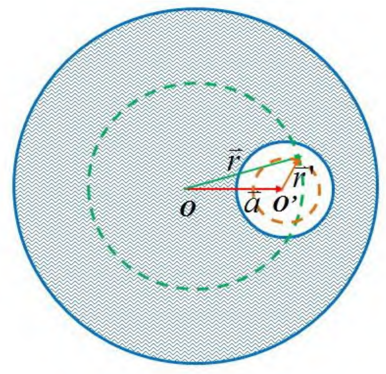
第1章 电力与电场

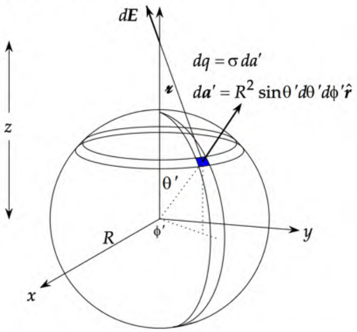
练习

- 均匀带电球壳的电场分布
- 均匀带电实心球的电场分布
- 均匀带电实心球中挖掉一个球的电场分布



Coulomb's Law example: field from a uniformly-charged spherical shell

Griffiths, problem 2.7: What is the electric field a distance z away from the center of a spherical shell with radius R and uniform surface charge density σ?

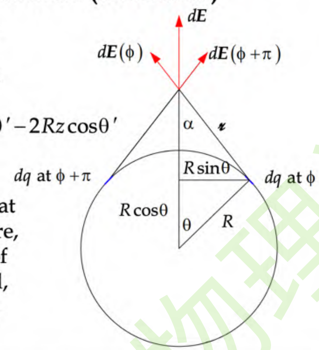


Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

In the plane at azimuth ϕ, it can be seen more easily that

$$\mathscr{r}^2 = R^2\sin^2\theta' + (z - R\cos\theta')^2$$
$$= R^2\sin^2\theta' + z^2 + R^2\cos^2\theta' - 2Rz\cos\theta'$$
$$= R^2 + z^2 - 2Rz\cos\theta'$$

Consider two area elements at azimuth ϕ and $\phi+\pi$ as before, the horizontal components of their contribution to E cancel, and the vertical components add.



Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

So $\;d\mathbf{E} = \hat{z}2\dfrac{dq}{\mathscr{r}^2}\cos\alpha$

$$= \hat{z}2\frac{\sigma R^2\sin\theta' d\theta' d\phi'}{R^2+z^2-2Rz\cos\theta'}\frac{z-R\cos\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= \hat{z}2\sigma R^2\frac{\sin\theta'(z-R\cos\theta')}{\left(R^2+z^2-2Rz\cos\theta'\right)^{3/2}}d\theta' d\phi'$$

$$\mathbf{E} = \hat{z}2\sigma R^2\int_0^{\pi}d\phi'\int_0^{\pi}\frac{\sin\theta'(z-R\cos\theta')}{\left(R^2+z^2-2Rz\cos\theta'\right)^{3/2}}d\theta'$$

The first integral is trivial: it just comes out to π.

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

For the second, substitute

$$w = \cos\theta', \quad dw = -\sin\theta' d\theta', \quad w = 1 \to -1:$$

$$\mathbf{E} = \hat{z}2\pi\sigma R^2\int_{-1}^{1}\frac{(z-Rw)}{\left(R^2+z^2-2Rzw\right)^{3/2}}dw$$

Break this integral in two. For the first one, substitute

$$u = R^2+z^2-2Rzw, \quad du = -2Rzdw,$$
$$u = R^2+z^2+2Rz \to R^2+z^2-2Rz$$

$$z\int_{-1}^{1}\frac{1}{\left(R^2+z^2-2Rzw\right)^{3/2}}dw = \frac{1}{2R}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}u^{-3/2}du$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\frac{1}{2R}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}u^{-3/2}du = \frac{1}{2R}\left[-2u^{-1/2}\right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}$$

$$= \frac{1}{R}\left[\frac{1}{\sqrt{R^2+z^2-2Rz}} - \frac{1}{\sqrt{R^2+z^2+2Rz}}\right]$$

The second half of the integral needs to be done by parts. Take

$$u = w \qquad dv = \frac{-Rdw}{\left(R^2+z^2-2Rzw\right)^{-3/2}}$$

$$du = dw \qquad v = \frac{1}{z}\frac{1}{\sqrt{R^2+z^2-2Rzw}} \quad \text{(as we just saw)}$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\int_C u\,dv = uv\big|_C - \int_C v\,du$$

$$\int_{-1}^{1} \frac{-Rw}{\left(R^2+z^2-2Rzw\right)^{3/2}}\,dw = \frac{1}{z}\left.\frac{w}{\sqrt{R^2+z^2-2Rzw}}\right|_{-1}^{1} - \frac{1}{z}\int_{-1}^{1}\frac{dw}{\sqrt{R^2+z^2-2Rzw}}$$

In the last term, use (again)

$$u = R^2+z^2-2Rzw, \quad du = -2Rz\,dw,$$
$$u = R^2+z^2+2Rz \to R^2+z^2-2Rz$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

and it becomes

$$-\frac{1}{z}\int_{-1}^{1}\frac{dw}{\sqrt{R^2+z^2-2Rzw}} = -\frac{1}{2Rz^2}\int_{R^2+z^2-2Rz}^{R^2+z^2+2Rz} u^{-1/2}\,du$$
$$= -\frac{1}{2Rz^2}\left[2\sqrt{u}\right]_{R^2+z^2-2Rz}^{R^2+z^2+2Rz}$$
$$= -\frac{1}{Rz^2}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)$$

So, putting all these terms together (and factoring out $1/z^2$ as we do), we get

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\mathbf{E} = \hat{\mathbf{z}}\frac{2\pi\sigma R^2}{z^2}\Bigg[\frac{z^2}{R}\left(\frac{1}{\sqrt{R^2+z^2-2Rz}} - \frac{1}{\sqrt{R^2+z^2+2Rz}}\right) + z\left(\frac{1}{\sqrt{R^2+z^2-2Rz}} + \frac{1}{\sqrt{R^2+z^2+2Rz}}\right) - \frac{1}{R}\left(\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right)\Bigg]$$

This looks like a mess until you notice that

$$\sqrt{R^2+z^2+2Rz} = \sqrt{(z+R)^2} = |z+R|$$
$$\sqrt{R^2+z^2-2Rz} = \sqrt{(z-R)^2} = |z-R|$$

Positive, since they represent the length of r, which is always positive.

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

$$\mathbf{E} = \hat{\mathbf{z}}\frac{2\pi\sigma R^2}{z^2}\Bigg[\frac{z^2}{R}\left(\frac{1}{|z-R|} - \frac{1}{|z+R|}\right) + z\left(\frac{1}{|z-R|} + \frac{1}{|z+R|}\right) - \frac{1}{R}\left(|z+R| - |z-R|\right)\Bigg]$$

$$= \hat{\mathbf{z}}\frac{2\pi\sigma R^2}{z^2}\Bigg[\frac{z^2}{R}\left(\frac{1}{|z-R|} - \frac{1}{|z+R|}\right) + z\left(\frac{1}{|z-R|} + \frac{1}{|z+R|}\right) - \frac{1}{R}\left(\frac{|z^2-R^2|}{|z-R|} - \frac{|z^2-R^2|}{|z+R|}\right)\Bigg]$$

Coulomb's Law example: field from a uniformly-charged spherical shell (continued)

which gives us, finally,

$$\mathbf{E} = \hat{\mathbf{z}}\frac{2\pi\sigma R^2}{z^2}\left[\frac{z-R}{|z-R|} + \frac{z+R}{|z+R|}\right]$$

Two cases: z larger than, or smaller than, R. (P outside, inside)

