Stability analysis and error estimates of discontinuous Galerkin methods for linear hyperbolic and convection-diffusion equations: semi-discrete

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Background

- The discontinuous Galerkin (DG) method is widely used to solve the time-dependent hyperbolic equations:
  - Proposed firstly for linear equation by Reed and Hill (1973);
  - Developed to nonlinear equation, by Cockburn and Shu (1989)
    - the numerical flux at element interfaces;
    - the explicit (TVD/SSP) Runge-Kutta time-marching;
    - the slope limiter . . .

  It is named the RKDG method.
- The local discontinuous Galerkin (LDG) method is widely-used to solve those PDEs with high order derivatives:
  - proposed firstly by Bassi and Rebay (1997) to solve the Navier-Stokes equation;
  - developed and firstly analyzed by Cockburn and Shu (1998) for convection diffusion equations;
  - extended to many PDEs with higher order derivatives: J. Yan(Iowa State U), Y. Xu (USTC), . . .
- Compared with wide applications, there is relatively less work on theory analysis, even for simple model equation.

Background

- The semi-discrete DG method:
  - local cell entropy inequality (1994), and hence the L^2-norm of the numerical solution does not increase v.s. time.
  - optimal error estimate,
  - superconvergence analysis, and post-processing,
  - ...
- The fully-discrete RKDG method:
  - total-variation-diminishing in the means, with the strong-stability-preserving (SSP) time-marching;
  - lower (time) order RKDG methods:
    - L^2-norm stabilities for linear hyperbolic equation;
    - L^2-norm error estimates for linear/nonlinear eqaution(s), with the sufficiently smooth solution;
    - local analysis of L^2-normerror estimates for the linear equation, when the initial solution has a discontinuity.
  - arbitrary order RKDG method for linear equation (reported in this talk):
    - L^2-norm stability for arbitrary RKDG methods;
    - optimal error estimate and superconvergence analysis.
  - ...

Outline

Outline

Semi-discrete DG method

Let us start from the 1d nonlinear hyperbolic equation

$$U_t + f(U)_x = 0, \quad (x,t) \in (0,1) \times (0,T], \tag{1}$$

equipped with the periodic boundary condition. f: physical flux.

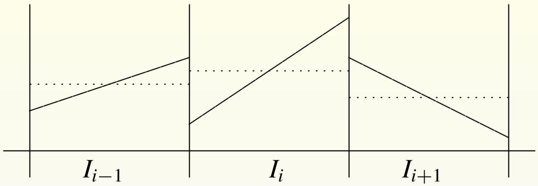
- Let $I_h = \{I_i\}_{i=1}^N$ be the quasi-uniform partition, where h is the maximum length of every element.
- The discontinuous finite element space is defined as the piecewise polynomials of degree at most $k \geq 0$, namely

$$V_h = \{v \colon v \in L^2(I), v|_{I_i} \in \mathcal{P}^k(I_i), i = 1, \ldots, N\}.$$

- jump and average at the interface point:

$$[\![v]\!] = v^+ - v^-, \quad \{\!\{v\}\!\} = \frac{1}{2}(v^- + v^+).$$

The semi-discrete DG method



- The semi-discrete DG method for the model equation is defined as follows: find the map $u\colon [0,T] \to V_h$ such that

$$(u_t, v) = \mathcal{H}(u, v), \quad \forall\, v \in V_h, \quad t \in (0, T], \tag{2}$$

  with the initial solution $u(x, 0) \in V_h$.

- Here $(\cdot, \cdot)$ is the usual L^2 inner product, and the spatial DG discretization

$$\mathcal{H}(u, v) = \sum_{1 \le i \le N} \left[\int_{I_j} f(u) v_x \, \mathrm{d}x + \hat{f}(u^-_{i+\frac{1}{2}}, u^+_{i+\frac{1}{2}}) [\![v]\!]_{i+\frac{1}{2}} \right] \tag{3}$$

  involves the numerical flux $\hat{f}(u^-, u^+)$.

Numerical flux

- Consistence: $\hat{f}(p,p) = f(p)$;
- Lip. continuous with two arguments;
- Stability demand:
  - Monotone: $\hat{f}(\uparrow,\downarrow)$

$$[f(p) - \hat{f}(u^-,u^+)][\![u]\!] \geq 0, \quad \forall p \in \text{inter}\{u^-,u^+\}.$$

    This ensures the local entropy inequality and hence the L^2-norm stability.
    Example: Lax-Fredrichs flux

$$\hat{f}(u^-,u^+) = \frac{1}{2}[f(u^-) + f(u^+)] - \frac{1}{2}C[\![u]\!],$$

    where $C = \max |f'(u)|$.
  - For linear case $f(u) = \beta u$, the numerical flux $\hat{f}(u^-,u^+)$ is allowed to be upwind-biased, namely

$$\hat{f}(u^-,u^+) = \beta \{\!\{u\}\!\}^{(\theta)} = \beta[\theta u^- + (1-\theta)u^+],$$

    where $\beta(\theta - 1/2) > 0$. In general, it is not an monotone flux.

Contents in this talk

- The model equations is simple:
  - linear constant hyperbolic equation (and convection-diffusion equation);
  - periodic boundary condition;
  - the upwind-biased numerical flux (and generalized alternating numerical flux);
  - the linear scheme without any nonlinear treatments.
- Stability analysis and error estimates (in L^2-norm) by energy technique:
  - semi-discrete DG/LDG method

    property of DG discretization, GGR projection, multi-dimension, . . .

  - fully discrete RKDG method

    stability performance, temporal differences of stage solutions, matrix transferring process, reference function at stage time, incomplete correction function technique, . . .

Outline

The semi-discrete DG method

Consider the 1d hyperbolic equation with nonzero constant β

$$U_t + \beta U_x = 0, \quad x \in (0,1), \quad t \in (0,T], \tag{4}$$

equipped with the periodic boundary condition.

- The DG method is defined as follows: find $u\colon [0,T] \to V_h$ such that

$$(u_t, v) = \mathcal{H}^\theta(u,v), \quad \forall\, v \in V_h, \quad t \in (0,T], \tag{5}$$

  with $u(x,0) \in V_h$ approximating the initial solution.
- The spatial DG discretization is given in the form

$$\mathcal{H}^\theta(u,v) = \sum_{1\le i\le N} \left[\int_{I_i} \beta u v_x \,\mathrm{d}x + \beta \{\!\{u\}\!\}^{(\theta)}_{i+\frac12} [\![v]\!]_{i+\frac12} \right], \tag{6}$$

  with the upwind-biased numerical flux, since $\beta(\theta - 1/2) > 0$.

