Space-Time Discontinuous Galerkin Finite Element Methods

I. Scalar Conservation Laws

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Motivation of research:

Aerodynamical applications, such as helicopters, maneuvering aircraft and fluid-structure interaction require:

- Moving and deforming flow domains.
- Improved capturing of vortical structures and flow discontinuities, such as shocks.
- Capability to deal with complex geometries.
- High computational efficiency for unsteady flow simulations.

Motivation of Present Research

Other free surface problems, e.g.:

- Stefan problems
- Two-phase flows with free surfaces
- Water waves

Objectives

To develop a numerical scheme for hyperbolic and parabolic conservation laws with the following properties:

- Conservative numerical discretization on moving and deforming meshes (satisfy geometric conservation law)
- Improve accuracy using *hp*-adaptation
- Maintain accuracy on irregular meshes
- Efficient capturing of discontinuities, interfaces and vortices
- Easy to parallelize

These requirements have been the primary motivation to develop space-time discontinuous Galerkin finite element methods.

Overview

- One-dimensional example: hyperbolic scalar conservation laws
  - space-time formulation
  - numerical flux
  - solution of non-linear coefficient equations
  - stability analysis of pseudo-time integration
- Multi-dimensional parabolic scalar conservation laws:
  - space-time discontinous Galerkin discretization
  - ALE formulation
- Concluding remarks

References

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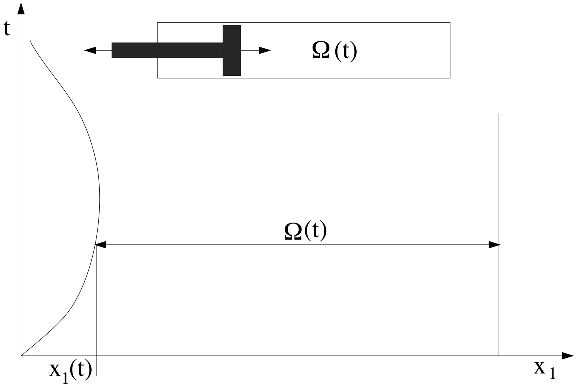
2. H. van der Ven and J.J.W. van der Vegt, Space-time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flows. II. Efficient flux quadrature, *Comput. Meth. Appl. Mech. Engrg.* **191**, pp. 4747-4780 (2002).

3. J.J. Sudirham, J.J.W. van der Vegt and R.M.J. van Damme, Space-time discontinuous Galerkin method for advection-diffusion problems on time-dependent domains, *Applied Numerical Mathematics*, **56**, pp. 1491-1518 (2006).

Time-Dependent Flow Domain



Example of a time dependent flow domain $\Omega(t)$.

Scalar Conservation Laws

- Consider the scalar conservation law in the time dependent flow domain $\Omega \subseteq \mathbb{R}$:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x_1} = 0, \qquad x_1 \in \Omega(t),\ t \in (t_0, T),$$

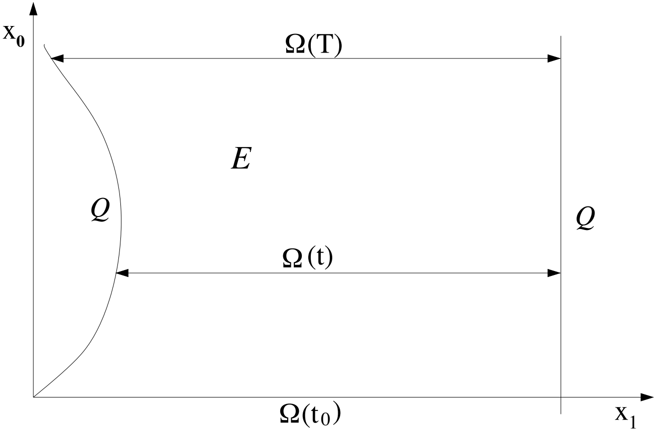
  with boundary conditions:

$$u(x_1, t) = \mathcal{B}(u, u_w), \qquad x_1 \in \partial\Omega(t),\ t \in (t_0, T),$$

  and initial condition:

$$u(x_1, 0) = u_0(x_1), \qquad x_1 \in \Omega(t_0).$$

Space-Time Domain



Example of a space-time domain $\mathcal{E}$.

Definition of Space-Time Domain

- Let $\mathcal{E} \subset \mathbb{R}^2$ be an open domain.
- A point $x \in \mathbb{R}^2$ has coordinates (x_0, x_1), where x_0 represents time and x_1 the spatial coordinate.
- Define the flow domain Ω at time t as:

$$\Omega(t) := \{x_1 \in \mathbb{R} \,|\, (t, x_1) \in \mathcal{E}\}.$$

- Define the boundary $\mathcal{Q}$ as:

$$\mathcal{Q} := \{x \in \partial\mathcal{E} \,|\, t_0 < x_0 < T\}.$$

- **Note:** The space-time domain boundary $\partial\mathcal{E}$ is equal to:

$$\partial\mathcal{E} = \Omega(t_0) \cup \mathcal{Q} \cup \Omega(T).$$

Space-Time Formulation of Scalar Conservation Laws

- Define the space-time flux vector: $\mathcal{F}(u) := (u, f(u))^T$, then scalar conservation laws can be written as:

  $$\operatorname{div} \mathcal{F}(u(x)) = 0, \qquad x \in \mathcal{E}$$

  with boundary conditions:

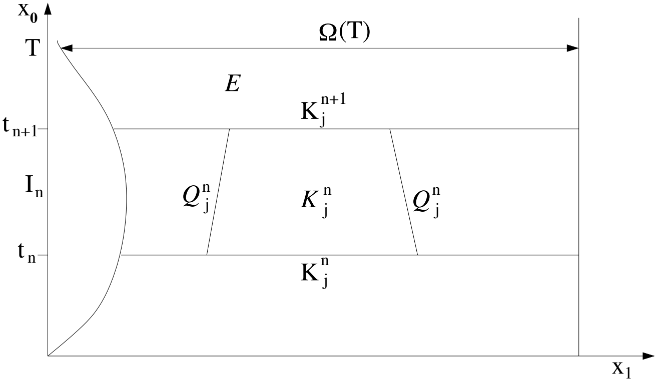
  $$u(x) = \mathcal{B}(u, u_w), \qquad x \in \mathcal{Q},$$

  and initial condition:

  $$u(x) = u_0(x), \qquad x \in \Omega(t_0).$$

- The div operator is defined as: $\operatorname{div} \mathcal{F} = \frac{\partial \mathcal{F}_i}{\partial x_i}$.

Space-Time Slab

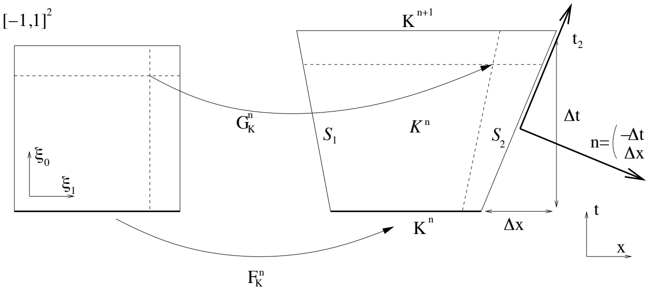


Space-time slab in space-time domain $\mathcal{E}$.

