Space-Time Discontinuous Galerkin Finite Element Methods

Part II Compressible Navier-Stokes Equations

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Space-Time Discontinuous Galerkin Finite Element Methods

Motivation of research

- In many applications one encounters moving and deforming time-dependent flow domains:
  - Aerodynamics: helicopters, maneuvering aircraft, wing control, surfaces
  - Fluid structure interaction,
  - Multi-Fluid flows,
  - Free surface problems,
  - Local time-stepping (not discussed).
- A key requirement for these applications is to obtain an accurate and conservative discretization on moving and deforming meshes.

Motivation of Research

Other requirements

- Improved capturing of vortical structures and flow discontinuities, such as shocks and interfaces, using *hp*-adaptation.
- Capability to deal with complex geometries.
- Excellent computational efficiency for unsteady flow simulations.

These requirements have been the main motivation to develop a space-time discontinuous Galerkin method.

Overview of Lecture II

- Space-time discontinuous Galerkin finite element discretization for compressible Navier-Stokes equations
- Applications in aerodynamics
- Concluding remarks

Geometry of Space-Time Domain for Three-Dimensional Time-Dependent Problems

- Consider an open domain $\mathcal{E} \subset \mathbb{R}^4$.

- A point $x \in \mathcal{E}$ has coordinates $x = (x_0, \bar{x})$ with $x_0 = t$, t time, and $\bar{x}$ the spatial coordinates.

- The flow domain $\Omega(t)$ at time t is defined as

$$\Omega(t) := \{x \in \mathcal{E} \mid x_0 = t,\ t_0 < t < T\}.$$

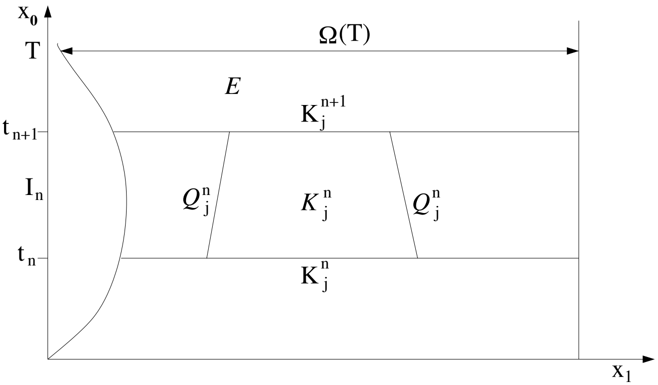
- The space-time domain boundary $\partial\mathcal{E}$ consists of the hypersurfaces

$$\begin{aligned}\Omega(t_0) :=& \{x \in \partial\mathcal{E} \mid x_0 = t_0\},\\ \Omega(T) :=& \{x \in \partial\mathcal{E} \mid x_0 = T\},\\ \mathcal{Q} :=& \{x \in \partial\mathcal{E} \mid t_0 < x_0 < T\}.\end{aligned}$$

Definition of Space-Time Slab

- Consider a partitioning of the time interval (t_0, T): $\{t_n\}_{n=0}^{N}$, and set $I_n = (t_n, t_{n+1})$.
- Define a space-time slab as $\mathcal{I}_n := \{x \in \mathcal{E} \mid x_0 \in I_n\}$.
- Split the space-time slab into non-overlapping elements $\mathcal{K}_j^n$.
- We will also use the notation $K_j^n = \mathcal{K}_j^n \cap \{t_n\}$ and $K_j^{n+1} = \mathcal{K}_j^n \cap \{t_{n+1}\}$ and $\mathcal{Q}_j^n = \partial\mathcal{K}_j^n \setminus (K_j^n \cup K_j^{n+1})$.

Space-Time Slab



Space-time slab in the space-time domain $\mathcal{E}$.

Compressible Navier-Stokes Equations

- Compressible Navier-Stokes equations in space-time domain $\mathcal{E}$

$$\frac{\partial U_i}{\partial x_0} + \frac{\partial F_k^e(U)}{\partial x_k} - \frac{\partial F_k^v(U, \nabla U)}{\partial x_k} = 0.$$

- Conservative variables $U \in \mathbb{R}^5$ and inviscid fluxes $F^e \in \mathbb{R}^{5\times 3}$

$$U = \begin{bmatrix} \rho \\ \rho u_j \\ \rho E \end{bmatrix}, \qquad F_k^e = \begin{bmatrix} \rho u_k \\ \rho u_j u_k + p\delta_{jk} \\ (\rho E + p) u_k \end{bmatrix}.$$

Compressible Navier-Stokes Equations

- Viscous flux $F^v \in \mathbb{R}^{5\times 3}$

$$F^v_k = \begin{bmatrix} 0 \\ \tau_{jk} \\ \tau_{kj}u_j - q_k \end{bmatrix},$$

with the total stress tensor τ is defined as

$$\tau_{jk} = \lambda \frac{\partial u_i}{\partial x_i}\delta_{jk} + \mu\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right).$$

- The dynamic viscosity coefficient μ given by Sutherland's law

$$\frac{\mu}{\mu_\infty} = \frac{T_\infty + T_S}{T + T_S}\left(\frac{T}{T_\infty}\right)^{3/2},$$

where T is the temperature, T_S a constant and $(\cdot)_\infty$ denotes free-stream values.