- **Larger (outside):**

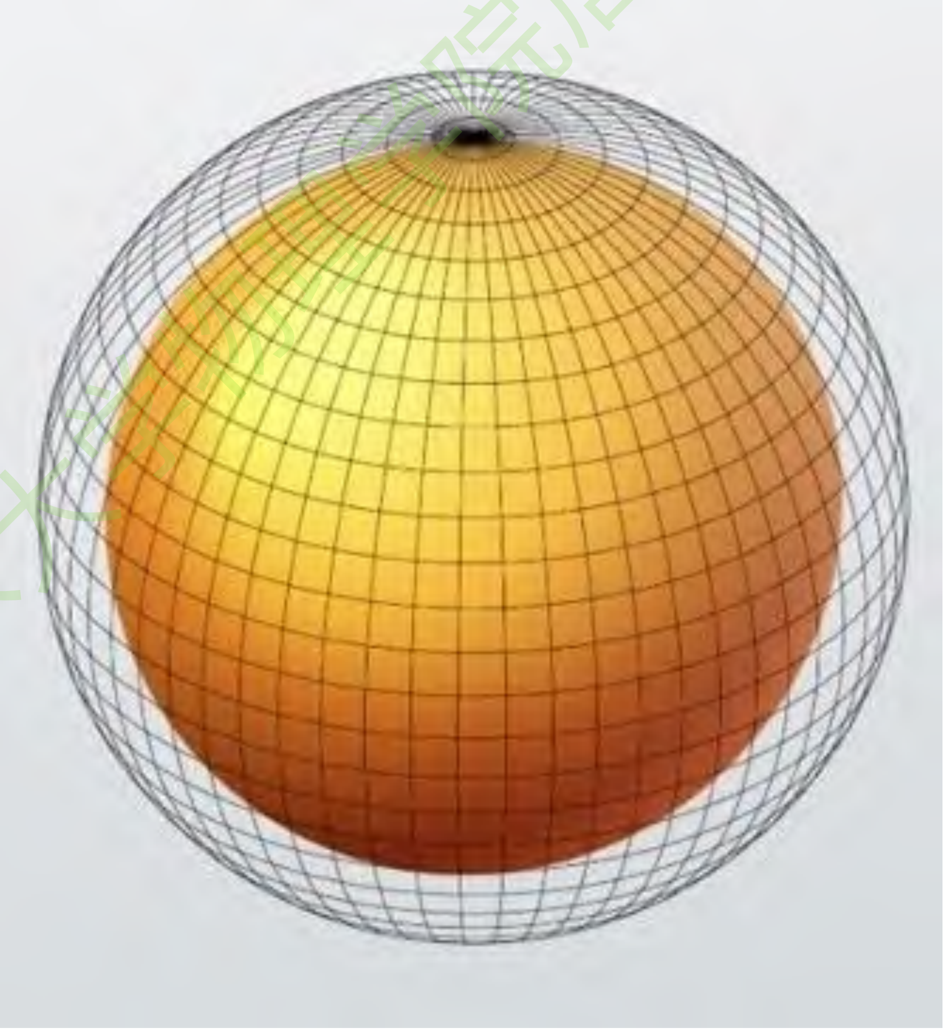
$$\frac{z-R}{|z-R|} = 1 = \frac{z+R}{|z+R|} \quad\Rightarrow\quad \boxed{\mathbf{E} = \hat{\mathbf{z}}\frac{4\pi\sigma R^2}{z^2} = \hat{\mathbf{z}}\frac{Q}{z^2}}$$

Behaves like a point charge at the sphere's center.

- **Smaller (inside):** means $|z-R| = R-z$, so

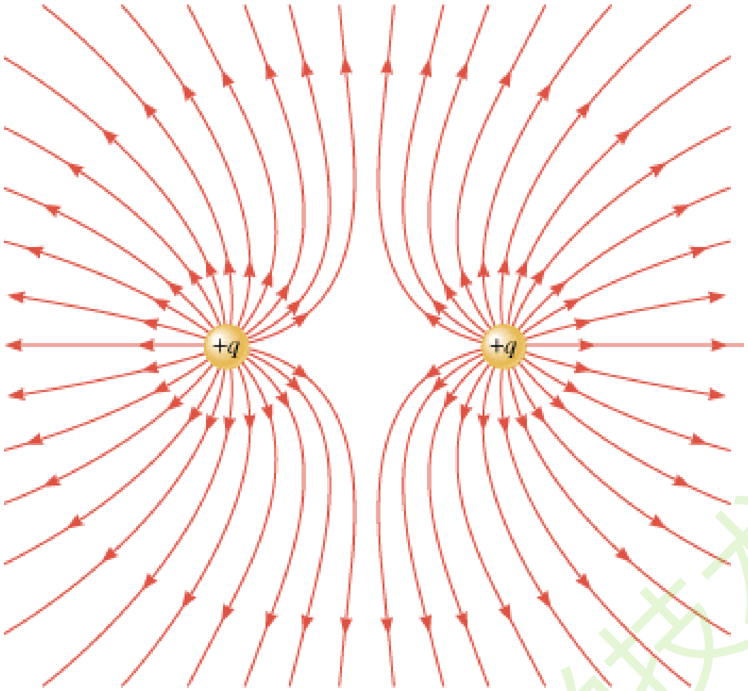
$$\frac{z-R}{R-z} + \frac{z+R}{z+R} = \frac{z^2-R^2+R^2-z^2}{R^2-z^2} = 0 \quad\Rightarrow\quad \boxed{\mathbf{E} = 0}$$

§ 1.4 高斯定理 (Gauss's Law)



§ 1.4.1 电场线与电通量

1. 电场线



用电场线来形象地描述电场：

- 场强**方向**：电场线切线方向
- 场强**大小**：电场线数密度

$$E = \frac{\Delta N}{\Delta S_{\perp}}$$

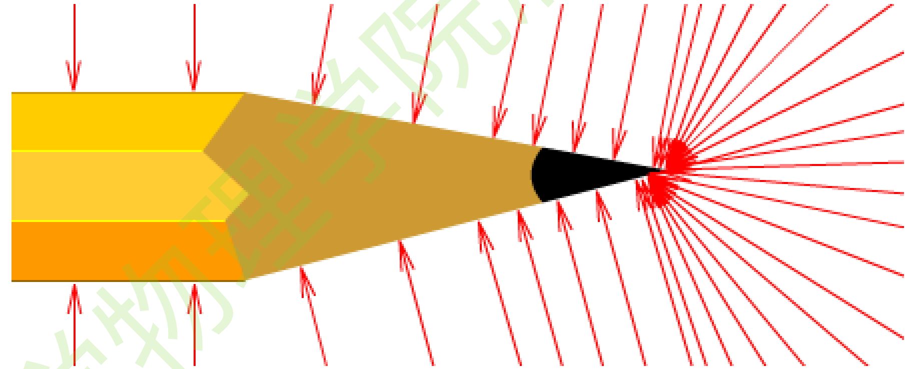
数密度：通过垂直于场强方向的单位面积的电场线条数

电场线密集的地方，电场强度就大

电场线的性质

- 电场线起自正电荷或者无限远
- 电场线终止于负电荷或者无限远
- 若体系正负电荷一样多，则正电荷发出的电场线全部终止于负电荷
- 两条电场线不会相交
- 静电场的电场线不会形成闭合曲线