Definition of Space-Time Slab

- Consider a partitioning of the time interval (t_0, T): $\{t_n\}_{n=0}^N$, and set $I_n = (t_n, t_{n+1})$.
- Define a space-time slab as: $\mathcal{I}_n := \{x \in \mathcal{E} \mid x_0 \in I_n\}$
- Split the space-time slab into non-overlapping elements: $\mathcal{K}_j^n$.
- We will also use the notation: $K_j^n = \mathcal{K}_j^n \cap \{t_n\}$ and $K_j^{n+1} = \mathcal{K}_j^n \cap \{t_{n+1}\}$

Geometry of Space-Time Element



Geometry of 2D space-time element in both computational and physical space.

Element Mappings

Definition of the mapping $G^n_{\mathcal{K}}$ which the connects the space-time element $\mathcal{K}^n$ to the reference element $\hat{\mathcal{K}} = (-1,1)^2$:

- Define a smooth, orientation preserving and invertible mapping Φ^n_t in the interval I_n as:

$$\Phi^n_t : \Omega(t_n) \to \Omega(t) : x_1 \mapsto \Phi^n_t(x_1), \quad t \in I_n.$$

- Split $\Omega(t_n)$ into the tessellation $\bar{\mathcal{T}}^n_h$ with non-overlapping elements K_j.

- Define $\chi_i(\xi_1), \xi_1 \in (-1,1)$ as the standard linear finite element shape functions:

$$\chi_1(\xi_1) = \tfrac{1}{2}(1-\xi_1),$$
$$\chi_2(\xi_1) = \tfrac{1}{2}(1+\xi_1).$$

Element Mappings

- The mapping F_K^n is defined as:

$$F_K^n : (-1, 1) \rightarrow K^n : \xi_1 \longmapsto \sum_{i=1}^{2} x_i(K^n)\chi_i(\xi_1),$$

  with $x_i(K^n)$ the spatial coordinates of the space-time element at time $t = t_n$.

- Similarly we define the mapping F_K^{n+1}:

$$F_K^{n+1} : (-1, 1) \rightarrow K^{n+1} : \xi_1 \longmapsto \sum_{i=1}^{2} \Phi_{t_{n+1}}^n(x_i(K^n))\chi_i(\xi_1).$$

Element Mappings

- The space-time element is defined by linear interpolation in time:

$$G_{\mathcal{K}}^n : (-1,1)^2 \to \mathcal{K}^n : (\xi_0, \xi_1) \longmapsto (x_0, x_1),$$

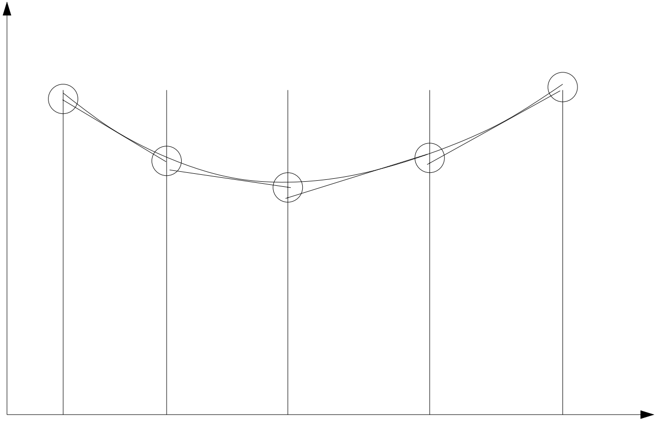
with:

$$(x_0, x_1) = \Big(\tfrac{1}{2}(t_n + t_{n+1}) - \tfrac{1}{2}(t_n - t_{n+1})\xi_0, \\ \tfrac{1}{2}(1-\xi_0)F_K^n(\xi_1) + \tfrac{1}{2}(1+\xi_0)F_K^{n+1}(\xi_1) \Big).$$

- The space-time tessellation is now defined as:

$$\mathcal{T}_h^n := \{\mathcal{K} = G_{\mathcal{K}}^n(\hat{\mathcal{K}}) \,|\, K \in \bar{\mathcal{T}}_h^n\}.$$

Discontinous Galerkin Approximation



Discontinous Galerkin approximation of a function

Note: The polynomial expansions are discontinuous at element faces.

Basis Functions

- Define the basis functions $\hat{\phi}_m, (m = 1, \cdots, (p+1)^2)$, in the master element $\hat{\mathcal{K}}$ as:

$$\hat{\phi}_m(\xi_0, \xi_1) = \xi_0^{i_0} \xi_1^{i_1}.$$

Remark: In practice the best option is to use orthogonal basis functions, e.g. Legendre polynomials or (generalized) Jacobi polynomials.

- Define the basis functions ϕ_m in an element $\mathcal{K}$ as:

$$\phi_m(x) = \hat{\phi}_m \circ G_{\mathcal{K}}^{-1}(x).$$

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Basis Functions

- Introduce the basis functions $\psi_m : \mathcal{K} \to \mathbb{R}$ and split the test and trial functions into an element mean at time t_{n+1} and a fluctuating part:

$$
\begin{aligned}
\psi_m(x) &= 1, && m = 1, \\
&= \phi_m(x) - \frac{1}{|K(t_{n+1})|} \int_{K(t_{n+1})} \phi_m dK, && m \geq 2.
\end{aligned}
$$

- This splitting is beneficial to show the relation between the space-time DG discretization and the Arbitrary Lagrangian Eulerian (ALE) method, but not essential in practice.

- A more general approach is to use orthogonal basis functions.

Finite Element Space

- Define the finite element space $V_h^p(\mathcal{T}_h^n)$ as:

$$V_h^p(\mathcal{T}_h^n) := \left\{ v_h \,\middle|\, v_h|_{\mathcal{K}} \in \mathcal{Q}^p(\mathcal{K}),\ \forall \mathcal{K} \in \mathcal{T}_h^n \right\},$$

  with $\mathcal{Q}^p(\mathcal{K}) = \mathrm{span}\{\phi_m, m = 1, \cdots, (p+1)^2\}$ a tensor product basis.

- The trial functions $u_h : \mathcal{T}_h^n \to \mathbb{R}^2$ are defined in each element $\mathcal{K} \in \mathcal{T}_h^n$ as:

$$u_h(x) = \mathcal{P}(u(x)|_{\mathcal{K}}) = \sum_{m=1}^{(p+1)^2} \hat{U}_m(\mathcal{K})\psi_m(x), \quad x \in \mathcal{K},$$

  with $\mathcal{P}$ the projection operator to the finite element space $V_h^p(\mathcal{T}_h^n)$ and $\hat{U}_m$ the DG expansion coefficients.

Finite Element Space

- **Note :** Since $\int_{K(t_{n+1})} \psi_m(x)dK = 0$ for $m \geq 2$, we have the relation:

  $$\bar{u}_h(K(t_{n+1})) := \frac{1}{|K(t_{n+1})|} \int_{K(t_{n+1})} u_h dK = \hat{U}_1,$$

  and we can write:

  $$u_h(x) = \bar{u}_h(K(t_{n+1})) + \tilde{u}_h(x),$$

  with $\int_{K(t_{n+1})} \tilde{u}(x)dK = 0$.

- One of the main benefits of this splitting is that the equation for $\hat{U}_1$ is very similar to a first order finite volume discretization and is only weakly coupled to the equations for $\tilde{u}_h$.

- This splitting is beneficial for the definition of the stabilization operator, which should only act on $\tilde{u}_h$.