- The second viscosity coefficient λ is related to μ following the Stokes hypothesis $3\lambda + 2\mu = 0$.

Compressible Navier-Stokes Equations

- The heat flux vector q is defined as

$$q_k = -\kappa \frac{\partial T}{\partial x_k},$$

  with κ the thermal conductivity coefficient.

- The system is closed using the equations of state for a calorically perfect gas.

$$p = (\gamma - 1)(\rho E - \frac{1}{2} u_i u_i), \quad T = \frac{1}{c_v}(E - \frac{1}{2} u_i u_i),$$

  with $\gamma = c_p / c_v$.

Compressible Navier-Stokes Equations

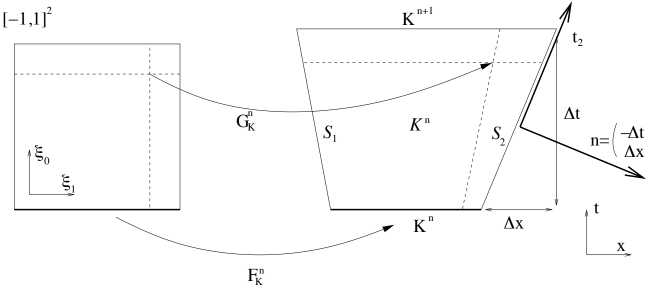
- The viscous flux F^v is homogeneous with respect to the gradient of the conservative variables ∇U

$$F^v_{ik}(U, \nabla U) = A_{ikrs}(U)\frac{\partial U_r}{\partial x_s},$$

with the homogeneity tensor $A \in \mathbb{R}^{5\times 3\times 5\times 3}$ defined as

$$A_{ikrs}(U) := \frac{\partial F^v_{ik}(U, \nabla U)}{\partial(\nabla U)}.$$

Geometry of Space-Time Element



Geometry of 2D space-time element in both computational and physical space.

Space-Time Element Definition

- Definition of the mapping G_K^n, which the connects the space-time element $\mathcal{K}^n$ to the reference element $\hat{\mathcal{K}} = [-1,1]^4$

$$G_K^n : [-1,1]^4 \to \mathcal{K}^n : \xi \longmapsto x,$$

with

$$(x_0, \bar{x}) = \Big(\tfrac{1}{2}(t_n + t_{n+1}) + \tfrac{1}{2}(t_n - t_{n+1})\xi_0,$$

$$\tfrac{1}{2}(1-\xi_0)F_K^n(\bar{\xi}) + \tfrac{1}{2}(1+\xi_0)F_K^{n+1}(\bar{\xi})\Big).$$

- Here $F_K^n : [-1,1]^3 \to K^n$, $F_K^{n+1} : [-1,1]^3 \to K^{n+1}$ are the mappings for the space elements, with

$$F_K^n : \hat{K} \to K^n : \bar{\xi} \mapsto \bar{x} = \sum_{i=1}^{8} x_i(K^n)\chi_i(\bar{\xi}),$$

with $x_i(K^n) \in \mathbb{R}^3$, $1 \le i \le 8$, the spatial coordinates of the vertices of the hexahedron K^n at time t_n.

- For F_K^{n+1} we have a similar expression using the vertices at $t = t_{n+1}$.

Space-Time Element Definition

- The space-time tessellation is now defined as
$$\mathcal{T}_h^n := \{\mathcal{K} = G_k^n(\hat{\mathcal{K}}) \mid K \in \bar{\mathcal{T}}_h^n\},$$
with $\bar{\mathcal{T}}_h^n$ the tessellation of $\Omega(t_n)$.

- The space-time normal vector at an element boundary point moving with velocity v is given by
$$n = \begin{cases} (1,0,0,0) & \text{at } K(t_{n+1}^-), \\ (-1,0,0,0) & \text{at } K(t_n^+), \\ (-v_k\bar{n}_k, \bar{n}) & \text{at } Q^n. \end{cases}$$

Approximation Spaces

- The finite element space associated with the tessellation $\mathcal{T}_h$ is given by

$$W_h := \left\{ W \in (L^2(\mathcal{E}_h))^5 \ : \ W|_{\mathcal{K}} \circ G_{\mathcal{K}} \in (P^k(\hat{\mathcal{K}}))^5, \quad \forall \mathcal{K} \in \mathcal{T}_h \right\}.$$

- We will also use the space

$$V_h := \left\{ V \in (L^2(\mathcal{E}_h))^{5\times 3} \ : \ V|_{\mathcal{K}} \circ G_{\mathcal{K}} \in (P^k(\hat{\mathcal{K}}))^{5\times 3}, \quad \forall \mathcal{K} \in \mathcal{T}_h \right\}.$$

- Note the fact that $\nabla_h W_h \subset V_h$ is essential for the discretization.