2. 电通量



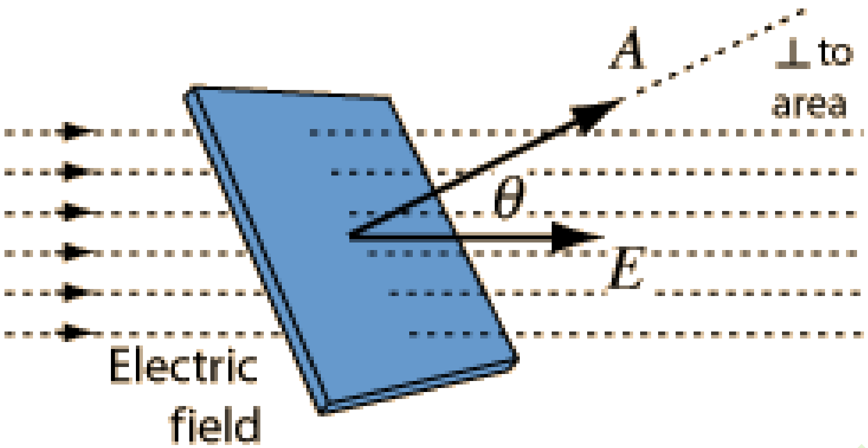
水流量：

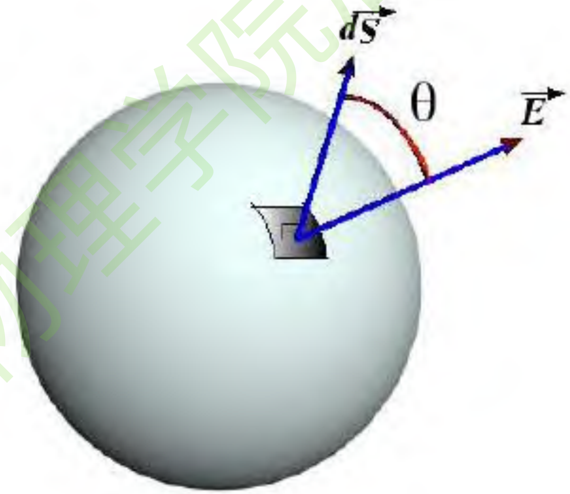
单位时间通过某个曲面的水量

$$\Delta\Phi_v = \vec{v}\Delta S\cos\theta = \vec{v}\cdot\Delta\vec{S}$$

电通量：

通过某个曲面的电场线根数



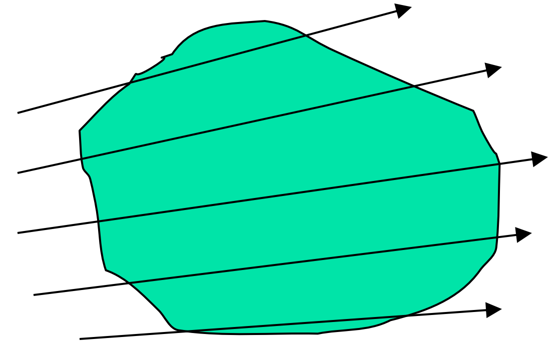


$$E = \frac{\Delta N}{\Delta S_{\perp}}$$

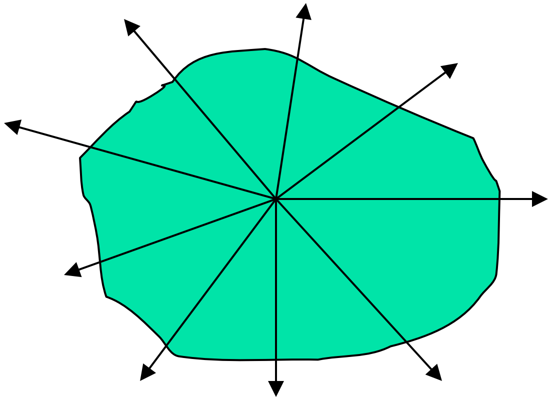
$$\Delta N = E\Delta S_{\perp} = \vec{E} \cdot \Delta\vec{S}$$

$$\Delta\Phi = \vec{E} \cdot \Delta\vec{S}$$

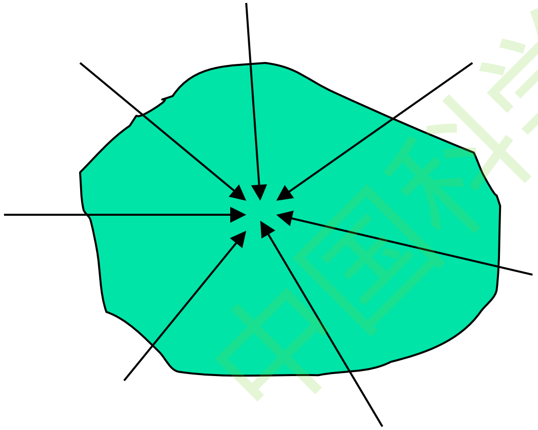
$$\Phi = \iint_S \vec{E} \cdot d\vec{S}$$



流入电通量=流出电通量



类似喷泉，有“源”



类似地漏，有“汇”

电通量的叠加原理

$$\Phi = \iint_S \vec{E} \cdot d\vec{S} \qquad \vec{E} = \sum_i \vec{E}_i$$

电场叠加原理

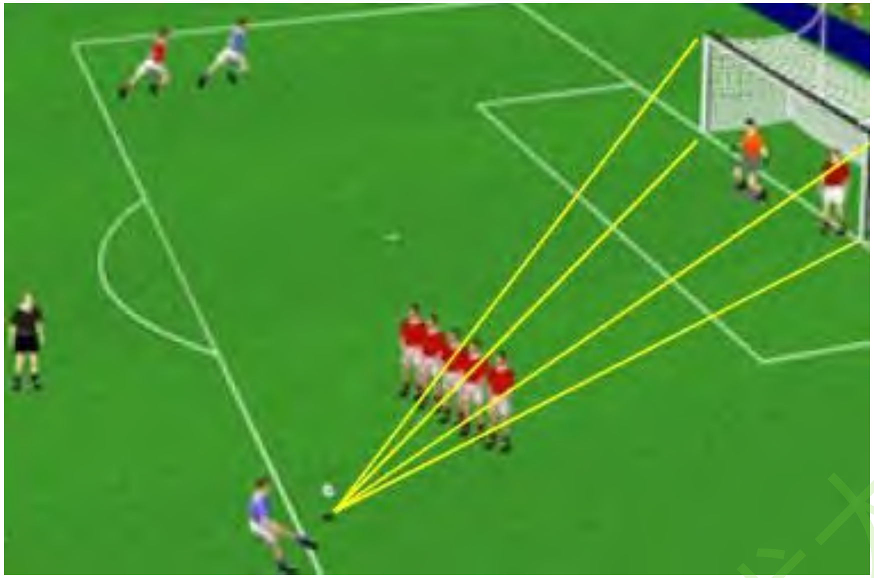
$$\Phi = \iint_S \sum_i \vec{E}_i \cdot d\vec{S}$$

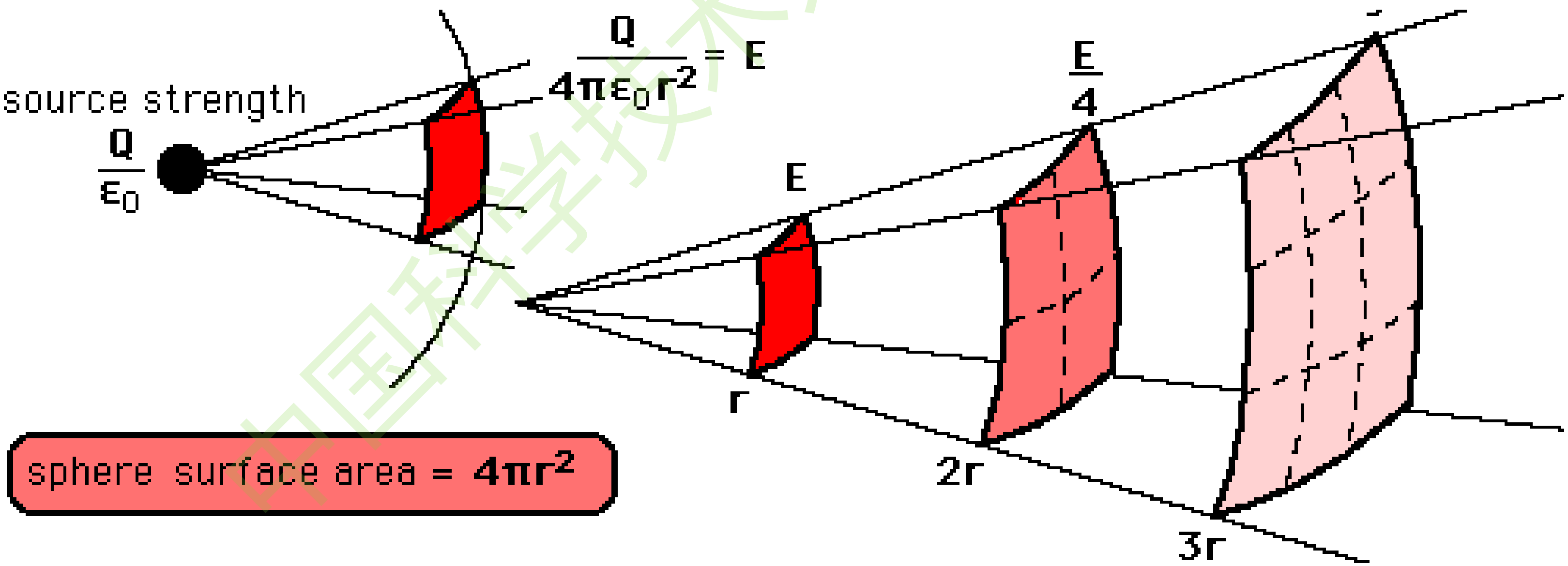
$$= \sum_i \iint_S \vec{E}_i \cdot d\vec{S}$$