Weak Formulation for STDG Method

The scalar conservation laws can be transformed into a weak formulation:

- Find a $u_h \in V_h^p$, such that for all $w_h \in V_h^p$, we have:

$$\sum_{n=0}^{N_T} \sum_{j=1}^{N_n} \Big(\int_{\mathcal{K}_j^n} w_h \, \mathrm{div} \mathcal{F}(u_h) d\mathcal{K} + \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h)^T \, \mathfrak{D}(u_h) \, \mathrm{grad}\, u_h d\mathcal{K} \Big) = 0.$$

- The second integral with $\mathfrak{D}(u_h) \in \mathbb{R}^2$ is the stabilization operator necessary to obtain monotone solutions near discontinuities.

Weak Formulation

After integration by parts we obtain the following weak formulation:

- Find a $u_h \in V_h^p$, such that for all $w_h \in V_h^p$, we have:

$$\sum_{n=0}^{N_T} \sum_{j=1}^{N_n} \Big(- \int_{\mathcal{K}_j^n} \mathrm{grad}\, w_h \cdot \mathcal{F}(u_h) d\mathcal{K} + \int_{\partial\mathcal{K}_j^n} w_h^- n^- \cdot \mathcal{F}(u_h^-) d(\partial\mathcal{K})$$
$$+ \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h)^T \mathfrak{D}(u_h)\, \mathrm{grad}\, u_h d\mathcal{K} \Big) = 0.$$

Numerical Fluxes

- We can transform the boundary integrals into:

$$\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}^- d(\partial\mathcal{K}) = \sum_{\mathcal{S}} \int_{\mathcal{S}} \Big((w_h^- n^- + w_h^+ n^+) \cdot \tfrac{1}{2}(\mathcal{F}^- + \mathcal{F}^+) + \tfrac{1}{2}(w_h^- + w_h^+)(\mathcal{F}^- \cdot n^- + \mathcal{F}^+ \cdot n^+) \Big) d\mathcal{S}, \quad (1)$$

with $\mathcal{F}^\pm = \mathcal{F}(u_h^\pm)$, and n^-, n^+ the normal vectors at each side of the face $\mathcal{S}$, which satisfy $n^+ = -n^-$.

Numerical Fluxes

- The formulation must be conservative, which imposes the condition:

$$\int_{\mathcal{S}} w_h n^- \cdot \mathcal{F}^- d\mathcal{S} = - \int_{\mathcal{S}} w_h n^+ \cdot \mathcal{F}^+ d\mathcal{S}, \qquad \forall w_h \in V_h^p(\mathcal{T}_h^n),$$

  hence the second contribution in (1) must be zero.

- The boundary integrals therefore are equal to:

$$\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}^- d(\partial\mathcal{K}) = \sum_{\mathcal{S}} \int_{\mathcal{S}} \tfrac{1}{2}(w_h^- - w_h^+) n^- \cdot (\mathcal{F}^- + \mathcal{F}^+) d\mathcal{S},$$

  using the relation $n^+ = -n^-$.

Numerical Fluxes

- Replace the multi-valued trace of the flux at $\mathcal{S}$ with a numerical flux function:

$$H(u_h^-, u_h^+, n) \cong \frac{1}{2} n \cdot (\mathcal{F}^- + \mathcal{F}^+),$$

then we obtain the relation:

$$\begin{aligned}\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- n^- \cdot \mathcal{F}^- d(\partial\mathcal{K}) &= \sum_{\mathcal{S}} \int_{\mathcal{S}} (w_h^- - w_h^+) H(u_h^-, u_h^+, n^-) d\mathcal{S} \\ &= \sum_{\mathcal{K}} \int_{\partial\mathcal{K}} w_h^- H(u_h^-, u_h^+, n^-) d(\partial\mathcal{K}),\end{aligned}$$

using the relation $H(u_h^-, u_h^+, n^-) = -H(u_h^+, u_h^-, n^+)$.

Numerical Fluxes

- The numerical flux at the boundary faces $K(t_n)$ and $K(t_{n+1})$, which have as normal vectors $n^- = (\mp 1, 0)^T$, respectively, is defined as:

$$\begin{aligned} H(u_h^-, u_h^+, n^-) &= u_h^+ \qquad \text{at } K(t_n) \\ &= u_h^- \qquad \text{at } K(t_{n+1}). \end{aligned}$$

- The numerical flux at the boundary faces $\mathcal{Q}^n$ is a monotone Lipschitz $H(u_h^-, u_h^+, n)$, which is consistent:

$$H(u, u, n) = n \cdot \mathcal{F}(u)$$

and conservative:

$$H(u_h^-, u_h^+, n^-) = -H(u_h^+, u_h^-, n^+).$$

Riemann Problem

- The monotone Lipschitz flux $H(u_h^-, u_h^+, n)$ is obtained by (approximately) solving the Riemann problem with initial states u_h^- and u_h^+ at the element faces $\mathcal{Q}^n$.

- This procedure introduces upwinding into the discontinuous Galerkin finite element method.

Upwind Fluxes

Consistent, monotone Lipschitz fluxes are:

- Godunov flux

$$H^G(u_h^-, u_h^+, n) = \begin{cases} \min\limits_{u \in [u_h^-, u_h^+]} \hat{f}(u), & \text{if } u_h^- \leq u_h^+ \\ \max\limits_{u \in [u_h^+, u_h^-]} \hat{f}(u), & \text{otherwise,} \end{cases}$$

  with $\hat{f}(u) = \mathcal{F}(u) \cdot n$.

Upwind Fluxes

- Local Lax-Friedrichs flux

$$H^{LLF}(u_h^-, u_h^+, n) = \frac{1}{2}(\hat{f}(u_h^-) + \hat{f}(u_h^+) - C(u_h^+ - u_h^-)),$$

  with

$$C = \max_{\inf(u_h^-, u_h^+) \leq s \leq \sup(u_h^-, u_h^+)} |\hat{f}'(s)|,$$

- Roe flux with entropy fix
- HLLC flux
- The choice which numerical flux should be used depends on many aspects, e.g. accuracy, robustness, computational complexity, and personal preference.

Arbitrary Lagrangian Eulerian Formulation

- The space-time normal vector at $\mathcal{Q}$ can be expressed as:

$$n = (-u_g \cdot \bar{n}, \bar{n}),$$

  with u_g the mesh velocity.

- If we introduce this relation into the numerical fluxes then

$$\hat{f}(u) = \mathcal{F}(u) \cdot n = f(u) \cdot \bar{n} - u_g \cdot \bar{n} u,$$

  which is exactly the flux in an ALE formulation.

Weak Formulation for DG Discretization

After introducing the numerical fluxes we can transform the weak formulation into:

- Find a $u_h \in V_h^p(\mathcal{T}_h^n)$, such that for all $w_h \in V_h^p(\mathcal{T}_h^n)$, the following variational equation is satisfied:

$$\sum_{j=1}^{N_n} \Big(-\int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h) \cdot \mathcal{F}(u_h) d\mathcal{K} + \int_{K_j(t_{n+1})} w_h^- u_h^- dK - \int_{K_j(t_n)} w_h^- u_h^+ dK + \int_{\mathcal{Q}_j^n} w_h^- H(u_h^-, u_h^+; u_g, n^-) d\mathcal{Q} + \int_{\mathcal{K}_j^n} (\mathrm{grad}\, w_h)^T \mathfrak{D}(u_h)\, \mathrm{grad}\, u_h \, d\mathcal{K} \Big) = 0.$$

- **Note:** Due to the causality of the time-flux the solution in a space-time slab only depends explicitly on the data from the previous space-time slab.

