HDG for convection-dominated problems

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Lectures Series on High-Order Numerical Methods

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Outline

1. Scalar linear convection-diffusion equation
   - CG
   - DG
   - HDG
2. Incompressible Navier-Stokes equations
   - Mixed FEM for Stokes
   - HDiv-DG/HDiv-HDG for Stokes
   - HDiv-HDG for Navier-Stokes

The linear convection-diffusion equation

The boundary value problem:

$$-\epsilon \triangle u + \boldsymbol{w} \cdot \nabla u = f, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega.$$

- $\epsilon > 0$ is the diffusivity coefficient
- $\boldsymbol{w} : \Omega \to \mathbb{R}^d$ is a velocity vector field (satisfying $\nabla \cdot \boldsymbol{w} = 0$)
- $f : \Omega \to \mathbb{R}$ is the source term

- We are interested in the convection dominated case where $\epsilon \ll \|\boldsymbol{w}\|_\infty$

Example 1: boundary layer.

Consider the simple 1D case:

$$-\epsilon u'' + u' = 1 \text{ in } (0,1), \text{ with } u(0) = u(1) = 0.$$

- Exact solution:

$$u(x) = x - \frac{1 - e^{x/\epsilon}}{1 - e^{1/\epsilon}} = x - \frac{e^{(x-1)/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}$$

- For $\epsilon \ll 1$ the solution behaves essentially like $\bar{u}(x) = x$ expect for x being close to the right boundary:

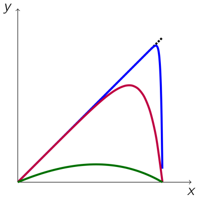

Figure: Solution for $\epsilon \in \{0, 0.01, 0.1, 1\}$

Weak formulation and coercivity

- Weak formulation: Find $u \in V = H_0^1(\Omega)$ such that, for all $v \in V$,

$$A(u,v) = \epsilon\, a(u,v) + b(\boldsymbol{w}; u, v) = \int_\Omega \epsilon \nabla u \cdot \nabla v + \int_\Omega (\boldsymbol{w} \cdot \nabla u) v = \int_\Omega f v.$$

- Since velocity $\boldsymbol{\omega}$ is divergence-free, we have

$$b(\boldsymbol{w}; u, u) = \int_\Omega (\boldsymbol{w} \cdot \nabla u) u = \frac{1}{2} \int_\Omega \nabla \cdot (\boldsymbol{w}\, u^2) = 0$$

- Hence, bilinear form $A(\cdot,\cdot)$ is coercive and bounded:

$$A(u,u) \geq c_0 \epsilon \|u\|_1^2, \quad A(u,v) \leq (\epsilon + \|\boldsymbol{w}\|_\infty) \|u\|_1 \|v\|_1,$$

where $c_0 > 0$ is related to the Poincare' constant

Galerkin FEM and Céa Lemma

- Galerkin FEM: Find $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$A(u_h, v) = F(v), \quad \forall v \in V_h.$$

- Energy estimate (Cea's lemma):

$$\|u - u_h\|_1 \leq \frac{\epsilon + \|\boldsymbol{w}\|_\infty}{c_0\,\epsilon} \inf_{v \in V_h} \|u - v\|_1 = \frac{1 + \frac{\|\boldsymbol{w}\|_\infty}{\epsilon}}{c_0} \inf_{v \in V_h} \|u - v\|_1$$

Proof: Galerkin orthogonality: $A(u - u_h, v) = 0 \quad \forall v \in V_h$.
Then, for any $v_h \in V_h$, we have:

$$\begin{aligned} \|u - u_h\|_1^2 \leq & \ \frac{1}{c_0\,\epsilon} A(u - u_h, u - u_h) \\ = & \ \frac{1}{c_0\,\epsilon} A(u - u_h, u - v_h) \leq \frac{\epsilon + \|\boldsymbol{w}\|_\infty}{c_0\,\epsilon} \|u - u_h\|_1 \|u - v_h\|_1 \end{aligned}$$

Example 2: Hemker problem. Show code [cd_CG.ipynb]

Page 3400 in [CMAME, 200(2011), pp. 3395-3409]

- Domain: $[-3,9]\times[-3,3]\backslash\{(x,y): x^2+y^2<1\}$
- Dirichlet boundary condition $u=0$ on left boundary, Dirichlet boundary condition $u=1$ on the disk boundary. Homogeneous Neumann BC on other boundaries.
- PDE: $-\epsilon\triangle u+\partial_x u=0$,
- Boundary layer can be observed around the inner disk boundary. Small $\epsilon \ll 1$ leads to large oscillations near the left half of the disk boundary.
- Denote the *local mesh Péclet number:* $Pe_T = \frac{\|\boldsymbol{w}\|_\infty h_T}{2r\epsilon}$. The challenging case is when $Pe_T \gg 1$. Here r is the polynomial degree.
- For $Pe_T \gg 1$, suitable convective stabilization is needed

Convective stabilization I: SUPG (skip)

SUPG stabilization

- Keep the same H^1-conforming finite element space, but replace $A(u_h, v) = F(v)$ with a stabilized version:

$$A_h(u_h, v) = A(u_h, v) + s_h(u_h, v) = F(v) + f_s(v).$$

- The (residual-based) stabilization term ($\gamma_T \geq 0$):

$$s_h(u_h, v) - f_s(v) = \sum_{T \in \Omega_h} \gamma_T \int_T (-\epsilon \Delta u_h + \boldsymbol{w} \cdot \nabla u_h - f)(\boldsymbol{w} \cdot \nabla v) \mathrm{dx}$$

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Convective stabilization II: upwinding DG

DG upwinding stabilization

- Relax H^1-conformity in the finite element space:

$$V_h^{dg} = \{v \in L^2(\Omega): \ v|_T \in \mathcal{P}^r(T), \ \forall T \in \Omega_h\} \not\subset H^1(\Omega)$$

- Upwinding DG for convection (upwinding numerical flux $\widehat{u}_h$):

$$b_h(\boldsymbol{w}; u_h, v) = \sum_{T \in \Omega_h} -(\boldsymbol{w} u_h, \nabla v)_T + \langle \boldsymbol{w} \cdot \boldsymbol{n}\, \widehat{u}_h, v \rangle_{\partial T}$$

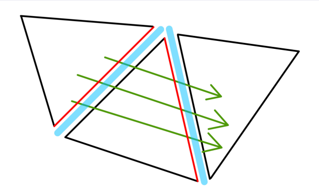


Figure: The flow direction is indicated by the green arrows. On every facet (blue) the numerical flux is chosen according to the upwind element (red).