Properties of DG discretization (arbitrary α)

- accurate skew-symmetric

$$\mathcal{H}^{1-\alpha}(\varphi,\psi)+\mathcal{H}^{\alpha}(\psi,\varphi)=0.$$

- approximating skew-symmetric

$$\mathcal{H}^{\alpha}(\psi,\varphi)+\mathcal{H}^{\alpha}(\varphi,\psi)=-(2\alpha-1)\sum_{i=1}^{N}[\![\varphi]\!]_{i+\frac{1}{2}}[\![\psi]\!]_{i+\frac{1}{2}}.$$

- negative semidefinite

$$\mathcal{H}^{\alpha}(\varphi,\varphi)=-\frac{1}{2}(2\alpha-1)\sum_{i=1}^{N}[\![\varphi]\!]_{i+\frac{1}{2}}^{2}=-\frac{1}{2}(2\alpha-1)\|[\![\varphi]\!]\|_{\Gamma_h}^{2}\leq 0.$$

- boundedness in the finite element space

$$\mathcal{H}^{\alpha}(\varphi,\psi)\leq Mh^{-1}\|\varphi\|\|\psi\|.$$

Stability analysis

Theorem 2.1

The DG method is stable in L^2-norm, namely

$$\|u(t)\| \leq \|u(0)\|.$$

- The proof is trivial.
- Taking $v = u$ in (6) and using the negative semidefinite property, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{1}{2}(2\theta - 1)\|[\![u]\!]\|_{\Gamma_h}^2 \leq 0,$$

  which implies the above stability.
- An additional stability mechanism is provided by the square of jumps, which is better than the standard FEM.

General framework of error estimate

- Error splitting: let χ be a reference function in V_h, consider

$$e = u - U = \xi - \eta,$$

where $\xi = u - \chi \in V_h$ and $\eta = U - \chi$.

- Estimate ξ by η: for example, we can do it by using the error equation

$$(\xi_t, v) - \mathcal{H}^\theta(\xi, v) = (\eta_t, v) - \mathcal{H}^\theta(\eta, v), \tag{7}$$

with the test function $v = \xi$.

  - The lower bound of LHS is usually given by the stability result.
  - Sharply estimate RHS by introducing a suitable χ, which is often defined as a well-defined projection.

- Applications of the Gronwall inequality and the triangular inequality.

Quasi-optimal error estimate

Definition 1 (L^2 projection)

Let $w \in L^2(I)$ be any given function. The L^2 projection, denoted by $\mathbb{P}_h w$, is the unique element in V_h such that

$$(w - \mathbb{P}_h w, v) = 0, \quad \forall v \in V_h. \tag{8}$$

- The projection is well-defined, and
- there holds the approximation property

$$\|\mathbb{P}_h^\perp w\| + h^{\frac{1}{2}} \|\mathbb{P}_h^\perp w^\pm\|_{\Gamma_h} \le C h^{k+1} \|w\|_{k+1}.$$

- Since V_h is discontinuous finite element space, (8) is equal to

$$(w - \mathbb{P}_h w, v)_{I_i} = 0, \quad \forall v \in \mathcal{P}^k(I_i), \quad i = 1, 2, \dots, N.$$

Hence this projection is also called the local L^2 projection.

Quasi-optimal error estimate

Theorem 2.2

Assume that the initial solution $U_0 \in H^{k+1}(I)$, then for any $t \in [0, T]$ we have

$$\|U(t) - u(t)\| \le C\|U_0\|_{k+1} h^{k+\frac{1}{2}},$$

with suitable setting of the initial solution (e.g., the L^2 projection).

- By the help of the enhanced stability mechanism and the definition of L^2 projection, we can get from (7) that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \frac{1}{2}(2\theta - 1)\|[\![\xi]\!]\|_{\Gamma_h}^2 &\le \|[\![\xi]\!]\|_{\Gamma_h}\|\{\!\{\eta\}\!\}^{(\theta)}\|_{\Gamma_h} \\ &\le \frac{1}{2}(2\theta - 1)\|[\![\xi]\!]\|_{\Gamma_h}^2 + C\|\{\!\{\eta\}\!\}^{(\theta)}\|_{\Gamma_h}^2.\end{aligned}$$

- By the approximation property of L^2 projection, we can yield

$$\|\xi\| \le \|\xi(0)\| + C\|U_0\|_{k+1} h^{k+\frac{1}{2}}. \tag{9}$$

- Noticing the initial setting, the triangular inequality ends the proof.

Optimal error estimate?

- However, the numerical experiments shows the optimal order. To obtain the sharp error estimate, we have to introduce a better projection.
- For $\theta = 0, 1$:
  - Interpolation on the Gauss-Radau points:

    P. LESAINT AND P. A. RAVIART, Mathematical Aspects of finite elements in PDEs, 89-145 (1974)
  - Gauss-Radau projection:

    P. CASTILLO AND B. COCKBURN, Math. Comp., **71**, 455-478 (2002)
- For general value of θ, the Generalized Gauss-Radau (GGR) projection is introduced.

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  H. L. Liu and N. Polymaklam, Numer. Math., **129**, 321-351 (2015)

  X. Meng, C. -W. Shu and B. Y. Wu, Math. Comp., **85**, 1225-1261 (2016)

  Y. Cheng, X. Meng and Q. Zhang, Math. Comp., **86**, 1233-1267 (2017)

1d GGR projection

Definition 2 (1d GGR)

Assume that $\theta \neq 1/2$. For any given periodic function $w \in H^1(I_h)$, the GGR projection, denoted by $\mathbb{G}_h^\theta w$, is the unique element in V_h such that

$$((\mathbb{G}_h^\theta)^\perp w, v)_{I_i} = 0, \ \forall v \in \mathcal{P}^{k-1}(I_i); \quad \{\!\{(\mathbb{G}_h^\theta)^\perp w\}\!\}_{i+\frac{1}{2}}^{(\theta)} = 0,$$

for $i = 1, 2, \ldots, N$. Here $(\mathbb{G}_h^\theta)^\perp w = w - \mathbb{G}_h^\theta w$ is the projection error.

- In general, the GGR projection is globally defined.

Lemma 2.1

The 1d GGR projection is well-defined, and satisfies

$$\|(\mathbb{G}_h^\theta)^\perp z\| + h^{\frac{1}{2}}, \|(\mathbb{G}_h^\theta)^\perp z\|_{\Gamma_h} \leq C\|z\|_{k+1} h^{k+1}. \qquad (10)$$

- Prove it later.

Application of GGR projection

Theorem 2.3

Assume that the initial solution $U_0 \in H^{k+2}(I)$, then for any $t \in [0, T]$ we have

$$\|U(t) - u(t)\| \leq C\|U_0\|_{k+2} h^{k+1},$$

with suitable setting of the initial solution (e.g., the L^2/GGR projection).

- The 1d GGR projection implies for any $v \in V_h$,

$$\mathcal{H}^{\theta}(\eta, v) = \sum_{1 \leq i \leq N} \left[\int_{I_i} \beta \eta v_x \, \mathrm{d}x + \beta \{\!\{\eta\}\!\}^{(\theta)}_{i+\frac{1}{2}} [\![v]\!]_{i+\frac{1}{2}} \right] = 0.$$

- It follows from (7) and Lemma 2.1 that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \frac{1}{2}(2\theta - 1)\|[\![\xi]\!]\|^2_{\Gamma_h} = (\eta_t, \xi) \leq \|\xi\|\|\eta_t\| \leq Ch^{k+1}\|\xi\|\|U_t\|_{k+1}.$$

- Integration and application the triangular inequality end the proof.