DG-Expansion Coefficient Equations for Element Mean

- Introduce the polynomial expansions for u_h and w_h into the weak formulation and use the fact that the coefficients $\hat{W}_m$ are arbitrary, then the following set of equations for the element mean $\bar{u}_h(K_j(t_{n+1}))$ is obtained:

$$\left|K_j(t_{n+1})\right|\bar{u}_h(K_j(t_{n+1})) - \left|K_j(t_n)\right|\bar{u}_h(K_j(t_n)) + \int_{\mathcal{Q}_j^n} H(u_h^-, u_h^+; u_g, n^-)\,d\mathcal{Q} = 0.$$

- These equations are equivalent to a first order accurate finite volume formulation, except that more accurate data are used at the element faces.

DG Equations for Element Fluctuations

- The equations for the coefficients $\hat{U}_m(\mathcal{K}_j^n)$, $(m \geq 2)$ for the fluctuating part of the flow field $\tilde{u}_h$ are equal to:

$$
\begin{aligned}
\sum_{m=2}^{(p+1)^2} \hat{U}_m(\mathcal{K}_j^n)\Big(-\int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial x_0}\psi_m d\mathcal{K} + \int_{K_j^{n+1}} \psi_l(t_{n+1}^-, x_1)\psi_m(t_{n+1}^-, x_1)dK \\
+ \int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial x_k}\mathfrak{D}_{kp}(u_h)\frac{\partial \psi_m}{\partial x_p} d\mathcal{K}\Big) \\
-\int_{K_j^n} u_h(t_n^-, x_1,)\psi_l(t_n^+, x_1)dK - \bar{u}_h(K_j^{n+1})\int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial x_0} d\mathcal{K} \\
+\int_{\mathcal{Q}_j^n} \psi_l H(u_h^-, u_h^+; u_g, n^-)d\mathcal{Q} - \int_{\mathcal{K}_j^n} \frac{\partial \psi_l}{\partial x_1}\mathcal{F}_1(u_h)d\mathcal{K} = 0, \qquad l = 2, \cdots, (p+1)^2.
\end{aligned}
$$

Solution of DG Expansion Coefficient Equations

- The space-time DG formulation results in an implicit time-integration scheme.
- The equations for the DG expansion coefficients are represented as:

$$\mathcal{L}(\hat{U}^n; \hat{U}^{n-1}) = 0.$$

- The non-linear equations for the expansion coefficients $\hat{U}^n$ can be solved by introducing a pseudo-time τ and marching the solution with a Runge-Kutta method to a steady state:

$$\frac{\partial \hat{U}^*}{\partial \tau} = -\frac{1}{\triangle t}\mathcal{L}(\hat{U}^*; \hat{U}^{n-1}).$$

- Convergence to steady state in pseudo-time can be accelerated using a multigrid procedure.

Runge-Kutta Scheme for Pseudo-Time Integration

For the pseudo-time integration we use a point-implicit five stage Runge-Kutta scheme:

- Initialize the first Runge-Kutta stage: $\hat{V}^{(0)} = \hat{U}^{n-1}$.
- Do for all stages $s = 1$ to 5:

$$(1 + \alpha_s \bar{\lambda}) \hat{V}^{(s)} = \hat{V}^{(0)} + \alpha_s \bar{\lambda} \Big(\hat{V}^{(s-1)} - \mathcal{L}^k (\hat{V}^{(s-1)}; \hat{U}^{n-1}) \Big)$$

- End do
- Update solution: $\hat{U}^n = \hat{V}^{(5)}$.

with $\bar{\lambda} = \triangle\tau / \triangle t$. The Runge-Kutta coefficients are:

$\alpha_1 = 0.0791451$, $\alpha_2 = 0.163551$, $\alpha_3 = 0.283663$, $\alpha_4 = 0.5$, and $\alpha_5 = 1.0$.

Stability Analysis of Pseudo-Time Integration for Linear Advection Equation

- Consider the linear advection equation:

$$u_t + au_x = 0,$$

with $a > 0$.

- If we assume a uniform mesh size then the space-time discontinuous Galerkin discretization is equal to:

$$\mathcal{L}(\hat{U}(\mathcal{K}^n); \hat{U}(\mathcal{K}^{n-1})) = \mathcal{A}\hat{U}(\mathcal{K}_j^n) - \mathcal{B}\hat{U}(\mathcal{K}_{j-1}^n) - \mathcal{C}\hat{U}(\mathcal{K}_j^{n-1}).$$

Stability Analysis of Pseudo-Time Integration for Linear Advection Equation

- The matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ in the space-time discontinuous Galerkin discretization are defined as:

$$\mathcal{A} = \begin{pmatrix} 1+\delta & \delta & -\delta \\ -\delta & \frac{1}{3}+\delta & \delta \\ -2-\delta & -\delta & 2+\frac{4}{3}\delta \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \delta & \delta & -\delta \\ -\delta & -\delta & \delta \\ -\delta & -\delta & \frac{4}{3}\delta \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ -2 & 0 & 0 \end{pmatrix},$$

with $\delta = a\triangle t/\triangle x$.

Fourier Analysis for Linear Advection Equation

- Consider the spatial Fourier mode: $\hat{U}(\mathcal{K}_j^n) = e^{i\theta j}\hat{U}^F$, the stability of the pseudo-time integration algorithm then is determined by the equation:

$$\frac{d\hat{U}^F}{d\tau} = -\frac{1}{\Delta t}\mathcal{P}(\theta)\hat{U}^F$$

  with $\mathcal{P}(\theta) = \mathcal{A} - e^{-i\theta}\mathcal{B}$.

- The matrix $\mathcal{P}$ can be written as: $\mathcal{P} = QMQ^{-1}$, with Q the matrix of the right eigenvectors and M the diagonal matrix with the eigenvalues $\mu_m(\theta)$ of $\mathcal{P}(\theta)$.

- Introducing a new vector $\hat{V}^F = Q^{-1}\hat{U}^F$, we obtain a system of independent ODEs:

$$\frac{d\hat{V}_m^F}{d\tau} = -\frac{\mu_m(\theta)}{\triangle t}\hat{V}_m^F, \qquad \text{for } m = 0, 1, 2.$$

Fourier Analysis for Linear Advection Equation

- This system of ordinary differential equations is solved with a semi-implicit Runge-Kutta scheme with an amplification factor $G(z)$, which is defined recursively as:

$$G(z) = 1$$

For $s = 1$ to 5

$$G(z) = \frac{1.0 + \alpha_s(\bar{\lambda} + z)G(z)}{1.0 + \alpha_s \bar{\lambda}}$$