Upwinding flux introduce numerical dissipation

$$
\begin{aligned}
b_h(\boldsymbol{w}; v_h, v_h) &= \sum_{T\in\Omega_h} -(\boldsymbol{w} v_h, \nabla v_h)_T + \langle \boldsymbol{w}\cdot\boldsymbol{n}\,\widehat{v}_h, v_h\rangle_{\partial T} \\
&= \sum_{T\in\Omega_h} -\frac{1}{2}\int_T \nabla\cdot(\boldsymbol{w} v_h^2) + \langle \boldsymbol{w}\cdot\boldsymbol{n}\,\widehat{v}_h, v_h\rangle_{\partial T} \\
&= \sum_{T\in\Omega_h} \int_{\partial T} \left(-\frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n}) v_h^2 + \boldsymbol{w}\cdot\boldsymbol{n}\widehat{v}_h v_h\right) \mathrm{ds} \\
&= \sum_{T\in\Omega_h} \int_{\partial T} \left(-\frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})(v_h - \widehat{v}_h)^2 + \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{n}\widehat{v}_h^2\right) \mathrm{ds} \\
&= \cdots \\
&= \sum_F \int_F \frac{1}{2}|\boldsymbol{w}\cdot\boldsymbol{n}| [\![v_h]\!]^2 \,\mathrm{ds} \ge 0.
\end{aligned}
$$

Discontinuous Galerkin: interior penalty for diffusion[1]

- Symmetric interior penalty DG (S-IPDG) for diffusion:

$$a_h(u_h, v) = \sum_{T\in\Omega_h} (\nabla u_h, \nabla v)_T - \underbrace{\langle \{\!\{\nabla u\}\!\} \cdot \boldsymbol{n}, v\rangle_{\partial T}}_{\text{consistency}}$$

$$\underbrace{-\langle \{\!\{\nabla v\}\!\} \cdot \boldsymbol{n}, u_h\rangle_{\partial T}}_{\text{for symmetry}} \; \underbrace{+\langle \frac{\alpha r^2}{h} [\![u_h]\!] \cdot \boldsymbol{n}, v\rangle_{\partial T}}_{\text{for stability}}$$

$$= \sum_{T\in\Omega_h} (\nabla u_h, \nabla v)_T$$

$$- \sum_{F\in\mathcal{E}_h^i} \left(\langle \{\!\{\nabla u_h\}\!\}, [\![v]\!]\rangle_F + \langle \{\!\{\nabla v\}\!\}, [\![u_h]\!]\rangle_F - \langle \frac{\alpha r^2}{h} [\![u_h]\!], [\![v]\!]\rangle_F \right)$$

$$- \sum_{F\in\mathcal{E}_h^\partial} \left(\langle \nabla u_h, v\rangle_F + \langle \nabla v, u_h\rangle_F - \langle \frac{\alpha r^2}{h} u_h, v\rangle_F \right)$$

- Stabilization parameter $\alpha > 0$ needs to be big enough for stability. In practice, taking $\alpha = 4$ is usually good enough.

[1] You can also use LDG/DDG...

DG scheme for convection-diffusion

DG scheme: Find $u_h \in V_h^{dg}$ such that, for all $v \in V_h^{dg}$,

$$A_h(u_h, v) = \epsilon\, a_h(u_h, v) + b_h(\boldsymbol{w}; u_h, v) = F(v).$$

- Consistency: $A_h(u - u_h, v) = 0$ for all $v \in V_h$.
  Proof: Integration by parts, for any $v \in V_h$,

$$A_h(u, v) = \sum_{T \in \Omega_h} \int_T (-\epsilon \Delta u + \boldsymbol{w} \cdot \nabla u) v \mathrm{dx} = F(v) = A_h(u_h, v).$$

- Coercivity: $A_h(u_h, u_h) \geq \frac{1}{2} \|\!|u_h|\!\|^2$ where

$$\|\!|v_h|\!\|^2 := \epsilon \underbrace{\sum_T (\|\nabla u_h\|_T^2 + \frac{\alpha r^2}{2h} \|[\![u_h]\!]\|_{\partial T}^2)}_{=\|u_h\|_a^2} + \underbrace{\sum_F \int_F |\boldsymbol{w} \cdot \boldsymbol{n}| [\![u_h]\!]^2 \,\mathrm{ds}}_{=|u_h|_b^2}$$

Proof of coercivity: diffusion part

- Trace inequality: On each element T, there exists a positive constant c_{tr} such that $\|u_T\|^2_{\partial T} \leq c_{tr}(r,T)\, h_T^{-1}\|u_T\|^2_T, \quad \forall u_T \in P^r(T)$.

For any $v \in V_h^{dg}$, we get

$$
\begin{aligned}
a_h(v,v) &= \sum_{T\in\Omega_h}\int_T \nabla v\cdot\nabla v + \sum_{F\in\mathcal{E}_h}\int_F 2\{\!\{-\nabla v\}\!\}[\![v]\!] + \frac{\alpha r^2}{h}[\![v]\!]^2 \\
&\geq \sum_{T\in\Omega_h}\left\{\|\nabla v\|^2_T - \int_{\partial T}|\nabla v|\cdot|[\![v]\!]| + \frac{\alpha r^2}{2h}\int_{\partial T}[\![v]\!]^2\right\} \\
&\geq \sum_{T\in\Omega_h}\left\{\|\nabla v\|^2_T + \frac{\alpha r^2}{2h}\|[\![v]\!]\|^2_{\partial T} - \frac{h}{\alpha r^2}\|\nabla v\|^2_{\partial T} - \frac{\alpha r^2}{4h}\|[\![v]\!]\|^2_{\partial T}\right\} \\
&\geq \sum_{T\in\Omega_h}\left\{\|\nabla v\|^2_T + \frac{\alpha r^2}{4h}\|[\![v]\!]\|^2_{\partial T} - \frac{c_{tr}}{\alpha r^2}\|\nabla v\|^2_T\right\}
\end{aligned}
$$

Take α big enough, e.g., $\alpha \geq 2c_{tr}/r^2$, we get $a_h(v,v) \geq \frac{1}{2}\|v\|^2_a$. In practice, $c_{tr} \propto r^2$, and taking $\alpha = 4$ is usually good enough.

Error analysis

We present the error analysis in a simplified case where the velocity field $\boldsymbol{w}$ is a polynomial of degree at most 1 on each element T.

Denote $\Pi u \in V_h^{dg}$ the L^2-projection of u. Write $\mathsf{e}_u = \Pi u - u_h$ and $\delta_u = \Pi u - u$. Then, by consistency we have the following error equation

$$A_h(\mathsf{e}_u, v) = A_h(\delta_u, v), \quad \forall v \in V_h^{dg}.$$

Taking $v = \mathsf{e}_u$ in the above equation and using coercivity result, we get

$$\frac{1}{2}\|\!|\mathsf{e}_u\|\!|^2 \leq A_h(\mathsf{e}_u, \mathsf{e}_u) = A_h(\delta_u, \mathsf{e}_u) = \epsilon a_h(\delta_u, \mathsf{e}_u) + b_h(\boldsymbol{w}; \delta_u, \mathsf{e}_u)$$

Next, we control each term in the above left hand side.