Trace Operators

- The jump of f in the Cartesian coordinate direction k is defined at internal faces as

$$[\![f]\!]_k = f^L n_k^L + f^R n_k^R.$$

- The average of f is defined at internal faces as

$$\{\!\{f\}\!\} = \tfrac{1}{2}(f^L + f^R).$$

- The jump operator satisfies the following product rule at internal faces

$$[\![g_i f_{ik}]\!]_k = \{\!\{g_i\}\!\}[\![f_{ik}]\!]_k + [\![g_i]\!]_k\{\!\{f_{ik}\}\!\},$$

- Relation between element boundary and face integrals

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{Q}} g_i^L f_{ik}^L \bar{n}_k^L \, d\mathcal{Q} = \sum_{\mathcal{S}\in\mathcal{S}_I^n} \int_{\mathcal{S}} [\![g_i f_{ik}]\!]_k \, d\mathcal{S} + \sum_{\mathcal{S}\in\mathcal{S}_B^n} \int_{\mathcal{S}} g_i^L f_{ik}^L \bar{n}_k^L \, d\mathcal{S}.$$

Space-Time Formulation Compressible Navier-Stokes Equations

- The compressible Navier-Stokes equations in the domain $\mathcal{E} \subset \mathbb{R}^4$ can be expressed as

$$\begin{cases} U_{i,0} + \frac{\partial F^e_{ik}}{\partial x_k} - \frac{\partial}{\partial x_k}\left(A_{ikrs}\frac{\partial U_r}{\partial x_s}\right) = 0 & \text{on } \mathcal{E}, \\ U = U_0 & \text{on } \Omega(t_0), \\ U = \mathcal{B}(U, U^b) & \text{on } \mathcal{Q}, \end{cases}$$

  for $i, r = 1, \ldots, 5$ and $k, s = 1, \ldots, 3$.

- The initial flow field is denoted by $U_0 : \Omega(t_0) \rightarrow \mathbb{R}^5$, with U_0 the initial condition.

- The boundary operator is denoted by $\mathcal{B} : \mathbb{R}^{5\times5} \rightarrow \mathbb{R}^5$ and is a function of the internal data U and the boundary data U^b derived from the boundary conditions.

- At the far-field boundary, suitable in- and out-flow conditions can be derived using local characteristics.

First Order System

- Rewrite the compressible Navier-Stokes equations as a first-order system using the auxiliary variable Θ

$$\frac{\partial U_i}{\partial x_0} + \frac{\partial F^e_{ik}(U)}{\partial x_k} - \frac{\partial \Theta_{ik}(U)}{\partial x_k} = 0,$$

$$\Theta_{ik}(U) - A_{ikrs}(U)\frac{\partial U_r}{\partial x_s} = 0.$$

- Note, this results in 5×3 additional equations for auxiliary variables Θ, which will be eliminated later using a lifting operator.

Weak Formulation

- Weak formulation for the compressible Navier-Stokes equations

Find a $U \in W_h$, $\Theta \in V_h$, such that for all $W \in W_h$ and $V \in V_h$, the following holds

$$
\begin{aligned}
-\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \Big(\frac{\partial W_i}{\partial x_0} U_i + \frac{\partial W_i}{\partial x_k}(F^e_{ik} - \Theta_{ik})\Big)\, d\mathcal{K} \\
+ \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} W^L_i(\widehat{U}_i + \widehat{F}^e_{ik} - \widehat{\Theta}_{ik}) n^L_k \, d(\partial\mathcal{K}) = 0,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\mathcal{K}\in\mathcal{T}^n_h} \int_{\mathcal{K}} V_{ik}\Theta_{ik}\, d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}^n_h} \int_{\mathcal{K}} V_{ik} A_{ikrs} \frac{\partial U_r}{\partial x_s}\, d\mathcal{K} \\
+ \sum_{\mathcal{K}\in\mathcal{T}^n_h} \int_{\mathcal{Q}} V^L_{ik} A^L_{ikrs}(\widehat{U}_r - U^L_r)\bar{n}^L_s \, d\mathcal{Q}.
\end{aligned}
$$

Transformation to Arbitrary Lagrangian Eulerian form

- The space-time normal vector on a grid moving with velocity $\vec{v}$ is

$$n = \begin{cases} (1,0,0,0)^T & \text{at } K(t_{n+1}^-), \\ (-1,0,0,0)^T & \text{at } K(t_n^+), \\ (-v_k \bar{n}_k, \bar{n})^T & \text{at } Q^n. \end{cases}$$

- The boundary integral then transforms into

$$\begin{aligned} &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} W_i^L(\widehat{U}_i + \widehat{F}_{ik}^e - \widehat{\Theta}_{ik}) n_k^L \, d(\partial\mathcal{K}) \\ &\qquad = \sum_{K\in\mathcal{T}_h} \Big(\int_{K(t_{n+1}^-)} W_i^L \widehat{U}_i \, dK + \int_{K(t_n^+)} W_i^L \widehat{U}_i \, dK \Big) \\ &\qquad + \sum_{K\in\mathcal{T}_h} \int_{\mathcal{Q}} W_i^L(\widehat{F}_{ik}^e - \widehat{U}_i v_k - \widehat{\Theta}_{ik}) \bar{n}_k^L \, d\mathcal{Q}. \end{aligned}$$

Numerical Fluxes

- The numerical flux $\widehat{U}$ at $K(t_{n+1}^-)$ and $K(t_n^+)$ is defined as an upwind flux to ensure causality in time

$$\widehat{U} = \begin{cases} U^L & \text{at } K(t_{n+1}^-), \\ U^R & \text{at } K(t_n^+). \end{cases}$$