$$= \sum_i \Phi_i$$

电通量叠加原理

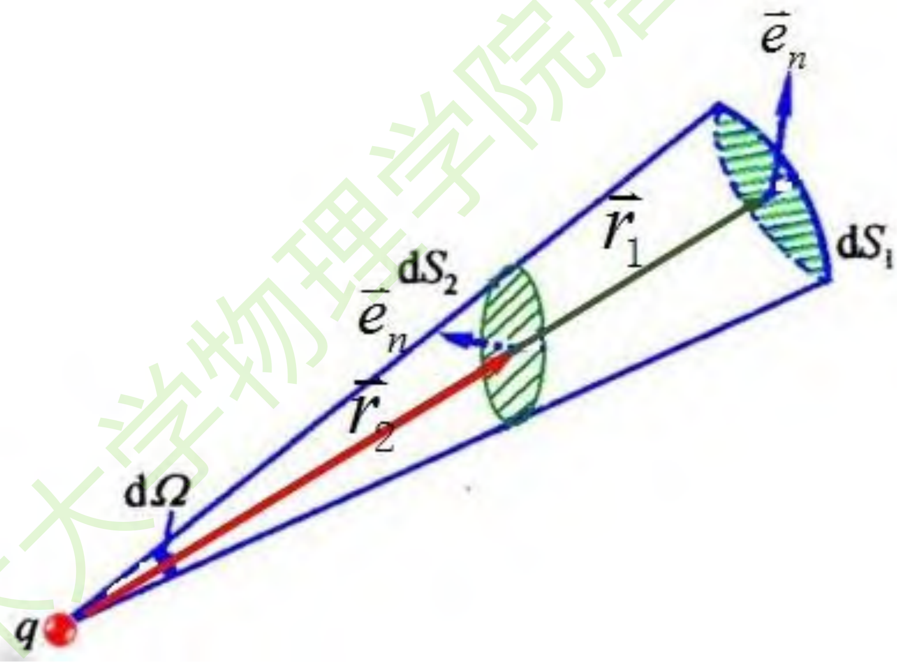
立体角




source strength
$\frac{Q}{\varepsilon_0}$
$\frac{Q}{4\pi\varepsilon_0 r^2} = E$
$\frac{E}{4}$
E
r
2r
3r
sphere surface area = $4\pi r^2$

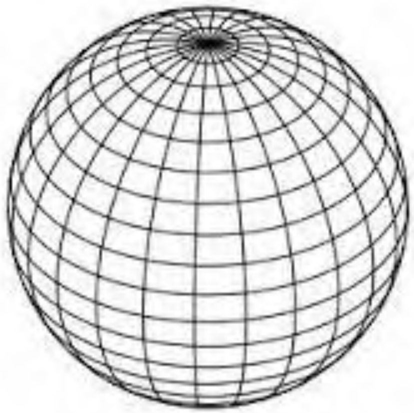
立体角

$$d\Omega = \frac{dS_{\perp}}{r^2} = \frac{\vec{e}_r \cdot d\vec{S}}{r^2}$$



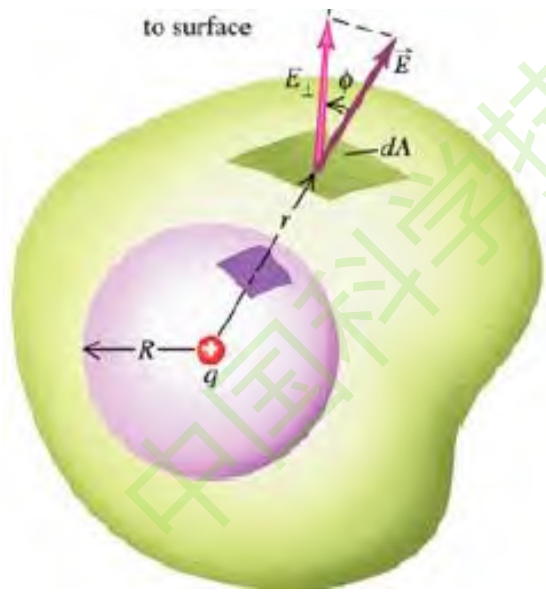
对同一圆锥，不同位置面元所张立体角相等

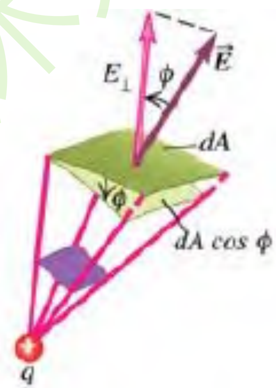
- 球面的立体角



$$\Omega = \oiint_S d\Omega = \frac{\oiint_S \vec{e}_r \cdot d\vec{S}}{r^2} = 4\pi$$

- 任意曲面的立体角



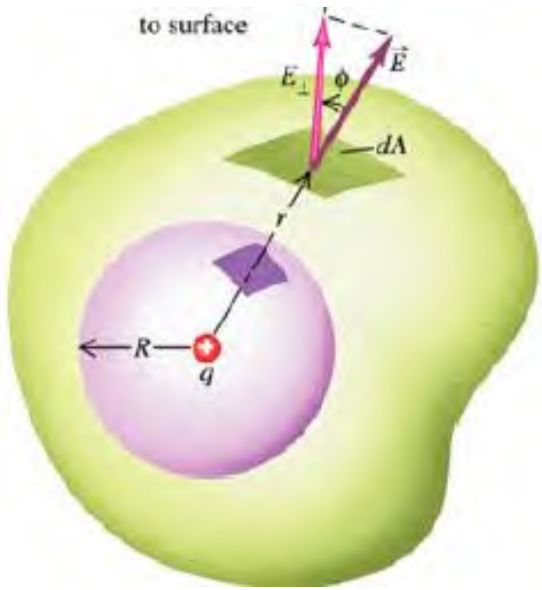


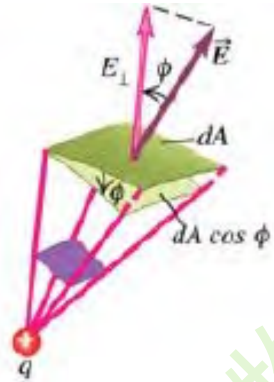
$$\Omega = \oiint_S d\Omega = 4\pi$$

- 点电荷对任意曲面的电通量

$$\Phi = \frac{q}{4\pi\varepsilon_0}\iint_S \frac{\vec{e}_r \cdot d\vec{S}}{r^2} = \frac{q}{4\pi\varepsilon_0}\iint_S d\Omega$$