Proof of Lemma 2.1

- Let $E = \mathbb{G}_h^\theta z - \mathbb{P}_h z$, where $\mathbb{P}_h z \in V_h$ is the local L^2-projection.
- Show below that $E \in V_h$ exists uniquely and satisfies

$$\|E\|_{L^2(\Omega_h)} + h^{\frac{1}{2}} \|E\|_{L^2(\Gamma_h)} \leq Ch^{\min(k+1,s+1)} \|z\|_{H^{s+1}(\Omega_h)}. \tag{11}$$

- These purposes can be achieved by direct manipulations through a linear system, since

$$\int_{I_i} Ev\mathrm{d}x = 0, \quad \forall v \in \mathcal{P}^{k-1}(I_i), \quad i = 1, \ldots, N, \tag{12a}$$

$$\{\!\{E\}\!\}_{i+\frac{1}{2}}^{(\theta)} = \{\!\{z - \mathbb{P}_h z\}\!\}_{i+\frac{1}{2}}^{(\theta)} \equiv b_i, \quad i = 1, \ldots, N, \tag{12b}$$

where b_i is the projection error resulting from the L^2-projection.

Proof of Lemma 2.1

- Due to the orthogonality of the rescaled Legendre polynomials, it is easy to see from (12a) that

$$E(x) = \alpha_{i,k} L_{i,k}(x) = \alpha_{i,k} \widehat{L}_k(\hat{x}),$$

in the element I_i, where $\hat{x} = 2(x - x_i)/h_i \in [-1, 1]$ and

$$L_{i,l}(x) \equiv \widehat{L}_l\Big(\frac{2(x - x_i)}{h_i}\Big) \equiv \widehat{L}_l(\hat{x}).$$

Here $\widehat{L}_l(\hat{x})$ is the standard Legendre polynomial in $[-1, 1]$ of degree l.

- Since $\widehat{L}_k(\pm 1) = (\pm 1)^k$, it follows from (12b) that

$$\theta \alpha_{i,k} + \widetilde{\theta}(-1)^k \alpha_{i+1,k} = b_i, \quad i = 1, \cdots, N. \tag{13}$$

Note that $\alpha_{N+1,k} = \alpha_{1,k}$ and $\widetilde{\theta} = 1 - \theta$.

Proof of Lemma 2.1

- The unknowns $\vec{\alpha}_N = (\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{N,k})^\top$ can be determined from the following linear algebra system

$$\mathbb{A}_N \vec{\alpha}_N = \vec{b}_N, \tag{14}$$

where $\vec{b}_N = (b_1, b_2, \ldots, b_N)^\top$ and

$$\mathbb{A}_N = \begin{bmatrix} \theta & \widetilde{\theta} & & & \\ & \theta & \widetilde{\theta} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ \widetilde{\theta} & & & & \theta \end{bmatrix}. \tag{15}$$

- It is easy to work out that

$$\det(\mathbb{A}_N) = \theta^N(1 - \zeta^N) \neq 0, \quad \text{with} \;\; \zeta = (-1)^{k+1}\tilde{\theta}/\theta \neq 1.$$

Hence E and $\mathbb{G}_h^\theta z$ is determined uniquely.

Proof of Lemma 2.1

- Easy to see that $\mathbb{A}_N^{-1}$ is a circulant matrix with the (i,j)-th entry

$$(\mathbb{A}_N^{-1})_{ij} = \frac{1}{\theta(1-\zeta^N)}\zeta^{\text{mod}(j-i,N)}.$$

- Both the row-norm and the column-norm satisfy

$$\|\mathbb{A}_N^{-1}\|_1 = \|\mathbb{A}_N^{-1}\|_\infty \le \frac{1}{|\theta||1-\zeta^N|}\frac{|1-|\zeta|^N|}{|1-|\zeta||} \le \frac{1}{|\theta||1-|\zeta||},$$

hence the spectral norm is bounded above by

$$\|\mathbb{A}_N^{-1}\|_2^2 \le \|\mathbb{A}_N^{-1}\|_1\|\mathbb{A}_N^{-1}\|_\infty \le \frac{1}{\theta^2(1-|\zeta|)^2}. \qquad (16)$$

- Note that this inequality holds independently of the element number N.

Proof of Lemma 2.1

- Owing to the approximation property of L^2-projection, we have

$$\begin{aligned}\|\vec{\alpha}_N\|_2^2 &= \|\mathbb{A}_N^{-1}\vec{b}_N\|_2^2 \le \|\mathbb{A}_N^{-1}\|_2^2\|\vec{b}_N\|_2^2 \\ &\le C\|\vec{b}_N\|_2^2 \le C\|z-\pi_h z\|_{\Gamma_h}^2 \le Ch^{2\min(k+1,s+1)-1}\|z\|_{H^{s+1}(\Omega_h)}^2.\end{aligned} \tag{17}$$

- Finally, noticing the simple facts

$$\|E\|_{L^2(\Omega_h)}^2 = \sum_{i=1}^{N}\alpha_{i,k}^2\|L_{i,k}(x)\|_{L^2(I_i)}^2 = \sum_{i=1}^{N}\frac{h_i\alpha_{i,k}^2}{2k+1} \le Ch\|\vec{\alpha}_N\|_2^2, \tag{18a}$$

$$\|E\|_{L^2(\Gamma_h)}^2 = \sum_{i=1}^{N}\alpha_{i,k}^2 = \|\vec{\alpha}_N\|_2^2, \tag{18b}$$

as well as (17), we can obtain (11) and finish the proof.

Remarks for $\theta = 1/2$

- Actually, $\theta = 1/2$ can be also used in the semi-discrete DG method.
- In general, the convergence order is k.
- However, the convergence order can achieve $k+1$ if the mesh is uniform and the degree k is even.
  - It can be proved by the super-convergence attribution of mesh construction and the L^2 projection.
  - Also by using the GGR projection in some sense.
- However, when directly taking $\theta = 1/2$ in Definition 2, the GGR projection uniquely exists only if
  - the degree k is even , and
  - the number of elements is odd.

  Hence, the GGR projection for $\theta = 1/2$ is defined to be the L^2 projection, namely

$$\mathbb{G}_h^{\frac{1}{2}} w = \mathbb{P}_h w.$$

Outline

Discontinuous finite element space

Consider the 2d linear constant hyperbolic equation

$$U_t + \beta_1 U_x + \beta_2 U_y = 0, \quad (x, y, t) \in (0,1)^2 \times (0, T], \tag{19}$$

equipped with the periodic boundary condition.