- At the space-time faces $\mathcal{Q}$ we introduce the HLLC approximate Riemann solver as numerical flux

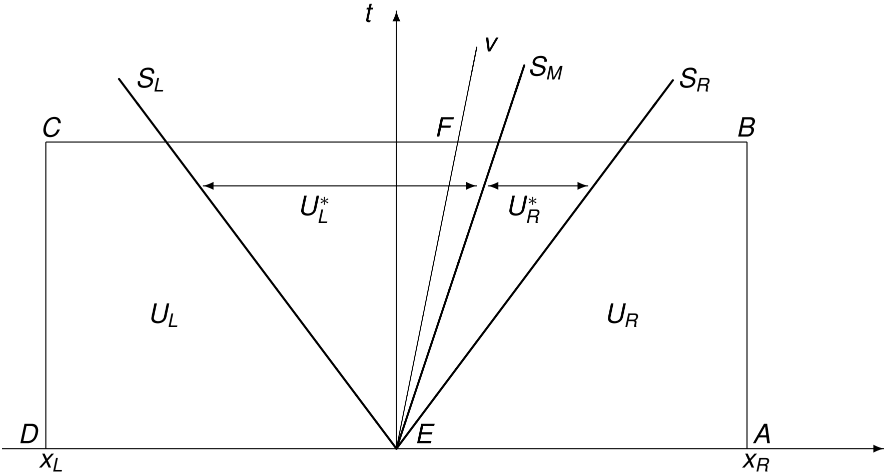
$$\bar{n}_k(\widehat{F}_{ik}^e - \widehat{U}_i v_k)(U^L, U^R) = H_i^{\mathrm{HLLC}}(U^L, U^R, v, \bar{n})$$

HLLC Flux on Moving Meshes

- The HLLC scheme was proposed by Toro (Shock Waves **4**(1994), 25) and Batten e.a. (SIAM J. Sci. Stat. Comput. **18**(1997), 1553), but only for non-moving meshes.

- The extension to moving meshes is most easily accomplished by considering the structure of the wave pattern in the Riemann problem that is assumed in the HLLC scheme.

HLLC Flux on Moving Meshes



Wave pattern in HLLC-flux

HLLC Flux on Moving Meshes

- The HLLC scheme assumes we have two averaged intermediate states U_L^* and U_R^* in the star region, which is the region bounded by the waves with the slowest and fastest signal speeds S_L and S_R, respectively.

- The star region is divided into two parts by a contact wave, which moves with velocity S_M.

- Outside the star region the solution still is at its initial values at time t_m, which are denoted U_L and U_R and are equal to the traces $U_h^-(t_m)$ and $U_h^+(t_m)$, respectively.

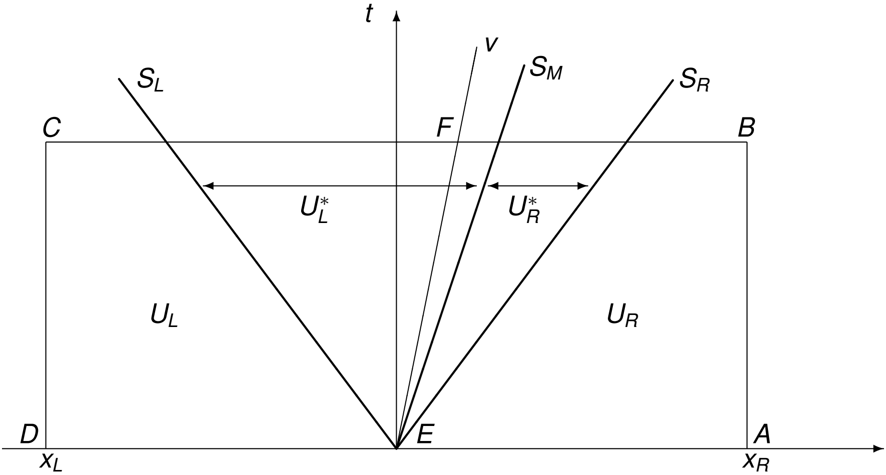
- In the time interval $[t_m, t_m + \triangle t)$ the solution U_{HLLC} at an element face which moves with the velocity v then is equal to

$$U_{HLLC} = \begin{cases} U_L \equiv U_h^-(t_m) & \text{if } S_L > v, \\ U_L^* & \text{if } S_L \le v < S_M, \\ U_R^* & \text{if } S_M \le v < S_R, \\ U_R \equiv U_h^+(t_m) & \text{if } S_R \le v, \end{cases}$$

  where depending on the grid velocity v we have to consider four different cases.

- The time interval $\triangle t$ is chosen such that there is no interaction with waves coming from other Riemann problems.

HLLC Flux on Moving Meshes



Wave pattern in HLLC-flux

HLLC Flux on Moving Meshes

- Assume that $S_L < v$, $S_R > v$, and $S_M \geq v$, then we can calculate the flux $H_{HLLC}(U_L, U_R)$ in the time interval $[t_m, t_m + \triangle t)$ by integrating the Euler equations over the control volumes $\square DEFC$ and $\square EABF$.