Error analysis: convection part

$$
\begin{aligned}
b_h(\boldsymbol{w};\delta_u,\mathsf{e}_u) &= \sum_{T\in\Omega_h} -\underbrace{(\boldsymbol{w}\delta_u,\nabla\mathsf{e}_u)_T}_{=0} + \langle \boldsymbol{w}\cdot\boldsymbol{n}\,\widehat{\delta_u},\mathsf{e}_u\rangle_{\partial T} \\
&= \sum_{F\in\mathcal{E}_h}\int_F |\boldsymbol{w}\cdot\boldsymbol{n}|\,\widehat{\delta_u}[\![\mathsf{e}_u]\!]\mathrm{ds} \\
&\le \Big(\sum_{F\in\mathcal{E}_h}\int_F |\boldsymbol{w}\cdot\boldsymbol{n}|\,\widehat{\delta_u}^{\,2}\mathrm{ds}\Big)^{1/2}|[\![\mathsf{e}_u]\!]|_b \\
&\lesssim \|\boldsymbol{w}\|_\infty h^{r+1/2}\|u\|_{r+1}\,|[\![\mathsf{e}_u]\!]|_b
\end{aligned}
$$

Error analysis: diffusion part

Repeatly using Cauchy-Schwarz inequality to obtain

$$a_h(\delta_u, \mathsf{e}_u) \lesssim \|\delta_u\|_{a*}\|\mathsf{e}_u\|_a$$

Here $\|\cdot\|_{a*}$ is a stronger norm:

$$\|v\|_{a*}^2 := \|v\|_a^2 + \sum_T h/r^2 \|\nabla v\|_{\partial T}^2$$

Note: by trace inequality, we easily get the following norm equavilence

$$\|v\|_a \leq \|v\|_{a*} \lesssim \|v\|_a, \quad \forall v \in V_h^{dg}$$

Standard approximation theory implies that

$$\|\delta_u\|_{a*} \lesssim h^r \|u\|_{r+1}$$

Putting it together

Hence

$$|||\mathsf{e}_u|||^2 \lesssim (\epsilon + h^{1/2}\|\boldsymbol{w}\|_\infty)h^r\|u\|_{r+1}|||\mathsf{e}_u|||$$

Finally, by triangle inequality, we get

$$|||u - u_h||| \lesssim |||u - \Pi u||| + |||\Pi u - u_h||| \lesssim (\epsilon + h^{1/2}\|\boldsymbol{w}\|_\infty)h^r\|u\|_{r+1}$$

- Taking $v = \boldsymbol{w} \cdot \nabla \mathsf{e}_u$, one can further control the L^2-norm of the directional derivative $\|\boldsymbol{w} \cdot \nabla \mathsf{e}_u\|_0$. We skip the detailed derivation.
- With more advanced techniques, we can also prove the L^2-norm of the error e_u converges at a rate of of $h^{r+1/2}$. [Ayuso&Marini SINUM2009]

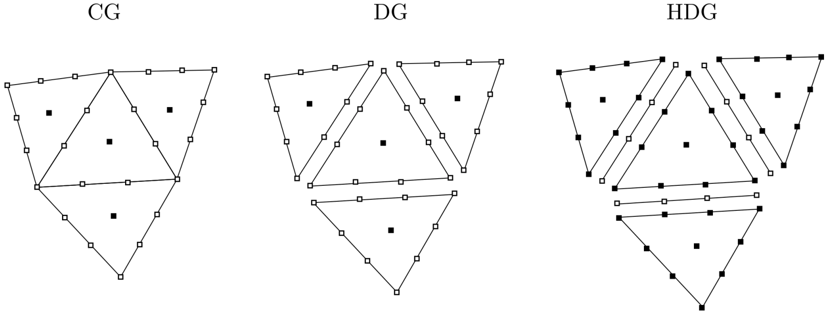
Show code [cd_DG.ipynb]

Outline

Hybridizable Discontinuous Galerkin (HDG) Methods[2]

- Major advantages of DG: robust in both diffusion-dominated (interior penalty) and convection-dominated (upwinding) regimes.
- Major drawback of DG: Computationally more expensive to solve compared with CG/SUPG. (more DOFs and more DOF coupling in the linear system)

HDG introduce new DOFs on the mesh skeleton to reduce the computational cost of a DG scheme while keeping all its advantages.



[2] Also known as hybrid DG/hybridized DG in the literature

HDG: facet FE space

- Finite element space on the mesh:

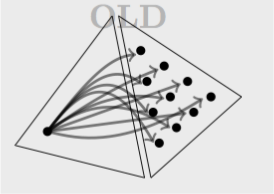
$$V_h^r = \{v \in L^2(\Omega_h): \;\; v|_T \in \mathcal{P}^r(T), \;\; \forall T \in \Omega_h\}$$

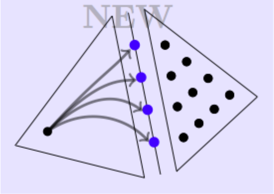
- Finite element space on mesh skeleton only:

$$M_h^r = \{\widehat{v} \in L^2(\mathcal{E}_h): \;\; \widehat{v}|_F \in \mathcal{P}^r(F), \;\; \forall F \in \mathcal{E}_h, \quad \widehat{v}|_{\mathcal{E}^\partial} = 0\}$$

Note: Dirichlet boundary condition is incoporated in M_h^r.
HDG method approximation the solution u using two finite element spaces

$$(u_h, \widehat{u}_h) \in V_h^r \times M_h^r$$





HDG: interior penalty for diffusion

Derivation of Symmetric interior penalty HDG:

- Integration by parts: for any $v \in V_h$,

$$\sum_T \int_T -\nabla \cdot (\nabla u)\, v \, \mathrm{dx} = \sum_T \int_T \nabla u \cdot \nabla v \, \mathrm{dx} - \int_{\partial T} \frac{\partial u}{\partial n} v \mathrm{ds}$$

- Normal continuity: for any $\widehat{v} \in M_h$,

$$\sum_T \int_{\partial T} \frac{\partial u}{\partial n} \widehat{v} \mathrm{ds} = \sum_{F \in \mathcal{E}^i} \int_F \underbrace{(\frac{\partial u}{\partial n^+} + \frac{\partial u}{\partial n^-})}_{=0} \widehat{v}\, \mathrm{ds} = 0$$

- Add symmetry and stability terms, we get the HDG diffusion operator:

$$a_h((u, \widehat{u}), (v, \widehat{v})) = \sum_T \int_T \nabla u \cdot \nabla v \, \mathrm{dx} - \int_{\partial T} \frac{\partial u}{\partial n} [\![v]\!] \mathrm{ds}$$

$$- \int_{\partial T} \frac{\partial v}{\partial n} [\![u]\!] \mathrm{ds} + \int_{\partial T} \frac{\alpha r^2}{h} [\![u]\!] [\![v]\!] \mathrm{ds}$$

Here $[\![v]\!] := v - \widehat{v}$.