- Let $\Omega_h = I_h \times J_h = \{K_{ij}\}_{i=1,\ldots,N_x}^{j=1,\ldots,N_y}$ denote a quasi-uniform tessellation with the rectangular element

$$K_{ij} \equiv I_i \times J_j \equiv (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}),$$

  of the length $h_i^x = x_{i+1/2} - x_{i-1/2}$ and the width $h_j^y = y_{j+1/2} - y_{j-1/2}$.
- The associated finite element space is defined as

$$V_h \equiv V_h^{(2)} \equiv \{v \in L^2(\Omega) : v|_K \in \mathcal{Q}^k(K_{ij}), \forall K_{ij} \in \Omega_h\}, \tag{20}$$

  where $\mathcal{Q}^k(K_{ij})$ denotes the space of polynomials on K_{ij} of degree at most $k \geq 0$ in each variable.

Notations of averages and jumps

- Similar to the one-dimensional case, we use

$$[\![v]\!]_{i+1/2,y} = v^{+}_{i+1/2,y} - v^{-}_{i+1/2,y}, \quad [\![v]\!]_{x,j+1/2} = v^{+}_{x,j+1/2} - v^{-}_{x,j+1/2} \tag{21}$$

  to denote the jumps on vertical and horizontal edges, where

$$v^{\pm}_{i+\frac{1}{2},y} = \lim_{x \to x_{i+\frac{1}{2}}^{\pm}} v(x,y), \quad v^{\pm}_{x,j+\frac{1}{2}} = \lim_{y \to y_{j+\frac{1}{2}}^{\pm}} v(x,y)$$

  are the traces along two different directions.
- The weighted averages on vertical and horizontal edges are respectively denoted by

$$\{\!\{v\}\!\}^{\theta_1,y}_{i+\frac{1}{2},y} = \theta_1 v^{-}_{i+\frac{1}{2},y} + \widetilde{\theta}_1 v^{+}_{i+\frac{1}{2},y}, \quad \{\!\{v\}\!\}^{x,\theta_2}_{x,j+\frac{1}{2}} = \theta_2 v^{-}_{x,j+\frac{1}{2}} + \widetilde{\theta}_2 v^{+}_{x,j+\frac{1}{2}}, \tag{22}$$

  with the given parameters θ_1 and θ_2.
- Here and below denote $\widetilde{\theta}_1 = 1 - \theta_1$ and $\widetilde{\theta}_2 = 1 - \theta_2$.

The semi-discrete DG method

- Similarly, the semi-discrete DG method is defined as follows: find the map $u\colon [0,T] \to V_h$ such that

$$(u_t, v) = \mathcal{H}(u, v), \quad \forall\, v \in V_h, \quad t \in (0, T], \tag{23}$$

with $u(x,y,0) \in V_h$ approximating the initial solution.

- The spatial DG discretization in 2d case is given in the form

$$\mathcal{H}(u,v) = \mathcal{H}^{1,\theta_1}(u,v) + \mathcal{H}^{2,\theta_2}(u,v),$$

with the DG discretization in two directions

$$\mathcal{H}^{1,\theta_1}(u,v) = \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \left[\int_{K_{ij}} \beta_1 u v_x \,\mathrm{d}x\,\mathrm{d}y + \int_{J_j} \beta_1 \{\!\{u\}\!\}^{\theta_1,y}_{i+\frac{1}{2},y} [\![v]\!]_{i+\frac{1}{2},y} \,\mathrm{d}y \right],$$

$$\mathcal{H}^{2,\theta_2}(u,v) = \sum_{1\le i\le N_x} \sum_{1\le j\le N_y} \left[\int_{K_{ij}} \beta_2 u v_y \,\mathrm{d}x\,\mathrm{d}y + \int_{I_i} \beta_2 \{\!\{u\}\!\}^{x,\theta_2}_{x,j+\frac{1}{2}} [\![v]\!]_{x,j+\frac{1}{2}} \,\mathrm{d}x \right].$$

- Here $\beta_\kappa(\theta_\kappa - 1/2) > 0$ is demanded for $\kappa = 1, 2$.

Optimal error estimate

Theorem 3.1 (multi-dimension)

Assume that the initial solution $U_0 \in H^{k+2}(\Omega) \cap C(\bar{\Omega})$, then for any $t \in [0, T]$ we have

$$\|U(t) - u(t)\| \leq C\|U_0\|_{k+2}h^{k+1},$$

with suitable setting of the initial solution (e.g., the L^2/GGR projection).

- This theorem can be proved by applying two-dimensional GGR projection

$$\mathbb{G}_h^{\theta_1,\theta_2} = \mathbb{X}_h^{\theta_1} \otimes \mathbb{Y}_h^{\theta_2}, \qquad (24)$$

  where $\mathbb{X}_h^{\theta_1}$ and $\mathbb{Y}_h^{\theta_2}$ are one-dimensional GGR projections, corresponding the x- and y- direction respectively.

- The detailed definitions of 2d GGR projection depend on the values of parameters, denoted by γ_1 and γ_2 below.

- For given function z, denote the GGR projection error by $\eta = z - \mathbb{G}_h^{\gamma_1,\gamma_2} z$.

Detailed definition (I)

- $\gamma_1 \neq 1/2$ and $\gamma_2 \neq 1/2$: Let $z \in H_2(\Omega_h)$ be a given periodic function, there holds for all i and j that

$$\int_{K_{ij}} \eta v \mathrm{d}x\mathrm{d}y = 0, \quad \forall v \in \mathcal{Q}^{k-1}(K_{ij}), \tag{25a}$$

$$\int_{J_j} \{\!\{\eta\}\!\}_{i+\frac{1}{2},y}^{\gamma_1,y} v \mathrm{d}y = 0, \quad \forall v \in \mathcal{P}^{k-1}(J_j), \tag{25b}$$

$$\int_{I_i} \{\!\{\eta\}\!\}_{x,j+\frac{1}{2}}^{x,\gamma_2} v \mathrm{d}x = 0, \quad \forall v \in \mathcal{P}^{k-1}(I_i), \tag{25c}$$

$$\eta_{i+\frac{1}{2},j+\frac{1}{2}}^{\gamma_1,\gamma_2} = 0, \tag{25d}$$

  with the weighted average at the corner point of element

$$\eta^{\gamma_1,\gamma_2} = \gamma_1\gamma_2\eta^{-,-} + \gamma_1\widetilde{\gamma}_2\eta^{-,+} + \widetilde{\gamma}_1\gamma_2\eta^{+,-} + \widetilde{\gamma}_1\widetilde{\gamma}_2\eta^{+,+}.$$

- This projection is firstly proposed and discussed in

X. Meng, C. -W. Shu and B. Y. Wu, Math. Comp., **85**, 1225-1261 (2016)

Detailed definition (II)

- $\gamma_1 \neq 1/2$ and $\gamma_2 = 1/2$: Let $z \in H_1(\Omega_h)$. There holds for all i and j that

$$\int_{K_{ij}} \eta v \,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall v \in \mathcal{P}^{k-1}(I_i) \otimes \mathcal{P}^k(J_j), \tag{26a}$$

$$\int_{J_j} \{\!\{\eta\}\!\}_{i+\frac{1}{2},y}^{\gamma_1,y} v \,\mathrm{d}y = 0, \quad \forall v \in \mathcal{P}^k(J_j). \tag{26b}$$

- $\gamma_1 = 1/2$ and $\gamma_2 \neq 1/2$: Let $z \in H_1(\Omega_h)$. There holds

$$\int_{K_{ij}} \eta v \,\mathrm{d}x\,\mathrm{d}y = 0, \quad \forall v \in \mathcal{P}^k(I_i) \otimes \mathcal{P}^{k-1}(J_j), \tag{27a}$$

$$\int_{I_i} \{\!\{\eta\}\!\}_{x,j+\frac{1}{2}}^{x,\gamma_2} v \,\mathrm{d}x = 0, \quad \forall v \in \mathcal{P}^k(I_i). \tag{27b}$$

- The above two projections have been proposed for $\gamma_1 = \gamma_2 = 1$ in

  B. DONG AND C. -W. SHU, SINUM **47**, 3240-3268(2009)

- $\gamma_1 = \gamma_2 = 1/2$: we define it to be the 2d L^2-projection.

