_{L^2(I)}$		$\|\xi_{xx}\|_{L^2(I)}$	
$r = 3$	1000	1.96E-10		1.23E-09		1.70E-08	
	2000	2.40E-11	3.03	1.51E-10	3.03	2.22E-09	2.94
	4000	2.98E-12	3.01	1.87E-11	3.01	3.10E-10	2.84
	8000	3.71E-13	3.01	2.33E-12	3.01	4.44E-11	2.81
	16000	4.61E-14	3.01	2.90E-13	3.01	5.93E-12	2.90
$r = 4$	1000	4.22E-14		1.95E-11		2.36E-08	
	2000	2.48E-15	4.09	1.64E-12	3.57	3.58E-09	2.72
	4000	1.53E-16	4.02	1.39E-13	3.56	5.46E-10	2.71
	8000	9.51E-18	4.01	1.31E-14	3.40	9.11E-11	2.58
	16000	5.96E-19	4.00	1.18E-15	3.48	1.45E-11	2.65
$r = 5$	1000	1.63E-14		1.80E-11		2.18E-08	
	2000	8.42E-16	4.27	1.72E-12	3.39	3.76E-09	2.53
	4000	4.11E-17	4.36	1.44E-13	3.57	5.53E-10	2.76
	8000	2.05E-18	4.32	1.29E-14	3.48	8.95E-11	2.63
	16000	1.04E-19	4.31	1.16E-15	3.48	1.43E-11	2.64
$r = 6$	1000	1.66E-14		1.80E-11		2.15E-08	
	2000	8.05E-16	4.37	1.60E-12	3.49	3.48E-09	2.62
	4000	3.95E-17	4.35	1.41E-13	3.50	5.47E-10	2.67
	8000	2.00E-18	4.31	1.27E-14	3.47	8.82E-11	2.63
	16000	1.10E-19	4.19	1.23E-15	3.36	1.51E-11	2.55

Accuracy enhancement of post-processed solution

	N	$q_{\rm nt} = k$		$q_{\rm nt} = 0$ (GGR)		L^2 projection	
$r = 3$	1000	3.67E-10		3.67E-10		3.67E-10	
	2000	4.59E-11	3.00	4.59E-11	3.00	4.59E-11	3.00
	4000	5.74E-12	3.00	5.74E-12	3.00	5.74E-12	3.00
	8000	7.17E-13	3.00	7.17E-13	3.00	7.17E-13	3.00
	16000	8.97E-14	3.00	8.97E-14	3.00	8.97E-14	3.00
$r = 4$	1000	9.38E-14		9.29E-14		9.24E-14	
	2000	5.82E-15	4.01	5.79E-15	4.00	5.77E-15	4.00
	4000	3.62E-16	4.01	3.61E-16	4.00	3.61E-16	4.00
	8000	2.26E-17	4.00	2.26E-17	4.00	2.25E-17	4.00
	16000	1.41E-18	4.00	1.41E-18	4.00	1.41E-18	4.00
$r = 5$	1000	3.46E-15		3.18E-15		3.15E-15	
	2000	1.06E-16	5.03	9.73E-17	5.03	9.61E-17	5.03
	4000	3.28E-18	5.01	3.00E-18	5.02	2.97E-18	5.02
	8000	1.02E-19	5.01	9.33E-20	5.01	9.22E-20	5.01
	16000	3.18E-21	5.00	2.91E-21	5.00	2.87E-21	5.00
$r = 6$	1000	3.48E-15		3.20E-15		3.17E-15	
	2000	1.07E-16	5.03	9.79E-17	5.03	9.67E-17	5.03
	4000	3.30E-18	5.01	3.02E-18	5.02	2.99E-18	5.02
	8000	1.03E-19	5.01	9.39E-20	5.01	9.28E-20	5.01
	16000	3.20E-21	5.00	2.93E-21	5.00	2.89E-21	5.00

Sharpness of regularity assumption (1)

Table: Superconvergence results. In each group, the regularity parameter is $\epsilon - 1$ for the left column, and is ϵ for the right column.