  Using Gauss' theorem we obtain for the control volume $\square DEFC$ the relation

$$\int_{x_L}^{S_L \triangle t} U_L \, dx + \int_{S_L \triangle t}^{v \triangle t} U_h(x, t_m + \triangle t) \, dx$$

$$= \int_{x_L}^{0} U_h(x, t_m) \, dx + \int_{t_m}^{t_m + \triangle t} \hat{F}(U_h(x_L, t)) \, dt - \int_{t_m}^{t_m + \triangle t} \hat{F}(U_h^-(vt, t)) \, dt, \qquad (2)$$

  and for the control volume $\square EABF$

$$\int_{v \triangle t}^{S_M \triangle t} U_h(x, t_m + \triangle t) \, dx + \int_{S_M \triangle t}^{S_R \triangle t} U_h(x, t_m + \triangle t) \, dx + \int_{S_R \triangle t}^{x_R} U_R \, dx$$

$$= \int_{0}^{x_R} U_h(x, t_m) \, dx + \int_{t_m}^{t_m + \triangle t} \hat{F}(U_h^+(vt, t)) dt - \int_{t_m}^{t_m + \triangle t} \hat{F}(U_h(x_R, t)) \, dt, \qquad (3)$$

  with $\hat{F}(U_h) = \bar{n}_{\mathcal{K}} \bar{F}(U_h)$.

HLLC Flux on Moving Meshes

- If we introduce the averaged solutions U_L^* and U_R^*, which are defined as

$$U_L^* = \frac{1}{(S_M - S_L)\triangle t} \int_{S_L \triangle t}^{S_M \triangle t} U_h(x, t_m + \triangle t) dx,$$

$$U_R^* = \frac{1}{(S_R - S_M)\triangle t} \int_{S_M \triangle t}^{S_R \triangle t} U_h(x, t_m + \triangle t) dx.$$

- Use the fact that $U_h^{\pm}$ is constant along the line $x = vt$ in the Riemann problem then we obtain after subtracting (2) from (3) the following expression for the HLLC flux at the interface in the time interval $[t_m, t_m + \triangle t)$

$$H_{HLLC}(U_L, U_R) = \tfrac{1}{2}(\hat{F}(U_L) + \hat{F}(U_R) + ((S_L - v) + (S_M - v))U_L^* + \\ ((S_R - v) - (S_M - v))U_R^* - S_L U_L - S_R U_R).$$

- For the other three cases: $(S_L < v, S_R > v, S_M \leq v)$, $(S_L < v, S_R < v, S_M < v)$, and $(S_L > v, S_R > v, S_M > v)$ a similar analysis can be made.

HLLC Flux on Moving Meshes

- If we combine the four cases then we obtain the following expression for the HLLC flux at a moving interface in the time interval $[t_m, t_m + \triangle t)$

$$H_{HLLC}(U_L, U_R) = \frac{1}{2}\big(\hat{F}(U_L) + \hat{F}(U_R) - (|S_L - v| - |S_M - v|)U_L^* + (|S_R - v| - |S_M - v|)U_R^* + |S_L - v|U_L - |S_R - v|U_R - v(U_L + U_R)\big).$$

- In order to completely define the HLLC flux we still need to define the star states U_L^* and U_R^*, and the wave speeds S_L, S_R and S_M.
- This can be done in various ways, but since there is no difference with the HLLC scheme for non-moving meshes, we only state the final results.

HLLC Flux on Moving Meshes

- We will follow the approach of Batten e.a. (SIAM J. Sci. Stat. Comput. **18**(1997), 1553) which assumed that

$$S_M = \hat{u}_L^* = \hat{u}_R^* = \hat{u}^*,$$

with $\hat{u}_{L,R} = \bar{n}_{\mathcal{K}} \cdot u_{L,R}$, and $\hat{u}^*$ the normal velocity calculated from the HLL approximation.

- This results in the following expression for S_M

$$S_M = \frac{\rho_R \hat{u}_R (S_R - \hat{u}_R) - \rho_L \hat{u}_L (S_L - \hat{u}_L) + p_L - p_R}{\rho_R (S_R - \hat{u}_R) - \rho_L (S_L - \hat{u}_L)}.$$

- The star states are obtained using the Rankine-Hugoniot relations across the waves moving with the velocities S_L and S_R

$$U_L^* = \frac{S_L - \hat{u}_L}{S_L - S_M} U_L + \frac{1}{S_L - S_M} \begin{pmatrix} 0 \\ (p^* - p_L)\bar{n}_{\mathcal{K}} \\ p^* S_M - p_L \hat{u}_L \end{pmatrix},$$

with an identical relation for U_R^*, only with L replaced with R.

HLLC Flux on Moving Meshes

- The intermediate pressures are equal to

$$
\begin{aligned}
p_L^* &= \rho_L(S_L - \hat{u}_L)(S_M - \hat{u}_L) + p_L, \\
p_R^* &= \rho_R(S_R - \hat{u}_R)(S_M - \hat{u}_R) + p_R.
\end{aligned}
$$

- The definition of S_M ensures that $p_L^* = p_R^* = p^*$, as is required for a contact discontinuity.