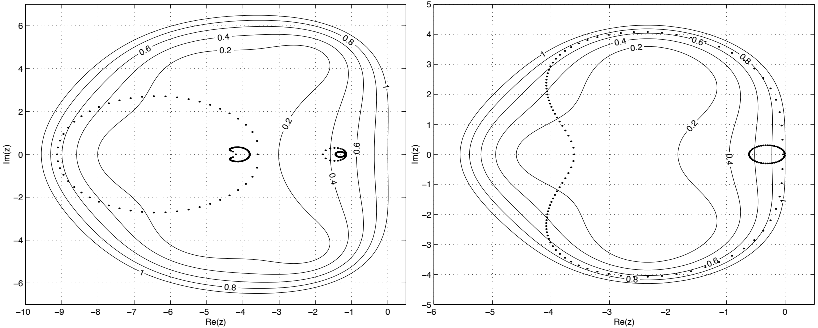
End for

- The pseudo-time integration method is stable if:

$$|G(z_m(\theta))| \leq 1$$

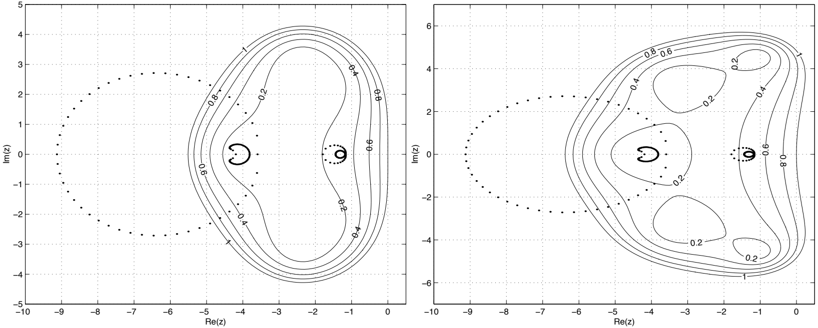
with $z_m(\theta) = -\frac{\Delta\tau}{\Delta t}\mu_m(\theta)$.

Stability Analysis of Pseudo-Time Integration for Linear Advection Equation



Locus of eigenvalues z_m (dots) of DG discretization of $u_t + au_x = 0$ and stability domain of 5-stage semi-implicit Runge-Kutta method with optimized coefficients. $CFL_{\triangle t} = 1.0$, $CFL_{\triangle \tau} = 1.8$ (left), $CFL_{\triangle t} = 100.0$, $CFL_{\triangle \tau} = 1.8$ (right).

Stability Analysis of Pseudo-Time Integration for Linear Advection Equation



Locus of eigenvalues z_m (dots) of DG discretization of $u_t + au_x = 0$ and stability domain of 5-stage explicit Runge-Kutta method with optimized coefficients (left) and 5-stage semi-implicit Jameson Runge-Kutta scheme (right). $CFL_{\triangle t} = 1.0$, $CFL_{\triangle \tau} = 1.8$.

Parabolic Scalar Conservation Laws

- Parabolic scalar conservation laws on a time-dependent domain $\Omega_t \subset \mathbb{R}^d$:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i(u(t,\bar{x})) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(D_{ij}(t,\bar{x})\frac{\partial u}{\partial x_i}\right) = 0, \text{ in } \Omega_t,$$

- with:
  - u a scalar quantity
  - f_i, $i = 1, \cdots, d$ real-valued flux functions
  - $D \in \mathbb{R}^{d\times d}$ a symmetric positive definite matrix of diffusion coefficients

Space-Time Formulation

- Introduce the convective flux $\mathcal{F} \in \mathbb{R}^{d+1}$ and the symmetric matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$ as:

$$\mathcal{F}(u) = \left(u, f_1(u), \cdots, f_d(u)\right),$$
$$A = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

- The parabolic scalar conservation law can be transformed into a space-time formulation as:

$$-\nabla \cdot (-\mathcal{F}(u) + A\nabla u) = 0 \quad \text{in} \quad \mathcal{E},$$

where $\nabla = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$ denotes the gradient operator in $\mathbb{R}^{d+1}$.

Boundary Conditions

- The boundary $\partial\mathcal{E}$ is divided into disjoint boundary subsets Γ_S, Γ_-, and Γ_+, where each subset is defined as follows:

$$\Gamma_S := \{x \in \partial\mathcal{E} : \bar{n}^T D \bar{n} > 0\},$$

$$\Gamma_- := \{x \in \partial\mathcal{E} \setminus \Gamma_S : \lambda(u) < 0\},$$

$$\Gamma_+ := \{x \in \partial\mathcal{E} \setminus \Gamma_S : \lambda(u) \geq 0\},$$

with:

- n the space-time normal vector at $\partial\mathcal{E}$
- $\bar{n}$ the spatial part of the space-time normal vector n
- $\lambda(u) = \frac{d}{du}(\mathcal{F}(u) \cdot n)$

Boundary Conditions

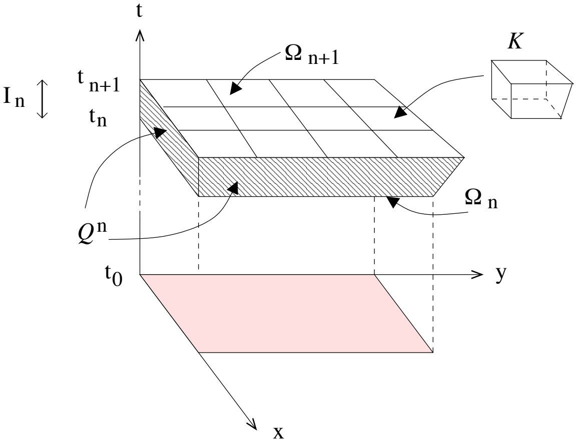
- The boundary conditions on different parts of $\partial\mathcal{E}$ are written as

$$\begin{aligned} u &= u_0 \quad \text{on } \Omega_0, \\ u &= g_D \quad \text{on } \Gamma_D, \\ \alpha u + n \cdot (A \nabla u) &= g_M \quad \text{on } \Gamma_M, \end{aligned}$$

- $\alpha \geq 0$ and u_0, g_D, g_M given functions defined on the boundary.

- There is no boundary condition imposed on Γ_+.

Space-Time Slab



Space-time slab $\mathcal{E}^n$ with space-time element $\mathcal{K}$.

Finite Element Spaces

- To each element $\mathcal{K}$ we assign a pair of nonnegative integers $p_{\mathcal{K}} = (p_{t,\mathcal{K}}, p_{s,\mathcal{K}})$ as local polynomial degrees

- Define $\mathcal{Q}_{p_{t,\mathcal{K}},p_{s,\mathcal{K}}}(\hat{\mathcal{K}})$ as the set of tensor-product polynomials on $\hat{\mathcal{K}}$ of degree $p_{t,\mathcal{K}}$ in the time direction and degree $p_{s,\mathcal{K}}$ in each spatial coordinate direction

- Define the finite element spaces of discontinuous piecewise polynomial functions as:

$$V_h^{(p_t,p_s)} := \{v \in L^2(\mathcal{E}) : v|_{\mathcal{K}} \circ G_{\mathcal{K}} \in \mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h\}$$

$$\Sigma_h^{(p_t,p_s)} := \{\tau \in L^2(\mathcal{E})^{d+1} : \tau|_{\mathcal{K}} \circ G_{\mathcal{K}} \in [\mathcal{Q}_{(p_{t,\mathcal{K}},p_{s,\mathcal{K}})}(\hat{\mathcal{K}})]^{d+1}, \forall \mathcal{K} \in \mathcal{T}_h\}$$

Trace Operators

- The so called traces of $v \in V_h^{(p_t,p_s)}$ on $\partial\mathcal{K}$ are defined as:

$$v_{\mathcal{K}}^{\pm} = \lim_{\epsilon \downarrow 0} v(x \pm \epsilon n_{\mathcal{K}})$$

- The traces of $\tau \in \Sigma_h^{(p_t,p_s)}$ are defined similarly.