J	$\|\|\|e\|\|\|_{L^2(S_h^{\mathrm{R}})}$, $q_{\mathrm{nt}} = 0, r = 4$, $\epsilon = 4$				$\|\|\|e_x\|\|\|_{L^2(S_h^{\mathrm{L}})}$, $q_{\mathrm{nt}} = 0, r = 3$, $\epsilon = 4$			
1000	6.53E-10		8.42E-11		1.42E-06		4.92E-07	
2000	5.65E-11	3.53	4.96E-12	4.09	2.57E-07	2.46	6.35E-08	2.95
4000	4.91E-12	3.52	3.10E-13	4.00	4.52E-08	2.51	7.76E-09	3.03
8000	4.31E-13	3.51	1.91E-14	4.02	8.29E-09	2.45	9.87E-10	2.98
16000	3.84E-14	3.49	1.21E-15	3.98	1.54E-09	2.43	1.22E-10	3.02
N	$\|\|\|\{\!\{e\}\!\}^{(\theta)}\|\|\|_{L^2(S_h^{\mathrm{B}})}$, $q_{\mathrm{nt}} = k, r = 5$, $\epsilon = 5$				$\|\|\|\bar{e}\|\|\|_{L^2(S_h^{\mathrm{E}})}$, $q_{\mathrm{nt}} = k, r = 5$, $\epsilon = 5$			
1000	1.72E-11		5.12E-12		1.64E-11		5.12E-12	
2000	8.20E-13	4.39	1.61E-13	4.99	7.88E-13	4.38	1.61E-13	4.99
4000	4.11E-14	4.32	5.06E-15	4.99	3.97E-14	4.31	5.07E-15	4.99
8000	2.02E-15	4.34	1.57E-16	5.01	1.97E-15	4.33	1.57E-16	5.01
16000	9.99E-17	4.34	4.94E-18	4.99	9.80E-17	4.33	4.94E-18	4.99

Sharpness of regularity assumption (2)

Table: Supraconvergence results. In each group, the regularity parameter is $\epsilon - 1$ for the left column, and is ϵ for the right column.

N	$\|\xi\|_{L^2(I)}$, $q_{\text{nt}} = k, r = 5$, $\epsilon = 5$				$\|\xi_x\|_{L^2(I)}$, $q_{\text{nt}} = k, r = 5$, $\epsilon = 6$			
1000	1.72E-11		5.12E-12		6.38E-10		2.99E-10	
2000	8.32E-13	4.37	1.61E-13	4.99	3.04E-11	4.39	9.39E-12	4.99
4000	4.08E-14	4.35	5.06E-15	4.99	1.48E-12	4.36	2.95E-13	4.99
8000	2.01E-15	4.34	1.57E-16	5.01	7.23E-14	4.35	9.15E-15	5.01
16000	9.96E-17	4.33	4.94E-18	4.99	3.58E-15	4.34	2.87E-16	4.99

Sharpness of regularity assumption (3)

Table: Accuracy enhancement of post-processed solution. The regularity parameter is $\epsilon - 1$ for the left column, and is ϵ for the right column.

N	$q_{\text{nt}} = k, r = 5, \quad \epsilon = 5$			
1000	1.59E-11		4.50E-12	
2000	7.59E-13	4.39	1.38E-13	5.03
4000	3.67E-14	4.37	4.25E-15	5.02
8000	1.79E-15	4.36	1.32E-16	5.01
16000	8.82E-17	4.35	4.12E-18	5.00

Outline

Concluding remarks

- Stability and error estimates of RKDG methods when solving linear hyperbolic equation:
  - a flexible framework is proposed to analyze the L^2-norm stability;
  - different stability mechanisms are pointed out;
  - a systematical theory discussions on the termination index and the contribution index.
  - the detailed performances are investigated for many popular schemes, especially for the same-stage-and-same-order RK time-marching.
  - the optimal L^2-norm error estimate and superconvergence analysis for the fully discrete RKDG methods.
- Future work
  - generalize to the other kinds of time-marching?
  - apply to the RKDG method for linear hyperbolic equations with variable coefficients and/or nonlinear conservation laws?
  - the other kinds of error estimates?

Concluding remarks

- We have proposed a uniform framework to obtain the L^2-norm stability performance of the RKDG method, by the help of the matrix transferring process, based on the temporal difference of stage solutions.
- In the error estimate, the additional tool is the definition of the reference function at every stage time.
- By the above technique, the superconvergence analysis can be easily carried out for the RKDG method.

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Thanks for your attention!