- The wave speeds S_L and S_R are computed according as

$$
S_L = \min(\hat{u}_L - a_L, \hat{u}_R - a_R), \qquad S_R = \max(\hat{u}_L + a_L, \hat{u}_R + a_R),
$$

with $a = \sqrt{\gamma p/\rho}$ the speed of sound.

ALE Weak Formulation

- The ALE flux formulation of the compressible Navier-Stokes equations transforms now into

  Find a $U \in W_h$, such that for all $W \in W_h$, the following holds

$$
\begin{aligned}
&- \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{K}} \Big(\frac{\partial W_i}{\partial x_0} U_i + \frac{\partial W_i}{\partial x_k} (F_{ik}^e - \Theta_{ik}) \Big) \, d\mathcal{K} \\
&+ \sum_{K \in \mathcal{T}_h^n} \Big(\int_{K(t_{n+1}^-)} W_i^L U_i^L \, dK - \int_{K(t_n^+)} W_i^L U_i^R \, dK \Big) \\
&+ \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}} W_i^L \big(H_i^{\mathrm{HLLC}}(U^L, U^R, v, \bar{n}) - \widehat{\Theta}_{ik} \bar{n}_k^L \big) \, d\mathcal{Q} = 0.
\end{aligned}
$$

Auxiliary Equation for Θ

- Recall the auxiliary equation for Θ.

  Find a $\Theta \in V_h$, such that for all $V \in V_h$ the following holds

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\Theta_{ik}\, d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik} A_{ikrs} \frac{\partial U_r}{\partial x_s}\, d\mathcal{K}$$
$$+ \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{Q}} V_{ik}^L A_{ikrs}^L (\widehat{U}_r - U_r^L)\bar{n}_s^L\, d\mathcal{Q}.$$

- The following relation holds for the element boundary integrals

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{Q}} g_i^L f_{ik}^L \bar{n}_k^L \, d\mathcal{Q} = \sum_{\mathcal{S}\in\mathcal{S}_I^n}\int_{\mathcal{S}} [\![g_i f_{ik}]\!]_k \, d\mathcal{S} + \sum_{\mathcal{S}\in\mathcal{S}_B^n}\int_{\mathcal{S}} g_i^L f_{ik}^L \bar{n}_k^L \, d\mathcal{S}.$$

- Transform the element boundary integrals into face integrals in the auxiliary equation

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{Q}} V_{ik}^L A_{ikrs}^L(\widehat{U}_r - U_r^L)\bar{n}_s^L \, d\mathcal{Q} = & \sum_{\mathcal{S}\in\mathcal{S}_I^n}\int_{\mathcal{S}} [\![V_{ik} A_{ikrs}(\widehat{U}_r - U_r)]\!]_s \, d\mathcal{S} \\ & + \sum_{\mathcal{S}\in\mathcal{S}_B^n}\int_{\mathcal{S}} V_{ik}^L A_{ikrs}^L(\widehat{U}_r - U_r^L)\bar{n}_s^L \, d\mathcal{S}.\end{aligned}$$

Numerical Fluxes in Auxiliary Equation

- Introduce the numerical flux proposed by Bassi and Rebay

$$\widehat{U} = \begin{cases} \{\!\{U\}\!\} & \text{at internal faces,} \\ U^b & \text{at boundary faces.} \end{cases}$$

- Use the relation

$$[\![g_i f_{ik}]\!]_k = \{\!\{g_i\}\!\}[\![f_{ik}]\!]_k + [\![g_i]\!]_k\{\!\{f_{ik}\}\!\},$$

  then we obtain

$$[\![V_{ik}A_{ikrs}(\widehat{U}_r - U_r)]\!]_s = -\{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s.$$

- The weak formulation for the auxiliary variable Θ then becomes

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{K}} V_{ik}\Theta_{ik}\,d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{K}} V_{ik}A_{ikrs}\frac{\partial U_r}{\partial x_s}\,d\mathcal{K} - \sum_{\mathcal{S}\in\mathcal{S}_I^n}\int_{\mathcal{S}}\{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s\,d\mathcal{S}$$
$$- \sum_{\mathcal{S}\in\mathcal{S}_B^n}\int_{\mathcal{S}} V_{ik}^L A_{ikrs}^L(U_r^L - U_r^b)\bar{n}_s^L\,d\mathcal{S}.$$

Lifting Operator

- Introduce the global lifting operator $\mathcal{R} \in \mathbb{R}^{5\times 3}$, defined in a weak sense as

  Find an $\mathcal{R} \in V_h$, such that for all $V \in V_h$

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\mathcal{R}_{ik}\, d\mathcal{K} = \sum_{\mathcal{S}\in\mathcal{S}_I^n} \int_{\mathcal{S}} \{\!\{ V_{ik} A_{ikrs} \}\!\} [\![U_r]\!]_s \, d\mathcal{S} + \sum_{\mathcal{S}\in\mathcal{S}_B^n} \int_{\mathcal{S}} V_{ik}^L A_{ikrs}^L (U_r^L - U_r^b) \bar{n}_s^L \, d\mathcal{S}.$$

- The weak formulation for the auxiliary variable is now transformed into

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\Theta_{ik}\, d\mathcal{K} = \sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\Big(A_{ikrs}\frac{\partial U_r}{\partial x_s} - \mathcal{R}_{ik}\Big)\, d\mathcal{K}, \quad \forall V \in V_h.$$

Θ Equation

- The primal formulation can be obtained by eliminating the auxiliary variable Θ using

$$\Theta_{ik} = A_{ikrs} \frac{\partial U_r}{\partial x_s} - \mathcal{R}_{ik}, \qquad \text{a.e. in } \mathcal{E}_h^n.$$

- Note, this is possible since $\nabla_h W_h \subset V_h$.