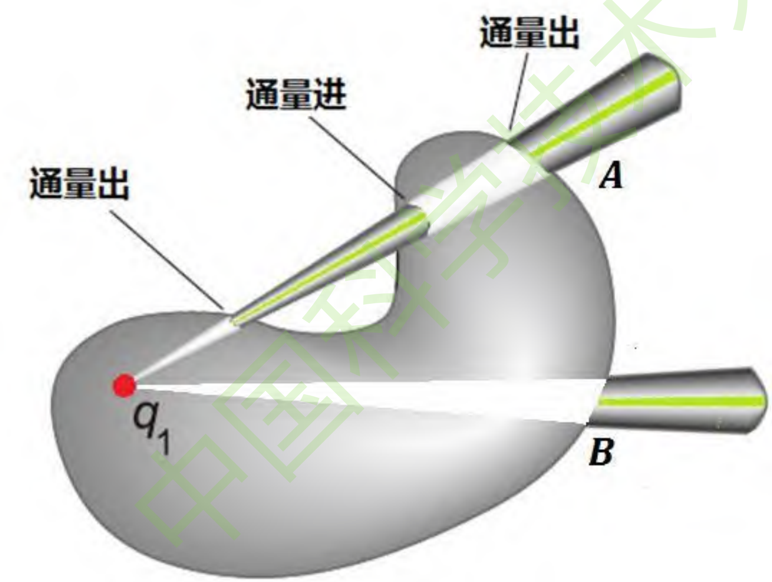
点电荷在封闭曲面内部



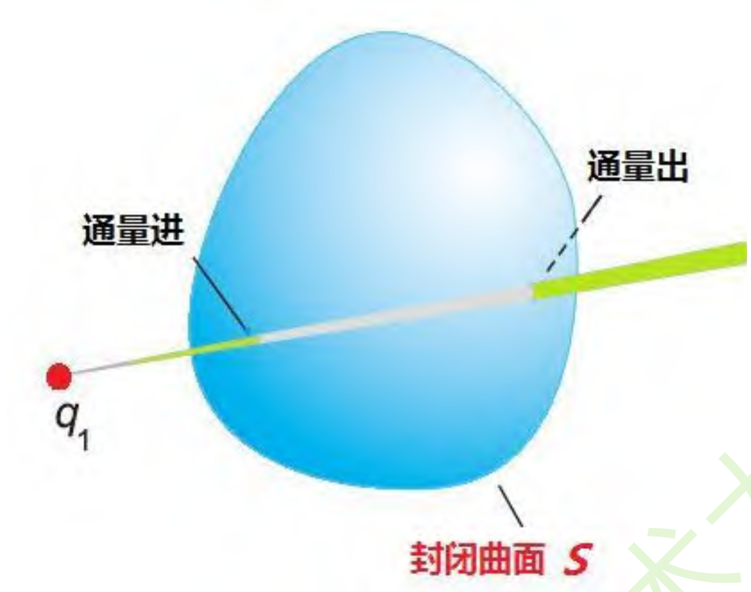


$$\Omega = \oiint_S d\Omega = 4\pi$$

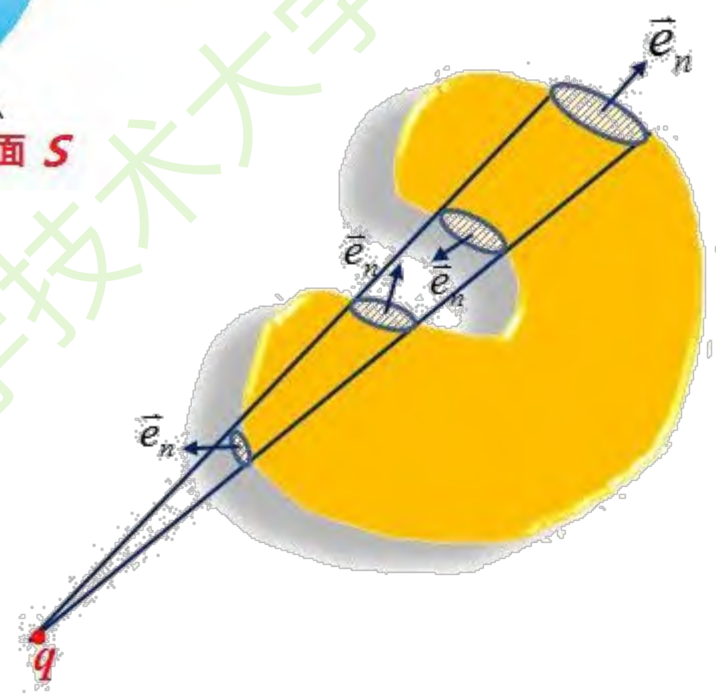
$$\Phi = \frac{q}{4\pi\varepsilon_0}\oiint_S d\Omega = \frac{q}{\varepsilon_0}$$