- Note that functions $v \in V_h^{(p_t,p_s)}$ and $\tau \in \Sigma_h^{(p_t,p_s)}$ are in general multivalued on a face $S \in \mathcal{F}_{\text{int}}$.

Average and Jump Operators

- Introduce the functions $v_i := v|_{\mathcal{K}_i}$, $\tau_i := \tau|_{\mathcal{K}_i}, n_i := n|_{\partial\mathcal{K}_i}$

- The average operator on $S \in \mathcal{F}_{\text{int}}$ is defined as:

$$\{\!\{v\}\!\} = \frac{1}{2}(v_i^- + v_j^-), \quad \{\!\{\tau\}\!\} = \frac{1}{2}(\tau_i^- + \tau_j^-), \quad \text{on } S \in \mathcal{F}_{\text{int}},$$

- The jump operator on $S \in \mathcal{F}_{\text{int}}$ is defined as:

$$[\![v]\!] = \ v_i^- \ n_i \ + \ v_j^- \ n_j, \quad [\![\tau]\!] = \tau_i^- \cdot n_i + \tau_j^- \cdot n_j, \text{ on } S \in \mathcal{F}_{\text{int}},$$

  with i and j the indices of the elements $\mathcal{K}_i$ and $\mathcal{K}_j$ which connect to the face $S \in \mathcal{F}_{\text{int}}$.

Average and Jump Operators

- On a face $S \in \mathcal{F}_{\text{bnd}}$, the average and jump operators on $S \in \mathcal{F}_{\text{bnd}}$ are defined as:

$$\{\!\{v\}\!\} = v^-, \qquad \{\!\{\tau\}\!\} = \tau^-,$$
$$[\![v]\!] = v^- \, n, \qquad [\![\tau]\!] = \tau^- \cdot n$$

- Note that the jump $[\![v]\!]$ is a vector parallel to the normal vector n and the jump $[\![\tau]\!]$ is a scalar quantity.

- We also need the spatial jump operator $\langle\!\langle \cdot \rangle\!\rangle$ for functions $v \in V_h^{(p_t,p_s)}$, which is defined as:

$$\langle\!\langle v \rangle\!\rangle = v_i^- \bar{n}_i + v_j^- \bar{n}_j, \quad \text{on } S \in \mathcal{F}_{\text{int}}, \qquad \langle\!\langle v \rangle\!\rangle = v^- \, \bar{n}, \quad \text{on } S \in \mathcal{F}_{\text{bnd}}.$$

Space-Time DG Discretization

- Introduce an auxiliary variable $\sigma = A\nabla u$ to obtain the following system of first order equations:

$$\begin{aligned} \sigma &= A\nabla u, \\ -\nabla \cdot (-\mathcal{F}(u) + \sigma) &= 0. \end{aligned}$$

Weak Formulation for Auxiliary Variable

- Multiply the auxiliary equation with an arbitrary test function $\tau \in \Sigma_h^{(p_t,p_s)}$ and integrate over an element $\mathcal{K} \in \mathcal{T}_h$

$$\int_{\mathcal{K}} \sigma \cdot \tau \, \mathrm{d}\mathcal{K} = \int_{\mathcal{K}} A \nabla u \cdot \tau \, \mathrm{d}\mathcal{K}, \quad \forall \tau \in \Sigma_h^{(p_t,p_s)}$$

- Substitute σ and u with their numerical approximation and integrate by parts twice and sum over all elements:

$$\int_{\mathcal{E}} \sigma_h \cdot \tau \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}} A \nabla_h u_h \cdot \tau \, \mathrm{d}\mathcal{E} + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial \mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}$$

- The variable $\hat{u}_h$ is the *numerical flux* that must be introduced to account for the multivalued trace on $\partial\mathcal{K}$.

Weak Formulation for Auxiliary Variable

- The following relation holds for vectors τ and scalars ϕ, piecewise smooth on $\mathcal{T}_h$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} (\tau \cdot n)\phi \, \mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}} \int_S \{\!\{\tau\}\!\} \cdot [\![\phi]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S [\![\tau]\!] \{\!\{\phi\}\!\} \, \mathrm{d}S$$

- Using the symmetry of the matrix A, the last contribution in the auxiliary equation then results in

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}$$
$$= \sum_{S\in\mathcal{F}} \int_S \{\!\{A\tau\}\!\} \cdot [\![\hat{u}_h - u_h]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S \{\!\{\hat{u}_h - u_h\}\!\} [\![A\tau]\!] \, \mathrm{d}S$$

Numerical Fluxes for Auxiliary Equation

- The following numerical fluxes result in a consistent and conservative scheme with a sparse matrix:

$$
\begin{aligned}
\hat{u}_h &= \{\!\{u_h\}\!\} && \text{on } S \in \mathcal{F}_{\mathrm{int}}, \\
\hat{u}_h &= g_D && \text{on } S \in \cup_n \mathcal{S}_D^n, \\
\hat{u}_h &= u_h^- && \text{elsewhere.}
\end{aligned}
$$

- Note that on faces $S \in \mathcal{S}_S^n$, which are the element boundaries K^n and K^{n+1}, the normal vector n has values $n = (\pm 1, \underbrace{0, \ldots, 0}_{d\,\times})$ and thus $An = (\underbrace{0, \ldots, 0}_{(d+1)\,\times})$. Hence there is no coupling between the space-time slabs.

Numerical Fluxes for Auxiliary Equation

- Substitute the numerical flux into the auxiliary equation and use that A contains continuous functions, we obtain for each space-time slab $\mathcal{E}^n$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K}$$
$$= - \sum_{S\in\mathcal{S}_{ID}^n} \int_S [\![u_h]\!] \cdot A\{\!\{\tau\}\!\} \, \mathrm{d}S + \sum_{S\in\mathcal{S}_D^n} \int_S g_D n \cdot A\tau \, \mathrm{d}S.$$

- Summing over all space-time slabs and using the symmetry of matrix A we can introduce the lifting operator to obtain

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} A(\hat{u}_h - u_h^-) n \cdot \tau^- \, \mathrm{d}\partial\mathcal{K} = \int_{\mathcal{E}} A R_{ID}([\![u_h]\!]) \cdot \tau \, \mathrm{d}\mathcal{E}$$

Lifting Operators

- Define the global lifting operator $R_{ID} : (L^2(\cup_n \mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$R_{ID}(\phi) = R(\phi) - R(\mathcal{P} g_D n)$$

- Define the global lifting operator $R : (L^2(\cup_n \mathcal{S}_{ID}^n))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$\int_{\mathcal{E}} R(\phi) \cdot q \, \mathrm{d}\mathcal{E} = -\sum_S \int_S \phi \cdot \{\!\{q\}\!\} \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}, \forall S \in \cup_n \mathcal{S}_{ID}^n.$$

Lifting Operators

- Using the symmetry of the matrix A, the lifting operator R_{ID} satisfies the relation:

$$
\begin{aligned}
&\int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \tau \, \mathrm{d}\mathcal{E} \\
&\quad = - \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \int_S A[\![u_h]\!] \cdot \{\!\{\tau\}\!\} \, \mathrm{d}S + \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S A g_D n \cdot \tau \, \mathrm{d}S
\end{aligned}
$$