ALE Weak Formulation for Primal Variables

- Recall the ALE flux formulation of the compressible Navier-Stokes equations

  Find a $U \in W_h$, such that for all $W \in W_h$, the following holds

$$
\begin{aligned}
& - \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{K}} \Big(\frac{\partial W_i}{\partial x_0} U_i + \frac{\partial W_i}{\partial x_k} (F_{ik}^e - \Theta_{ik}) \Big) \, d\mathcal{K} \\
& + \sum_{K \in \mathcal{T}_h^n} \Big(\int_{K(t_{n+1}^-)} W_i^L U_i^L \, dK - \int_{K(t_n^+)} W_i^L U_i^R \, dK \Big) \\
& + \sum_{\mathcal{K} \in \mathcal{T}_h^n} \int_{\mathcal{Q}} W_i^L (H_i^{\mathrm{HLLC}} (U^L, U^R, v, \bar{n}) - \widehat{\Theta}_{ik} \bar{n}_k^L) \, d\mathcal{Q} = 0.
\end{aligned}
$$

Numerical Fluxes for Θ

- The numerical flux $\widehat{\Theta}$ in the primary equation is defined following Brezzi as a central flux $\widehat{\Theta} = \{\!\{\Theta\}\!\}$

$$\widehat{\Theta}_{ik}(U^L, U^R) = \begin{cases} \{\!\{A_{ikrs}\frac{\partial U_r}{\partial x_s} - \eta \mathcal{R}^{\mathcal{S}}_{ik}\}\!\} & \text{for internal faces,} \\ A^b_{ikrs}\frac{\partial U^b_r}{\partial x_s} - \eta \mathcal{R}^{\mathcal{S}}_{ik} & \text{for boundary faces,} \end{cases}$$

- The local lifting operator $\mathcal{R}^{\mathcal{S}} \in \mathbb{R}^{5\times 3}$ is defined as follows

Find an $\mathcal{R}^{\mathcal{S}} \in V_h$, such that for all $V \in V_h$

$$\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} V_{ik}\mathcal{R}^{\mathcal{S}}_{ik}\, d\mathcal{K} = \begin{cases} \displaystyle\int_{\mathcal{S}} \{\!\{V_{ik}A_{ikrs}\}\!\}[\![U_r]\!]_s\, d\mathcal{S} & \text{for internal faces,} \\ \displaystyle\int_{\mathcal{S}} V^L_{ik}A^L_{ikrs}(U^L_r - U^b_r)\bar{n}_s\, d\mathcal{S} & \text{for external faces.} \end{cases}$$

Space-Time Formulation for Compressible Navier-Stokes Equations

Find a $U \in W_h$, such that for all $W \in W_h$

$$
\begin{aligned}
&-\sum_{\mathcal{K}\in\mathcal{T}_h^n}\int_{\mathcal{K}}\Big(\frac{\partial W_i}{\partial x_0}U_i+\frac{\partial W_i}{\partial x_k}\big(F^e_{ik}-A_{ikrs}\frac{\partial U_r}{\partial x_s}+\mathcal{R}_{ik}\big)\Big)\,d\mathcal{K} \\
&+\sum_{K\in\mathcal{T}_h^n}\Big(\int_{K(t^-_{n+1})}W_iU_i^L\,dK-\int_{K(t^+_n)}W_iU_i^R\,dK\Big) \\
&+\sum_{\mathcal{S}\in\mathcal{S}^n_{IB}}\int_{\mathcal{S}}(W_i^L-W_i^R)H_i(U^L,U^R,v,\bar{n}^L)\,d\mathcal{S} \\
&-\sum_{\mathcal{S}\in\mathcal{S}^n_{I}}\int_{\mathcal{S}}[\![W_i]\!]_k\{\!\{A_{ikrs}\frac{\partial U_r}{\partial x_s}-\eta\mathcal{R}^{\mathcal{S}}_{ik}\}\!\}\,d\mathcal{S} \\
&-\sum_{\mathcal{S}\in\mathcal{S}^n_{B}}\int_{\mathcal{S}}W_i^L\big(A^b_{ikrs}\frac{\partial U^b_r}{\partial x_s}-\eta\mathcal{R}^{\mathcal{S}}_{ik}\big)\bar{n}^L_k\,d\mathcal{S}=0,
\end{aligned}
$$

Basis Functions

- The basis functions are polynomials of degree k to represent the trial function U and the test function W in each element $\mathcal{K} \in \mathcal{T}_h^n$:

$$U_i(t, \bar{x})|_{\mathcal{K}} = \hat{U}_{im}\psi_m(t, \bar{x}),$$
$$W_l(t, \bar{x})|_{\mathcal{K}} = \psi_l(t, \bar{x}).$$

  with ψ the basis functions.