Properties of 2d GGR projection

- As 1d case, we also have the optimal approximation result

Lemma 3.1 (Approximation property of 2d case)

For any γ_1 and γ_2, the 2d GGR projection is well-defined, and

$$\|(\mathbb{G}_h^{\gamma_1,\gamma_2})^{\perp} w\| + h^{\frac{1}{2}} \|(\mathbb{G}_h^{\gamma_1,\gamma_2})^{\perp} w\|_{\Gamma_h} \leq C\|w\|_{k+1} h^{k+1}. \tag{28}$$

  - The proof line is very similar as that for the 1d case.
  - Discuss each term on the RHS of the following decomposition

$$\mathbb{G}_h^{\gamma_1,\gamma_2} w - \mathbb{G}_h^{\frac{1}{2},\frac{1}{2}} w = E_x + E_y + E_{xy},$$

    through the matrix analysis.

- However, the 2d GGR projection can not completely eliminate the errors emerged in the interior or on the boundary of every element.

Superconvergence property

- Fortunately, there holds the following superconvergence property.

Lemma 3.2 ($\theta_1 \neq 1/2$ and $\theta_2 \neq 1/2$)

Let $\ell = 1, 2$ *and* $U \in H^{k+2}(\Omega_h) \cap C(\bar{\Omega})$. *For any* $v \in V_h$, *there holds*

$$\left| \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} H_{ij}^{\ell,\theta_\ell} \left(U - \mathbb{G}_h^{\theta_1,\theta_2} U, v \right) \right| \leq Ch^{k+1} \|U\|_{H^{k+2}(\Omega_h)} \|v\|,$$

on the quasi-uniform Cartesian mesh, where (for example)

$$\mathcal{H}_{ij}^{1,\theta_1}(w, z) = \int_{K_{ij}} w \frac{\partial z}{\partial x} \, \mathrm{d}x \, \mathrm{d}y - \int_{J_j} \left[(w^{\theta_1,y} z^-)_{i+\frac{1}{2},y} - (w^{\theta_1,y} z^+)_{i-\frac{1}{2},y} \right] \mathrm{d}y.$$

- By this result, it is easy to prove Theorem 3.1.

Proof of Lemma 3.2

- Main points:
  - New representation of LHS
  - Kernel space of new representation
  - Rough boundedness
- More clear statements are given in this talk.
- References:

P. Castillo and B. Cockburn, Math. Comp., **71**, 455-478 (2002)

X. Meng, C. -W. Shu and B. Y. Wu, Math. Comp., **85**, 1225-1261 (2016)

Y. Cheng, X. Meng and Q. Zhang, Math. Comp., **86**, 1233-1267 (2017)

- For notational simplicity, below we denote

$$\mathbb{G}_h^{\theta_1,\theta_2} = \mathbb{G}_h, \quad \mathbb{G}_h^{\theta_1} = \mathbb{X}_h, \quad \mathbb{G}_h^{\theta_2} = \mathbb{Y}_h.$$

Step1: new representation of LHS

Proposition 3.1

If the function U is continuous everywhere, then

$$\{\!\{\mathbb{G}_h U\}\!\}^{\theta_1,y}_{i+\frac{1}{2},y} = \mathbb{Y}_h U(x_{i+\frac{1}{2}},\cdot), \quad \{\!\{\mathbb{G}_h U\}\!\}^{x,\theta_2}_{x,j+\frac{1}{2}} = \mathbb{X}_h U(\cdot, y_{j+\frac{1}{2}}).$$

- Take the edge $x = x_{i+1/2}$ as an example. Two items of the 2d GGR projection imply for $\forall j$ that

$$\int_{J_j} \{\!\{\mathbb{G}_h U\}\!\}^{\theta_1,y}_{i+\frac{1}{2},y} v \,\mathrm{d}y = \int_{J_j} U(x_{i+\frac{1}{2}},\cdot) v \,\mathrm{d}y, \quad \forall v \in \mathcal{P}^{k-1}(J_j),$$

$$\{\!\{\mathbb{G}_h U\}\!\}^{\theta_1,\theta_2}_{i+\frac{1}{2},j+\frac{1}{2}} = U_{i+\frac{1}{2},j+\frac{1}{2}}.$$

- The 1d GGR projection implies for any j that

$$\int_{J_j} \mathbb{Y}_h U(x_{i+\frac{1}{2}},\cdot) v \,\mathrm{d}x = \int_{J_j} U(x_{i+\frac{1}{2}},\cdot) v \,\mathrm{d}y, \quad \forall v \in \mathcal{P}^{k-1}(J_j),$$

$$\{\!\{\mathbb{Y}_h U(x_{i+\frac{1}{2}},\cdot)\}\!\}^{x,\theta_2}_{i+\frac{1}{2},j+\frac{1}{2}} = U_{i+\frac{1}{2},j+\frac{1}{2}}.$$

- The uniqueness of 1d GGR projection yields the first conclusion.

Step 1: new representation of LHS

- Hence, for any continuous function U, we have

$$
\begin{aligned}
&\mathcal{H}_{ij}^{1,\theta_1}(\mathbb{G}_h^{\perp}U, v)\\
&= \int_{K_{ij}} \mathbb{G}_h^{\perp}U v_x \,\mathrm{d}x\,\mathrm{d}y - \int_{J_j} (\mathbb{Y}_h^{\perp}U v^-)_{i+\frac{1}{2},y}\,\mathrm{d}y + \int_{J_j} (\mathbb{Y}_h^{\perp}U v^+)_{i-\frac{1}{2},y}\,\mathrm{d}y\\
&= \int_{K_{ij}} \mathbb{G}_h^{\perp}U v_x \,\mathrm{d}x\,\mathrm{d}y - \int_{J_j} (\mathbb{Y}_h^{\perp}U^- v^-)_{i+\frac{1}{2},y}\,\mathrm{d}y + \int_{J_j} (\mathbb{Y}_h^{\perp}U^+ v^+)_{i-\frac{1}{2},y}\,\mathrm{d}y\\
&\equiv \mathcal{E}_{ij}^{1}(U, v),
\end{aligned} \tag{29}
$$

and similarly have

$$
\begin{aligned}
&\mathcal{H}_{ij}^{2,\theta_2}(\mathbb{G}_h^{\perp}U, v)\\
&= \int_{K_{ij}} \mathbb{G}_h^{\perp}U v_y \,\mathrm{d}x\,\mathrm{d}y - \int_{I_i} (\mathbb{X}_h^{\perp}U^- v^-)_{x,j+\frac{1}{2}}\,\mathrm{d}x + \int_{I_i} (\mathbb{X}_h^{\perp}U^+ v^+)_{x,j-\frac{1}{2}}\,\mathrm{d}x\\
&\equiv \mathcal{E}_{ij}^{2}(U, v).
\end{aligned} \tag{30}
$$

- Extent the above representations to broken Sobolev space.