点电荷在封闭曲面外面



$$\Phi = \frac{q}{4\pi\varepsilon_0}\oiint_S d\Omega = 0$$



高斯定理

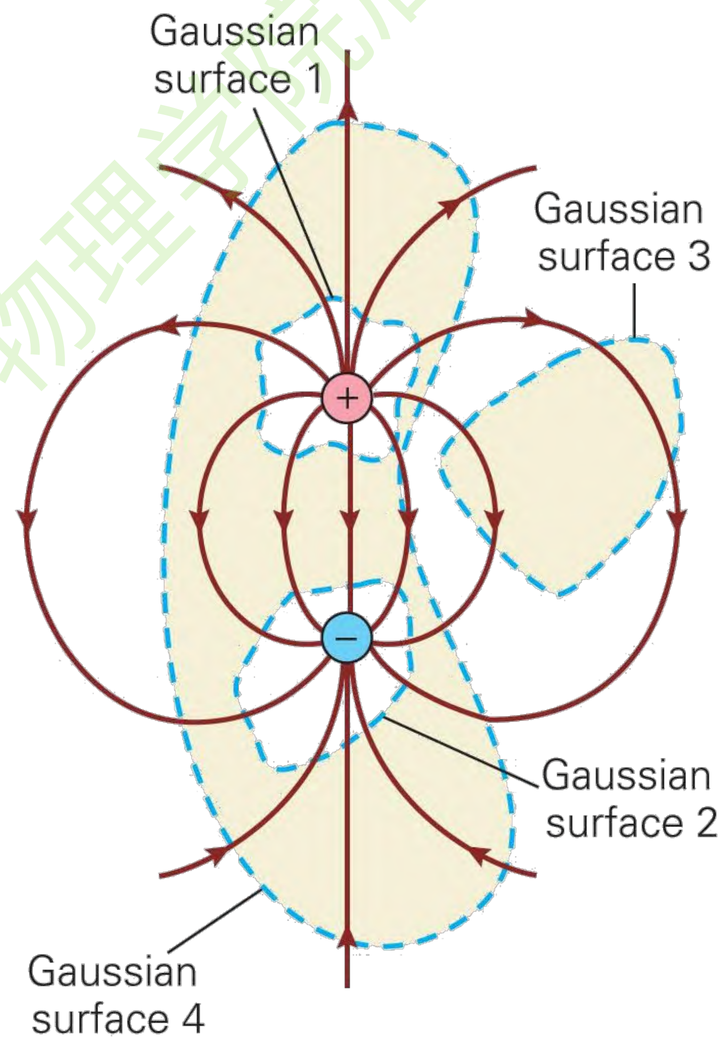
一个点电荷的电通量：

- 当封闭曲面包含点电荷时

$$\Phi = \frac{q}{\varepsilon_0}$$

- 当封闭曲面不包含点电荷时

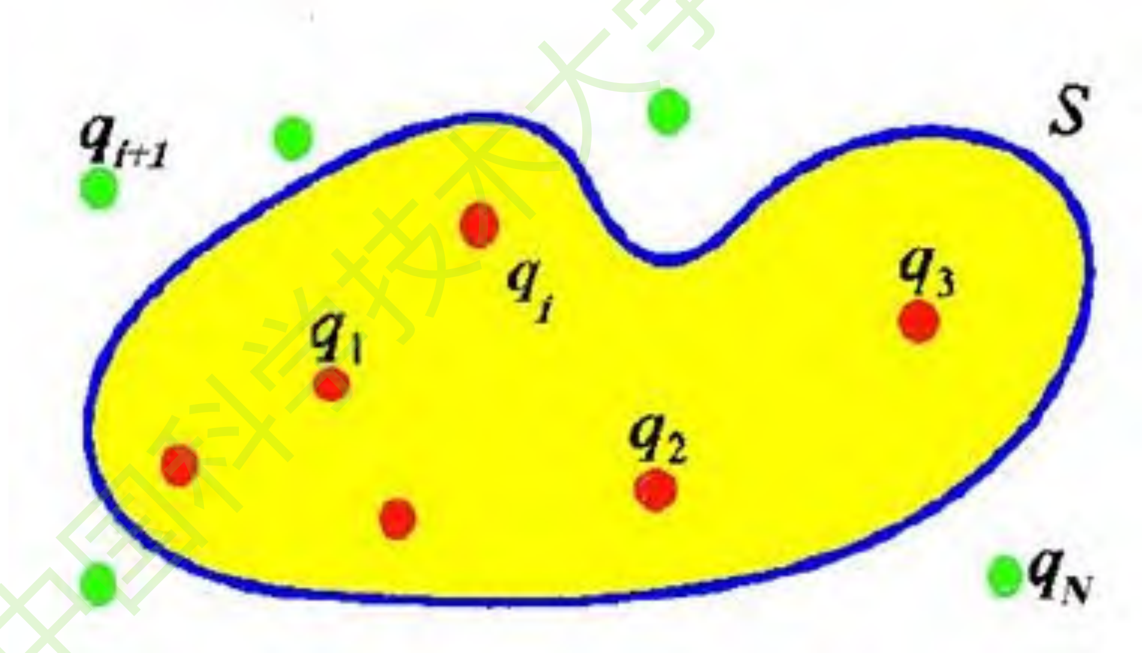
$$\Phi = 0$$



若空间中有一组点电荷$q_1, q_2, q_3, \ldots, q_N$

对任意形状的闭合曲面，根据电通量叠加原理

$$\Phi = \sum_i \Phi_i = \sum_{S_{in}} \Phi_i + \sum_{S_{out}} \Phi_i = \sum_{S_{in}} \frac{q_i}{\varepsilon_0}$$



电场对任意封闭曲面的电通量只决定于被包围在封闭曲面内部的电荷，且等于包围在封闭曲面内电量代数和除以ε_0，与封闭曲面外的电荷无关。

这一结论就是静电场的**高斯定理（Gauss's Law）**。

$$\oint\!\!\!\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\sum_{S_{in}} q_i$$

或

$$\oint\!\!\!\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\iiint_V \rho dV$$

由数学的高斯定理，可得：

$$\oiint_S \vec{E}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{E}\,dV = \frac{1}{\varepsilon_0}\iiint_V \rho dV$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

高斯定理的微分形式

电场强度的散度等于电荷密度除以真空介电常数

高斯定理的讨论

- 高斯定理表明静电场是有源场
  - 高斯定理给出了场和场源的一种联系。
  - 电荷是静电场的源。

$$\oiint_S \vec{E}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{E}\,dV = \frac{1}{\varepsilon_0}\iiint_V \rho\,dV$$

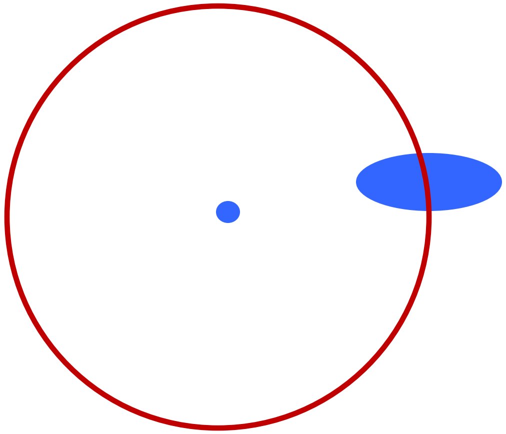
$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

高斯定理的讨论

- 高斯面上的电荷问题

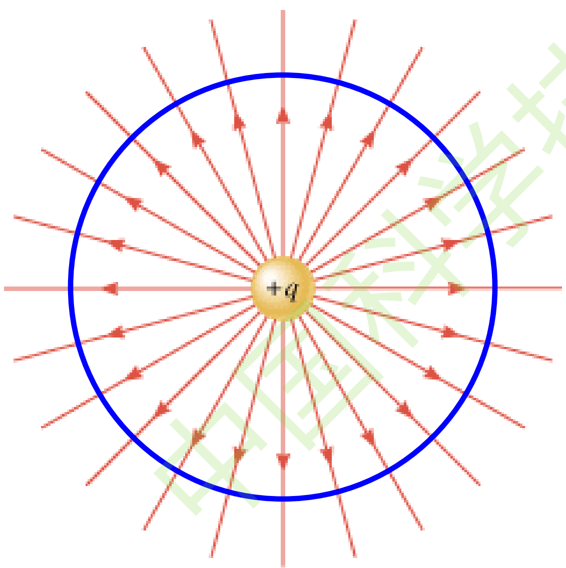
电荷被高斯面分割成内部电荷和外部电荷，那么是否存在恰好在高斯面上的电荷？

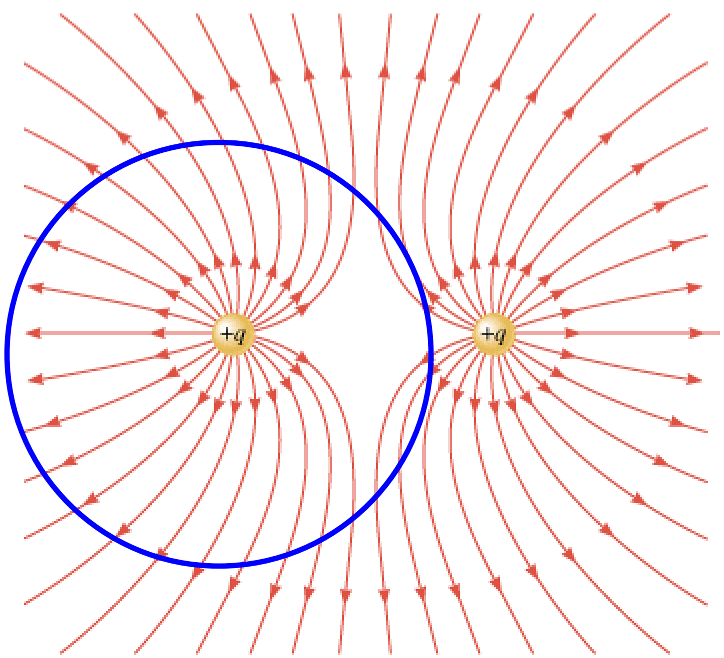
这是不存在的。



高斯定理的讨论

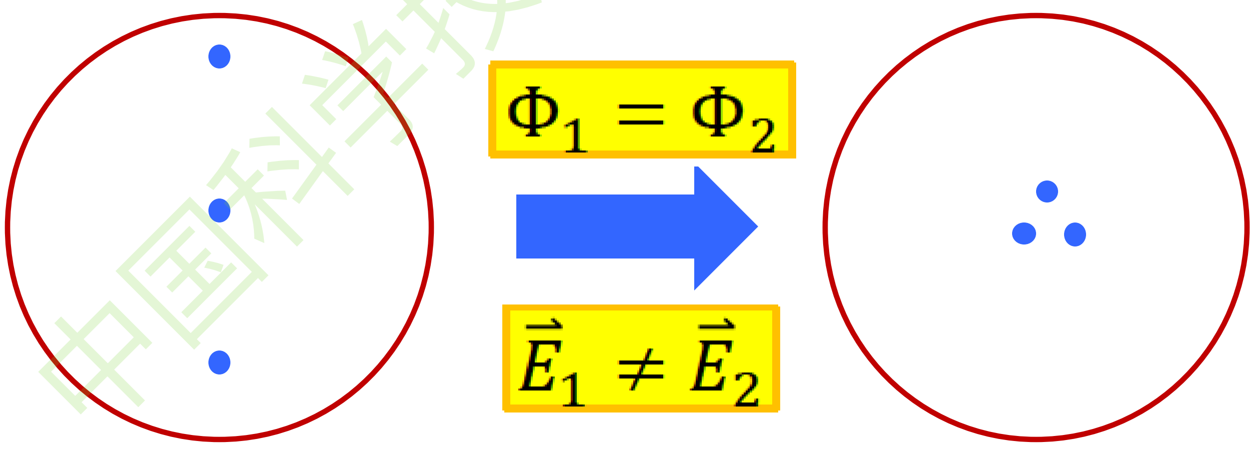
- 高斯面上的电场强度问题
  - 高斯面上的电场强度是空间全部电荷所产生，不仅仅是高斯面内部的电荷
  - 同一高斯面上不同地方的电场强度大小可能相同，也可能不相同