Coercivity of HDG diffusion operator

Denote HDG norm

$$\|(u, \widehat{u})\|_a^2 := \sum_T \{\|\nabla u\|_T^2 + \frac{\alpha r^2}{h}\|[\![u]\!]\|_{\partial T}^2\}$$

We can easily show that

$$a_h((u, \widehat{u}), (u, \widehat{u})) \geq \frac{1}{2}\|(u, \widehat{u})\|_a^2$$

for α sufficiently large. (Again, taking $\alpha = 4$ is usually enough)

Advantage of HDG: static condensation

Consider the pure diffusion problem $-\Delta u = f$ on Ω with homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$.
HDG scheme: find $(u_h, \widehat{u}_h) \in V_h \times M_h$ s.t.

$$a_h((u_h, \widehat{u}_h), (v, \widehat{v})) = f(v) = \sum_T \int_T f\, v \mathrm{dx}$$

The above discretization gives rise to a matrix equation of the form

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} U \\ \widehat{U} \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \rightsquigarrow \begin{array}{l} U = -A^{-1}B\widehat{U} + A^{-1}F \\ (-B^TA^{-1}B + C)\widehat{U} = -B^TA^{-1}F \end{array}$$

Here U and $\widehat{U}$ represent the coefficient vector of the DOFs for u_h and $\widehat{u}_h$.

$$A \leftrightarrow a_h((u_h, 0), (v, 0)), \quad B \leftrightarrow a_h((0, \widehat{u}_h), (v, 0)),$$

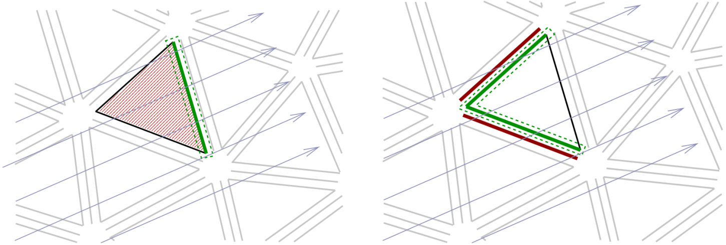
$$C \leftrightarrow a_h((0, \widehat{u}_h), (0, \widehat{v})), \qquad F \leftrightarrow f(v).$$

Key observation: A is block-diagonal, very easy to invert. [sparsity.ipynb]

Upwinding HDG for convection:

$$b_h(\boldsymbol{w}; (u_h, \widehat{u}_h), (v, \widehat{v})) = \sum_{T \in \Omega_h} -(\boldsymbol{w} u_h, \nabla v)_T + \langle \boldsymbol{w} \cdot \boldsymbol{n}\, \widehat{u}_h^{up}, (v - \widehat{v}) \rangle_{\partial T}$$

HDG Upwinding numerical flux: $\widehat{u}_h^{up} := \begin{cases} u_h & \text{if } \boldsymbol{w} \cdot \boldsymbol{n} > 0, \\ \widehat{u}_h & \text{if } \boldsymbol{w} \cdot \boldsymbol{n} \leq 0. \end{cases}$



(a) at the outflow of an element (green, surrounded by dashed line) the element value (red, hatched) is taken. $u^{up} = u$

(b) at the inflow of an element (green, surrounded by dashed line) the facet value (red, bold) is taken. $u^{up} = u_F$

Coercivity of upwinding HDG on a semi-norm

$$
\begin{aligned}
b_h(\boldsymbol{w};(v_h,\widehat{v}_h),(v_h,\widehat{v}_h)) &= \sum_{T\in\Omega_h} -(\boldsymbol{w}v_h,\nabla v_h)_T + \langle \boldsymbol{w}\cdot\boldsymbol{n}\,\widehat{v}_h^{up}, [\![v_h]\!]\rangle_{\partial T} \\
&= \sum_{T\in\Omega_h} -\frac{1}{2}\int_T \nabla\cdot(\boldsymbol{w}v_h^2) + \langle \boldsymbol{w}\cdot\boldsymbol{n}\,\widehat{v}_h^{up}, [\![v_h]\!]\rangle_{\partial T} \\
&= \sum_{T\in\Omega_h} \int_{\partial T} \left(-\frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n})v_h^2 + \boldsymbol{w}\cdot\boldsymbol{n}\widehat{v}_h^{up}[\![v_h]\!]\right)\,\mathrm{ds} \\
&= \cdots \\
&= \sum_T \int_{\partial T} \frac{1}{2}|\boldsymbol{w}\cdot\boldsymbol{n}|[\![v_h]\!]^2\,\mathrm{ds} := \frac{1}{2}|(v_h,\widehat{v}_h)|_b^2
\end{aligned}
$$

Note: the term $|(v_h,\widehat{v}_h)|_b$ is non negative, and is usually called a numerical dissipation term.

HDG scheme for convection-diffusion

HDG scheme: Find $(u_h, \widehat{u}_h) \in V_h \times M_h$ such that, for all $(v, \widehat{v}) \in V_h \times M_h$,

$$A_h((u_h, \widehat{u}_h), (v, \widehat{v})) = \epsilon\, a_h((u_h, \widehat{u}_h), (v, \widehat{v})) + b_h(\boldsymbol{w}; (u_h, \widehat{u}_h), (v, \widehat{v})) = F(v).$$

- Consistency: $A_h((u, u) - (u_h, \widehat{u}_h), (v, \widehat{v})) = 0$ for all $(v, \widehat{v}) \in V_h \times M_h$.
- Coercivity: $A_h((v, \widehat{v}), (v, \widehat{v})) \geq \frac{1}{2} \|\!|(v, \widehat{v})|\!\|^2$ for all $(v, \widehat{v}) \in V_h \times M_h$ where

$$\|\!|(v, \widehat{v})|\!\|^2 := \underbrace{\epsilon \sum_T (\|\nabla v\|_T^2 + \frac{\alpha r^2}{2h} \|[\![v]\!]\|_{\partial T}^2)}_{= \|(v,\widehat{v})\|_a^2} + \underbrace{\sum_F \int_F |\boldsymbol{w} \cdot \boldsymbol{n}| [\![v]\!]^2 \, \mathrm{ds}}_{= |(v,\widehat{v})|_b^2}$$

- Convergence: (Exercise. hint: follow the steps in DG)

$$\|\!|(u - u_h, u - \widehat{u}_h)|\!\| \lesssim (\epsilon + h^{1/2} \|\boldsymbol{w}\|_\infty) h^r \|u\|_{r+1} \qquad (1)$$

A more efficient hybrid DG scheme: EDG

We can further save computational cost of the HDG scheme by replacing the discontinuous facet space M_h to be continuous on the mesh skeleton:

$$\widetilde{M_h} = M_h \cap C^0(\mathcal{E}_h)$$

The resulting scheme is called the embedded DG (EDG) method.

All the previous analysis still holds for EDG (verify it yourself), and the resulting global linear system has exactly the same sparsity pattern as the CG method (after static condensation).

Show code [cd_HDG.ipynb]

References: DG/HDG for convection-diffusion

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The Stokes problem

We first consider the following Stokes problem, which models *creeping flow*:

$$-2\mu\nabla\cdot\epsilon(u) + \nabla p = f, \quad \text{in } \Omega \tag{2a}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega \tag{2b}$$

$$u = 0, \quad \text{on } \partial\Omega \tag{2c}$$

$$\int_\Omega p\,\mathrm{dx} = 0. \tag{2d}$$

- u: velocity, p: pressure, $\mu \geq 0$: dynamic viscosity. Deformation tensor:
$$\epsilon(u) := \frac{1}{2}(\nabla u + \nabla u^T)$$
- The last equation is a pressure *average-zero* constrain for the uniqueness of pressure.