Numerical Fluxes for Auxiliary Equation

- Combine all terms, then we obtain for all $\tau \in \Sigma_h^{(p_t,p_s)}$:

$$\int_{\mathcal{E}} \sigma_h \cdot \tau \, \mathrm{d}\mathcal{E} = \int_{\mathcal{E}} A\nabla_h u_h \cdot \tau \, \mathrm{d}\mathcal{E} + \int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \tau \, \mathrm{d}\mathcal{E},$$

- This implies that we can express $\sigma_h \in \Sigma_h^{(p_t,p_s)}$ as:

$$\sigma_h = A\nabla_h u_h + AR_{ID}([\![u_h]\!]) \quad \text{a.e. } \forall x \in \mathcal{E}.$$

Weak Formulation for Parabolic Scalar Conservation Laws

- The weak formulation for parabolic scalar conservation laws can be expressed as:

  Find a $u_h \in V_h^{(p_t,p_s)}$, such that $\forall v \in V_h^{(p_t,p_s)}$ the following relation is satisfied:

$$\int_{\mathcal{E}} (-\mathcal{F}(u_h) + \sigma_h) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\mathcal{K}} (-\hat{\mathcal{F}}_h + \hat{\sigma}_h) \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = 0.$$

- Here we replaced $\mathcal{F}(u_h)$, σ_h on $\partial\mathcal{K}$ with the numerical fluxes $\hat{\mathcal{F}}_h$, $\hat{\sigma}_h$, to account for the multivalued traces on $\partial\mathcal{K}$.

Numerical Fluxes

- Separate the numerical fluxes into an convective flux $\hat{\mathcal{F}}_h$ and a diffusive flux $\hat{\sigma}_h$.
- For the convective flux, the obvious choice is an upwind flux. Here we use the Local Lax-Friedrichs flux for convenience:

$$\hat{\mathcal{F}}_h(u_h^-, u_h^+) = \{\!\{\mathcal{F}(u_h)\}\!\} + C_S [\![u_h]\!]$$

- The parameter C_S is chosen as:

$$C_S = \max_{u \in [u_h^-, u_h^+]} |\lambda(u)| \quad \text{on } S \in \mathcal{F}_{\text{int}}$$

  with $\lambda(u) = \frac{d}{du}(\mathcal{F}(u) \cdot n)$.

Convective Numerical Fluxes

- After summation over all elements we obtain:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!]) \cdot nv^- \, \mathrm{d}\partial\mathcal{K}$$

$$= \sum_{S\in\mathcal{F}_{\mathrm{int}}} \int_S (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!]) \cdot [\![v]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\mathrm{bnd}}} \int_S \mathcal{F}(u_h) \cdot nv \, \mathrm{d}S$$

Numerical Fluxes for Auxiliary Variable

- Introduce, the diffusive flux $\hat{\sigma}_h = \{\!\{\sigma_h\}\!\}$, then after summation over all elements we obtain:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \{\!\{\hat{\sigma}_h\}\!\} \cdot n v^- \, \mathrm{d}\partial\mathcal{K} = \sum_{S\in\mathcal{F}} \int_S \{\!\{\sigma_h\}\!\} \cdot [\![v]\!] \, \mathrm{d}S$$

- Recall also the relation

$$\sigma_h = A\nabla_h u_h + AR_{ID}([\![u_h]\!]) \quad \text{a.e. } \forall x \in \mathcal{E}.$$

DG Discretization for Primal Variable

- Combining all terms and eliminating σ_h, we obtain the DG formulation for u_h:

$$\int_{\mathcal{E}} \big(-\mathcal{F}(u_h) + A\nabla_h u_h + AR_{ID}([\![u_h]\!])\big) \cdot \nabla_h v \, \mathrm{d}\mathcal{E}$$

$$+ \sum_{S\in\mathcal{F}_{\text{int}}} \int_S (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!]) \cdot [\![v]\!] \, \mathrm{d}S + \sum_{S\in\mathcal{F}_{\text{bnd}}} \int_S \mathcal{F}_h(u_h) \cdot nv \, \mathrm{d}S$$

$$- \sum_{S\in\mathcal{F}} \int_S \big(A\{\!\{\nabla_h u_h\}\!\} + A\{\!\{R_{ID}([\![u_h]\!])\}\!\}\big) \cdot [\![v]\!] \, \mathrm{d}S = 0$$

Simplifying the DG Discretization

- The DG discretization can be simplified using the following steps.

- Recall the lifting operator R_{ID} satisfies the relation

$$\begin{aligned}
&\int_{\mathcal{E}} AR_{ID}([\![u_h]\!]) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} \\
&\quad = - \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \int_S A[\![u_h]\!] \cdot \{\!\{\nabla_h v\}\!\} \, \mathrm{d}S + \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S A g_D n \cdot \nabla_h v \, \mathrm{d}S
\end{aligned}$$

- The lifting operator R_{ID} has nonzero values only on faces $S \in \mathcal{S}_{ID}^n$.

Simplifying the DG Discretization

- Using the lifting operators R and R_{ID} we obtain:

$$
\begin{aligned}
&-\sum_{S\in\mathcal{F}}\int_S A\{\!\{R_{ID}([\![u_h]\!])\}\!\}\cdot[\![v]\!]\,\mathrm{d}S \\
&=\int_{\mathcal{E}} AR([\![u_h]\!])\cdot R([\![v]\!])\,\mathrm{d}\mathcal{E}-\int_{\mathcal{E}} AR(\mathcal{P}g_D n)\cdot R([\![v]\!])\,\mathrm{d}\mathcal{E}
\end{aligned}
$$

Lifting Operators

- Define the local lifting operator $r_S : (L^2(S))^{d+1} \to \Sigma_h^{(p_t,p_s)}$ as:

$$\int_{\mathcal{E}} r_S(\phi) \cdot q \, \mathrm{d}\mathcal{E} = -\int_S \phi \cdot \{\!\{q\}\!\} \, \mathrm{d}S, \quad \forall q \in \Sigma_h^{(p_t,p_s)}, \forall S \in \cup_n \mathcal{S}_{ID}^n.$$

- The support of the operator r_S is limited to the element(s) that share the face S.

Simplifying the DG Discretization

- Following the approach of Brezzi we replace each global lifting operator with the local lifting operators r_S, and make the following simplifications:

$$\int_{\mathcal{E}} AR([\![u_h]\!]) \cdot R([\![v]\!])\, \mathrm{d}\mathcal{E} \cong \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} Ar_S([\![u_h]\!]) \cdot r_S([\![v]\!])\, \mathrm{d}\mathcal{K},$$

$$\int_{\mathcal{E}} AR(\mathcal{P} g_D n) \cdot R([\![v]\!])\, \mathrm{d}\mathcal{E} \cong \sum_{S \in \cup_n \mathcal{S}_{D}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} Ar_S(\mathcal{P} g_D n) \cdot r_S([\![v]\!])\, \mathrm{d}\mathcal{K}$$

- A sufficient condition for the constant $\eta_{\mathcal{K}}$ to guarantee a stable and unique solution is $\eta_{\mathcal{K}} > n_f$, with n_f the number of faces of an element.

- The advantage of this replacement is that the discretization matrix is considerably sparser than when the global lifting operators are used.