Step 2: kernel space of new representation

Proposition 3.2

Note that $\Omega_h = I_h \times J_h$. *For any* $w \in \mathcal{P}^{k+1}(\Omega_h)$, *there holds*

$$\mathcal{E}_{ij}^{\ell}(w, v) = 0, \quad \forall v \in V_h = \mathcal{Q}^k. \tag{31}$$

- Start the proof from the special function

$$w = p(x)q(y),$$

where $p(x) \in H^1(I_h)$ and $q(y) \in H^1(J_h)$. The definitions imply that

$$\mathbb{G}_h w(x, y) = \mathbb{X}_h p(x) \cdot \mathbb{Y}_h q(y). \tag{32}$$

- Furthermore, it is easy to see have

$$\mathbb{X}_h^{\perp} w(x, y^{\pm}) = \mathbb{X}_h^{\perp} p(x) \cdot q(y^{\pm}), \tag{33a}$$

$$\mathbb{Y}_h^{\perp} w(x^{\pm}, y) = p(x^{\pm}) \cdot \mathbb{Y}_h^{\perp} q(y). \tag{33b}$$

Step 2: kernel space of new representation

- Now additionally assume $q(y) \in \mathcal{P}^k(J_h)$. Then (32) implies

$$\mathbb{G}_h^{\perp} w = \mathbb{X}_h^{\perp} p(x) \cdot q(y). \tag{34}$$

- It follows from (33b) that $\mathbb{Y}_h^{\perp} w(x_{i\mp 1/2}^{\pm}, y) = 0$. Together with (34), we have

$$\mathcal{E}_{ij}^1(w, v) = 0, \quad \forall v \in V_h. \tag{35}$$

  since three terms are all equal to zero.

- An application of integration by part along y-direction, together with (34) and (33a), yield

$$\mathcal{E}_{ij}^2(w, v) = -\int_{K_{ij}} \mathbb{X}^{\perp} p(x) q_y(y) v(x, y) \, \mathrm{d}x \, \mathrm{d}y, \quad \forall v \in V_h. \tag{36}$$

Step 2: kernel space of new representation

- It is easy to see that (31) holds for any $w \in \mathcal{Q}^k(\Omega_h)$, and hence we just need to verify it for two kinds of functions

$$w = L_{i',k+1}(x)\mathbf{1}_K, \quad w = L_{j',k+1}(y)\mathbf{1}_K,$$

where $K = K_{i'j'}$ goes through all elements.

- Take the first type as an example. It is easy to see that

$$w(x,y) = p(x)q(y),$$

with the separation

$$p(x) = L_{i',k+1}(x)\mathbf{1}_{I_{i'}} \in H^1(I_h), \quad q(y) = \mathbf{1}_{J_{j'}} \in P^k(J_h).$$

As a result, we can get (31) by using (35) and (36).

- Now we can compete the proof of Proposition 3.2.

Step 3: rough boundedness

- Assume $z \in H^1(\Omega_h)$.
- Using the inverse inequality and the approximations of GGR projections, and we have

$$\begin{aligned}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\mathcal{E}_{ij}^{\ell}(z,v) &\le Ch\|z\|_{H^1(\Omega_h)}\|\nabla v\|_{H^1(\Omega_h)} + Ch\|z^{\pm}\|_{H^1(\Gamma_h)}\|v\|_{\Gamma_h} \\ &\le C\|z\|_{H^1(\Omega_h)}\|v\| + Ch\|z\|_{H^2(\Omega_h)}\|v\|.\end{aligned} \tag{37}$$

- Here we have used the trace inequality in each element

$$\|z\|_{H^1(\partial K_{ij})} \le C\|z\|_{H^1(K_{ij})}^{\frac{1}{2}}\|z\|_{H^2(K_{ij})}^{\frac{1}{2}} \le Ch^{-\frac{1}{2}}\|z\|_{H^1(K_{ij})} + Ch^{\frac{1}{2}}\|z\|_{H^2(K_{ij})},$$

where the bounding constant C is independent of h and K_{ij}.

Final proof of Lemma 3.2

- Summing up the above three conclusions, we have

$$
\begin{aligned}
&\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\mathcal{H}_{ij}^{\ell,\theta_\ell}(U-\mathbb{G}_h^{\theta_1,\theta_2}U,v)\\
=&\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\mathcal{E}_{ij}^{\ell}(U,v)\\
=&\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\mathcal{E}_{ij}^{\ell}\Big(U-w,v\Big)\\
\le& C\|U-w\|_{H^1(\Omega_h)}\|v\|+Ch\|U-w\|_{H^2(\Omega_h)}\|v\|,
\end{aligned}
$$

  for any $w\in\mathcal{P}^{k+1}(\Omega_h)$.
- Now we can complete the proof by using the simple approximation property.

Outline

Convection diffusion equation

Consider the 1d linear constant convection diffusion equation

$$U_t + cU_x = dU_{xx} + f(x,t), \quad (x,t) \in (0,1) \times (0,T], \tag{38}$$

equipped with the periodic boundary condition. Here $d \geq 0$ and assume $c \geq 0$ for simplicity.

- Introduce the auxiliary variable $P = \sqrt{d}U_x$, and U is called the prime variable.
- Consider the equivalent first-order system

$$\frac{\partial U}{\partial t} + \frac{\partial h_u}{\partial x} = f, \qquad P + \frac{\partial h_p}{\partial x} = 0,$$

with the physical flux

$$(h_u, h_p) = (cU - \sqrt{d}P, -\sqrt{d}U).$$

Semi-discrete LDG scheme

- The semi-discrete scheme is defined in each element: find u and p in the finite element space V_h, such that

$$\int_{I_i} \frac{\partial u}{\partial t} v \mathrm{d}x - \int_{I_i} h_u \frac{\partial v}{\partial x} \mathrm{d}x + (\hat{h}_u v^-)_{i+\frac{1}{2}} - (\hat{h}_u v^+)_{i-\frac{1}{2}} = \int_{I_i} f v \mathrm{d}x, \tag{39a}$$

$$\int_{I_i} p r \mathrm{d}x - \int_{I_i} h_p \frac{\partial r}{\partial x} \mathrm{d}x + (\hat{h}_p r^-)_{i+\frac{1}{2}} - (\hat{h}_p r^+)_{i-\frac{1}{2}} = 0, \tag{39b}$$

  hold for any $i = 1, 2, \ldots, N$ and for any test function $(v, r) \in V_h \times V_h$.