Weak formulation

- The weak form: find $(u, p) \in V \times Q$ s.t.

$$a(u, v) - b(v, p) = F(v), \quad b(u, q) = G(q), \quad \forall (v, q) \in V \times Q. \quad (8.2.5)$$

- Bilinear/linear forms:

$$a(u, v) = \int_\Omega 2\mu\epsilon(u) \cdot \epsilon(v) \,\mathrm{dx}, \quad b(v, p) = \int_\Omega (\nabla \cdot v) p \,\mathrm{dx}$$
$$F(v) = \int_\Omega f \cdot v \,\mathrm{dx}, \quad G(q) = \int_\Omega g\, q \,\mathrm{dx}.$$

  For the Stokes equations (2), we have $G(q) = 0$.

- Function spaces:

$$V = [H_0^1(\Omega)]^d, \quad Q = L_0^2(\Omega) = \{q \in L^2 : \int_\Omega q = 0\}$$

Well-posedness and mixed FEM

(Boffi,Brezzi,Fortin, 2018)

Theorem 8.2.1. *Let $\underline{f}$ be given in $(H^{-1}(\Omega))^n$ and g in $Q = L_0^2(\Omega)$. Then, there exists a unique $(\underline{u}, p) \in V \times Q$, solution to problem (8.2.5), which satisfies*

$$||\underline{u}||_V + ||p||_Q \leq C(||\underline{f}||_{H^{-1}} + \|g\|_Q). \tag{8.2.10}$$

- Mixed FEM:

Now, choosing an approximation $V_h \subset V$ and $Q_h \subset Q$ yields the discrete problem

$$\begin{cases} 2\mu \int_\Omega \underline{\underline{\varepsilon}}(\underline{u}_h) : \underline{\underline{\varepsilon}}(\underline{v}_h)\, dx - \int_\Omega p_h \operatorname{div} \underline{v}_h\, dx = \int_\Omega \underline{f} \cdot \underline{v}_h\, dx & \forall\, \underline{v}_h \in V, \\ \int_\Omega q_h \operatorname{div} \underline{u}_h\, dx = (g, q_h) & \forall\, q_h \in Q_h. \end{cases} \tag{8.2.11}$$

Stability of mixed FEM

Proposition 8.2.1. *Let* $(\underline{u}, p) \in V \times Q$ *be the solution of* (8.2.5) *and suppose the following inf-sup condition holds true*

$$\inf_{q_h \in Q_h} \sup_{\underline{v}_h \in V_h} \frac{\int_\Omega q_h \operatorname{div} \underline{v}_h \, dx}{||q_h||_Q ||\underline{v}_h||_V} \geq k_h. \tag{8.2.16}$$

Then, there exists a unique $(\underline{u_h}, p_h) \in V_h \times Q_h$*, solution to* (8.2.11)*, and the following estimate holds*

$$\|u_h - u\|_V \leq \left(\frac{2\|a\|}{\alpha} + \frac{2\|a\|^{1/2}\|b\|}{(\alpha)^{1/2} k_h} \right) E_u + \frac{\|b\|}{\alpha} E_p, \tag{8.2.17}$$

$$\|p_h - p\|_Q \leq \left(\frac{2\|a\|^{3/2}}{(\alpha)^{1/2} k_h} + \frac{\|a\| \, \|b\|}{k_h^2} \right) E_u + \frac{3\|a\|^{1/2}\|b\|}{(\alpha)^{1/2} k_h} E_p \tag{8.2.18}$$

with α *given by* (8.2.9). □

Notation: $E_u := \inf_{v_h \in V_h} \|u - v_h\|_V$, $E_p := \inf_{q_h \in Q_h} \|p - q_h\|_V$

Stability of mixed FEM

We consider the case where

$$k_h \geq k_0 > 0 \qquad (8.2.19)$$

The following quasi-optimal estimate is an immediate consequence.

Proposition 8.2.2. *With the same hypotheses as in Proposition 8.2.1, let us suppose that (8.2.19) holds. Then, there exists C, independent of h, such that*

$$\|u_h - u\|_V + \|p_h - p\|_Q \leq C(E_u + E_p). \qquad (8.2.20)$$

□

Some examples: check out code [stokes_mixed.ipynb]

The following claims will not be proved, but only verified numerially

Continuous Pressure pairs (velocity-pressure)

- The $P_1 - P_1$ element [NO]
- The mini element: $(P_1\text{-}bubble) - P_1$ pair [YES]
- The Taylor-Hood element: $P_k - P_{k-1}\,(k \geq 2)$ [YES]
- ...

Discontinuous Pressure pairs (velocity-pressure)

- The $P_1 - P_0$ element [NO]
- The $P_2 - P_0$ element: [YES]
- The Scott-Vogelius element: $P_k - P_{k-1}^{dc}$ pair [YES/NO][3]
- ...

[3] Stability holds for $k \geq d$ on special meshes (Alfeld splits), and for $k \geq 2d$ on generial meshes. d: space dimension

Beyond inf-sup stability: pressure robustness

John et al. *SIAM Review*, 59 (2017), pp. 492–544

Two fundamentaion obervations for the Stokes equations (2):

1. For solutions to exist, the divergence operator must possess a certain surjectivity property, the fundamental inf-sup compatibility condition: There exists a constant β such that

$$\inf_{q \in L^2_0(\Omega)\setminus\{0\}} \sup_{\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)\setminus\{\boldsymbol{0}\}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta > 0. \tag{1.3}$$

   Otherwise, the constraint $-\nabla \cdot \boldsymbol{u} = g$ cannot hold.

2. A fundamental invariance property holds: Changing the external force by a gradient field changes only the pressure solution, and not the velocity; in symbols,

$$\boldsymbol{f} \to \boldsymbol{f} + \nabla \psi \quad \Longrightarrow \quad (\boldsymbol{u}, p) \to (\boldsymbol{u}, p + \psi), \tag{1.4}$$

   since the additional force field $\nabla \phi$ is balanced by the pressure gradient, and the no-slip boundary conditions do not involve the pressure.