DG Discretization for Parabolic Scalar Conservation Laws

- Define the form $a_a : V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$ $a_d : V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$:

$$a_a(u_h, v) = -\int_{\mathcal{E}} \mathcal{F}(u_h) \cdot \nabla_h v \, \mathrm{d}\mathcal{E} + \sum_{S \in \mathcal{F}_{\text{int}}} \int_S (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S [\![u_h]\!]) \cdot [\![v]\!] \, \mathrm{d}S$$
$$+ \sum_{S \in (\cup_n \mathcal{S}^n_{MDSp} \cup \Gamma_+)} \int_S \mathcal{F}(u_h) \cdot n v \, \mathrm{d}S,$$

DG Discretization for Parabolic Scalar Conservation Laws

- Define the bilinear form $a_d : V_h^{(p_t,p_s)} \times V_h^{(p_t,p_s)} \to \mathbb{R}$:

$$
\begin{aligned}
a_d(u_h, v) = &\int_{\mathcal{E}} D\overline{\nabla}_h u_h \cdot \overline{\nabla}_h v \, \mathrm{d}\mathcal{E} \\
&- \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \int_S \left(D\langle\!\langle u_h \rangle\!\rangle \cdot \{\!\{\overline{\nabla}_h v\}\!\} + D\{\!\{\overline{\nabla}_h u_h\}\!\} \cdot \langle\!\langle v \rangle\!\rangle \right) \mathrm{d}S \\
&+ \sum_{S \in \cup_n \mathcal{S}_{ID}^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S([\![u_h]\!]) \cdot \bar{r}_S([\![v]\!]) \, \mathrm{d}\mathcal{K} \\
&+ \sum_{S \in \cup_n \mathcal{S}_M^n} \int_S \alpha u_h v \, \mathrm{d}S,
\end{aligned}
$$

DG Discretization for Parabolic Scalar Conservation Laws

- Define $\ell : V_h^{(p_t,p_s)} \to \mathbb{R}$ as:

$$\begin{aligned}
\ell(v) = &- \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S g_D D\bar{n} \cdot \overline{\nabla}_h v \, \mathrm{d}S \\
&+ \sum_{S \in \cup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\mathcal{P} g_D n) \cdot \bar{r}_S([\![v]\!]) \, \mathrm{d}\mathcal{K} + \sum_{S \in \cup_n \mathcal{S}_M^n} \int_S g_M v \, \mathrm{d}S \\
&- \sum_{S \in \cup_n \mathcal{S}_{DBSm}^n} \int_S \mathcal{F}(g_D) \cdot n v \, \mathrm{d}S + \int_{\Omega_0} c_0 v \, \mathrm{d}\Omega.
\end{aligned}$$

DG Discretization for Parabolic Scalar Conservation Laws

- Note, we introduced the following boundary and initial conditions in the DG discretization:

$$
\begin{aligned}
D\overline{\nabla}_h u_h \cdot \bar{n} &= g_M - \alpha u_h && \text{on } S \in \cup_n \mathcal{S}_M^n, \\
u_h &= g_D && \text{on } S \in \cup_n \mathcal{S}_{DBSm}^n, \\
u_h &= u_0 && \text{on } \Omega_0,
\end{aligned}
$$

- The space-time DG discretization for the parabolic scalar conservation law can now be formulated as:

Find a $u_h \in V_h^{(p_t,p_s)}$, such that $\forall v \in V_h^{(p_t,p_s)}$ the following relation is satisfied:

$$a(u_h, v) = \ell(v)$$

ALE DG Formulation

- On faces $S \in \mathcal{S}_S^n$, the space-time normal vector is equal to:

$$n = (\pm 1, \underbrace{0, \ldots, 0}_{d\ \times})$$

  and is not affected by the mesh velocity.

- On the faces $S \in \mathcal{S}_I^n$ the space-time normal vector depends on the mesh velocity u_g:

$$n = (-u_g \cdot \bar{n}, \bar{n}),$$

  which also holds on the boundary faces $S \in \mathcal{F}_{\text{bnd}} \setminus (\Omega_0 \cup \Omega_T)$.

ALE DG Formulation

- On $S \in \cup_n \mathcal{S}_I^n$, the flux can be written in the ALE formulation as:

$$\{\!\{\mathcal{F}(u_h)\}\!\} \cdot [\![v]\!] = \{\!\{f(u_h) - u_g u_h\}\!\} \cdot \langle\!\langle v \rangle\!\rangle,$$

- All other contributions are not affected by the mesh velocity.

ALE DG Formulation

- The form $a_a(\cdot,\cdot)$ in the ALE formulation is now equal to:

$$
\begin{aligned}
a_a(u_h, v) = & -\int_{\mathcal{E}} \mathcal{F}(u_h)\cdot\nabla_h v \,\mathrm{d}\mathcal{E} \\
& + \sum_{S\in\cup_n \mathcal{S}_I^n} \int_S (\{\!\{ f(u_h) - u_g u_h \}\!\}\cdot\langle\!\langle v\rangle\!\rangle + C_S [\![u_h]\!]\cdot[\![v]\!])\,\mathrm{d}S \\
& + \sum_{S\in\cup_n \mathcal{S}_S^n} \int_S (\{\!\{\mathcal{F}(u_h)\}\!\} + C_S[\![u_h]\!])\cdot[\![v]\!]\,\mathrm{d}S \\
& + \sum_{S\in(\cup_n \mathcal{S}_{MDSp}^n \cup \Gamma_+)} \int_S (f(u_h) - u_g u_h)\cdot\bar{n} v \,\mathrm{d}S,
\end{aligned}
$$

ALE DG Formulation

- The linear form $\ell(\cdot)$ in the ALE formulation is now equal to:

$$
\begin{aligned}
\ell(v) = &- \sum_{S \in \cup_n \mathcal{S}_D^n} \int_S g_D D\bar{n} \cdot \overline{\nabla}_h v \, \mathrm{d}S \\
&+ \sum_{S \in \cup_n \mathcal{S}_D^n} \sum_{\mathcal{K} \in \mathcal{T}_h} \eta_{\mathcal{K}} \int_{\mathcal{K}} D\bar{r}_S(\mathcal{P} g_D n) \cdot \bar{r}_S([\![v]\!]) \, \mathrm{d}\mathcal{K} + \sum_{S \in \cup_n \mathcal{S}_M^n} \int_S g_M v \, \mathrm{d}S \\
&- \sum_{S \in \cup_n \mathcal{S}_{DBSm}^n} \int_S (f(g_D) - g_D u_g) \cdot \bar{n} v \, \mathrm{d}S + \int_{\Omega_0} c_0 v \, \mathrm{d}\Omega,
\end{aligned}
$$

- The bilinear form $a_d(\cdot,\cdot)$ is not influenced by the mesh velocity.

Conclusions

The main properties of space-time discontinuous Galerkin finite elements methods can be summarized as:

- The space-time discontinuous Galerkin finite element method results in a very local, element wise discretization, which has as benefits:
  - the space-time discretization automatically satisfies the geometric conservation law for deforming elements
  - efficient grid adaptation using local grid refinement, no complications caused by hanging nodes and gradient reconstruction
  - combines very well with unstructured grids
  - boundary conditions can be easily implemented

Conclusions

- no special numerical treatment is required to achieve higher order accuracy
- no interpolation is necessary after remeshing or local mesh refinement, only time fluxes need to be transferred
- maintains accuracy on irregular grids
- efficient parallelization

References

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