- The generalized alternating numerical fluxes is defined as

$$(\hat{h}_u, \hat{h}_p) = (c\{\!\{u\}\!\}^{(\theta)} - \sqrt{d}\{\!\{p\}\!\}^{(\widetilde{\gamma})}, -\sqrt{d}\{\!\{u\}\!\}^{(\gamma)}), \tag{40}$$

  where θ and γ are two given parameters.

- Assume in addition $\theta \geq \frac{1}{2}$ for upwind-biased.
  - $\gamma = 1$: the purely alternating numerical flux is used for diffusion.
  - $\theta = 1$: the purely upwind flux is used for convection.

Stability

Theorem 4.1 (stability)

The semi-discrete LDG scheme is stable in the L^2-norm, namely

$$\|u(T)\| \leq \|u(0)\| + \int_0^T \|f(t)\| \mathrm{d}t. \tag{41}$$

- The proof is trivial and standard.
- Note that the jumps of u and p do not provide for diffusion any stability contribution.

Proof of Theorem 4.1

- Adding up two equations in (39) and summing them over all elements, the LDG scheme (39) can be written in the form:

$$\int_\Omega \frac{\partial u_h}{\partial t} v \mathrm{d}x + G_h(u,p;v,r) = \int_\Omega f v \mathrm{d}x \tag{42}$$

for any test function $(v, r) \in V_h \times V_h$, where

$$\begin{aligned} & G_h(u,p;v,r) \\ = & \int_\Omega pr \mathrm{d}x + \sum_{i=1}^{N} \left\{ -c\mathcal{H}_i^{(\theta)}(u,v) + \sqrt{d}\mathcal{H}_i^{(\widetilde{\theta})}(p,v) + \sqrt{d}\mathcal{H}_i^{(\theta)}(u,r) \right\} \end{aligned} \tag{43}$$

with the locally-defined functional for the given parameter α,

$$\mathcal{H}_i^{(\alpha)}(w,z) = \int_{I_i} w \frac{\partial z}{\partial x} \mathrm{d}x - \{\!\{w\}\!\}_{i+\frac{1}{2}}^{(\alpha)} z_{i+\frac{1}{2}}^{-} + \{\!\{w\}\!\}_{i-\frac{1}{2}}^{(\alpha)} z_{i-\frac{1}{2}}^{+}. \tag{44}$$

Proof of Theorem 4.1

- Using the negative semidefinite property and the skew-symmetric property, we have the following identities:

$$G_h(u,p;u,p) = \|p\|^2 + c\left(\theta - \frac{1}{2}\right)\|[\![u_h]\!]\|^2_{\Gamma_h}. \tag{45}$$

- Taking the test function $(v,r) = (u,p)$ in (42), and using (45) and Cauchy–Schwarz inequality, we can easily get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \le \|f\|\|u\|. \tag{46}$$

- The L^2-norm stability (41) immediately follows by canceling $\|u\|$ on both sides and integrating the inequality with respect to the time.

Error estimate

Theorem 4.2 (error estimate)

Assume $u \in L^{\infty}(H^{k+1}) \cap L^2(H^{k+2})$ *and* $\frac{\partial u}{\partial t} \in L^2(H^{k+1})$*, then the LDG scheme satisfies the optimal and uniform error estimate*

$$\|u(T) - u_h(T)\| \leq C(1+T)\left[h^{k+1} + \sqrt{c}\min\left(\frac{\sqrt{ch}}{\sqrt{d}}, \frac{\sqrt{d}}{\sqrt{ch}}, 1\right)|\gamma - \theta| h^{k+\frac{1}{2}}\right],$$

if the initial solution is good enough to ensure

$$\|U(0) - u(0)\| \leq Ch^{k+1}.$$

Here the bounding constant $C > 0$ *is independent of mesh size* h *and the reciprocal of the diffusion coefficient* d*.*

- A nice application of the GGR projection;
- Do not adopt the dual technique or the elliptic projection.
- Easily extended to 2d problem, by using the 2d GGR projection with the modifications similar as below.

Proof line

- Denote the error with the decomposition

$$e_u = u - u = (u - \chi_u) - (u - \chi_u) \equiv \eta_u - \xi_u, \tag{47a}$$
$$e_p = p - p = (p - \chi_p) - (p - \chi_p) \equiv \eta_p - \xi_p, \tag{47b}$$

where χ_u and χ_p are the element in V_h.
- The energy identity is given as

$$\int_\Omega \frac{\partial \xi_u}{\partial t}\xi_u \mathrm{d}x + G_h(\xi_u, \xi_p; \xi_u, \xi_p) = \int_\Omega \frac{\partial \eta_u}{\partial t}\xi_u \mathrm{d}x + G_h(\eta_u, \eta_p; \xi_u, \xi_p) \tag{48}$$

where $G_h(\cdot, \cdot; \cdot, \cdot)$ are the LDG space discretization, and the left-hand side satisfies the following inequality

$$\text{LHS} \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \|\xi_p\|^2 + c\left(\theta - \frac{1}{2}\right)\sum_{i=1}^{N}[\![\xi_u]\!]^2_{i+\frac{1}{2}}. \tag{49}$$

Proof line

- The rest work is to establish the optimal boundedness for the right-hand side of (48), with

$$
\begin{aligned}
& G_h(\eta_u, \eta_p; v, r) \\
= & \int_\Omega \eta_p r \mathrm{d}x + \sum_{i=1}^{N} \left\{ -c\mathcal{H}_i^{(\theta)}(\eta_u, v) + \sqrt{d}\mathcal{H}_i^{(\widetilde{\gamma})}(\eta_p, v) + \sqrt{d}\mathcal{H}_i^{(\gamma)}(\eta_u, r) \right\}.
\end{aligned} \tag{50}
$$

- Below we would like to adopt the GGR projection to simultaneously eliminate the projection error in

$$
\mathcal{H}_i^{(\alpha)}(w, z) = \int_{I_i} w \frac{\partial z}{\partial x} \mathrm{d}x - \{\!\{w\}\!\}_{i+\frac{1}{2}}^{(\alpha)} z_{i+\frac{1}{2}}^{-} + \{\!\{w\}\!\}_{i-\frac{1}{2}}^{(\alpha)} z_{i-\frac{1}{2}}^{+},
$$

with $w = \eta_u, \eta_p$ and $\alpha = \theta, \gamma$.