Beyond inf-sup stability: pressure robustness

- The significance of the first observation is well known and forms a cornerstone of mixed FEM for the Stokes and NavierStokes equations: finite element spaces for velocity and pressure shall satisfy a discrete inf-sup condition, c.f. (8.2.16).
- However, almost all mixed/stabilized finite elements violate the condition (1.4) in the discrete level:

$$f \to f + \nabla\psi \;\not\Rightarrow\; (u_h, p_h) \to (u_h, p_h + \psi_h)$$

  and also violate local mass conservation: $\nabla \cdot u_h \neq 0$
- A scheme that fulfills the condition (1.4) in the discrete level is called *pressure-robust*.
- **Claim:** FEM satisfies a strong divergence-free property $\nabla \cdot u_h = 0$ is pressure-robust.
  Show code [stokes_pr.ipynb]

Improve mass conservation of classical mixed FEM

Grad-Div stabilization

- Adding the term $0 = -\gamma\nabla(\nabla \cdot u)$ to the momentum equation:

$$-2\mu\nabla \cdot \epsilon(u) + \nabla p - \gamma\nabla(\nabla \cdot u) = f,$$

Mixed FEM with grad-div stabilization

The mixed FEM with grad-div stabilization is to find $(u_h, p_h) \in V_h \times Q_h$ such that $A_{\rm gd}((u_h, p_h), (v_h, q_h)) = L_{\rm gd}(v_h, q_h) \quad \forall (v_h, q_h) \in V_h \times Q_h$, where

$$\begin{aligned} A_{\rm gd}((u_h, p_h), (v_h, q_h)) &= a(u_h, v_h) - b(v_h, p_h) + b(u_h, q_h) + \gamma(\nabla \cdot u, \nabla \cdot v), \\ L_{\rm gd}((v_h, q_h)) &= F(v_h), \end{aligned}$$

Here $\gamma \geq 0$ is a proper chosen stabilization parameter.

Grad-div stabilization penalizes for lack of mass conservation. However, it is not a complete remedy as the resulting scheme is still not pressure-robust.

Strongly divergence-free mixed FEM

- The construction of mixed FEM that satisfies the divergence-free constraint strongly is a much harder task
- Existing divergence-free mixed FEM is usually more complex to implement than classical mixed FEM like the Taylor-Hood elements
- Perhaps, the most popular choice of div-free mixed FEM is the the Scott-Vogelius element on Alfeld splits (barycentric refined mesh)

Scott-Vogelius elements on Alfeld splits

Let Ω_h be an Alfeld splitted simplicial mesh. The Scott-Vogelius finite elements

$$V_h = \{v \in H_0^1(\Omega) : \; v|_T \in \mathcal{P}^k(T), \quad \forall T \in \mathcal{T}_h\},$$

$$Q_h = \{q \in L^2(\Omega) : \; q|_T \in \mathcal{P}^{k-1}(T), \quad \forall T \in \mathcal{T}_h\}$$

is inf-sup stable if $k \geq 2$ in 2D or $k \geq 3$ in 3D, and its velocity approximation is strongly divergence-free: $\nabla \cdot u_h = 0$.

See (Arnold&Qin,92) for the proof in 2D, and (Zhang, 05) for the proof in 3D.

Outline

1. Scalar linear convection-diffusion equation
   - CG
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   - HDG
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   - HDiv-HDG for Navier-Stokes

Relax H^1-conformity: the $H(\mathrm{div})$-DG scheme

(Cockburn, Kanschat, & Schötzau, JSC, 31(2007), pp. 61–73)

- Use mixed FE spaces for Darcy flow:

$$V_h^{\mathrm{div}} \subset H(\mathrm{div};\Omega) \quad \text{and} \quad Q_h = \nabla \cdot V_h^{\mathrm{div}} \subset L^2(\Omega)$$

e.g. $V_h^{\mathrm{div}} = \{v \in H(\mathrm{div};\Omega) : v|_T \in \mathcal{P}^r(T)\}, Q_h = \{q \in L^2(\Omega) : q|_T \in \mathcal{P}^{r-1}(T)\}$

- Because $Q_h = \nabla \cdot V_h^{\mathrm{div}}$, the divergence-free constrain in velocity is automatically satisfied: $\int_\Omega (\nabla \cdot u_h) q \mathrm{dx} = 0, \quad \forall q \in Q_h \rightsquigarrow \nabla \cdot u_h \equiv 0.$
- $V_h^{\mathrm{div}} \not\subset H^1(\Omega) \rightsquigarrow$ Apply DG for the viscous term.

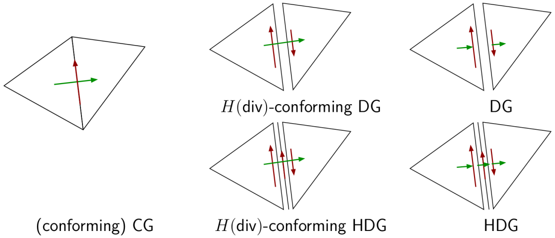

Figure: tangential and normal continuity for different methods.

The $H(\text{div})$-DG scheme. check out [stokes_hdivdg.ipynb]

Symmetric-interior penalty DG (S-IPDG) with $H(\text{div})$-conforming space V_h^{div} for second-order viscous term $-2\mu\nabla\cdot(\epsilon(u))$[4]:

$$
\begin{aligned}
a_h(u_h, v_h) = & \sum_{T\in\Omega_h}(2\mu\epsilon(u_h), \epsilon(v_h))_T - \sum_{F\in\mathcal{E}_h}\underbrace{\langle 2\mu\{\!\{\epsilon(u_h)\}\!\}, [\![v_h]\!]\rangle_F}_{\text{consistency}} \\
& - \sum_{F\in\mathcal{E}_h}\underbrace{\langle 2\mu\{\!\{\epsilon(v_h)\}\!\}, [\![u_h]\!]\rangle_F}_{\text{symmetry}} + \sum_{F\in\mathcal{E}_h}\underbrace{\langle\frac{2\mu\alpha r^2}{h}[\![u_h]\!], [\![v_h]\!]\rangle_F}_{\text{stability}}
\end{aligned}
$$

Here average $\{\!\{\epsilon(u)\}\!\} := \frac{1}{2}(\epsilon(u^+) + \epsilon(u^-))n^+$, and jump $[\![u]\!] = u^+ - u^-$. Stability parameter α needs to be big enough for coercivity. We take $\alpha = 4$ in practice.

The $H(\text{div})$-DG scheme: find $(u_h, p_h) \in V_h^{\text{div}} \times Q_h$ s.t.

$$
a_h(u_h, v_h) - b(v_h, p_h) = F(v_h),\ b(u_h, q_h) = 0,\ \forall (v_h, q_h) \in V_h^{\text{div}} \times Q_h.
$$

[4] Compare with DG for diffusion in page 12

Improve computational efficiency: $H(\mathrm{div})$-DG $\rightsquigarrow$ $H(\mathrm{div})$-HDG

Hybridization and static condensation

- Functions in $H(\mathrm{div})$ is continuous along normal direction across element boundaries, but discontinuous along tangential directions:

$$[\![u_h]\!] = \mathrm{tang}([\![u_h]\!]), \qquad \text{where } \mathrm{tang}(v) = v - (v \cdot n)n.$$

- For $H(\mathrm{div})$-HDG, we shall further introduce a facet finite element space that *only lives on the mesh skeleton and is only active on the tangential component*:

$$\widehat{V}_h := \{\widehat{v} \in L^2(\mathcal{E}_h) : \ \widehat{v}|_F \in [\mathcal{P}^r(F)]^d \times n, \ \ \forall F \in \mathcal{E}_h\}$$

- Then, we can replace the element-element coupling of the jump term $[\![u_h]\!]$ in the $H(\mathrm{div})$-DG scheme with the following *HDG-jump term*:

$$[\![u_h]\!] \rightsquigarrow \mathrm{tang}(u_h - \widehat{u}_h)$$

$\rightsquigarrow$ all element-wise calculation can be locally static condensed out.