高斯定理的讨论

- 高斯定理只表明了电通量与电荷的关系
  - 高斯定理表明：只要高斯面内总的电荷数不变，高斯面上的电通量不变
  - 但是，如果电荷（无论内部还是外部）分布改变高斯面上的电场强度会发生改变



高斯定理与库仑定律的关系

- 高斯定理来源于库仑定律
  - 主要反映了库仑定律的平方反比定律

$$\Phi = \frac{q}{4\pi\varepsilon_0}\iint_S \frac{\vec{e}_r \cdot d\vec{S}}{r^2} = \frac{q}{4\pi\varepsilon_0}\iint_S d\Omega$$

高斯定理与库仑定律的关系

- 高斯定理比库仑定律更普遍
  - 运动电荷产生的电场
  - 变化的磁场产生的涡旋电场
- 从高斯定理不能导出库仑定律
  - 高斯定理不能反映电场力的有心力特征

高斯面的选取

- 通过感兴趣的点做一个高斯面（封闭的）
- 高斯面要么与电场线平行，要么与电场线垂直且匀强

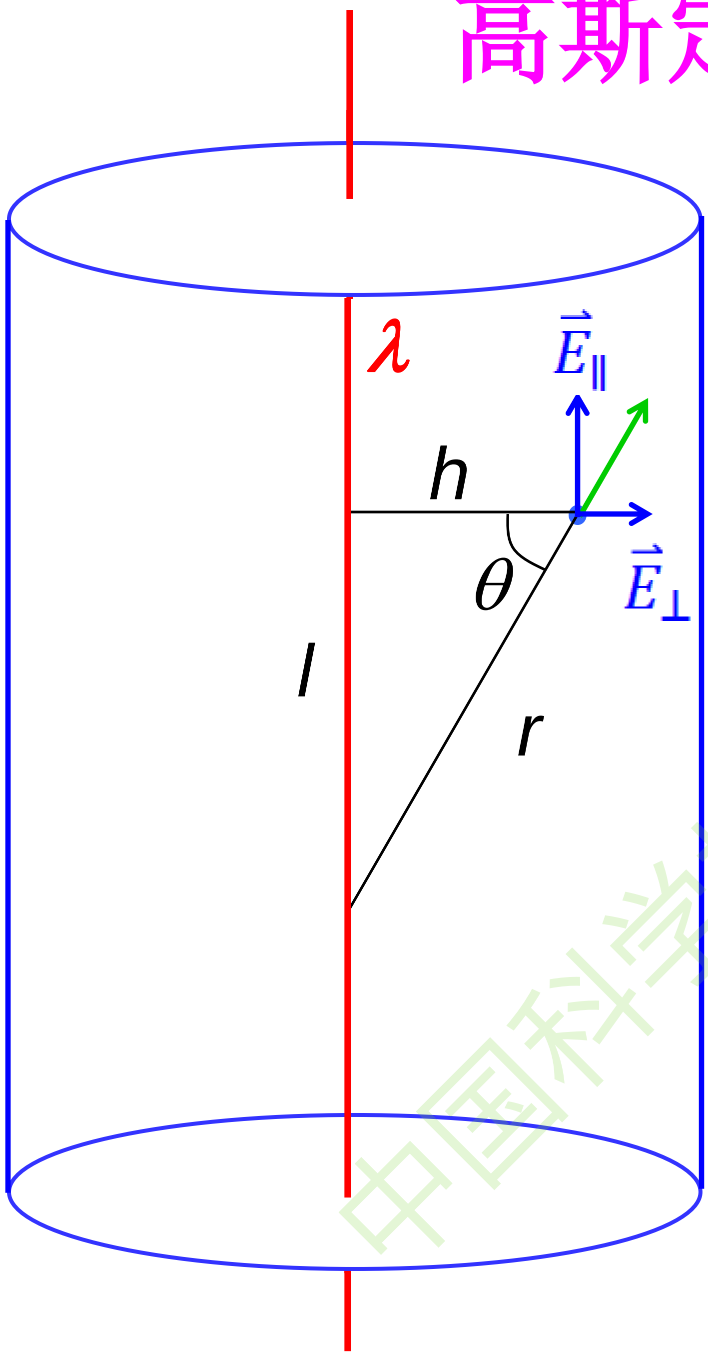
➔ 方便将***E***从积分号里提取出来

$$\oiint_S \vec{E}\cdot d\vec{S}=\frac{1}{\varepsilon_0}\iiint_V \rho dV$$

- 高斯面的对称性往往与电荷分布的对称性相同

➔ 高斯面的形状往往和电荷分布的形状一样，大小按需要任取

高斯定理应用举例(1)



无限长均匀线电荷的电场

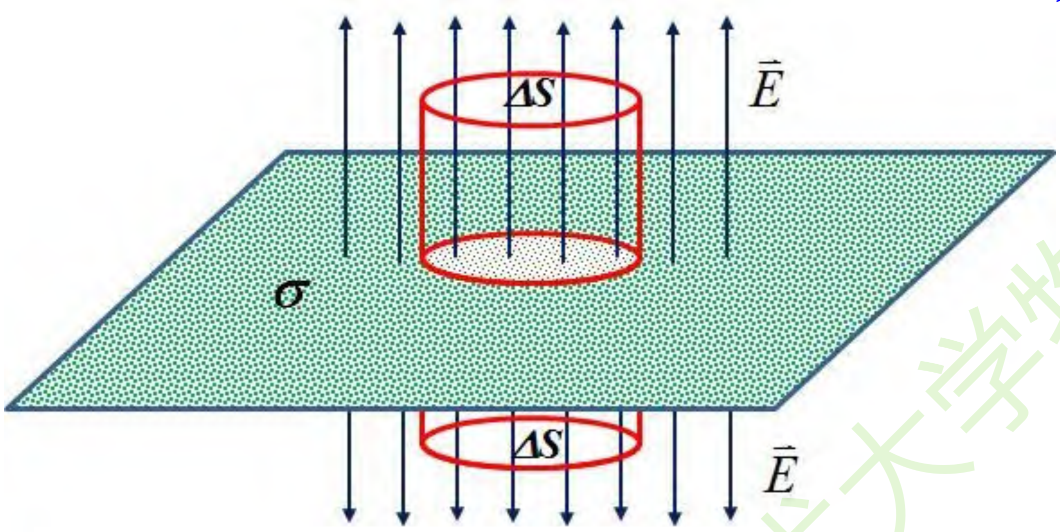
$$\oiint_S \vec{E}\cdot d\vec{S} = E_{\perp}(h)\cdot 2\pi h L$$

$$= \frac{1}{\varepsilon_0}\lambda L$$

$$E_{\perp}(h) = \frac{\lambda}{2\pi\varepsilon_0 h}$$

高斯定理应用举例(2)