- The basis functions are defined such that the test and trial functions are split into an element mean at time t_{n+1} and a fluctuating part.

- This construction facilitates the definition of the artificial dissipation operator and of the multigrid convergence acceleration method.

- The basis functions ψ are given by

$$\begin{aligned}\psi_m &= 1, && m = 0,\\ &= \phi_m(t, \bar{x}) - \frac{1}{|K_j(t_{n+1}^-)|}\int_{K_j(t_{n+1}^-)} \phi_m(t, \bar{x})\, dK, && m = 1, \dots, N,\end{aligned}$$

  where the basis functions ϕ are given by

$$\phi_m = \hat{\phi}_m \circ G_{\mathcal{K}}^{-1} \quad \text{with} \quad \hat{\phi}_m(\xi) \in P^k(\hat{\mathcal{K}}),$$

  with ξ the local coordinates in the master element $\hat{\mathcal{K}}$.

Lifting operators

- The DG coefficients of global and local lifting operators need to be expressed in terms of the DG coefficients of the primal variable U.
- Recall the expression for the lifting operator

$$
\begin{aligned}
\sum_{\mathcal{K}\in\mathcal{T}_h^n} \int_{\mathcal{K}} W_{i,k}\mathcal{R}_{ik}\, d\mathcal{K} = {} & \sum_{\mathcal{S}\in\mathcal{S}_I^n} \int_{\mathcal{S}} \{\!\{W_{i,k}A_{ikrs}\}\!\} [\![U_r]\!]_s \, d\mathcal{S} \\
& + \sum_{\mathcal{S}\in\mathcal{S}_B^n} \int_{\mathcal{S}} W_{i,k}^L A_{ikrs}^L (U_r^L - U_r^b)\bar{n}_s^L \, d\mathcal{S}.
\end{aligned}
$$

- The face integrals can be directly computed by replacing the test and trial functions by their polynomial expansions.

Lifting operators

- The local lifting are similarly expressed as

$$\mathcal{R}^{\mathcal{S}}(t,\bar{x})|_{\mathcal{K}} = \hat{R}_j \psi_j(t,\bar{x}).$$

  and a small linear system must be solved for the expansion coefficients $\hat{R}_j$.

- The local lifting operator is only non-zero on the two elements $\mathcal{K}^L$ and $\mathcal{K}^R$ connected to the face $\mathcal{S} \in \mathcal{S}_I^n$, hence

$$\int_{\mathcal{K}^R} V_{ik} \mathcal{R}_{ik}^{\mathcal{S}} \, d\mathcal{K} + \int_{\mathcal{K}^L} V_{ik} \mathcal{R}_{ik}^{\mathcal{S}} \, d\mathcal{K} = \int_{\mathcal{S}} \{\!\{ V_{ik} A_{ikrs} \}\!\} [\![U_r]\!]_s \, d\mathcal{S}.$$

- This is equivalent with the two following equations:

$$\int_{\mathcal{K}^{L,R}} V_{ik} \mathcal{R}_{ik}^{\mathcal{S}} \, d\mathcal{K} = \frac{1}{2} \int_{\mathcal{S}} V_{ik}^{L,R} A_{ikrs}^{L,R} [\![U_r]\!]_s \, d\mathcal{S},$$

  where the superscript L, R refers to the traces from either the left or right element.

Lifting operators

- Replacing $\mathcal{R}^{\mathcal{S}}$ by its polynomial approximation leads to two systems of linear equations for the expansion coefficients $\hat{R}_{ikj}$ of $\mathcal{R}^{\mathcal{S}}_{ik}$ on $\mathcal{S} \in \mathcal{S}_I$:

$$\hat{R}^{L,R}_{ikj} \int_{\mathcal{K}^{L,R}} \psi_l \psi_j \, d\mathcal{K} = \frac{1}{2} \int_{\mathcal{S}} \psi_l^{L,R} A^{L,R}_{ikrs} [\![U_r]\!]_s \, d\mathcal{S}.$$

- The element mass matrices on the l.h.s. are denoted by $M^{L,R}_{lj}$ and can easily be inverted leading to following expression for the expansion coefficients of the local lifting operator on $\mathcal{S} \in \mathcal{S}_I$:

$$\hat{R}^{L,R}_{ikj} = \frac{1}{2} (M^{-1})^{L,R}_{jl} \int_{\mathcal{S}} \psi_l^{L,R} A^{L,R}_{ikrs} [\![U_r]\!]_s \, d\mathcal{S}.$$

- Similarly, the expression for the expansion coefficients of the local lifting operator for the faces $\mathcal{S} \in \mathcal{S}_B$ is:

$$\hat{R}^{L}_{ikj} = (M^{-1})^{L}_{jl} \int_{\mathcal{S}} \psi_l^{L} A^{L}_{ikrs} (U^L_r - U^b_r) \bar{n}^L_s \, d\mathcal{S}.$$

- The expressions for the local lifting operator can now be introduced into the DG formulation, resulting in the primal formulation without auxiliary variables.