The $H(\mathrm{div})$-HDG scheme. check out [stokes_hdivhdg.ipynb]

Symmetric-interior penalty HDG (S-IPHDG) with $H(\mathrm{div})$-conformity for second-order viscous term $-2\mu\nabla\cdot(\epsilon(u))$[5]:

$$a_h((u,\widehat{u}),(v,\widehat{v})) = \sum_{T\in\Omega_h}(2\mu\epsilon(u_h),\epsilon(v_h))_T - \underbrace{\langle 2\mu\,\epsilon(u_h)n, [\![v_h]\!]\rangle_{\partial T}}_{\text{consistency}}$$

$$- \underbrace{\langle 2\mu\,\epsilon(v_h)n, [\![u_h]\!]\rangle_{\partial T}}_{\text{symmetry}} + \underbrace{\langle \frac{2\mu\alpha r^2}{h}[\![u_h]\!], [\![v_h]\!]\rangle_{\partial T}}_{\text{stability}}$$

Here $[\![v]\!] := \mathrm{tang}(v-\widehat{v})$. Again, stability parameter α needs to be big enough for coercivity. We take $\alpha = 4$ in practice.

The $H(\mathrm{div})$-HDG scheme: find $(u_h,\widehat{u}_h,p_h)\in V_h^{\mathrm{div}}\times\widehat{V}_h\times Q_h$ s.t.

$$a_h((u_h,\widehat{u}_h),(v_h,\widehat{v}_h)) - b(v_h,p_h) = F(v_h), \quad b(u_h,q_h) = 0.$$

[5] Compare with HDG for diffusion in page 22

$H(\mathrm{div})$-HDG: a glance over the error analysis

Lehrenfeld, 2010, Diploma thesis

Proposition 2.3.3 (Galerkin Orthogonality). *Let $U_h = (\underline{\mathbf{u}}_h, p_h) \in Z_{h,D}$ be the solution of* (2.3.10) *and $U = (\underline{\mathbf{u}}, p) \in Z$ be the solution of* (2.3.1)*. Then there holds*

$$\mathcal{B}_h(\underline{\mathbf{u}}_h - \underline{\mathbf{u}}, \underline{\mathbf{v}}_h) + \mathcal{D}_h(\underline{\mathbf{u}}_h - \underline{\mathbf{u}}, q_h) + \mathcal{D}_h(\underline{\mathbf{v}}_h, p_h - p) = 0 \quad \forall\ (\underline{\mathbf{v}}_h, q_h) \in S_{h,0} \times Q_h \qquad (2.3.22a)$$

$$\iff \qquad \mathcal{K}_h(U_h - U, V_h) = 0 \quad \forall\ V_h \in Z_{h,0} \qquad (2.3.22b)$$

Proposition 2.3.4 (Coercivity). *For a shape regular mesh and $\tau_h h$ (with h the local mesh size) sufficiently large $\mathcal{B}_h(\cdot,\cdot)$ is coercive on S_h, that is*

$$\mathcal{B}_h(\underline{\mathbf{u}}, \underline{\mathbf{u}}) \geq c\nu \|\!|\underline{\mathbf{u}}|\!\|_{1,*}^2 \geq \alpha_{\mathcal{B}_h} \nu \|\!|\underline{\mathbf{u}}|\!\|_1^2 \qquad \forall\ \underline{\mathbf{u}} \in S_h \qquad (2.3.26)$$

with c, $\alpha_{\mathcal{B}_h} \in \mathbb{R}$ independent of the mesh size.

Proposition 2.3.5 (discrete LBB-condition for $\mathcal{D}_h$). *For $\mathcal{D}_h(\cdot,\cdot)$ there holds*

$$\sup_{\underline{\mathbf{u}} \in S_h^{k+1}} \frac{\mathcal{D}_h(\underline{\mathbf{u}}, q)}{\|\!|\underline{\mathbf{u}}|\!\|_1} \geq \alpha_{\mathcal{D}_h} \|q\|_{L^2} \qquad \forall\ q \in Q_h^k \qquad (2.3.27)$$

(as long as $\Gamma_D \neq \partial\Omega$ (see also the subsequent Remark 2.3.3)).

$H(\mathrm{div})$-HDG: a glance over the error analysis

Proposition 2.3.6 (Boundedness). *For all* $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in S$ *and* $q \in Q$ *there holds:*

$$|\mathcal{B}_h(\underline{\mathbf{u}}, \underline{\mathbf{v}})| \leq \beta_{\mathcal{B}_h} \ \nu \ \|\underline{\mathbf{u}}\|_1 \ \|\underline{\mathbf{v}}\|_1 \tag{2.3.32}$$

with $\beta_{\mathcal{B}_h} = \sup_{\underline{x} \in \mathcal{F}_h}(1 + \tau_h h)$ *and*

$$|\mathcal{D}_h(\underline{\mathbf{u}}, q)| \leq \underbrace{\sqrt{d}}_{= \beta_{\mathcal{D}_h}} \ \|q\|_{L^2} \ \|\underline{\mathbf{u}}\|_1 \tag{2.3.33}$$

Lemma 2.3.11 (Cea's Lemma for Stokes). *Let* $(\underline{\mathbf{u}}, p) \in Z$ *be the solution of* (2.3.1) *and* $(\underline{\mathbf{u}}_h, p_h) \in Z_h^{k+1}$ *the solution of* (2.3.10). *Then there holds*

$$\|p - p_h\|_{L^2} + \sqrt{\nu}\|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_1 \leq c\left\{\inf_{q_h \in Q_h^k} \|p - q_h\|_{L^2} + \sqrt{\nu} \inf_{\underline{\mathbf{v}}_h \in S_h^{k+1}} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_1\right\} \tag{2.3.42}$$

for c *independent of the mesh size.*

Lemma 2.3.12 (An estimate in the $\|\cdot\|_1$ norm). *Let* $\mathcal{T}_h$ *be a quasi-uniform shape regular mesh,* $(\underline{\mathbf{u}}_h, p_h) \in Z_h^k$ *be the solution of* (2.3.10) *and* $(\underline{u}, p) \in [H^1(\Omega)]^d \cap [H^m(\mathcal{T}_h)]^d \times H^{m-1}(\mathcal{T}_h)$, $m \geq 2$ *the solution of* (2.3.1). *Then there holds the following error estimate:*

$$\|p - p_h\|_{L^2} + \sqrt{\nu}\|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_1 \leq Ch^s\left(\sqrt{\nu}|\underline{u}|_{H^s(\mathcal{T}_h)} + |p|_{H^{s-1}(\mathcal{T}_h)}\right) \qquad s = \min(k, m-1) \tag{2.3.47}$$