无限大面电荷的电场



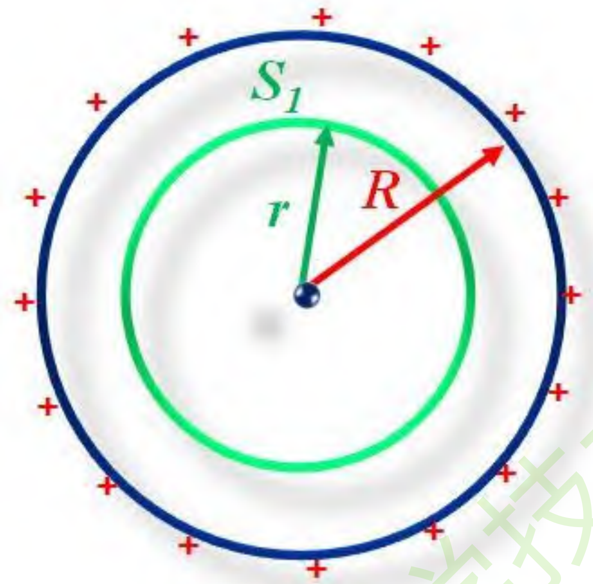
$$\oiint_S \vec{E}\cdot d\vec{S} = 2E\Delta S$$

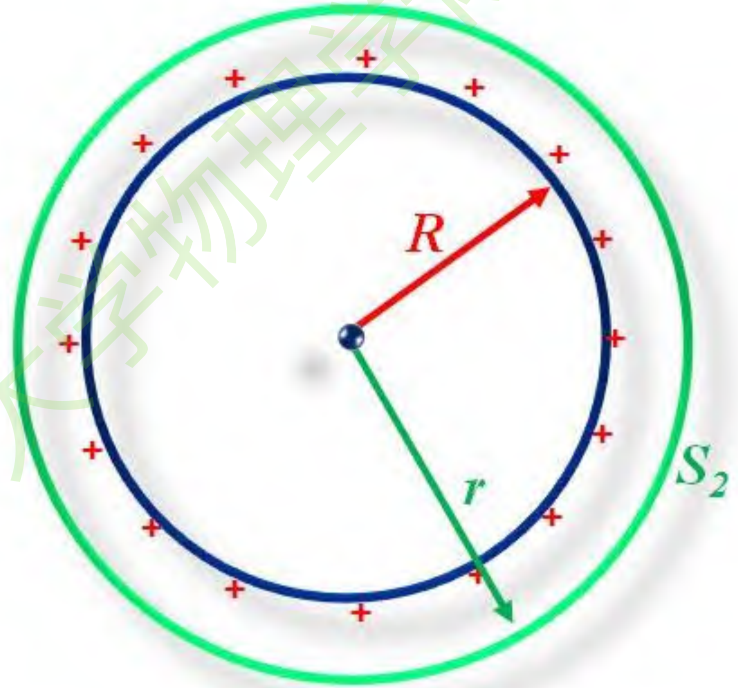
$$E = \frac{\sigma}{2\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0}\sigma\Delta S$$

高斯定理应用举例(3)

均匀带电球面的电场

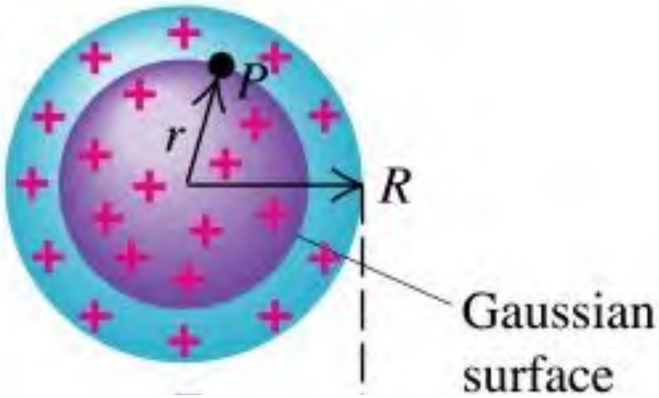




$$E = 0$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

高斯定理应用举例(4)



均匀带电球体的电场

- 球内：

$$4\pi r^2 E = \frac{1}{\varepsilon_0}\frac{r^3}{R^3}Q$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{r}{R^3}Q$$

- 球外：

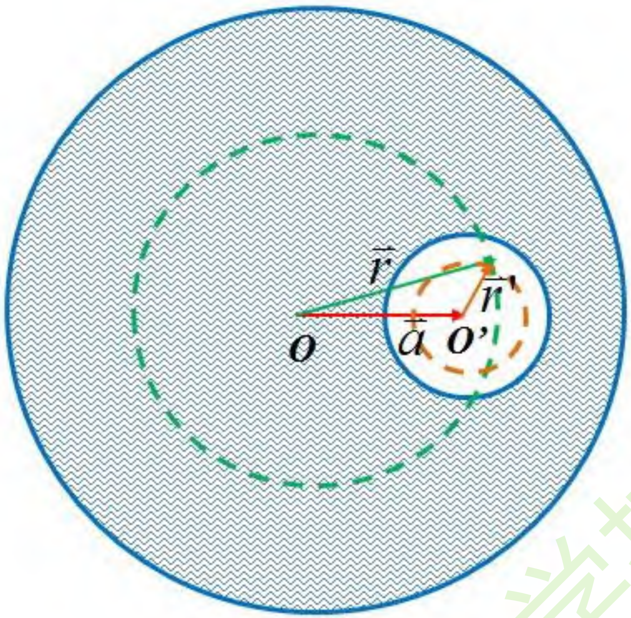
$$4\pi r^2 E = \frac{1}{\varepsilon_0}Q$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}Q$$

高斯定理应用举例(5)

带电球体中球形空腔的电场

- 将空腔看成同时填满密度为+ρ和-ρ的电荷



- 电荷为+ρ的实心大球产生的电场：

$$4\pi r^2 E_+ = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho$$

$$\vec{E}_+ = \frac{\rho}{3\varepsilon_0}\vec{r}$$

- 电荷为-ρ的实心小球产生的电场：

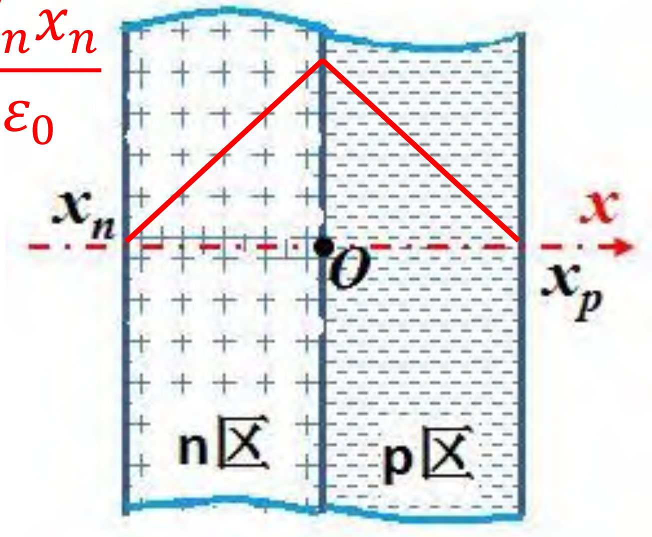
$$\vec{E}_- = \frac{-\rho}{3\varepsilon_0}\vec{r}'$$

- 球形空腔里的电场：

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r}') = \frac{\rho}{3\varepsilon_0}\vec{a}$$ **匀强电场！**

高斯定理应用举例(6)

半导体PN结



n区： $\rho = N_n e$

p区： $\rho = -N_p e$

电中性：$N_n x_n = N_p x_p$

求电场分布

根据对称性，电场强度沿x轴

$$\nabla \cdot \vec{E} = \frac{dE}{dx} = \frac{\rho}{\varepsilon_0}$$

$$\frac{dE}{dx} = \begin{cases} 0 & x < x_n \parallel x > x_p \\ \frac{N_n e}{\varepsilon_0} & x_n < x < 0 \\ -\frac{N_p e}{\varepsilon_0} & 0 < x < x_p \end{cases}$$

$$E = \begin{cases} 0 & x < x_n \parallel x > x_p \\ \frac{N_n e}{\varepsilon_0}(x - x_n) & x_n < x < 0 \\ -\frac{N_p e}{\varepsilon_0}(x - x_p) & 0 < x < x_p \end{cases}$$

作业

- 1.16
- 1.19
- 1.21
- 1.32