Proof when parameters are the same

- Let $\chi_u = \mathbb{G}_h^{\theta} U$ and $\chi_p = \mathbb{G}_h^{\widetilde{\theta}} P$. Since $\theta = \gamma$, the GGR projection implies

$$\mathcal{H}_i^{(\theta)}(\eta_u, \xi_u) = 0, \quad \mathcal{H}_i^{(\widetilde{\theta})}(\eta_p, \xi_u) = 0, \quad \mathcal{H}_i^{(\theta)}(\eta_u, \xi_p) = 0. \tag{51}$$

- Hence, due to the approximation property of GGR projection, we have

$$\begin{aligned}\text{RHS} &= \int_\Omega \frac{\partial \eta_u}{\partial t} \xi_u \mathrm{d}x + G_h(\eta_u, \eta_p; \xi_u, \xi_p) = \int_\Omega \frac{\partial \eta_u}{\partial t} \xi_u \mathrm{d}x + \int_\Omega \eta_p \xi_p \mathrm{d}x \\ &\leq Ch^{k+1} \left\| \frac{\partial u}{\partial t} \right\|_{H^{k+1}} \|\xi_u\| + Ch^{k+1} \|p\|_{H^{k+1}} \|\xi_p\|,\end{aligned}$$

- It follows from the energy equation (48) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|^2 + \|\xi_p\|^2 + c\left(\theta - \frac{1}{2}\right)\sum_{i=1}^{N} [\![\xi_u]\!]_{i+\frac{1}{2}}^2 \\ &\qquad \leq Ch^{k+1} \left\| \frac{\partial u}{\partial t} \right\|_{H^{k+1}} \|\xi_u\| + Ch^{k+1} \|p\|_{H^{k+1}} \|\xi_p\|.\end{aligned}$$

- Application of the Gronwall's inequality can complete the proof.

Proof when parameters are not the same (cont.)

- If $\theta \neq \gamma$, the above treatment can not eliminate completely the projection errors at the same time. For example, it follows from (51) that

$$\mathcal{H}_i^{(\gamma)}(\eta_u, \xi_p) \neq \mathcal{H}_i^{(\theta)}(\eta_u, \xi_p) = 0.$$

- Below we consider three treatments!
- If we want to eliminate completely the boundary errors coming from the diffusion part, we can define

$$\eta_u = U - \mathbb{G}_h^{\gamma} U \quad \text{and} \quad \eta_p = P - \mathbb{G}_h^{\widetilde{\gamma}} P,$$

  Along the same line as before, we can obtain the error estimate

$$\|u(T) - u(T)\| \leq C(1+T)\left[h^{k+1} + \sqrt{c}|\gamma - \theta| h^{k+\frac{1}{2}}\right], \tag{52}$$

Proof when parameters are not the same (cont.)

- If we want to eliminate completely the boundary errors coming from the convection part, we can define

$$\eta_u = U - \mathbb{G}_h^{\theta} U \quad \text{and} \quad \eta_p = P - \mathbb{G}_h^{\widetilde{\theta}} P.$$

- Along the same analysis as before, we can easily see that the RHS of error equation has a new term

$$\sum_{i=1}^{N} \sqrt{d}(\gamma - \theta)(\llbracket \eta_p \rrbracket \llbracket \xi_u \rrbracket - \llbracket \eta_u \rrbracket \llbracket \xi_p \rrbracket)_{i+\frac{1}{2}}.$$

- This term can be bounded by the stability and the approximation property, with the help of Young's inequality and the inverse inequality

$$\sum_{i=1}^{N} \llbracket \xi_p \rrbracket_{i+\frac{1}{2}}^2 \leq Ch^{-1} \|\xi_p\|^2.$$

- An application of the Cauchy–Schwarz inequality and the Gronwall inequality yields the error estimate

$$\|u(T) - u(T)\| \leq C(1+T) \left[h^{k+1} + \sqrt{d} |\gamma - \theta| h^k \right], \tag{53}$$

Proof when parameters are not the same (cont.)

- A new GGR projection is needed!

Definition 3

For any vector-valued function $z = (z_u, z_p) \in [C(\bar{\Omega}_h)]^2$, define

$$\mathbb{Q}_h^{\theta,\gamma}(z_u, z_p) = (\mathbb{G}_h^{\gamma} z_u, \mathbb{G}_h^{\widetilde{\gamma},\star} z_p) \in V_h \times V_h, \tag{54}$$

where

- $\mathbb{G}_h^{\gamma} z_u$ is the same as before, and
- $\mathbb{G}_h^{\widetilde{\gamma},\star} z_p$ depends on both z_p and z_u. For any $i = 1, \ldots, N$, there hold

$$\int_{I_i} (\mathbb{G}_h^{\widetilde{\gamma},\star} z_p) v \mathrm{d}x = \int_{I_i} z_p v \mathrm{d}x, \quad \forall v \in P^{k-1}(I_i), \tag{55a}$$

$$\{\!\{\mathbb{G}_h^{\widetilde{\gamma},\star} z_p\}\!\}_{i+\frac{1}{2}}^{(\widetilde{\gamma})} = \{\!\{z_p^{(\widetilde{\gamma})}\}\!\}_{i+\frac{1}{2}} - \frac{c}{\sqrt{d}}(\gamma - \theta)[\![z_u - \mathbb{G}_h^{\gamma} z_u]\!]_{i+\frac{1}{2}}. \tag{55b}$$

- Note that $\mathbb{G}_h^{\widetilde{\gamma},\star} z_p = \mathbb{G}_h^{\widetilde{\gamma}} z_p$ if $\gamma = \theta$.

Proof when parameters are not the same

- Similarly, we can derive the unique existence and

$$\|z_p - \mathbb{G}_h^{\widetilde{\gamma},\star} z_p\| \leq C h^{k+1} \left(\|z_p\|_{H^{k+1}(\Omega_h)} + \frac{c}{\sqrt{d}} |\gamma - \theta| \cdot \|z_u\|_{H^{k+1}(\Omega_h)} \right),$$

  since $z_u - \mathbb{G}_h^{\gamma} z_u$ is already known to be of order h^{k+1}.

- In order that projection errors on the element boundaries are eliminated completely and simultaneously, we define

$$\eta_u = U - \mathbb{G}_h^{\gamma} U, \quad \eta_p = P - \mathbb{G}_h^{\widetilde{\gamma},\star} P,$$

  which yields

$$G_h(\eta_u, \eta_p; \xi_u, \xi_p) = \int_\Omega \eta_p \xi_p \mathrm{d}x.$$

- Repeating the similar arguments as before, we can obtain

$$\|u(T) - u(T)\| \leq C(1+T) \left(1 + \frac{c}{\sqrt{d}} |\gamma - \theta| \right) h^{k+1}. \tag{56}$$

- The proof is completed by (52), (53) and (56).

Outline

Concluding remarks

- Stability analysis and optimal error estimates in L^2-norm are given in this talk for the DG/LDG method.
- The good stability comes from the numerical viscosity provided by the square of jumps on the element interface.
- In general, the strength of numerical viscosity is measured by

$$\alpha(\hat{f}; u^-, u^+) = \begin{cases} \dfrac{f(\{\!\{u\}\!\}) - \hat{f}(u^-, u^+)}{[\![u]\!]}, & [\![u]\!] \neq 0, \\ \frac{1}{2}|f'(\{\!\{u\}\!\})|, & \text{otherwise.} \end{cases}$$

References about this issue:

Q. Zhang and C. -W. Shu, SINUM 42(2004), 641-666.

J. Luo, C. -W. Shu and Q. Zhang. ESAIM 49(2015), 991-1018

- The GGR projection is good at obtaining the optimal error estimate.

Thanks for your attention!