$H(\mathrm{div})$-HDG: pressure-robust estimates

Velocity approximation is independent of pressure:

Lemma 2.3.13 (Ceá-like Lemma for the velocity)**.** *Let* $(\underline{\mathbf{u}}, p) \in Z$ *be the solution of* (2.3.1) *and* $(\underline{\mathbf{u}}_h, p_h) \in Z_h$ *the solution of* (2.3.10)*. Then there holds the following estimate not including the pressure field:*

$$\|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_1 \leq c_1 \inf_{\underline{\mathbf{w}}_h \in S_h^*} \|\underline{\mathbf{u}} - \underline{\mathbf{w}}_h\|_1 \leq c_2 \inf_{\underline{\mathbf{w}}_h \in S_h} \|\underline{\mathbf{u}} - \underline{\mathbf{w}}_h\|_1 \tag{2.3.48}$$

with c_1 *and* c_2 *independent of the mesh size.*

Duality argument to get L^2-estimate

Lemma 2.3.14 (Aubin Nitsche for the velocity)**.** *Let* $(\underline{\mathbf{u}}, p) \in Z$ *be the solution of* (2.3.1) *and* $(\underline{\mathbf{u}}_h, p_h) \in Z_h$ *the solution of* (2.3.10)*. Then there holds*

$$\|\underline{u} - \underline{u}_h\|_{L^2} \leq c\, h \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_1 \tag{2.3.51}$$

for c *independent of the mesh size.*

References: mixed FEM/HDiv-DG/HDiv-HDG for Stokes

D. Boffi, F. Brezzi, M. Fortin, *Mixed Finite Element Methods and Applications*, Springer(2018), **Chapter 8**.

V. John, *Finite Element Methods for Incompressible Flow Problems* Springer (2016), **Chapter 4.5**.

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B. Cockburn, G. Kanschat, D. Schötzau, *A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations* JSC, 31(2007), 61–73.

C. Lehrenfeld, *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*, 2010. Diploma Thesis. **Chapter 2**

Outline

1. Scalar linear convection-diffusion equation
   - CG
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The steady-state Navier-Stokes equations

Now we consider the following steady Navier-Stokes equations:

$$u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad \text{in } \Omega \tag{3a}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega \tag{3b}$$

$$u = 0, \quad \text{on } \partial\Omega \tag{3c}$$

$$\int_\Omega p \, \mathrm{dx} = 0. \tag{3d}$$

- We are interested in the high Reynolds number case where $Re = \frac{VL}{\nu} \gg 1$.
- We would expect some special treatment of the *nonlinear convection* term to make a scheme stable in the convection-dominated regime:
  - SUPG stabilization
  - upwinding via DG

$H(\mathrm{div})$-HDG: upwinding stabilization.

The $H(\mathrm{div})$-HDG scheme: find $(u_h, \widehat{u}_h, p_h) \in V_h^{\mathrm{div}} \times \widehat{V}_h \times Q_h$ s.t.

$$a_h((u_h, \widehat{u}_h), (v_h, \widehat{v}_h)) - b(v_h, p_h) + c_h(u_h; (u_h, \widehat{u}_h), (v_h, \widehat{v}_h)) = F(v_h),$$
$$b(u_h, q_h) = 0.$$

for all $(v_h, \widehat{v}_h, q_h) \in V_h^{\mathrm{div}} \times \widehat{V}_h \times Q_h$. Here the convection term

$$c_h = \sum_T - \int_T (u \otimes u) : \nabla v \, \mathrm{dx} + \int_{\partial T} (u \cdot n) \widehat{u}^{up} \cdot [\![v]\!] \, \mathrm{ds}$$

where $[\![v]\!] = \mathrm{tang}(v - \widehat{v})$, and $\widehat{u}^{up}$ is the hdg-upwinding flux base on $u \cdot n$.

Note: the convection term above is a standard upwinding (H)DG discretization of the operator $\nabla \cdot (u \otimes u) = (u_i u_j)_{,i} = u_i u_{j,i} = u \cdot \nabla u$.

HDiv-HDG scheme: properties

(1) Exact mass conservation/pressure robustness: $\nabla \cdot u_h \equiv 0$

(2) Natural *upwinding* discretization of convection term:
no need of additional stabilization or skew-symmetrization

$$c_h(u_h; (v_h, \widehat{v}_h), (v_h, \widehat{v}_h)) = \frac{1}{2} \sum_{F \in \mathcal{E}_h} \int_F |u_h \cdot n| [\![v_h]\!]^2 \, \mathrm{ds} \geq 0,$$

(3) Energy-identity:
minimal amount of numerical dissipation

$$\underbrace{c_h(u_h; (u_h, \widehat{u}_h), (u_h, \widehat{u}_h))}_{\text{num. disp. } \geq 0} + \underbrace{a_h((u_h, \widehat{u}_h), (u_h, \widehat{u}_h))}_{\text{phy. disp. } \geq 0} = F(u_h)$$

(4) Fairly general structured/unstructured meshes. High-order/low-order accuracy (vary polynomial degree).

Test case: driven cavity. Check out code sns_hdivhdg.ipynb

- $\Omega = [0,1] \times [0,1]$. No body forces $f = 0$.
- Dirichlet boundary condition: $u = 0$ on three sides, on top side (the cavity lid), $u = (1,0)$.

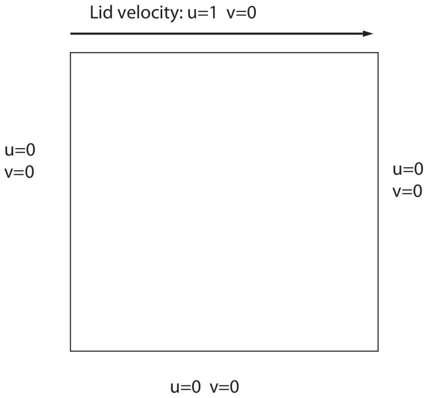

Figure 5.3. *Driven cavity domain and boundary conditions.*

References: FEM for Navier-Stokes

- Classical mixed FEM

W. LAYTON, *Introduction to the Numerical Analysis of Incompressible Viscous Flows*, SIAM (2008), **Chapter 7**.

- Stabilized FEM

V. JOHN, *Finite Element Methods for Incompressible Flow Problems*, Springer (2016), **Chapter 5.3/Chapter 8.8**.

- HDiv-HDG

C. LEHRENFELD, *Hybrid Discontinuous Galerkin methods for solving incompressible flow problems*, 2010. Diploma Thesis. **Chapter 2**

- (L2)-HDG

N.C. NGUYEN, J. PERAIRE, B. COCKBURN, *An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations*, JCP, 230(2011), 1147–1170.

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Conclusion

- HDG methods are well-suited for steady convection-dominated problems.
  - Upwinding DG stabilization for convection
  - Hybridization and static condensation for efficient linear system solver
  - Exact mass conservation for incompressible flow
  - **TODO:** Fast HDG linear system solver
- The HDG technique might also be used to speed-up a DG solver for unsteady problems with *implicit* time stepping.

Thank you for your attention! Any questions?